Spanning trees of graphs on surfaces and the intensity of loop-erased random walk on $\mathbb{Z}^2$

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Abstract

We show how to compute the probabilities of various connection topologies for uniformly random spanning trees on graphs embedded in surfaces. As an application, we show how to compute the "intensity" of the loop-erased random walk in $\mathbb{Z}^2$, that is, the probability that the walk from $(0,0)$ to ∞ passes through a given vertex or edge. For example, the probability that it passes through $(1,0)$ is $5/16$; this confirms a 15-year old conjecture about the stationary sandpile density on $\mathbb{Z}^2$. We do the analogous computation for the triangular lattice, honeycomb lattice and $\mathbb{Z} \times \mathbb{R}$, for which the probabilities are $5/18$, $13/36$, and $1/4 - 1/\pi^2$ respectively.

Contents

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1 Introduction

1.1 Response matrices and groves

Let $\mathcal{G}$ be a graph and $c\colon \mathcal{G} \to \mathbb{R}_{>0}$ a positive conductance on each edge. Let $\mathcal{N}$ be a subset of its vertices. The triple $(\mathcal{G}, c, \mathcal{N})$ is a **resistor network**. Associated to this data is the Dirichlet-to-Neumann matrix (also called response matrix) L, defined as follows. Given a function $f : \mathcal{N} \to \mathbb{R}$ find its harmonic extension h on $\mathcal{G}$, that is a function on the vertices of $\mathcal{G}$ that is harmonic on $\mathcal{G} \setminus \mathcal{N}$ and has values f on $\mathcal{N}$. Then $L(f) = -\Delta(h)|_{\mathcal{N}}$ is a linear function of f, where Δ is the (positive semidefinite) graph Laplacian. In electrical terms, $L(f)$ gives the current flow into the nodes $\mathcal{N}$ when they are held at f volts. While it is not obvious from this definition, L is a symmetric negative semidefinite matrix.

Planar resistor networks where $\mathcal{N}$ is a subset of the outer face were called **circular planar networks** in [CIM98], and were studied in [CdV94, CdVGV96, CIM98]. These authors classified which matrices occur as response matrices: they are precisely the matrices whose "non-interlaced" minors are nonnegative. (A non-interlaced minor is one in which there are no 4 indices $a < b < c < d$ for which a and c are rows and b and d are columns or vice versa.) Furthermore they showed how to construct a graph having a given matrix L.

On a circular planar graph a **grove** is a spanning forest (set of edges with no cycles) in which every component contains at least one vertex in $\mathcal{N}$. In [KW11a] we studied the natural probability measure on groves (where each grove occurs with probability proportional to the product of its edge weights), showing how to compute the probability that a random grove has a given connection topology in terms of the entries in L.

1.2 Graphs on surfaces

We study here the same problem for a graph $\mathcal{G}$ embedded on a surface Σ. Here the usual notion of response matrix is not rich enough to extract information about the underlying topological structure of a grove. Given a resistor network on a surface Σ, a natural generalization of the response matrix is a matrix-valued function $\mathscr{L}$ on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)$ of flat H-connections on $\mathcal{G}$; here $H = \mathbb{C}^*$ or $\mathrm{SL}_2(\mathbb{C})$. We show here how $\mathscr{L}$ can be used to compute connection probabilities of (certain types of) groves on $\mathcal{G}$.

The question of characterizing which matrices $\mathscr{L}$ occur as a function of the topology of Σ remains open. See Lam and Pylyavskyy [LP11a] for related work in the case when the surface Σ is an annulus.

We give special attention to the case where the surface Σ is an annulus; this is the easiest case beyond the planar one (but already quite involved) and also has applications to the study of spanning trees on planar graphs.

1.3 Applications to planar graphs

Using these techniques one can in principle compute the probability that the path of the uniform spanning tree from a to b in a planar graph passes through a given set of edges or vertices (as in Figure 1). We carry out this computation for $\mathbb{Z}^2$ for a single edge or vertex (see Figure 2); our method shows that the answer is in $\mathbb{Q}(\pi)$; we conjecture it to in fact be in $\mathbb{Q}[\frac{1}{\pi}]$. For the triangular and hexagonal lattices, we conjecture these probabilities to be in $\mathbb{Q}[\frac{\sqrt{3}}{\pi}]$ (see Figure 13 and Figure 15).

For example, we show that the probability that the loop-erased walk in $\mathbb{Z}^2$ from $(0,0)$ to ∞ contains the point $(1,0)$ is $5/16$. (See Figure 1.) This number was predicted by Levine and Peres [LP11b] and by Poghosyan and Priezzhev [PP11], by relating this probability to the average stationary density of the abelian sandpile model.

The connection between the spanning trees and the abelian sandpile model was discovered by Majumdar and Dhar [MD92], and Priezzhev [Pri94] used this connection to compute the height distribution of the abelian sandpile model, in terms of two integrals that could not be evaluated in closed form. Grassberger evaluated these integrals numerically, and conjectured that the stationary density of the sandpile on $\mathbb{Z}^2$ is $17/8$. Later Jeng, Piroux, and Ruelle [JPR06] showed how to express one of these two integrals in terms of the other, and determined the sandpile height distribution in closed form, under the assumption that the remaining integral, which numerically is 0.5 ± 10^{-12}, is exactly $1/2$. Our derivation of this probability that LERW passes through $(1,0)$ confirms these conjectures (although our methods are different), and shows that this aforementioned integral is exactly $1/2$.

Similar computations compute this probability on the triangular grid and honeycomb grid; the probabilities are $5/18$ and $13/36$, respectively. We do not know why these are rational numbers. On $\mathbb{Z} \times \mathbb{R}$ (with exponential horizontal jump rates) the probability is $1/4 - 1/\pi^2$.

We can also compute these probabilities for other vertices. For example, in the square lattice, the probability that the LERW from $(0,0)$ to ∞ passes through $(1,1)$ is $1/4 - 1/(4\pi) + 1/(2\pi^2)$, and the probability of that it passes through $(2,0)$ is $1/8 + 1/(4\pi) + 1/(4\pi^2) - 3/(2\pi^3) + 1/(2\pi^4)$ (see Figure 2).

While we were writing up our results, Poghosyan, Priezzhev, and Ruelle independently found another proof that the probability of visiting $(1,0)$ is $5/16$ [PPR11]. They also asked about the probability about visiting other points, and remarked that the probability that the LERW visits $(1,1)$ is numerically close to $2/9$. (This differs from the true value by about 10^{-3}.)

There are some intersting coincidences in the (undirected) edge intensities of loop-erased random walk. For each of the square, triangular, and hexagonal lattices, there are several groups of edges which are unrelated by any symmetry of the lattice for which the undirected edge intensities are identical (see Figures 11, 13, and 15). We do not have an explanation for this phenomenon.

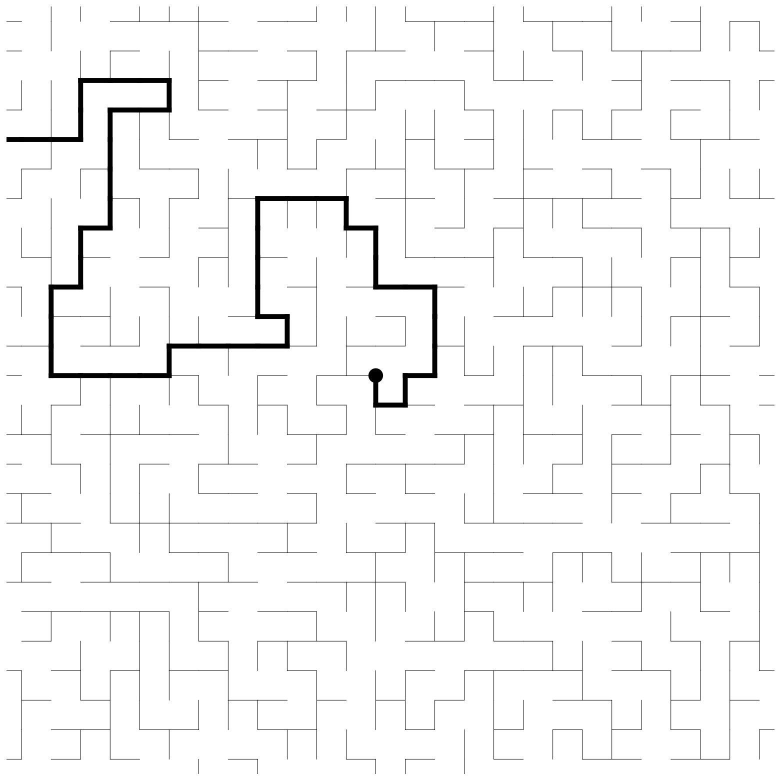


Figure 1: A portion of the uniform spanning tree on $\mathbb{Z}^2$ (actually a portion of the UST on a very large grid), with the path from $(0,0)$ to ∞ shown in bold. The uniform spanning tree on $\mathbb{Z}^2$ can be constructed as a weak limit of uniform spanning trees on large boxes. It is known that the limiting measure exists, is unique, and is supported on trees of $\mathbb{Z}^2$ [Pem91]. (This is in contrast to e.g., $\mathbb{Z}^5$, where the limiting measure is supported on forests with infinitely many trees [Pem91].) Almost surely, within the uniform spanning tree of $\mathbb{Z}^2$, each vertex has a unique infinite path starting from it [Pem91] (see also [BLPS01, LP11c] for modern proofs). The path to infinity (or in the case of a finite graph, the path to a boundary vertex) is a loop-erased random walk (LERW) [Pem91] (see also [Wil96]). In this example, the LERW from vertex $(0,0)$ to ∞ passed through the vertex $(1,0)$. The probability that this happens is the LERW intensity at $(1,0)$, which is $5/16$.

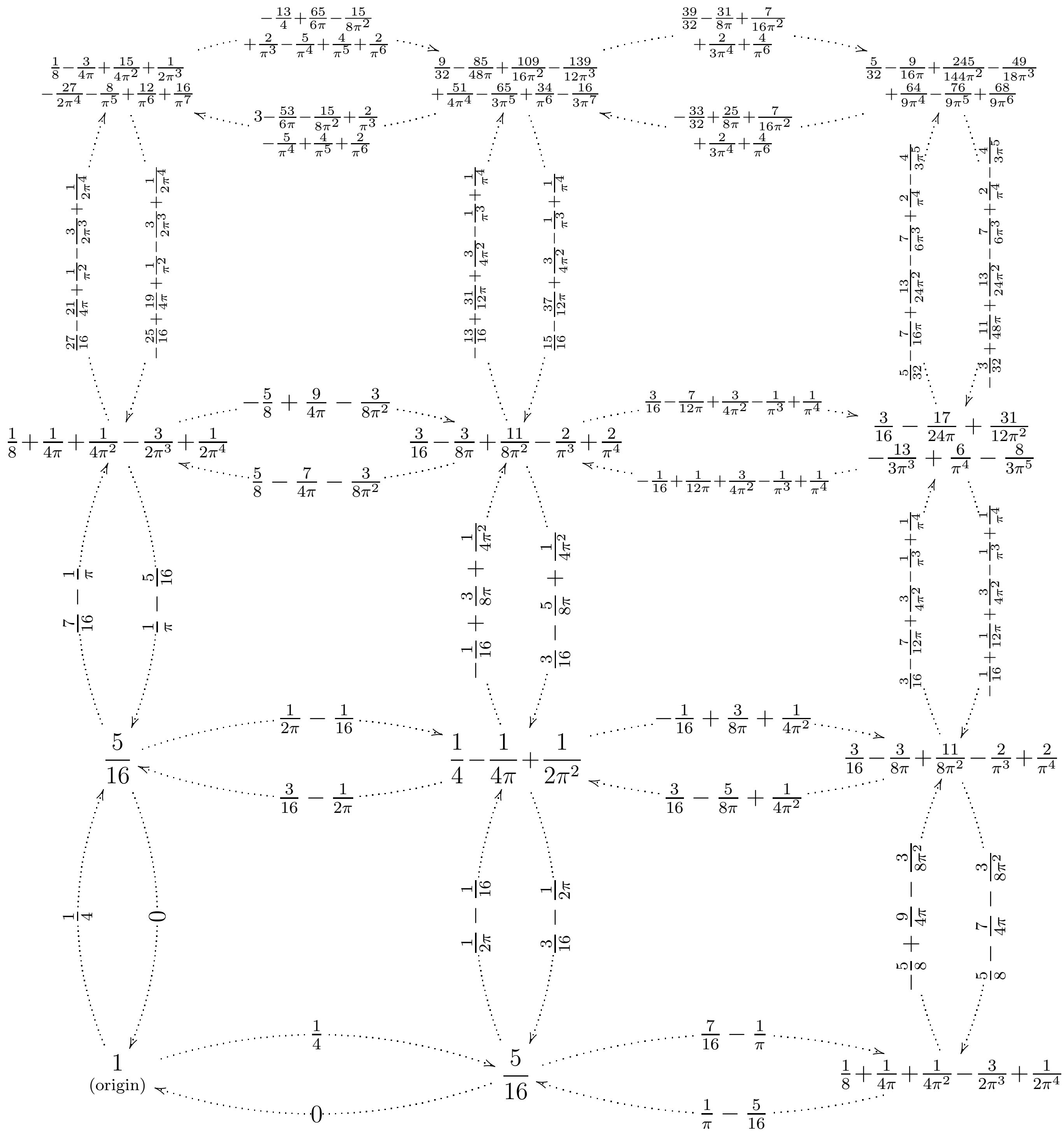


Figure 2: Intensity of loop-erased random walk on $\mathbb{Z}^2$. The origin is at the lower-left, and directed edge-intensities as well as vertex-intensities of the LERW are shown. (See also Figure 11.)

2 Bundles and connections

Let $\mathcal{G}$ be a graph. Given a fixed vector space V, a V**-bundle**, or simply a **vector bundle** B on $\mathcal{G}$ is the choice of a vector space V_v isomorphic to V for every vertex v of $\mathcal{G}$. A vector bundle can be identified with the vector space $V_{\mathcal{G}} = \oplus_v V_v \cong V^{|\mathcal{G}|}$. A **section** of a vector bundle B is an element of $V_{\mathcal{G}}$.

If H is a subgroup of $\mathrm{Aut}(V)$, an H**-connection** Φ is the choice for each directed edge $e = (v, w)$ of $\mathcal{G}$ of an isomorphism $\phi_{v,w} \in H$ between the corresponding vector spaces $\phi_{v,w} : V_v \to V_w$, with the property that $\phi_{v,w} = \phi_{w,v}^{-1}$. This isomorphism is called the **parallel transport** of vectors in V_v to vectors in V_w. Given an oriented cycle γ in $\mathcal{G}$ starting at v, the **monodromy** of the connection is the element of $\mathrm{Aut}(V_v)$ which is the product of these isomorphisms around γ. Monodromies starting at different vertices on γ are conjugate.

Two connections $\Phi = \{\phi_e\}$ and $\Phi' = \{\phi'_e\}$ are **gauge equivalent** if there are maps $\psi_v : V_v \to V_v$ such that $\phi_{v,w} \circ \psi_v = \psi_w \circ \phi'_{v,w}$ for all vertices v and w of $\mathcal{G}$. A connection is **trivializable** if it is gauge equivalent to a trivial bundle (in which all $\phi_{v,w}$ are the identity).

It is useful to extend the notion of bundle and connection to the edges as well: define for each edge e of $\mathcal{G}$ a copy V_e of V, and define maps $\phi_{v,e} : V_v \to V_e$ whenever v is an endpoint of e, with the property that $\phi_{e,w} \circ \phi_{v,e} = \phi_{v,w}$ whenever edge e joins vertices v and w.

A **line bundle** is a V-bundle where $V \cong \mathbb{C}$, the 1-dimensional complex vector space. In this case if we choose a basis for each $\mathbb{C}$ then the parallel transport is just multiplication by an element of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Furthermore, the monodromy of a cycle is in $\mathbb{C}^*$ and does not depend on the vertex of the cycle.

In this paper we will take $V = \mathbb{C}^1$ or $\mathbb{C}^2$, and use $H = \mathbb{C}^*$- or $H = \mathrm{SL}_2(\mathbb{C})$-connections. In some applications it is more appropriate to use $\mathrm{U}(1) \subset \mathbb{C}^*$ and $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$ as structure groups.

2.1 Laplacian Δ

Let $c\colon E \to \mathbb{R}_{>0}$ be a conductance associated to each edge. We then define $\Delta : V_{\mathcal{G}} \to V_{\mathcal{G}}$ acting on sections by the formula

$$\Delta f(v) = \sum_{w:(v,w)\in E} c_{v,w}(f(v) - \phi_{w,v} f(w)). \tag{1}$$

A section is said to be **harmonic** if it is in the kernel of Δ.

We define an operator d from sections of the bundle over vertices to sections over the edges, for an oriented edge $e = (v, w)$, by

$$df(e) = c_{v,w}(\phi_{v,e} f(v) - \phi_{w,e} f(w)),$$

and its "adjoint"

$$d^*\omega(v) = \sum_{e\sim v} \phi_{e,v}\omega(e).$$

Then the Laplacian can be written $\Delta = d^*d$ [Ken10].

A **cycle-rooted spanning forest (CRSF)** in a graph is a set of edges each of whose components contains a unique cycle, that is, has as many vertices as edges. A component of a CRSF is called a **cycle-rooted tree (CRT)**.

Theorem 2.1 ([For93, Ken10]). *For a $\mathbb{C}^*$-connection,*

$$\det \Delta = \sum_{CRSFs} \prod_{edges\ e} c(e) \prod_{cycles} \left(2 - w - \frac{1}{w}\right),$$

where the sum is over cycle-rooted spanning forests, where the first product is over all edges of the CRSF and the second is over cycles of the CRSF, and w is the monodromy of the cycle defined above.

For a U(1)-connection, Δ is Hermitian and positive semidefinite [Ken10]. The monodromy of a loop is in U(1) and so $2 - w - 1/w \geq 0$ and thus we can define a probability measure on CRSFs where a CRSF has a probability proportional to $\prod_e c(e) \prod(2 - w - \frac{1}{w})$.

A similar result holds for a $\mathrm{SL}_2(\mathbb{C})$-connection. Now Δ is a quaternion-Hermitian matrix, that is, a matrix with entries in $\mathrm{GL}_2(\mathbb{C})$ which satisfies $\Delta_{i,j} = \Delta^*_{j,i}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Its q-determinant counts CRSFs:

Theorem 2.2 ([Ken10]). *For an $\mathrm{SL}_2(\mathbb{C})$-connection,*

$$\operatorname{qdet} \Delta = \sum_{CRSFs} \prod_{edges\ e} c(e) \prod_{cycles} (2 - \operatorname{Tr} w),$$

where the sum is over cycle-rooted spanning forests, where the first product is over all edges of the CRSF and the second is over cycles of the CRSF, and w is the monodromy of the cycle.

For information on q-determinants, see [Dys70]; for the purposes of this paper one can define $\operatorname{qdet} M = \sqrt{\det \widetilde{M}}$, where if M is an $N \times N$ matrix with entries in $\mathrm{GL}_2(\mathbb{C})$ then $\widetilde{M}$ is the $2N \times 2N$ matrix with $\mathbb{C}$ entries obtained by replacing each entry of M by its 2×2 block of complex numbers. In the cases of primary interest M is a quaternion-Hermitian, or "self-dual", matrix; for self-dual matrices qdet is a polynomial in the matrix entries. Matrices with $\mathrm{GL}_2(\mathbb{C})$ entries enjoy many of the properties of usual matrices: for example, multiplication and addition work the same way. The inverse of a self-dual matrix is well-defined and is both a left- and right-inverse, see e.g., [Dys70].

For an SU(2)-connection, $2 - \operatorname{Tr} w \geq 0$ and this allows us to define a measure on CRSFs as above.

2.2 Dirichlet boundary conditions

If $B \subset \mathcal{G}$ is a set of vertices, the Laplacian with Dirichlet boundary conditions at B is defined on sections over $\mathcal{G} \setminus B$ by the same formula (1) above with $v \in \mathcal{G} \setminus B$ and the sum over all $\mathcal{G}$. In other words Δ is just a submatrix of the usual Laplacian on $\mathcal{G}$. Its determinant also has an interpretation. A CRSF on a graph with boundary B is a set of edges such that each component is either a CRT not containing any vertex of B or a tree containing a single vertex of B. In this setting Theorems 2.1 and 2.2 have the same statements (where tree components do not have any monodromy term). See [Ken10].

2.3 Green's function $\mathscr{G}$

The Green's function for the standard Laplacian (with Dirichlet boundary conditions) is the inverse of the Laplacian. It has the probabilistic interpretation that $G_{p,q}$ is $(\sum_r c_{q,r})^{-1}$ times the expected number of visits to q of a simple random walk started at p (and stopped at the boundary); equivalently, it is $(\sum_r c_{q,r})^{-1}$ times the sum over all paths from p to q which do not hit the boundary, of the probability of the path.

In the case of a graph with connection, the Green's function $\mathscr{G}$ is again the inverse of the Laplacian, and has a similar probabilistic interpretation:

Lemma 2.3. *$\mathscr{G}_{p,q}$ is $(\sum_r c_{q,r})^{-1}$ times the sum over all from p to q of the product of the parallel transports along the path (from q to p) times the path probability (from p to q). This will be matrix-valued for an $\mathrm{SL}_2(\mathbb{C})$-bundle.*

Proof. Using the above definition of $\mathscr{G}$ as a sum over paths,

$$\sum_r \mathscr{G}_{p,r}\Delta_{r,q} = \mathscr{G}_{p,q}\sum_{r\sim q} c_{q,r} - \sum_{r\sim q}\mathscr{G}_{p,r}c_{r,q}\phi_{q,r}$$

and since any nontrivial path to q must have last step from a neighbor of q, this equals zero unless $p = q$ and the path has length 0, in which case the second sum is zero and the first term is $(\sum_r c_{q,r})^{-1}\sum_r c_{r,q} = 1$. ☐

2.4 Response matrix $\mathscr{L}$

Let $\mathcal{N}$ be a nonempty set of nodes of $\mathcal{G}$, and $n = |\mathcal{N}|$. For each node v pick a preferred basis for V_v, the vector space over v.

We define an $n\times n$ matrix $\mathscr{L} = \mathscr{L}_\Phi$ (with entries in H), the response matrix, or Dirichlet-to-Neumann matrix, from this data: $\mathscr{L}\colon V^{\mathcal{N}} \to V^{\mathcal{N}}$ is (minus) the Schur reduction of the Laplacian Δ to $\mathcal{N}$. That is, $\mathscr{L}$ is defined as follows. Order the vertices so that $\mathcal{N}$ comes first. In this ordering the Laplacian is

$$\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

Then $\mathscr{L} = -A + BC^{-1}B^*$. Note that C is the Laplacian with Dirichlet boundary conditions at $\mathcal{N}$, and since $\mathcal{G}$ is connected, C is (for generic connections) nonsingular.

From the viewpoint of harmonic functions, $\mathscr{L}$ is the Dirichlet-to-Neumann matrix: given $f \in V^{\mathcal{N}}$, find the unique section h with boundary values f at the nodes and harmonic at the interior (non-node) vertices. Then $\mathscr{L}f$ is $-\Delta h$ evaluated at the nodes. To see this, let h_1 be h at the interior vertices, that is, $h = \begin{pmatrix} f \\ h_1 \end{pmatrix}$. If

$$\Delta h = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} f \\ h_1 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix},$$

then $B^*f + Ch_1 = 0$, i.e., $h_1 = -C^{-1}B^*f$, and then $c = Af - BC^{-1}B^*f = -\mathscr{L}f$.

The response matrix $\mathscr{L}$ has entries which are functions of the parallel transports. See Theorem 4.2 below for an explicit probabilistic interpretation of the entries of $\mathscr{L}$. In order to define $\mathscr{L}$ as a matrix one must choose a basis in V_v for each node v. Base changes in the V_v then act on $\mathscr{L}$ by conjugation by diagonal matrices with entries in H.

Lemma 2.4. *When the Laplacian Δ is nonsingular, the response matrix $\mathscr{L}$ is given by*

$$\mathscr{L} = -(\Delta^{-1}|_{\mathcal{N}})^{-1}.$$

Proof. Write $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ where A is the submatrix indexed by the nodes. Submatrix C is invertible since it is the Laplacian with Dirichlet boundary conditions. We have

$$\Delta = \begin{bmatrix} A - BC^{-1}B^* & BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B^* & C \end{bmatrix}$$

and using $\mathscr{L} = -A + BC^{-1}B^*$,

$$\Delta^{-1} = \begin{bmatrix} I & 0 \\ -C^{-1}B^* & C^{-1} \end{bmatrix} \begin{bmatrix} -\mathscr{L}^{-1} & \mathscr{L}^{-1}BC^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} -\mathscr{L}^{-1} & * \\ * & * \end{bmatrix}. \qquad \square$$

If $\mathscr{G}$ is the Green's function with boundary at node n, then $\mathscr{G} = \tilde{\Delta}^{-1}$, where the Dirichlet Laplacian $\tilde{\Delta}$ is obtained from Δ simply by removing row and column n. Since $\mathscr{L}$ has the same response as Δ on the set $\mathcal{N}$, the response matrix of $\tilde{\Delta}$ is just $\mathscr{L}$ with row and column n removed. Thus $[\mathscr{L}_{i,j}]_{i=1,\dots,n-1}^{j=1,\dots,n-1} = -\tilde{\Delta}^{-1}|_{\mathcal{N}\setminus\{n\}}$, or equivalently,

$$[\mathscr{L}_{i,j}]_{i=1,\dots,n-1}^{j=1,\dots,n-1} = -\left([\mathscr{G}_{i,j}]_{i=1,\dots,n-1}^{j=1,\dots,n-1}\right)^{-1}. \tag{2}$$

By perturbing Δ, we see that (2) holds even if the Laplacian Δ is singular, so long as the Dirichlet Laplacian is nonsingular.

3 Graphs on surfaces

Let Σ be an oriented surface, possibly with boundary, and $\mathcal{G}$ a graph embedded on Σ in such a way that complementary components (the connected components of the surface after it is cut along the edges of $\mathcal{G}$) are contractible or peripheral annuli (that is, an annular neighborhood of a boundary component). We call the pair $(\mathcal{G}, \Sigma)$ a **surface graph**.

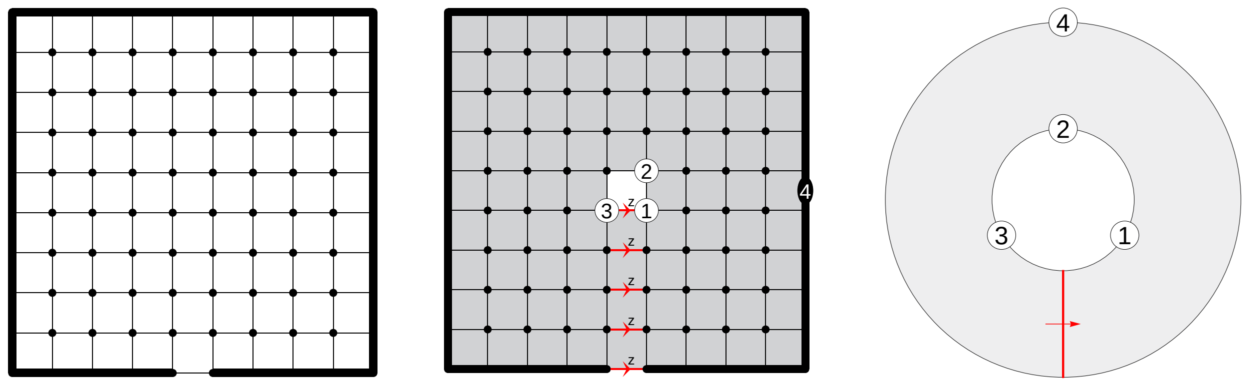


Figure 3: On the left is a graph $\mathcal{G}$, which is the square grid restricted to a finite box with wired boundary conditions: the outer boundary is one vertex (and the bottom edge is a self-loop). This graph will be relevant to our later loop-erased random walk computations. The middle diagram shows $\mathcal{G}$ embedded in a surface Σ, which in this case is an annulus whose inner boundary is one of the squares of the grid. There is a "zipper" (edges crossing a dual path) connecting the inner boundary to the outer boundary of the annulus, and edges crossing the zipper have parallel transport z from their left endpoint to their right endpoint. We have labeled four of the vertices on the boundary of Σ, which we call nodes. On the right is a schematic diagram of the surface graph which shows the surface Σ, the zipper, and the nodes of $\mathcal{G}$, but not the internal vertices and edges of $\mathcal{G}$.

3.1 Nodes and interior vertices

For each boundary component of the surface Σ there is a "peripheral" cycle on $\mathcal{G}$, consisting of those vertices and edges bounding the same complementary component as the boundary component. We select from this cycle a (possibly empty) set of vertices. The union of these special vertices over all boundary components will be the nodes $\mathcal{N}$; the non-node vertices are **interior vertices** (even though these may be on the boundary of Σ).

Planar maps, in which Σ is topological disk, are examples of surface graphs: these are called **circular planar graphs** in [CIM98]. In this case the nodes are a subset of the vertices on the outer face.

3.2 Flat bundles

Given a surface graph $(\mathcal{G}, \Sigma)$, a vector bundle on $\mathcal{G}$ with connection Φ is **flat** if it has trivial monodromy around any loop which is contractible on Σ. In this case, the monodromy around a loop only depends on the homotopy class of the (pointed) loop in $\pi_1(\Sigma)$, and so the monodromy determines a representation of $\pi_1(\Sigma)$ into $\mathrm{Aut}(V_p)$, where p is the base point. This representation depends on the base point p for π_1; choosing a different base point will conjugate the representation.

Conversely, let $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Aut}(V))$ be a representation of $\pi_1(\Sigma)$ into $\mathrm{Aut}(V)$; there is a unique (up to gauge equivalence) flat bundle with monodromy ρ. It is easy to construct: for example start with a trivial bundle on a spanning tree of $\mathcal{G}$; for each additional edge the parallel transport along it is determined by the topological type of the resulting cycle created.

In the case of a line bundle, $\mathrm{Aut}(\mathbb{C}) = \mathbb{C}^*$ is abelian and the monodromy of a loop is well defined without regard to base point. Moreover in this case $\rho(\gamma)$ only depends on the homology class of the loop γ, since any map from $\pi_1(\Sigma)$ to an abelian group factors through $H_1(\Sigma)$.

4 The response matrix and probabilities

4.1 Circular planar graphs

In the case of a planar graph, there is no monodromy and $\mathscr{L} = L$ is a matrix of real numbers. This case was analyzed by [CdV94], see also [CdVGV96, CIM98]. Colin de Verdière showed that response matrices L of planar graphs are characterized by having nonnegative "non-interlaced" minors. Given two disjoint subsets of nodes R and S, we say that R and S are non-interlaced if R and S are contained in disjoint intervals in the circular order on the nodes. When $|R| = |S|$, the corresponding minor is $\det(L_R^S) \geq 0$ (the determinant of the submatrix whose rows are indexed by R and columns by S).

In [KW11a, Proposition 2.8] (see also [CIM98, Lemma 4.1]), there is an interpretation of the entries of L in terms of groves. A **grove** is a spanning forest with the property that every component contains at least one node. The weight of a grove is the product of the conductances of its edges.

Theorem 4.1 ([CIM98, KW11a]). *When $i \neq j$, $L_{i,j}$ is a ratio of two terms: the numerator is the weighted sum of groves in which nodes i, j are in the same component and every other node is in a component by itself; the denominator is the weighted sum of groves in which every node is in its own component.*

The diagonal terms have a slightly different interpretation: $L_{i,i}$ has the same denominator as the others; but its numerator is minus the weighted sum of components in which nodes *different from i are in separate components, and i is in one of these components.*

More generally, for disjoint non-interlaced subsets $R, S \subset \mathcal{N}$ with $|R| = |S|$, $\det(L_R^S)$ is a ratio of two terms: the denominator is the weighted sum of groves in which every node is in its own component, and the numerator is the weighted sum of groves in which the nodes in R are connected pairwise with nodes in S, and other nodes are in their own component.

In particular the last statement proves that the non-interlaced minors of L are nonnegative.

Groves can be grouped into subsets according to the way they partition the nodes (that is, the way the nodes are connected in a grove). For example, a grove of type $1, 2 \mid 3, 4, 5 \mid 6$ is one in which nodes 1 and 2 are in a tree, nodes $3, 4, 5$ are in a second tree, and node 6 is in its own tree. For a partition σ of the nodes, we let $Z[\sigma]$ denote the weighted sum of groves of type σ. For circular planar graphs with n nodes on the boundary, we previously showed [KW11a] how to compute the ratio $Z[\sigma]/Z[1|2|\cdots|n]$ for any planar partition σ of $\{1, 2, \ldots, n\}$. This ratio is an integer-coefficient polynomial in the $L_{i,j}$ [KW11a].

4.2 $\mathscr{L}$ matrix entries

Like in Theorem 4.1, in the case of a flat bundle on a surface graph there is a combinatorial interpretation of the entries of $\mathscr{L}$. A collection of edges of a surface graph $(\mathcal{G}, \Sigma)$ is a **cycle-rooted grove (CRG)** if each component is either a CRT (a component containing one cycle) not containing a node, or a tree containing at least one node. Moreover for each CRT component, the cycle must be topologically nontrivial. A CRG is distinguished from a CRSF by the fact that in a CRG the tree components may contain several nodes, while in a CRSF the tree components contain a unique node.

A CRG has a weight which is the product of its edge conductances times the product over its cycles of $2 - w - 1/w$ (for a line bundle) or $2 - \mathrm{Tr}(w)$ (for a $\mathrm{SL}_2(\mathbb{C})$-bundle), where w is the monodromy around the cycle. For a partition σ of the nodes, we define

$$\mathscr{Z}[\sigma] := \text{weighted sum of cycle-rooted groves of type } \sigma$$
$$\mathscr{Z} := \text{weighted sum of cycle-rooted groves in which all nodes are connected}$$

For example, the weighted sum of CRSFs is $\mathscr{Z}[1|2|\cdots|n]$. Suppose the partition σ is a partial pairing, i.e., σ consists of doubleton and singleton parts, say $\sigma = r_1, s_1 \mid \cdots \mid r_k, s_k \mid t_1 \mid \cdots \mid t_\ell$. We can define

$$\mathscr{Z}[{}^{s_1}_{r_1}|\cdots|{}^{s_k}_{r_k}|t_1|\cdots|t_\ell] := \mathscr{Z}[\sigma] \times \prod_{i=1}^{k} \text{parallel transport to } r_i \text{ from } s_i$$

for line bundles (so that the structure group is commutative and the above product makes sense), and for vector bundles when σ has only one doubleton part.

Theorem 4.2. *If $i \neq j$, then*

$$\mathscr{L}_{i,j} = \frac{\mathscr{Z}\left[{}^{j}_{i}|\textit{(nodes other than } i \textit{ and } j \textit{ in singleton parts)}\right]}{\mathscr{Z}[1|2|\cdots|n]}.$$

Proof. Let us first do the line bundle case. Let $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ be the Laplacian of $\mathcal{G}$, with A indexed by the nodes $\mathcal{N}$.

We make a new graph $\tilde{\mathcal{G}}$ by adding an edge $e_{i,j}$ to $\mathcal{G}$ which connects i and j and has parallel transport z when directed from i to j. Let $\tilde{\Delta}$ be the line bundle Laplacian on the new graph $\tilde{\mathcal{G}}$, with Dirichlet boundary conditions at the nodes except nodes i and j, that is, $\tilde{\Delta} = \begin{bmatrix} a & b \\ b^* & C \end{bmatrix}$ where $a = \begin{bmatrix} A_{i,i}+1 & A_{i,j}-z^{-1} \\ A_{j,i}-z & A_{j,j}+1 \end{bmatrix}$ and b is the ith and jth column of B.

By Theorem 2.1 (and its extension discussed in section 2.2), $-[z](\det \tilde{\Delta})$ is a sum of CRSFs with each node except i, j in its own tree component, and nodes i, j in a cycle containing edge $e_{i,j}$, and the weight includes the parallel transport of the path in $\mathcal{G}$ from j to i. (Here $[z^\alpha]f(z)$ refers to the coefficient of z^α in $f(z)$.) We can write

$$\tilde{\Delta} = \begin{bmatrix} a - bC^{-1}b^* & bC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ b^* & C \end{bmatrix}, \tag{3}$$

$$\det \tilde{\Delta} = \det[a - bC^{-1}b^*] \det C$$

$$-[z] \det \tilde{\Delta} = -[z^0][a - bC^{-1}b^*]_{1,2} \det C$$

However

$$[z^0][a - bC^{-1}b^*]_{1,2} = [A - BC^{-1}B^*]_{i,j} = -\mathscr{L}_{i,j}.$$

Finally, $\det C$ is the sum of CRSFs.

The proof in the $\mathrm{SL}_2(\mathbb{C})$-bundle case is similar. Let $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ be the $\mathrm{SL}_2(\mathbb{C})$-bundle Laplacian of $\mathcal{G}$. Add an edge $e_{i,j}$ to $\mathcal{G}$ from node i to node j with parallel transport $M \in \mathrm{SL}_2(\mathbb{C})$. As above let $\tilde{\Delta} = \begin{bmatrix} a & b \\ b^* & C \end{bmatrix}$ where $a = \begin{bmatrix} A_{i,i}+I & A_{i,j}-M^{-1} \\ A_{j,i}-M & A_{j,j}+I \end{bmatrix}$ and b is the ith and jth column of B. Now $\det \tilde{\Delta}$ gives a weighted sum of CRSFs with each node except i, j in its own tree component, where the weight is the product of the monodromies along the cycles. If the CRSF contains a cycle that uses edge $e_{i,j}$, then the monodromy of this cycle will depend on M, and otherwise, the weight of the CRSF does not depend on M. We can write

$$\operatorname{qdet} \Delta' = C_0 + \sum_{\omega} C_\omega (2 - \operatorname{Tr}(K_\gamma M)), \tag{4}$$

where the sum is over configurations ω with a cycle γ containing edge $e_{i,j}$, K_γ is the parallel transport to i from j in the cycle γ, and C_0, C_γ and K_γ do not depend on M.

We have (3) in this case as well, where C does not depend on M. Letting $D = bC^{-1}b^*$, which does not depend on M, we can write

$$a - bC^{-1}b^* = \begin{bmatrix} A_{i,i} + I - D_{i,i} & A_{i,j} - D_{i,j} - M^{-1} \\ A_{j,i} - D_{j,i} - M & A_{j,j} + I - D_{j,j} \end{bmatrix}.$$

This is a 2×2 matrix with entries in $\mathrm{GL}_2(\mathbb{C})$, whose qdet equals

$$\mathrm{qdet}(a - bC^{-1}b^*) = \mathrm{Tr}((A_{i,j} - D_{i,j})M) + C_1$$

where C_1 does not depend on M. Comparing with (4) we see that, since M was arbitrary,

$$-\sum_{\omega} C_\omega K_\omega = A_{i,j} - D_{ij} = [A - BC^{-1}B^*]_{i,j} \det C = -\mathscr{L}_{i,j}. \qquad \square$$

Principal minors of $\mathscr{L}$ also have probabilistic interpretations:

Theorem 4.3. *Let $S \subset \mathcal{N}$, then the $S \times S$ minor $\det \mathscr{L}_S^S$ is the ratio of two terms: the denominator is the weighted sum of CRSFs; the numerator is $(-1)^{|S|}$ times the weighted sum of CRGs of $\mathcal{G}_S$, the graph $\mathcal{G}$ in which all nodes in S are considered interior, and in which the remaining nodes are in separate components. If $Q = \{q_1, \ldots, q_\ell\} = \mathcal{N} \setminus S$, then*

$$\det \mathscr{L}_S^S = (-1)^{|S|} \frac{\mathscr{Z}[q_1|q_2|\cdots|q_\ell]}{\mathscr{Z}[1|2|\cdots|n]}.$$

Proof. Order the vertices of G by first $\mathcal{N} \setminus S$, then S, then the internal nodes. In this order we have

$$\Delta = \begin{bmatrix} A_1 & A_2 & B_1 \\ A_2^* & A_3 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix}.$$

Then

$$\mathscr{L} = -\begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} C^{-1} \begin{bmatrix} B_1^* & B_2^* \end{bmatrix}$$

and $\mathscr{L}_S^S = \det(-A_3 + B_2 C^{-1} B_2^*)$. The proof follows from the identity

$$\begin{bmatrix} A_3 & B_2 \\ B_2^* & C \end{bmatrix} = \begin{bmatrix} A_3 - B_2 C^{-1} B_2^* & B_2 C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B_2^* & C \end{bmatrix}$$

upon taking determinants: the left-hand side determinant is the weighted sum of CRGs of $\mathcal{G}_S$, the right-hand side determinant is $(-1)^{|S|}$ times the product of $\mathscr{L}_S^S$ and $\det C$ which counts CRSFs. In the $\mathrm{SL}_2(\mathbb{C})$-case we used the fact that $\mathrm{qdet} \begin{bmatrix} X & Y \\ 0 & I \end{bmatrix} = \mathrm{qdet} \begin{bmatrix} X & 0 \\ Y & I \end{bmatrix} = \mathrm{qdet}\, X$. $\square$

For line bundles there is an interpretation of more general minors:

Theorem 4.4. *Let Q, R, S, T be disjoint sequences of $\mathcal{N}$ for which $\mathcal{N} = Q \cup R \cup S \cup T$ and $|R| = |S|$. Then $\det \mathscr{L}_{R \cup T}^{S \cup T}$ is the ratio of two terms: the denominator is the weighted sum of CRSFs; the numerator is a signed weighted sum of CRGs of $\mathcal{G}_T$, the graph $\mathcal{G}$ in which all nodes in T are considered interior, the nodes in Q are in singleton parts, and in which nodes in R are paired with nodes in S, with the sign being the sign of the pairing permutation, times the parallel transports from S to R:*

$$\det \mathscr{L}_{R,T}^{S,T} = \sum_{\textit{permutations } \rho} (-1)^\rho \frac{\mathscr{Z}\left[{}^{s_{\rho(1)}}_{r_1} | \cdots |{}^{s_{\rho(k)}}_{r_k} | q_1 | \cdots | q_\ell\right]}{\mathscr{Z}[1|2|\cdots|n]}$$

Proof. The previous proof shows how to reduce to the case T is empty. Furthermore $\det \mathscr{L}_R^S \det C = \det \Delta_{R \cup I}^{S \cup I}$ where I is the set of internal nodes. So we need to evaluate $\det \Delta_{R \cup I}^{S \cup I}$. The proof now follows the proof of Theorem 2.1 which is found in [Ken10, proof of Theorem 1]. Write $\Delta = d^* d$ where d is the operator from sections over the vertices to sections over the edges. Then $\Delta_{R \cup I}^{S \cup I} = d_{S \cup I}^* d_{R \cup I}$ where d_X is the restriction of d to sections over X. By the Cauchy-Binet theorem,

$$\det d_{S \cup I}^* d_{R \cup I} = \sum_Y \det (d_{S \cup I}^Y)^* \det d_{R \cup I}^Y,$$

where the sum is over collections of edges Y of cardinality $|S \cup I|$. The nonzero terms in the sum are collections of edges in which each component is a CRT if we glue r_i to s_i for each i. Equivalently, each component is either a CRT or a tree containing a unique $r \in R$ and $s \in S$. The weight of a component with a cycle is $2 - w - 1/w$ where w is the monodromy of the cycle; the weight of a path is the parallel transport to the r from the s. It remains to compute the signature of each configuration.

This signature is the same as the signature in the case of a trivial bundle, which is determined by [CIM98] to be the signature of the permutation from R to S determined by the pairing. ☐

4.3 Cycle-rooted grove probabilities

Theorem 4.5. *The probability of any topological type of CRG involving only two-node connections and loops is a function of $\mathscr{L}$ and the weighted sum of CRSFs.*

Proof. By a result of Kenyon [Ken10] (based on a theorem of Fock and Goncharov [FG06]), on a graph embedded on a surface with no nodes one can compute the probability of any topological type of CRSF (that is, the probability that a CRSF has a given set of homotopically nontrivial cycles up to isotopy) from the determinant of the Laplacian considered as a function on the space of flat $\mathrm{SL}_2(\mathbb{C})$-connections. Indeed, as X runs over all possible "finite laminations", that is, isotopy classes of collections of finite, pairwise disjoint, topologically nontrivial simple loops on the surface, the products $\prod_{\text{cycles in } X}(2 - \operatorname{Tr} w)$ form a basis for a vector space (the vector space of regular functions on the representation variety) and the Laplacian determinant is an element of this vector space. In other words, Theorem 2.2 above shows that $\det \Delta = \sum_X C_X \prod_{\text{cycles in } X}(2 - \operatorname{Tr} w)$, where X runs over finite laminations; such an expression determines each coefficient C_X uniquely.

To compute the probability that a random CRG Y on Σ has a fixed topology of node connections, add edges $e_{i,j}$ to $\mathcal{G}$ connecting endpoints of all two-node connections $i \to j$ of Y; the resulting graph $\mathcal{G}'$ can be embedded on a surface Σ' containing Σ, and the union of Y and the new edges is a CRSF on $\mathcal{G}'$. (We obtain Σ' from Σ by gluing a single strip running from i to j for each $e_{i,j}$; in this way cycles containing different $e_{i,j}$s are in different homotopy classes.)

Any CRG on $\mathcal{G}$ with the same node connection type as Y can be completed to a CRSF on Σ' by adding the edges $e_{i,j}$. Conversely (since each added edge $e_{i,j}$ is in a different homotopy class on Σ') each CRSF of this topological type comes from a CRG on $\mathcal{G}$ with the same connection type as Y.

The flat connection on $\mathcal{G}$ can be extended to a flat connection on $\mathcal{G}'$ by taking generic parallel transports $\phi_{i,j}$ along the $e_{i,j}$.

It remains to show that the Laplacian determinant of $\mathcal{G}'$ is a function of $\mathscr{L}$, the new parallel transports $\Phi = \{\phi_{i,j}\}$, and $\det C$. However $\Delta^{\mathcal{G}'} = \Delta + S_\Phi$ where S_Φ is supported on the nodes; using $\Delta = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ we have

$$\det \Delta^{\mathcal{G}'} = \det \begin{bmatrix} A + S_\Phi & B \\ B^* & C \end{bmatrix} = \det \begin{bmatrix} A + S_\Phi - BC^{-1}B^* & BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B^* & C \end{bmatrix}$$
$$= \det(-\mathscr{L} + S_\Phi) \det C. \qquad \square$$

5 Basic surface graphs

The simplest non-circular-planar case is the annulus. Since $\pi_1(\Sigma)$ is abelian in this case it usually suffices to consider a line bundle rather than a two-dimensional bundle. The $\mathscr{L}$ matrix then depends on a single variable $z \in \mathbb{C}^*$ which is the monodromy of a flat connection. For simplicity we choose a connection which is the identity on all edges except for the edges crossing a "zipper", that is, a dual path connecting the boundaries; these edges have parallel transport z.

Suppose $(\mathcal{G}, \Sigma)$ is a surface graph on an annulus with n_1 nodes on one boundary component and n_2 on the other. Then $\mathscr{L}$ is an $(n_1 + n_2)$-dimensional matrix with entries which are rational in z. Let $\mathscr{Z}_0 = \mathscr{Z}[1|2|\cdots|n]$ be the weighted sum of CRSFs of $\mathcal{G}$ (CRGs in which each node is in a separate component). We have

$$\mathscr{Z}(1|2|\cdots|n) = \sum_k \alpha_k (2 - z - z^{-1})^k,$$

where α_k is the weighted sum of CRSFs with k cycles winding around the annulus.

While we have not attempted to show that every connection probability can be computed via the $\mathscr{L}$-entries, we present here some case of small n_1, n_2.

5.1 Annulus with $(2,0)$ boundary nodes

Suppose there are two nodes on one boundary and none on the other. Then $\mathscr{L}_{1,2}(z)$ counts connections to 1 from 2. There are only two topologically different configurations, which are illustrated in Figure 4. We have the following theorem.

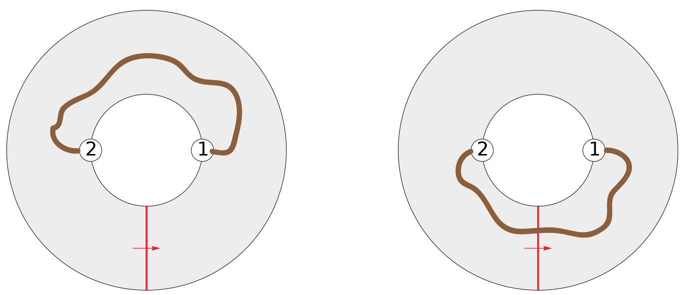


Figure 4: The two topologically distinct ways to connect the nodes on an annulus with $(2,0)$ boundary nodes.

Theorem 5.1. $\frac{\partial}{\partial z} \log \mathscr{L}_{1,2}(z)|_{z=1}$ *is the probability that the LERW to* 1 *from* 2 *crosses the zipper.*

Proof. Let A_k, B_k, and α_k (respectively) be the weighted sum of cycle-rooted groves which contain k cycles winding around the annulus, and in which nodes 1 and 2 are (respectively) connected by a path not crossing the zipper, connected by a path crossing the zipper, or are not connected. Then

$$\mathscr{Z}[\begin{smallmatrix}2\\1\end{smallmatrix}] = \sum_{k\geq 0} [A_k(2 - z - 1/z)^k + B_k z(2 - z - 1/z)^k] = A_0 + B_0 z + O((z-1)^2)$$

$$\mathscr{Z}[1|2] = \sum_{k\geq 0} \alpha_k (2 - z - 1/z)^k = \alpha_0 + O((z-1)^2)$$

By Theorem 4.2, $\mathscr{L}_{1,2} = \mathscr{Z}[\begin{smallmatrix}2\\1\end{smallmatrix}]/\mathscr{Z}[1|2]$, so

$$\frac{\partial}{\partial z} \log \mathscr{L}_{1,2}(z) = \frac{\partial_z \mathscr{Z}[\begin{smallmatrix}2\\1\end{smallmatrix}]}{\mathscr{Z}[\begin{smallmatrix}2\\1\end{smallmatrix}]} - \frac{\partial_z \mathscr{Z}[1|2]}{\mathscr{Z}[1|2]} = \frac{B_0}{A_0 + B_0} + O(z-1). \qquad \square$$

5.2 Annulus with $(1,1)$ boundary nodes

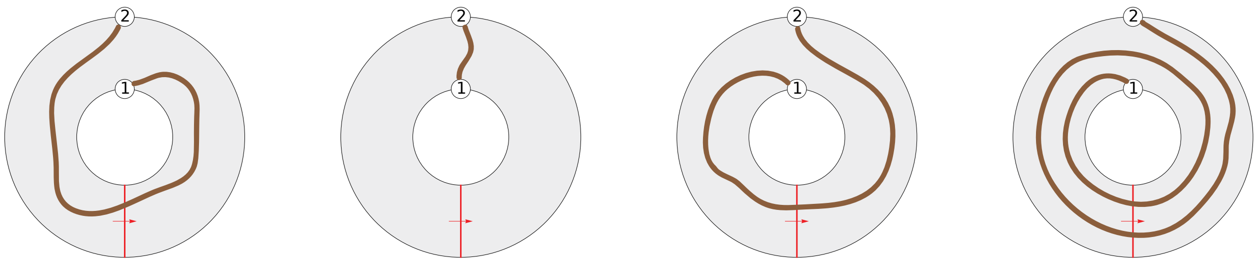


Figure 5: Different numbers of windings of the path to 1 from 2. These configurations contribute to c_1, c_0, c_{-1}, and c_{-2} respectively.

We let c_j denote the weighted sum of CRGs connecting 1 and 2 and such that the path from 1 to 2 crosses the zipper j times algebraically (see Figure 5). Then $\mathscr{Z}[{}^{2}_{1}] = \sum_{j\in\mathbb{Z}} c_j z^j$, from which one can extract c_j for each j. Theorem 4.2 shows that $\mathscr{L}_{1,2} = \mathscr{Z}[{}^{2}_{1}]/\mathscr{Z}[1|2]$, so $\mathscr{L}_{1,2}$ and $\mathscr{Z}[1|2]$ together determine the winding distribution (which we knew already from Theorem 4.5). But $\mathscr{Z}[1|2]$ is also a function of z, so $\mathscr{L}_{1,2}$ does not by itself determine the winding distribution. But from $\mathscr{L}_{1,2}$ we can extract the expected number of algebraic crossings of the zipper via

$$\mathbb{E}[\#\text{ algebraic crossing of zipper}] = \frac{\partial}{\partial z} \log \mathscr{L}_{1,2}\bigg|_{z=1}.$$

5.3 Annlus with $(2,1)$ boundary nodes

Suppose that nodes $1,2$ are on the inner boundary and 3 on the outer. Suppose the zipper starts between 1 and 2 (cclw) as in Figure 6.

This case can be reduced to the previous cases using Theorem 4.4. For example, consider the case that all three nodes are connected, and the path from 1 to 2 goes either left or right of the hole, as illustrated in Figure 6.

Here by Theorem 4.4 above, $\mathscr{L}^{2,3}_{1,3}$ is $1/\mathscr{Z}_0$ times the sum of CRGs in which 1 and 2 are connected by a path, and 3 is considered an interior vertex (that is, either connected to this component or connected to a CRT disjoint from it); such configurations are weighted by the power of z determined by the path from 1 to 2. Thus $\mathscr{L}^{2,3}_{1,3}\mathscr{Z}_0 = A + Bz + C(2 - z - 1/z)$ and A and B are the desired quantities.

The quantities $\mathscr{L}^{2,3}_{1,2}$ and $\mathscr{L}^{1,3}_{1,2}$ carry additional information about the winding of the path from 1 to 3 (respectively from 2 to 3) around the zipper.

Figure 6: The two main cases for connecting three boundary points on the annulus in the $(2,1)$ case (ignoring windings of the connection to vertex 3).

5.4 Annulus with $(3,0)$ boundary nodes

This is a case which can be derived from the $(2,0)$ case using Theorem 4.4. Suppose nodes $1,2,3$ are in counterclockwise order on the inner boundary, with a counterclockwise zipper between nodes 1 and 3. Consider the case when all nodes are connected; there are three possible configurations A_1, A_2, A_3 correspond to which complementary component of the triple connection the outer boundary component lies (see Figure 7). The numerator of $\mathscr{L}_{1,3}^{2,3}$ is $A_1 + A_2 + zA_3$, the numerator of $\mathscr{L}_{1,2}^{3,2}$ is $zA_1 + A_2 + zA_3$, and the numerator of $\mathscr{L}_{2,1}^{3,1}$ is $zA_1 + A_2 + A_3$. These three quantities suffice to determine A_1, A_2, A_3.

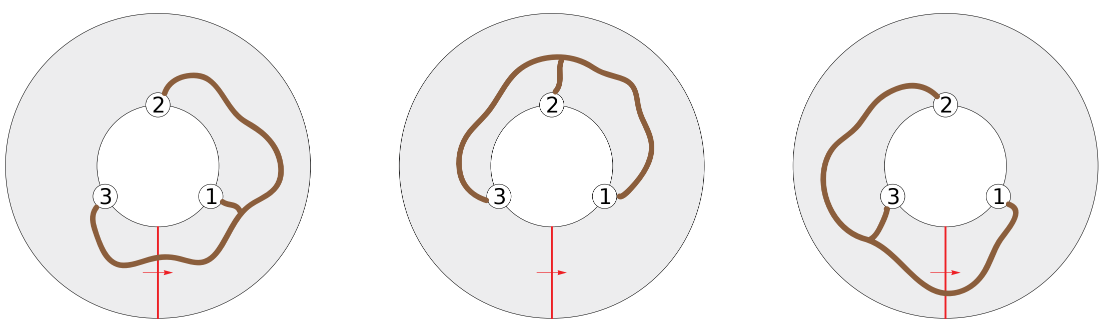
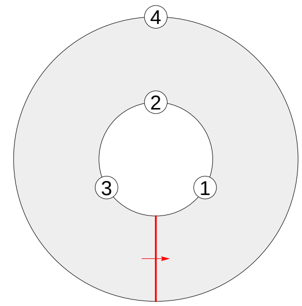


Figure 7: The three topologically distinct subcases when the three nodes are connected (for the annulus with $(3,0)$ nodes).

5.5 Annulus with $(3,1)$ boundary nodes



This is a case we will need when we do the LERW computations below. Suppose there are 4 nodes in all, with nodes $1,2,3$ on the inner boundary in counterclockwise order and 4 on the outer boundary. Suppose the zipper starts between 1 and 3 and is oriented counterclockwise, as in the figure. We wish to compute the ratios $Z[1,2|3,4]/Z[1|2|3|4]$ and $Z[1,2|3,4]/Z[1,2,3,4]$ (as before, $Z[\sigma]$ denotes the weighted sum of groves of type σ, for the trivial bundle).

Recall that $\mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}]$ denotes the weighted sum of cycle-rooted groves of type ${}^{2}_{1}|{}^{4}_{3}$, times the parallel transport of the path to node 1 from 2 and the path to node 3 from node 4, so that for a trivial bundle, $\mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}] = Z[1,2|3,4]$. Because of the path connecting nodes 3 and 4, in fact there will be no cycles in the CRG. Similarly, $\mathscr{Z}[{}^{3}_{1}|{}^{4}_{2}]$ and $\mathscr{Z}[{}^{3}_{2}|{}^{4}_{1}]$ denote the weighted sum of CRGs of type ${}^{3}_{1}|{}^{4}_{2}$ and ${}^{3}_{2}|{}^{4}_{1}$, times their respective parallel transports.

Now by Theorem 4.4, $\mathscr{L}^{3,4}_{1,2}$ has numerator counting connections ${}^{3}_{1}|{}^{4}_{2}$ and ${}^{4}_{1}|{}^{3}_{2}$ (with a minus sign). Similarly for $\mathscr{L}^{2,4}_{1,3}$ and $\mathscr{L}^{2,3}_{1,4}$. With $\mathscr{Z}_0$ denoting the sum of CRSFs, we have

$$\begin{aligned}
\mathscr{Z}_0 \det \mathscr{L}^{2,4}_{1,3} &= \mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}] - \mathscr{Z}[{}^{2}_{3}|{}^{4}_{1}] = \mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}] - \mathscr{Z}[{}^{3}_{2}|{}^{4}_{1}] \\
\mathscr{Z}_0 \det \mathscr{L}^{1,4}_{3,2} &= \mathscr{Z}[{}^{1}_{3}|{}^{4}_{2}] - \mathscr{Z}[{}^{1}_{2}|{}^{4}_{3}] = \mathscr{Z}[{}^{1}_{3}|{}^{4}_{2}] - \mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}] \\
\mathscr{Z}_0 \det \mathscr{L}^{3,4}_{2,1} &= \mathscr{Z}[{}^{3}_{2}|{}^{4}_{1}] - \mathscr{Z}[{}^{3}_{1}|{}^{4}_{2}] = \mathscr{Z}[{}^{3}_{2}|{}^{4}_{1}] - z^2 \mathscr{Z}[{}^{1}_{3}|{}^{4}_{2}]
\end{aligned} \tag{5}$$

As a consequence

$$\frac{\mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}]}{\mathscr{Z}_0} = \frac{\mathscr{L}^{2,4}_{1,3} + z^2 \mathscr{L}^{1,4}_{3,2} + \mathscr{L}^{3,4}_{2,1}}{1-z^2}. \tag{6}$$

When $z \to 1$ both the numerator and denominator of (6) converge to 0, so we evaluate the limiting ratio using l'Hôpital's rule. We can expand $\mathscr{L}_{u,v} = L_{u,v} + (z-1)L'_{u,v} + O((z-1)^2)$, where $L_{u,v}$ is symmetric and $L'_{u,v}$ is antisymmetric. Then in the limit $z \to 1$ this gives

$$\begin{aligned}
\frac{Z[1,2|3,4]}{Z[1|2|3|4]} &= \lim_{z\to 1} \frac{\mathscr{Z}[{}^{2}_{1}|{}^{4}_{3}]}{\mathscr{Z}[1|2|3|4]} = \lim_{z \to 1} \frac{\mathscr{L}^{2,4}_{1,3} + z^2 \mathscr{L}^{1,4}_{3,2} + \mathscr{L}^{3,4}_{2,1}}{1-z^2} \\
&= \frac{L_{1,2}L'_{3,4} + L'_{1,2}L_{3,4} - L_{1,4}L'_{3,2} - L'_{1,4}L_{3,2}}{-2} \\
&\quad + \frac{(L_{3,1}L'_{2,4} + L'_{3,1}L_{2,4} - L_{2,1}L'_{3,4} - L'_{2,1}L_{3,4}) + 2(L_{3,1}L_{2,4} - L_{2,1}L_{3,4})}{-2} \\
&\quad + \frac{L_{2,3}L'_{1,4} + L'_{2,3}L_{1,4} - L_{1,3}L'_{2,4} - L'_{1,3}L_{2,4}}{-2} \\
&= -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,2}L_{3,4} - L_{1,3}L_{2,4}.
\end{aligned} \tag{7}$$

It is useful to express this above formula in terms of the Green's function where the nth node is the boundary. We can express $L_{i,n} = -\sum_{j=1}^{n-1} L_{i,j}$ and $\mathscr{Z}[1,2,3,4]/\mathscr{Z}[1|2|3|4] = \det[\mathscr{L}_{i,j}]_{i=1,\dots,n-1}^{j=1,\dots,n-1}$ using Theorem 4.3. Let $\mathscr{G}_{i,j} = G_{i,j} + (z-1)G'_{i,j} + O((z-1)^2)$. Then using (2) to express $\mathscr{L}$ in terms of $\mathscr{G}$ and taking the limit $z \to 1$, some algebraic manipulation yields

$$\frac{Z[1,2|3,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{1,2} - G_{1,3}. \tag{8}$$

(In Corollary 6.5 we will see a much easier way to convert an L-formula into a G-formula.)

While the left-hand sides of these formulas (7) and (8) are symmetric in nodes 1 and 2, the right-hand sides are not. This asymmetry is due to the location of the zipper, and moving the zipper would change the values of the $L'_{i,j}$ and the $G'_{i,j}$ (albeit in a predictable way). If we keep the zipper between nodes 1 and 3, then we should expect a different formula for $Z[1,3|2,4]$ than what we would get by permuting the indices $1,2,3$ in the formula for $Z[1,2|3,4]$. Indeed, if we carry out the computations as above, we obtain

$$\frac{Z[1,2|3,4]}{Z[1|2|3|4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,2}L_{3,4} - L_{1,3}L_{2,4} \tag{9a}$$

$$\frac{Z[1,3|2,4]}{Z[1|2|3|4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} \tag{9b}$$

$$\frac{Z[2,3|1,4]}{Z[1|2|3|4]} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4} + L_{1,4}L_{2,3} - L_{1,3}L_{2,4} \tag{9c}$$

and

$$\frac{Z[1,2|3,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{1,2} - G_{1,3} \tag{10a}$$

$$\frac{Z[1,3|2,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} \tag{10b}$$

$$\frac{Z[2,3|1,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1} + G_{2,3} - G_{1,3} \tag{10c}$$

5.6 Pair of pants with $(2,0,0)$ boundary nodes

The remaining two annulus cases are most easily viewed as special cases of the case when the surface Σ is a pair of pants with 2 nodes on one boundary and no other nodes, see Figure 8. Put an $\mathrm{SL}_2(\mathbb{C})$ bundle with monodromies A and B around the two central holes C_A and C_B, and supported on zippers from the holes to the boundary between nodes 1 and 2.

The parallel transport of a path to node 1 from node 2 is of the form

1. I, if the path has both holes on its right

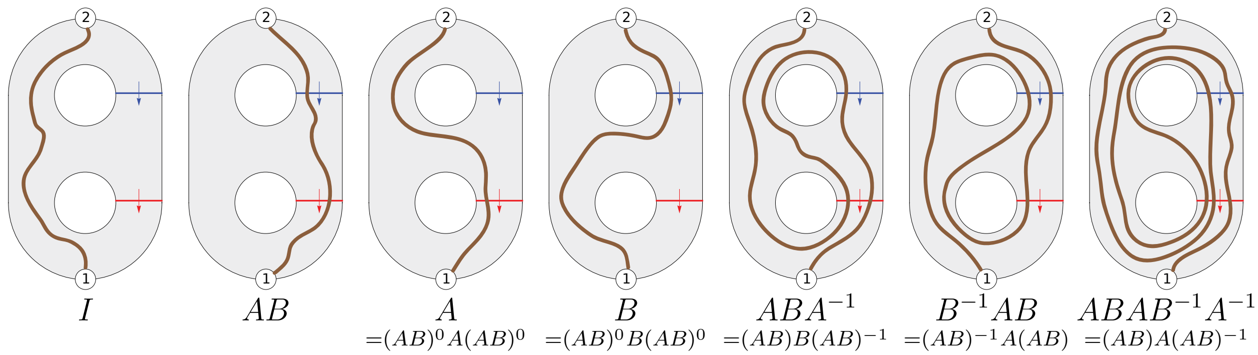


Figure 8: Some of the possible topological types for the path between nodes 1 and 2 when the surface is a pair of pants with both nodes on one boundary. The lower zipper (in red) has parallel transport A, and the upper zipper (in blue) has parallel transport B. For each diagram, the parallel transport of the path to 1 from 2 is shown.

2. AB, if the path has both holes on its left

3. $(AB)^{-k}A(AB)^k$ for some $k \in \mathbb{Z}$, if the path has the lower hole on its left and the upper hole on its right, and k is the algebraic number of crossings that a dual path from the lower hole to the left boundary makes across the upper zipper

4. $(AB)^{-k}B(AB)^k$ for some $k \in \mathbb{Z}$, if the path has the lower hole on its right and the upper hole on its left, and k is the algebraic number of crossings that a dual path from the upper hole to the left boundary makes across the lower zipper

We let $c^{(\mathrm{RR})}$, $c^{(\mathrm{LL})}$, $c_k^{(\mathrm{LR})}$, and $c_k^{(\mathrm{RL})}$ (for $k \in \mathbb{Z}$) denote the weighted sum of cycle-rooted groves of the above types. We further let $c_\ell^{(\mathrm{RR})}$ and $c_\ell^{(\mathrm{LL})}$ denote the number of cycle-rooted groves of type $c^{(\mathrm{RR})}$ and $c^{(\mathrm{LL})}$ in which there are $\ell \in \mathbb{N}$ loops that surround both holes.

We need to choose matrices A and B for which $\det A = 1$ and $\det B = 1$, and it is convenient to choose them so that $\mathrm{Tr}(A) = 2$ and $\mathrm{Tr}(B) = 2$ (so that loops which surround one hole but not the other have weight 0), and so that AB is diagonal. We can take

$$A = \begin{bmatrix} \frac{2x}{x+1} & y \\ -\frac{(x-1)^2}{y(x+1)^2} & \frac{2}{x+1} \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{2x}{x+1} & -\frac{y}{x} \\ \frac{(x-1)^2 x}{y(x+1)^2} & \frac{2}{x+1} \end{bmatrix}$$

for variables x and y. Then we have

$$AB = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix},$$

and since AB is diagonal, it is straightforward to evaluate

$$(AB)^{-k}A(AB)^k = \begin{bmatrix} \frac{2x}{x+1} & yx^{-2k} \\ -\frac{(x-1)^2x^{2k}}{y(x+1)^2} & \frac{2}{x+1} \end{bmatrix}$$
$$(AB)^{-k}B(AB)^k = \begin{bmatrix} \frac{2x}{x+1} & -yx^{-2k-1} \\ \frac{(x-1)^2x^{1+2k}}{y(x+1)^2} & \frac{2}{x+1} \end{bmatrix}.$$

$$\mathscr{Z}[^2_1] = I\sum_{\ell\in\mathbb{N}} c_\ell^{(\mathrm{RR})}(2-x-1/x)^\ell + AB\sum_{\ell\in\mathbb{N}} c_\ell^{(\mathrm{LL})}(2-x-1/x)^\ell + \sum_{k\in\mathbb{Z}} c_k^{(\mathrm{LR})}(AB)^{-k}A(AB)^k + \sum_{k\in\mathbb{Z}} c_k^{(\mathrm{RL})}(AB)^{-k}B(AB)^k.$$

Only the last two sums contribute to the $1,2$ entry of $\mathscr{Z}[^2_1]$:

$$\mathscr{Z}[^2_1]_{1,2} = y\sum_{k\in\mathbb{Z}} \left[c_k^{(\mathrm{LR})}x^{-2k} - c_k^{(\mathrm{RL})}x^{-2k-1}\right].$$

This is a Laurent series in x from which one can extract the coefficients $c_k^{(\mathrm{LR})}$ and $c_k^{(\mathrm{RL})}$. Once these are known, the coefficients $c_\ell^{(\mathrm{RR})}$ and $c_\ell^{(\mathrm{LL})}$ can be extracted from $\mathscr{Z}[^2_1]_{1,1}$ and $\mathscr{Z}[^2_1]_{2,2}$.

5.7 Annulus with $(2,2)$ boundary nodes

On the annulus with 4 nodes, put nodes $1,2$ on the outer boundary and $3,4$ on the inner boundary. Suppose we wish to compute the probability of the connections $13|24$ and $14|23$. This computation can be used to compute the probability that an edge e is on the LERW from 1 to 2 (an equivalent calculation was done in [Ken00a]).

Insert an extra edge e_{34} from 3 to 4; this "splits" the inner boundary into two (See Figure 9). This is then a special case of the construction of section 5.6. In the notation of that section, it suffices to use $x=-1$ and $x=1$ to distinguish crossings $13|24$ and $14|23$: $[\mathscr{Z}_{1,2}]_{1,2}$ in the case $x=-1$ gives the sum and in the case $x=1$ the difference of the two desired quantities.

5.8 Annulus with $(4,0)$ boundary nodes

When 4 nodes are on the outer boundary and none on the inner (and nodes $1,2,3,4$ are in cclw order), the case we have not yet discussed is the $14|23$ case: there are three subcases depending on whether the paths from 1 to 4 and 2 to 3 go left or right of the inner boundary.

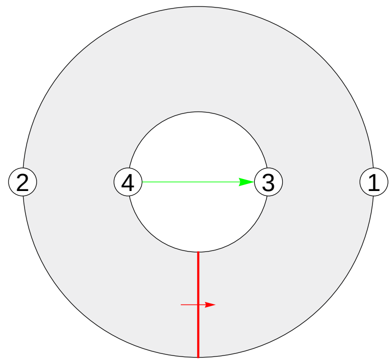


Figure 9: Computing crossings 13|24 and 14|23.

Again this is a special case of the $(2,0,0)$ case if we put an extra edge between nodes 3 and 4.

In this case the only possible parallel transports to 1 from 2 are (using the connection from that section)

$$AB, I, B, A, B^{-1}AB,$$

and only in the first two cases is there a possible extra loop surrounding both C_A and C_B. Thus

$$\mathscr{Z}_{1,2} = I(c_0^{(\mathrm{RR})}+c_1^{(\mathrm{RR})}(2-x-1/x))+AB(c_0^{(\mathrm{LL})}+c_1^{(\mathrm{LL})}(2-x-1/x))+Bc_0^{(\mathrm{RL})}+Ac_0^{(\mathrm{LR})}+B^{-1}ABc_1^{(\mathrm{LR})}.$$

The $1,2$ entry in $\mathscr{Z}_{1,2}$ is

$$[\mathscr{Z}_{1,2}]_{1,2} = -\frac{y}{x}c_0^{(\mathrm{RL})} + yc_0^{(\mathrm{LR})} + \frac{y}{x^2}c_1^{(\mathrm{LR})}.$$

From this we can extract the three cases of interest.

6 Annular-one surface graphs

Suppose n is even and that the first $n-1$ nodes are on the inner boundary of the annulus arranged in counterclockwise order, and node n is by itself on the outer boundary, and that the zipper is between nodes $n-1$ and 1 and directed in the counterclockwise direction (from 1 to $n-1$), as in section 5.5. We call these annular-one surface graphs; they are the next case after circular planar graphs, and they play an important role in our loop-erased random walk calculations in section 8. Annular-one surface graphs of course include the $(1,1)$ and $(3,1)$ cases that we did in the last section, but for expository purposes we treated those special cases in separately. We are interested in computing, for any partition σ, the weighted sum of groves of type σ, which we denote $Z[\sigma]$. We show how to compute $Z[\sigma]/Z[1|2|\cdots|n]$ in terms of the response matrix $\mathscr{L}$, and $Z[\sigma]/Z[1,2,\ldots,n]$ in terms of the Green's function $\mathscr{G}$. We treat first the case when σ is a complete pairing, and then show how to reduce the computation for general partitions to the computation for pairings.

6.1 Complete pairings

There are $n-1$ ways to connect the two boundaries (ignoring windings). When the annulus is cut along this connection, the domain becomes planar, so there are $C_{n/2-1}$ ways to pair up the remaining nodes. We have

$$(n-1)C_{n/2-1} = (n-1)\frac{(n-2)!}{(n/2-1)!(n/2)!} = \frac{1}{2}\frac{n!}{(n/2)!(n/2)!} = \frac{1}{2}\binom{n}{n/2}.$$

So the number of annular pairings equals the number of equations arising from the determinant formula. In fact, there is a natural bijection between the $\mathscr{L}$-determinants and the annular pairings which is based on the cycle lemma of Dvoretzky and Motzkin [DM47], which we use in Appendix A.1 to show that these equations are linearly independent for any even n.

The approach to computing the normalized probability of pairings is similar to the (3,1) case above. In any directed pairing, the connection between node n and the other boundary determines whether or not and in what direction that any other directed pair crosses the zipper. Reversing the direction of any directed pair (other than the pair containing n) that crosses the zipper introduces a factor of z^2. Since only even powers of z appear, it is convenient to change variables to

$$\zeta = z^2.$$

For example, $\mathscr{Z}[{5 \atop 3}|{2 \atop 1}|{6 \atop 4}] = \zeta \mathscr{Z}[{3 \atop 5}|{2 \atop 1}|{6 \atop 4}]$.

For more compact notation let $\ddddot{\mathscr{Z}}_\sigma := \frac{\mathscr{Z}_\sigma}{\mathscr{Z}[1|2|\ldots|n]}$. When expanding an $\mathscr{L}$-determinant into a signed sum of $\ddddot{\mathscr{Z}}_\sigma$'s, where σ is a directed pairing, $\ddddot{\mathscr{Z}}_\sigma$ can be put into a canonical form $\zeta^{\text{power}}\ddddot{\mathscr{Z}}_{\sigma'}$, where σ' is a directed pairing in which the pairs are directed counterclockwise around the annulus. We can then solve for any given $\ddddot{\mathscr{Z}}_\tau = \ddddot{\mathscr{Z}}_\tau(z)$ in terms of the $\mathscr{L}$-determinants and ζ and take the limit $\zeta \to 1$.

The system of linear equations can be represented by a matrix A_n. When recording the linear equation corresponding to $\det \mathscr{L}_R^S$, we can re-order R and S in any manner, and this would just scale row $\det \mathscr{L}_R^S$ of A_n by ± 1, which has no effect on our ability to solve for the $\ddddot{\mathscr{Z}}[\sigma]$s. But the signs in A_n^{-1} are surprisingly nice when we order R and S in a manner that corresponds to $\det \mathscr{L}_R^S$'s associated pairing in the aforementioned bijection.

The matrix for A_2 is just the 1×1 matrix whose entry is 1 (since $\mathscr{Z}[{2 \atop 1}]/\mathscr{Z}[1|2] = \mathscr{L}_{1,2} = \det \mathscr{L}_1^2$):

$$
\begin{array}{r|c|}
 & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] \\
\det \mathscr{L}_{1}^{2} & 1
\end{array}
$$

The matrix A_4 encodes the system of equations (5) we saw for the $(3,1)$ case:

$$
\begin{array}{r|ccc|}
 & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}2|4\\1|3\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}1|4\\3|2\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}3|4\\2|1\end{smallmatrix}\right] \\
\det \mathscr{L}_{1,3}^{2,4} & 1 & 0 & -1 \\
\det \mathscr{L}_{3,2}^{1,4} & -1 & 1 & 0 \\
\det \mathscr{L}_{2,1}^{3,4} & 0 & -\zeta & 1
\end{array}
$$

The next matrix A_6 is

$$
\begin{array}{r|cccccccccc|}
 & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}2|4|6\\1|3|5\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}4|3|6\\1|2|5\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}1|3|6\\5|2|4\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}3|2|6\\5|1|4\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}5|2|6\\4|1|3\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}2|1|6\\4|5|3\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}4|1|6\\3|5|2\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}1|5|6\\3|4|2\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}3|5|6\\2|4|1\end{smallmatrix}\right] & \dddot{\mathscr{Z}}\left[\begin{smallmatrix}5|4|6\\2|3|1\end{smallmatrix}\right] \\
\det \mathscr{L}_{1,3,5}^{2,4,6} & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
\det \mathscr{L}_{1,2,5}^{4,3,6} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta & -1 & 0 \\
\det \mathscr{L}_{5,2,4}^{1,3,6} & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
\det \mathscr{L}_{5,1,4}^{3,2,6} & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\det \mathscr{L}_{4,1,3}^{5,2,6} & 0 & 0 & \zeta & -\zeta & 1 & -\zeta & 0 & 0 & -1 & 0 \\
\det \mathscr{L}_{4,5,3}^{2,1,6} & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\det \mathscr{L}_{3,5,2}^{4,1,6} & -1 & 0 & 0 & 0 & 1 & -\zeta & 1 & -1 & 0 & 0 \\
\det \mathscr{L}_{3,4,2}^{1,5,6} & 0 & 0 & 0 & \zeta & -1 & 0 & 0 & 1 & 0 & 0 \\
\det \mathscr{L}_{2,4,1}^{3,5,6} & 0 & 0 & -\zeta & 0 & 0 & 0 & \zeta & -\zeta & 1 & -1 \\
\det \mathscr{L}_{2,3,1}^{5,4,6} & 0 & 0 & 0 & 0 & 0 & \zeta^2 & -\zeta & 0 & 0 & 1
\end{array}
$$

Each $\det \mathscr{L}_R^S$ is a signed-sum of $(n/2)!$ of the $\dddot{\mathscr{Z}}[\sigma]$'s, but not all of these σ's can be embedded in the annulus, so the rows generally have fewer than $(n/2)!$ nonzero entries. For each pairing σ that embeds in the annulus, the column $\dddot{\mathscr{Z}}[\sigma]$ contains $2^{n/2-1}$ nonzero entries: for each pair $\{i,j\}$ in σ, except the pair containing n, either $i \in R$ and $j \in S$ or else $j \in R$ and $i \in S$.

The inverses A_n^{-1} of these matrices are

$$
\begin{array}{c|c|}
 & \det \mathscr{L}^{2}_{1} \\
\ddddot{\mathscr{Z}}[{}^{2}_{1}] & 1
\end{array}
$$

$$
\begin{array}{c|ccc|}
 & \det \mathscr{L}^{2,4}_{1,3} & \det \mathscr{L}^{1,4}_{3,2} & \det \mathscr{L}^{3,4}_{2,1} \\
\ddddot{\mathscr{Z}}[{}^{2}_{1}|{}^{4}_{3}] & 1 & \zeta & 1 \\
\ddddot{\mathscr{Z}}[{}^{1}_{3}|{}^{4}_{2}] & 1 & 1 & 1 \\
\ddddot{\mathscr{Z}}[{}^{3}_{2}|{}^{4}_{1}] & \zeta & \zeta & 1
\end{array} \times \frac{1}{(1-\zeta)^1}
$$

$$
\begin{array}{c|cccccccccc|}
 & \det \mathscr{L}^{2,4,6}_{1,3,5} & \det \mathscr{L}^{4,3,6}_{1,2,5} & \det \mathscr{L}^{1,3,6}_{5,2,4} & \det \mathscr{L}^{3,2,6}_{5,1,4} & \det \mathscr{L}^{5,2,6}_{4,1,3} & \det \mathscr{L}^{2,1,6}_{4,5,3} & \det \mathscr{L}^{4,1,6}_{3,5,2} & \det \mathscr{L}^{1,5,6}_{3,4,2} & \det \mathscr{L}^{3,5,6}_{2,4,1} & \det \mathscr{L}^{5,4,6}_{2,3,1} \\
\ddddot{\mathscr{Z}}[{}^{2}_{1}|{}^{4}_{3}|{}^{6}_{5}] & \zeta+1 & \zeta+1 & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta^2+\zeta & 2\zeta & 2\zeta & \zeta+1 & 2 \\
\ddddot{\mathscr{Z}}[{}^{4}_{1}|{}^{3}_{2}|{}^{6}_{5}] & \zeta & 1 & \zeta & \zeta & \zeta & \zeta^2 & \zeta & \zeta & 1 & 1 \\
\ddddot{\mathscr{Z}}[{}^{1}_{5}|{}^{3}_{2}|{}^{6}_{4}] & \zeta+1 & 2 & \zeta+1 & \zeta+1 & \zeta+1 & 2\zeta & \zeta+1 & \zeta+1 & 2 & 2 \\
\ddddot{\mathscr{Z}}[{}^{3}_{5}|{}^{2}_{1}|{}^{6}_{4}] & 1 & 1 & \zeta & 1 & 1 & \zeta & \zeta & \zeta & 1 & 1 \\
\ddddot{\mathscr{Z}}[{}^{5}_{4}|{}^{2}_{1}|{}^{6}_{3}] & 2\zeta & 2\zeta & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta^2+\zeta & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta+1 \\
\ddddot{\mathscr{Z}}[{}^{2}_{4}|{}^{1}_{5}|{}^{6}_{3}] & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\ddddot{\mathscr{Z}}[{}^{4}_{3}|{}^{1}_{5}|{}^{6}_{2}] & \zeta+1 & \zeta+1 & 2\zeta & 2\zeta & \zeta+1 & 2\zeta & \zeta+1 & \zeta+1 & \zeta+1 & 2 \\
\ddddot{\mathscr{Z}}[{}^{1}_{3}|{}^{5}_{4}|{}^{6}_{2}] & \zeta & \zeta & \zeta & \zeta & 1 & \zeta & \zeta & 1 & 1 & 1 \\
\ddddot{\mathscr{Z}}[{}^{3}_{2}|{}^{5}_{4}|{}^{6}_{1}] & \zeta^2+\zeta & 2\zeta & \zeta^2+\zeta & \zeta^2+\zeta & 2\zeta & 2\zeta^2 & \zeta^2+\zeta & 2\zeta & \zeta+1 & \zeta+1 \\
\ddddot{\mathscr{Z}}[{}^{5}_{2}|{}^{4}_{3}|{}^{6}_{1}] & \zeta & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta^2 & \zeta & \zeta & \zeta & 1
\end{array} \times \frac{1}{(1-\zeta)^2}
$$

For example, the fifth row tells us

$$\frac{\mathscr{Z}\left[\begin{smallmatrix}5\\4\end{smallmatrix}\middle|\begin{smallmatrix}2\\1\end{smallmatrix}\middle|\begin{smallmatrix}6\\3\end{smallmatrix}\right]}{\mathscr{Z}[1|2|3|4|5|6]} = \frac{1}{(1-\zeta)^2}\left\{\begin{array}{c} 2\zeta\det\mathscr{L}^{2,4,6}_{1,3,5} + 2\zeta\det\mathscr{L}^{4,3,6}_{1,2,5} + (\zeta^2+\zeta)\det\mathscr{L}^{1,3,6}_{5,2,4} \\ +2\zeta\det\mathscr{L}^{3,2,6}_{5,1,4} + (\zeta+1)\det\mathscr{L}^{5,2,6}_{4,1,3} \\ +(\zeta^2+\zeta)\det\mathscr{L}^{2,1,6}_{4,5,3} + (\zeta^2+\zeta)\det\mathscr{L}^{4,1,6}_{3,5,2} \\ +2\zeta\det\mathscr{L}^{1,5,6}_{3,4,2} + (\zeta+1)\det\mathscr{L}^{3,5,6}_{2,4,1} + (\zeta+1)\det\mathscr{L}^{5,4,6}_{2,3,1} \end{array}\right\} \quad (11)$$

and the sixth row tells us

$$\frac{\mathscr{Z}\left[\begin{smallmatrix}2\\4\end{smallmatrix}\middle|\begin{smallmatrix}1\\5\end{smallmatrix}\middle|\begin{smallmatrix}6\\3\end{smallmatrix}\right]}{\mathscr{Z}[1|2|3|4|5|6]} = \frac{1}{(1-\zeta)^2}\left\{\begin{array}{c} \det\mathscr{L}^{2,4,6}_{1,3,5} + \det\mathscr{L}^{4,3,6}_{1,2,5} + \det\mathscr{L}^{1,3,6}_{5,2,4} \\ +\det\mathscr{L}^{3,2,6}_{5,1,4} + \det\mathscr{L}^{5,2,6}_{4,1,3} + \det\mathscr{L}^{2,1,6}_{4,5,3} + \det\mathscr{L}^{4,1,6}_{3,5,2} \\ +\det\mathscr{L}^{1,5,6}_{3,4,2} + \det\mathscr{L}^{3,5,6}_{2,4,1} + \det\mathscr{L}^{5,4,6}_{2,3,1} \end{array}\right\} \quad (12)$$

Theorem 6.1. *For positive even integers n, the matrix A_n is nonsingular.*

We postpone the proof of this theorem until the appendix.

There are some clear patterns in A_n^{-1} which warrant further investigation:

Conjecture 6.1. *For positive even integers n, the entries of the matrix $(1-\zeta)^{n/2-1}\times\mathsf{A}_n^{-1}$ are all polynomials in ζ, with nonnegative integer coefficients, and with degree at most $n/2-1$. These polynomials, when evaluated at $\zeta=1$, only depend on the row of the matrix, and always divide $(n/2-1)!$. These numbers count "Dyck tilings" (certain objects defined in [KW11b]) whose lower boundary is the Dyck path associated with the pairing that indexes the row of A_n^{-1}. (We have verified this for all $n\le 14$.)*

Conjecture 6.2. *For positive even integers n,*

$$\det\mathsf{A}_n = (1-\zeta)^{2^{n-2}-\frac{1}{2}\binom{n}{n/2}}.$$

(We have verified this for all $n\le 16$.)

Proposition 6.2. *For positive even integers n, the row in $(1-\zeta)^{n/2-1}\mathsf{A}_n^{-1}$ corresponding to*

$$\ddddot{\mathscr{Z}}\left[\begin{smallmatrix}n/2-1\\n/2+1\end{smallmatrix}\middle|\begin{smallmatrix}n/2-2\\n/2+2\end{smallmatrix}\middle|\cdots\middle|\begin{smallmatrix}1\\n-1\end{smallmatrix}\middle|\begin{smallmatrix}n\\n/2\end{smallmatrix}\right]$$

consists of all 1's.

Proof. We compute the all ones horizontal vector multiplied by A_n. The resulting vector is indexed by pairings σ. If the matrix entry $(\mathsf{A}_n)_{(A,B),\sigma}$ is nonzero, then σ pairs A to B. If σ contains a pair which does not cross the zipper, then let (i,j) be the lexicographically smallest such pair. We can "swap" the pair (i,j) and find that

$$(\mathsf{A}_n)_{((A\setminus\{i,j\})\cup\{i\},(B\setminus\{i,j\})\cup\{j\}),\sigma} = -(\mathsf{A}_n)_{((A\setminus\{i,j\})\cup\{j\},(B\setminus\{i,j\})\cup\{i\}),\sigma},$$

so the σ entry of the product vector is zero. If every pair of σ crosses the zipper, then σ must be $\sigma_0 = {}^{n/2-1}_{n/2+1}\big|^{n/2-2}_{n/2+2}\big|\cdots\big|^{1}_{n-1}\big|^{n}_{n/2}$. We have

$$(\mathsf{A}_n)_{((A\setminus\{i,j\})\cup\{i\},(B\setminus\{i,j\})\cup\{j\}),\sigma_0} = -\zeta \times (\mathsf{A}_n)_{((A\setminus\{i,j\})\cup\{j\},(B\setminus\{i,j\})\cup\{i\}),\sigma_0},$$

for $i < j$, and hence the product vector is $(1-\zeta)^{n/2-1}$ at σ_0. □

We can expand out the $\mathscr{L}$-determinants into sums of products of the $\mathscr{L}_{i,j}$, each of which depends on ζ, where $\mathscr{L}_{j,i}(\zeta) = \mathscr{L}_{i,j}(1/\zeta)$. Since the denominator is $(1-\zeta)^{n/2-1}$, to evaluate the limit $\zeta \to 1$, we can differentiate the numerator and denominator $n/2-1$ times with respect to ζ and then set ζ to 1. The denominator of course becomes $(-1)^{n/2-1}(n/2-1)!$. The numerator will consist of monomials of degree $n/2$ in the quantities

$$\mathscr{L}_{i,j}\big|_{\zeta=1}, \frac{\partial}{\partial\zeta}\mathscr{L}_{i,j}\bigg|_{\zeta=1}, \frac{\partial^2}{\partial\zeta^2}\mathscr{L}_{i,j}\bigg|_{\zeta=1}, \dots, \frac{\partial^{n/2-1}}{\partial\zeta^{n/2-1}}\mathscr{L}_{i,j}\bigg|_{\zeta=1}.$$

In each case all the terms involving higher order derivatives of $\mathscr{L}_{i,j}$ cancel upon setting ζ to 1. This is quite convenient, since there are fewer quantities that we need to evaluate later. We already have

$$L_{i,j} = \mathscr{L}_{i,j}\big|_{z=1} = \mathscr{L}_{i,j}\big|_{\zeta=1},$$

let us define

$$L'_{i,j} := \frac{\partial}{\partial z}\mathscr{L}_{i,j}\bigg|_{z=1} = 2\,\frac{\partial}{\partial\zeta}\mathscr{L}_{i,j}\bigg|_{\zeta=1}.$$

Theorem 6.3. *For all positive even n and pairings σ of $\{1,\dots,n\}$, $Z_\sigma/Z_{1|2|\cdots|n}$ is a polynomial of degree $n/2$ in the quantities*

$$\{L_{i,j} : 1 \le i < j \le n\} \quad \textit{and} \quad \{L'_{i,j} : 1 \le i < j \le n-1\}.$$

We prove this theorem in the appendix.

Conjecture 6.3. *The coefficients in the polynomials in Theorem 6.3 are all integers. (We have verified this for all σ for all $n \le 10$.)*

In the above two examples we obtain

$$
\begin{aligned}
\frac{Z[4,5|1,2|3,6]}{Z[1|2|3|4|5|6]} &= L_{1,4}L_{2,6}L_{3,5} - L_{1,4}L_{2,5}L_{3,6} - L_{1,3}L_{2,6}L_{4,5} + L_{1,2}L_{3,6}L_{4,5} + L_{1,3}L_{2,5}L_{4,6} \\
&\quad - L_{1,2}L_{3,5}L_{4,6} - L_{3,6}L_{4,5}L'_{1,2} + L_{3,5}L_{4,6}L'_{1,2} + L_{2,6}L_{4,5}L'_{1,3} - L_{2,5}L_{4,6}L'_{1,3} \\
&\quad - L_{2,6}L_{3,5}L'_{1,4} + L_{2,5}L_{3,6}L'_{1,4} - L_{1,6}L_{4,5}L'_{2,3} + 2L_{1,5}L_{4,6}L'_{2,3} - L_{1,4}L_{5,6}L'_{2,3} \\
&\quad + 2L_{5,6}L'_{1,4}L'_{2,3} - 2L_{4,6}L'_{1,5}L'_{2,3} + L_{1,6}L_{3,5}L'_{2,4} - 2L_{1,5}L_{3,6}L'_{2,4} + L_{1,3}L_{5,6}L'_{2,4} \\
&\quad - 2L_{5,6}L'_{1,3}L'_{2,4} + 2L_{3,6}L'_{1,5}L'_{2,4} + L_{1,4}L_{3,6}L'_{2,5} - L_{1,3}L_{4,6}L'_{2,5} + 2L_{4,6}L'_{1,3}L'_{2,5} \\
&\quad - 2L_{3,6}L'_{1,4}L'_{2,5} - L_{1,6}L_{2,5}L'_{3,4} + 2L_{1,5}L_{2,6}L'_{3,4} - L_{1,2}L_{5,6}L'_{3,4} + 2L_{5,6}L'_{1,2}L'_{3,4} \\
&\quad - 2L_{2,6}L'_{1,5}L'_{3,4} + 2L_{1,6}L'_{2,5}L'_{3,4} - L_{1,4}L_{2,6}L'_{3,5} + L_{1,2}L_{4,6}L'_{3,5} - 2L_{4,6}L'_{1,2}L'_{3,5} \\
&\quad + 2L_{2,6}L'_{1,4}L'_{3,5} - 2L_{1,6}L'_{2,4}L'_{3,5} + L_{1,3}L_{2,6}L'_{4,5} - L_{1,2}L_{3,6}L'_{4,5} + 2L_{3,6}L'_{1,2}L'_{4,5} \\
&\qquad - 2L_{2,6}L'_{1,3}L'_{4,5} + 2L_{1,6}L'_{2,3}L'_{4,5}
\end{aligned}
$$

$$
\frac{Z[4,2|5,1|3,6]}{Z[1|2|3|4|5|6]} = \begin{array}{l} L_{5,6}L'_{1,4}L'_{2,3} - L_{4,6}L'_{1,5}L'_{2,3} - L_{5,6}L'_{1,3}L'_{2,4} + L_{3,6}L'_{1,5}L'_{2,4} + L_{4,6}L'_{1,3}L'_{2,5} \\ -L_{3,6}L'_{1,4}L'_{2,5} + L_{5,6}L'_{1,2}L'_{3,4} - L_{2,6}L'_{1,5}L'_{3,4} + L_{1,6}L'_{2,5}L'_{3,4} - L_{4,6}L'_{1,2}L'_{3,5} \\ +L_{2,6}L'_{1,4}L'_{3,5} - L_{1,6}L'_{2,4}L'_{3,5} + L_{3,6}L'_{1,2}L'_{4,5} - L_{2,6}L'_{1,3}L'_{4,5} + L_{1,6}L'_{2,3}L'_{4,5} \end{array}
$$

This last formula can be written as a Pfaffian:

$$
\frac{Z[4,2|5,1|3,6]}{Z[1|2|3|4|5|6]} = \operatorname{Pf}\begin{bmatrix} 0 & L'_{1,2} & L'_{1,3} & L'_{1,4} & L'_{1,5} & L_{1,6} \\ -L'_{1,2} & 0 & L'_{2,3} & L'_{2,4} & L'_{2,5} & L_{2,6} \\ -L'_{1,3} & -L'_{2,3} & 0 & L'_{3,4} & L'_{3,5} & L_{3,6} \\ -L'_{1,4} & -L'_{2,4} & -L'_{3,4} & 0 & L'_{4,5} & L_{4,6} \\ -L'_{1,5} & -L'_{2,5} & -L'_{3,5} & -L'_{4,5} & 0 & L_{5,6} \\ -L_{1,6} & -L_{2,6} & -L_{3,6} & -L_{4,6} & -L_{5,6} & 0 \end{bmatrix}
$$

Notice that the derivatives appear everywhere except the last row and column.

6.2 Partial pairings

When we carry out our loop-erased random walk computations, we also need to handle partial pairings, where the grove partition has singleton components as well as pairs, so long as node n is in a pair. The determinant formulas from Theorem 4.4 are, as formal polynomials in the matrix entries of the response matrix $\mathscr{L}$, exactly the same as they would be if the singleton parts of σ were removed, and these singleton nodes were made internal vertices. As a result, the "L-polynomials" look the same, even though the underlying L-variables depend on the set of nodes. For example, to find $Z[1,2|3,5|4]$ we can take (9a) and simply replace the 4-indices with 5's:

$$
\frac{Z[1,2|3,5|4]}{Z[1|2|3|4|5]} = -L'_{1,2}L_{3,5} - L'_{2,3}L_{1,5} - L'_{3,1}L_{2,5} + L_{1,2}L_{3,5} - L_{1,3}L_{2,5}.
$$

The formula in terms of G and G' does however look different.

Theorem 6.4. *Let σ be a partial pairing, in which the nodes T are in singleton parts, with $n \notin T$. In the above formulas expressing $\mathscr{Z}[\sigma]/\mathscr{Z}[1|2|\cdots|n]$ in terms of $\mathscr{L}$-determinants, we can replace each $\det \mathscr{L}_R^S$ with $\det \hat{\mathscr{G}}_{R\cup T}^{S\cup T}$, where $\hat{\mathscr{G}}_{i,j} = \mathscr{G}_{i,j}$ for $j < n$ and $\hat{\mathscr{G}}_{i,n} = 1$, and the result will be a formula for $Z[\sigma]/Z[1, 2, \ldots, n]$.*

Proof. Since the higher derivatives of $L_{i,j}$ and the first derivative of $L_{i,n}$ always cancel out, we may compute $\dddot{Z}[\sigma]$ using any convenient choice of $L'_{i,n}$, and for present purposes it is convenient to make the choice for which $\sum_j \mathscr{L}_{i,j} = 0$ for each i. Then writing the sequence S as $S = S^*, n$, where $n \notin S^*$, we may express the determinant $\det \mathscr{L}_R^S$ as

$$\det \mathscr{L}_R^S = -\sum_{i \notin S} \det \mathscr{L}_R^{S^*,i}.$$

Let T be the sequence $\{1, \ldots, n-1\} \setminus (R \cap S^*)$ in sorted order. Suppose for now that R and S are also in sorted order, and that R, T and S^*, T are also in sorted order. We can use Jacobi's formula on each summand to obtain

$$\frac{\det \mathscr{L}_R^S}{\det \mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}} = -\sum_{i \notin S} (-1)^{\sum R + \sum S^* + i + |\{s \in S^* : s > i\}|} \det(\mathscr{L}^{-1})_{\{1,\ldots,n-1\}\setminus\{S^*,i\}}^{\{1,\ldots,n-1\}\setminus R}.$$

Now we use $\{1, \ldots, n-1\} \setminus R = S^*, T$ and $\{1, \ldots, n-1\} \setminus \{S^*, i\} = R, T \setminus \{i\}$, and the fact that $\mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}$ and $\mathscr{G}_{1,\ldots,n-1}^{1,\ldots,n-1}$ are negative inverses to write

$$\frac{\det \mathscr{L}_R^S}{\det \mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}} = (-1)^{\sum R + \sum S^* + |R| + |T|} \sum_{i \in R \cup T} (-1)^{i + |\{s \in S^* : s > i\}|} \det \mathscr{G}_{R,T\setminus\{i\}}^{S^*,T}.$$

When we expand $\det \hat{\mathscr{G}}_{R,T}^{S^*,n,T}$ along column n, we obtain

$$\det \hat{\mathscr{G}}_{R,T}^{S^*,n,T} = \sum_{j=1}^{|R\cup T|} (-1)^{j+|R|} \det \mathscr{G}_{R,T \text{ with } j\text{th item removed}}^{S^*,T}$$

If the jth item of R, T is i, then $i - j = |\{s \in S^* : s < i\}|$, so

$$\det \hat{\mathscr{G}}_{R,T}^{S^*,n,T} = (-1)^{|R|+|S^*|} \sum_{j=1}^{|R\cup T|} (-1)^{i+|\{s \in S^* : s > i\}|} \det \mathscr{G}_{R,T\setminus\{i\}}^{S^*,T}$$

and so

$$\frac{\det \mathscr{L}_R^S}{\det \mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}} = (-1)^{\sum R + \sum S^* + |R| + |T| + |R| + |S^*|} \det \hat{\mathscr{G}}_{R,T}^{S^*,n,T},$$

and since $R \cup S^* = \{1, 2, \ldots, 2|R| - 1\}$, which adds up to $|R|$ modulo 2, we have

$$\frac{\det \mathscr{L}_R^S}{\det -\mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}} = \det \hat{\mathscr{G}}_{R,T}^{S,T}. \tag{13}$$

Now observe that if we relax the assumption that R, S, R, T, and S^*, T are in sorted order, the left- and right-hand sides of the above equation change signs the same number of times.

Finally, recall that $\frac{\mathscr{Z}[1,2,\ldots,n]}{\mathscr{Z}[1|2|\ldots|n]} = \det -\mathscr{L}_{1,\ldots,n-1}^{1,\ldots,n-1}$. □

Corollary 6.5. *For a complete pairing σ, the Green's function formula for $Z[\sigma]/Z[1,2,\ldots,n]$ can be obtained from the response-matrix formula for $Z[\sigma]/Z[1|2|\cdots|n]$ simply by replacing each $L_{i,j}$ with $G_{i,j}$ and each $L'_{i,j}$ with $G'_{i,j}$, and then setting $G_{i,n} = 1$.*

Corollary 6.6. *For a partial pairing σ in which node n is paired, the Green's function formulas for $Z[\sigma]/Z[1,2,\ldots,n]$ are invariant under the addition of global constant to the Green's function.*

Proof. The column indexed by n in $\det \hat{\mathscr{G}}_{R,T}^{S,T}$ is all-ones. □

6.3 Windings

We can also extract information about the windings of the paths within a grove pairing in a manner similar to that described in the $(1,1)$ case. For a given directed pairing σ, we have

$$\mathscr{Z}[\sigma] = \sum_k z^k Z[\sigma,(k)],$$

where $Z[\sigma,(k)]$ is the weighted sum of groves of type σ in which the algebraic number of zipper crossings (involving all pairs in σ) is k. Then

$$\mathbb{E}[\text{algebraic number of zipper crossings for groves of type } \sigma] = \lim_{z\to 1} \frac{\partial}{\partial z} \log \frac{\mathscr{Z}[\sigma]}{\mathscr{Z}[1|2|\cdots|n]}.$$

6.4 General partitions

For more general partitions σ, we can express $Z[\sigma]$ as a linear combination of $Z[\tau]$'s where the τ's are partial pairings which may have unlisted nodes. We explain first how to do this with circular planar graphs (where the surface is a disk), and then for annular-one graphs. It is also possible to do this reduction for the annulus with 2 nodes on each boundary, but we do not know how to do it for the annulus with 2 nodes on one boundary and 3 on the other.

Consider the disk with nodes labeled $1,\ldots,n$ in counterclockwise order on the outer boundary. For a partition σ, let i be the smallest node label that is in a part of σ of size more than 2, and let s be the size of this part. We will measure the "badness" of the partition σ by the quantity $n(n-i)+s$. Let j be the next-smallest item in i's part of σ. Let σ^* denote the partition obtained from σ by "de-listing" j, i.e., by regarding j as an internal vertex which can occur in any of σ^*'s parts. If σ has k parts, then we can write $Z[\sigma^*] = \sum_{\ell=1}^k Z[\sigma^* \text{ with } j \text{ added to } \ell\text{th part}]$. One of these terms on the right will be $Z[\sigma]$, so

$$Z[\sigma] = Z[\sigma^*] - \sum_{\ell} Z[\sigma^* \text{ with } j \text{ added to } \ell\text{th part}].$$

where the sum runs over all parts of σ^* except the one containing i. Because the graph is circular planar, unless j is added to a part of σ^* that is "covered" by the part containing i, there will be no groves of that partition type. Thus each of the nonzero terms on the right has a smaller badness than σ, so we can iterate this process to eventually express $Z[\sigma]$ as a linear combination of $Z[\tau]$'s where τ is a partial pairing. For example,

$$\begin{aligned} Z[1,5,8|2,3,4|6,7] &= Z[1,8|2,3,4|6,7] - Z[1,8|2,3,4,5|6,7] - Z[1,8|2,3,4|5,6,7] \\ &= Z[1,8|2,4|6,7] - Z[1,8|2,5|6,7] - Z[1,8|2,4|5,7]. \end{aligned}$$

For an annular-one graph, we can do essentially the same procedure as for a circular planar graph, except that we start with the part containing node n, reducing its size if the part has more than two nodes. Once this part has size two, say that it is $\{h,n\}$, then we list the remaining nodes in the order $h+1, h+2, \ldots, n-1, 1, 2, \ldots, h-1$, and do reductions described above. If any node gets adjoined to the part $\{h,n\}$, then the term for that partition will be zero. If n started out in a singleton part, we can start out as in the circular planar case (with the order $1,\ldots,n-1$) until a node gets adjoined to n's part, and then cyclically re-order the nodes as in the doubleton case.

7 The Green's function and its monodromy-derivative

To carry out our loop-erased random walk computations for various lattices, we will use our formulas for the connection probabilities in annular-one graphs developed in section 6, and for this we need the Green's function G together with its derivative G' with respect to a zipper monodromy. We will need G and G' for both the full lattice, and the lattice after some of its edges have been cut.

7.1 Green's function and potential kernel

The Green's function $G_{u,v}$ is infinite for recurrent lattices such as $\mathbb{Z}^2$, but there is a quantity known as the potential kernel $A_{u,v}$ which behaves like a Green's function, except that $A_{u,u} = 0$, and $G_{u,v}$ and $A_{u,v}$ have the opposite sign convention (see [Spi76]). Suppose that a graph $\mathcal{G}$

is the intersection of $\mathbb{Z}^2$ or another lattice $\mathbb{L}$ with a region surrounding the origin, with "wired boundary conditions", i.e., all the lattice vertices in $\mathbb{L}$ that are not in the region are merged into a single vertex in $\mathcal{G}$ that plays the role of boundary. If R denotes the electrical resistance within $\mathcal{G}$ from the origin to the boundary, then

$$G^{\mathcal{G}}_{u,v} = R - A^{\mathbb{L}}_{u,v} + o(1),$$

where the error term tends to 0 for fixed u and v as $R \to \infty$. For translation invariant lattices, $A^{\mathbb{L}}_{u,v}$ depends only on $u - v$, and is written as $a^{\mathbb{L}}_{u-v}$.

Since all of our formulas for crossings of the annulus are invariant when a global constant is added to the Green's function (involving terms such as $G_{1,2} - G_{1,3}$), it is straightfoward to take the limit $\lim_{\mathcal{G}\to\mathbb{L}}$ of these formulas by replacing each $G^{\mathcal{G}}_{u,v}$ in the formula with $-A^{\mathbb{L}}_{u,v}$, which we shall also denote by $\bar{G}^{\mathbb{L}}_{u,v}$.

For convenience let us work with a modified finite graph $\bar{\mathcal{G}}$ approximating the lattice $\mathbb{L}$, obtained by adjoining an edge with conductance $-1/R$ to node n of $\mathcal{G}$, taking node n of $\bar{\mathcal{G}}$ to be the other endpoint of this edge. This has the effect of making the resistance in $\bar{\mathcal{G}}$ from the origin to node n exactly zero.

For any partition σ for which n is not in a singleton part, we have $Z^{\bar{\mathcal{G}}} = -Z/R$ and $Z^{\bar{\mathcal{G}}}[\sigma] = -Z[\sigma]/R$, so in particular we can compute $Z[\sigma]/Z = Z^{\bar{\mathcal{G}}}[\sigma]/Z^{\bar{\mathcal{G}}}$ by working with the Green's function $\mathscr{G}^{\bar{\mathcal{G}}}$ of this modified graph. We define $\bar{G}_{u,v}$ and $\bar{G}'_{u,v}$ by the expansion

$$\mathscr{G}^{\bar{\mathcal{G}}}_{u,v} = \bar{G}_{u,v} + (z-1)\bar{G}'_{u,v} + O((z-1)^2).$$

Then in the limit $\mathcal{G} \to \mathbb{L}$, we have $\bar{G}_{u,v} \to -A^{\mathbb{L}}_{u,v}$.

It is well-known how to compute the potential kernel on periodic lattices by taking the Fourier coefficients of the characteristic polynomial of the lattice [Spi76]. The potential kernel can also be computed for any "isoradial" graph by doing local computations [Ken02]. The square, triangular, and hexagonal lattices are both periodic and isoradial, so for these lattices either method can be employed.

We shall make use of the following smoothness result:

Lemma 7.1 ([Stö50, Ken02]). *For points $z = (z_1, z_2)$ far from $(0,0)$, the potential kernel on $\mathbb{Z}^2$ behaves like*

$$A^{\mathbb{Z}^2}_{0,z} = \frac{1}{2\pi}\log|z| + \frac{\frac{3}{2}\log 2 + \gamma}{2\pi} + O(1/|z|^2),$$

where γ is Euler's constant.

Similar formulas hold for other lattices [Ken02].

7.2 Derivative of the Green's function

7.2.1 Infinite sum formula

Let S be the adjacency matrix of the zipper, i.e.,

$$S_{k,\ell} = \begin{cases} 1 & \text{there is a zipper edge directed from } k \text{ to } \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Delta(z) = \Delta_0 + (1 - z^{-1})S + (1 - z)S^*$, so

$$\begin{aligned} \Delta(z)^{-1} &= \big(\Delta_0(1 + (1 - z^{-1})\Delta_0^{-1}S + (1 - z)\Delta_0^{-1}S^*)\big)^{-1} \\ &= \Delta_0^{-1} - (1 - z^{-1})\Delta_0^{-1}S\Delta_0^{-1} - (1 - z)\Delta_0^{-1}S^*\Delta_0^{-1} + O((z - 1)^2) \\ \mathscr{G}_{u,v} &= G_{u,v} - (z - 1) \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}G_{\ell,v} - G_{u,\ell}G_{k,v}) + O((z - 1)^2) \end{aligned}$$

The sum is over zipper edges (k, ℓ) in which the zipper direction is from k to ℓ, and $c_{k,\ell}$ is the conductance of edge (k, ℓ). The linear term in $z - 1$ gives us the desired derivative:

$$G'_{u,v} = \partial_z \mathscr{G}_{u,v}\big|_{z=1} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(G_{u,k}G_{\ell,v} - G_{u,\ell}G_{k,v}). \tag{14}$$

For the modified graph $\bar{\mathcal{G}}$, we of course have

$$\bar{G}'_{u,v} = \partial_z \mathscr{G}^{\bar{\mathcal{G}}}_{u,v}\Big|_{z=1} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(\bar{G}_{u,k}\bar{G}_{\ell,v} - \bar{G}_{u,\ell}\bar{G}_{k,v}). \tag{15}$$

For a vertical zipper in $\mathbb{Z}^2$ (or the triangular lattice or hexagonal lattice) started in the face whose lower-left corner is the origin, directed down towards infinity, we define

$$\bar{G}'^{\mathbb{L}}_{u,v} = - \sum_{\text{zipper edges } (k,\ell)} c_{k,\ell}(\bar{G}^{\mathbb{L}}_{u,k}\bar{G}^{\mathbb{L}}_{\ell,v} - \bar{G}^{\mathbb{L}}_{u,\ell}\bar{G}^{\mathbb{L}}_{k,v}). \tag{16}$$

For fixed u and v, for zipper edges (k, ℓ) at a distance r from the origin, it is straightforward to use the smoothness result in Lemma 7.1 to show that edge (k, ℓ) contributes $O(r^{-2}\log r)$ to the sum 16, so this sum is absolutely convergent.

We would like to know that $\bar{G}'^{\mathcal{G}}_{u,v}$ converges to $\bar{G}'^{\mathbb{L}}_{u,v}$ as defined in (16) for a sequence of $\mathcal{G}$'s converging to $\mathbb{L}$. For our purposes in section 8 when we analyze loop-erased random on the lattice, we do not need this convergence of $\bar{G}'$ for every sequence of $\mathcal{G}$'s converging to $\mathbb{L}$, it will suffice to have convergence for some sequence of $\mathcal{G}$'s tending to $\mathbb{L}$. Perhaps the easiest way to show this is to exploit the reflection symmetry that each of the square, triangular, and hexagonal lattices possess.

Lemma 7.2. *If $\mathbb{L}$ is the square, triangular, or hexagonal lattice, and $L \in \mathbb{N}$, let $\mathcal{G}_L = [-L^3, L^3] \times [-L, L^3] \cap \mathbb{L}$ be the off-center box surrounding the origin and zipper (as in Figure 3 except with the lower boundary much closer to the origin than the other boundaries), where the lower boundary of the box is aligned with an axis of reflection symmetry of the lattice $\mathbb{L}$. Let u, v be fixed points in $\mathbb{L}$. Then*

$$\lim_{L\to\infty} \bar{G}'^{\mathcal{G}_L}_{u,v} = \bar{G}'^{\mathbb{L}}_{u,v}.$$

Proof. We can approximate $G^{\mathcal{G}_L}_{u,w}$ and $G^{\mathcal{G}_L}_{v,w}$ (for w within distance L of the origin) using the Green's function of the lattice intersected with the upper-half plane. More precisely, we approximate $G^{\mathcal{G}_L}_{p,q}$ by

$$G^{\approx}_{p,q} := -A^{\mathbb{L}}_{p,q} + A^{\mathbb{L}}_{p^*,q},$$

where $p^* = p - (0, 2L)$ is the reflection of p through the lower boundary of the box, and $A^{\mathbb{L}}$ is the potential kernel of the lattice. By construction $G^{\approx}_{p,q}$ is zero for q along the lower side of the box, and by the smoothness result from Lemma 7.1, $G^{\approx}_{p,q} = O(1/L^2)$ along the other three sides of the box. Both $G^{\approx}_{p,q}$ and $G^{\mathcal{G}_L}_{p,q}$ are harmonic in both p and q within the box (except on the boundary), and $G^{\mathcal{G}_L}_{p,q}$ is zero along all four sides of the box. By the maximal principle for harmonic functions, for p and q within $\mathcal{G}_L$ we have

$$|G^{\approx}_{p,q} - G^{\mathcal{G}_L}_{p,q}| = O(1/L^2),$$

i.e.,

$$\bar{G}^{\mathcal{G}_L}_{p,q} = -A^{\mathbb{L}}_{p,q} + A^{\mathbb{L}}_{p^*,q} - A^{\mathbb{L}}_{0^*,0} + O(1/L^2).$$

Next we compare the contribution of a zipper edge (k, ℓ) to $\bar{G}'^{\mathcal{G}_L}_{u,v}$ and $\bar{G}'^{\mathbb{L}}_{u,v}$. We have

$$\begin{aligned}
-(\bar{G}^{\mathcal{G}_L}_{u,k}\bar{G}^{\mathcal{G}_L}_{\ell,v} - \bar{G}^{\mathcal{G}_L}_{u,\ell}\bar{G}^{\mathcal{G}_L}_{k,v}) = &- (-A^{\mathbb{L}}_{u,k} + A^{\mathbb{L}}_{u^*,k} - A^{\mathbb{L}}_{0^*,0})(-A^{\mathbb{L}}_{v,\ell} + A^{\mathbb{L}}_{v^*,\ell} - A^{\mathbb{L}}_{0^*,0}) \\
&+ (-A^{\mathbb{L}}_{u,\ell} + A^{\mathbb{L}}_{u^*,\ell} - A^{\mathbb{L}}_{0^*,0})(-A^{\mathbb{L}}_{v,k} + A^{\mathbb{L}}_{v^*,k} - A^{\mathbb{L}}_{0^*,0}) + O(L^{-2}\log L) \\
= &- (\bar{G}^{\mathbb{L}}_{u,k}\bar{G}^{\mathbb{L}}_{\ell,v} - \bar{G}^{\mathbb{L}}_{u,\ell}\bar{G}^{\mathbb{L}}_{k,v}) \\
&- (\bar{G}^{\mathbb{L}}_{u,k^*}\bar{G}^{\mathbb{L}}_{\ell^*,v} - \bar{G}^{\mathbb{L}}_{u,\ell^*}\bar{G}^{\mathbb{L}}_{k^*,v}) \\
&+ A^{\mathbb{L}}_{u,k}A^{\mathbb{L}}_{v^*,\ell} + A^{\mathbb{L}}_{u^*,k}A^{\mathbb{L}}_{v,\ell} - A^{\mathbb{L}}_{u,\ell}A^{\mathbb{L}}_{v^*,k} - A^{\mathbb{L}}_{u^*,\ell}A^{\mathbb{L}}_{v,k} \\
&+ A^{\mathbb{L}}_{0^*,0}\big[- A^{\mathbb{L}}_{u,k} + A^{\mathbb{L}}_{u^*,k} - A^{\mathbb{L}}_{v,\ell} + A^{\mathbb{L}}_{v^*,\ell} + A^{\mathbb{L}}_{u,\ell} - A^{\mathbb{L}}_{u^*,\ell} + A^{\mathbb{L}}_{v,k} - A^{\mathbb{L}}_{v^*,k}\big] \\
&+ O(L^{-2}\log L) \\
= &- (\bar{G}^{\mathbb{L}}_{u,k}\bar{G}^{\mathbb{L}}_{\ell,v} - \bar{G}^{\mathbb{L}}_{u,\ell}\bar{G}^{\mathbb{L}}_{k,v}) - (\bar{G}^{\mathbb{L}}_{u,k^*}\bar{G}^{\mathbb{L}}_{\ell^*,v} - \bar{G}^{\mathbb{L}}_{u,\ell^*}\bar{G}^{\mathbb{L}}_{k^*,v}) \\
&+ A^{\mathbb{L}}_{u,k}(A^{\mathbb{L}}_{v^*,\ell} - A^{\mathbb{L}}_{0^*,0}) - (A^{\mathbb{L}}_{u^*,\ell} - A^{\mathbb{L}}_{0^*,0})A^{\mathbb{L}}_{v,k} \\
&- A^{\mathbb{L}}_{u,\ell}(A^{\mathbb{L}}_{v^*,k} - A^{\mathbb{L}}_{0^*,0}) + (A^{\mathbb{L}}_{u^*,k} - A^{\mathbb{L}}_{0^*,0})A^{\mathbb{L}}_{v,\ell} \\
&+ O(L^{-2}\log L)
\end{aligned}$$

Recall that u and v are fixed, so they are within distance $O(1)$ of the origin. The second term is $O(L^{-2}\log L)$. If the zipper edge (k,ℓ) is at distance r from the origin, then the next four terms largely cancel one another and add up to $O(1/(rL))$. Upon summing over all zipper edges, we find $\bar{G}'^{\mathcal{G}_L}_{u,v} = \bar{G}'^{\mathbb{L}}_{u,v} + O(L^{-1}\log L)$. □

7.2.2 Zipper deformations

The next task we have is to evaluate in closed form the infinite sum in (16). This we can do for many lattices $\mathbb{L}$, including the square lattice, triangular lattice, and hexagonal lattice, although it is not clear how to do this for arbitrary lattices.

We shall need to deform the path that the zipper takes. In general deforming the zipper while keeping its endpoints fixed has no effect on $\bar{G}'_{u,v}$, unless the zipper is deformed across either u or v. If the zipper is moved across u in the direction of the arrow on the zipper, then $\bar{G}'_{u,v}$ decreases by $\bar{G}_{u,v}$, and similarly, moving the zipper across v (in the direction of the arrow) increases $\bar{G}'_{u,v}$ by $\bar{G}_{u,v}$. We can also move the endpoint of the zipper by adding a new zipper edge (k,ℓ) (or removing an old one) near the endpoint of the zipper, which of course just adds (or removes) one term to the summations (15) and (16).

7.2.3 Closed-form evaluation of $\bar{G}'$ on $\mathbb{Z}^2$

Next we evaluate $\bar{G}'$ for $\mathbb{Z}^2$. The first step is to rotate the entire lattice 180° about the terminal square of the zipper. The rotation of course preserves the lattice $\mathbb{Z}^2$, and maps u and v to $(1,1)-u$ and $(1,1)-v$ respectively, but now the zipper goes up to infinity rather than down to infinity. Let $\bar{G}'^{\uparrow}$ denote $\bar{G}'$ with the repositioned zipper. We have

$$\bar{G}'_{u,v} = \bar{G}'^{\uparrow}_{(1,1)-u,(1,1)-v}.$$

The next step is to deform the zipper so that it once again goes downwards. We can deform the initial segment of the zipper so that it goes downwards, then circles around back up along a large-radius circle, and then continues back up as before. By Lemma 7.2, the summands along the zipper starting with the large-radius circle and the subsequent path to infinity are negligible. So we have

$$\bar{G}'_{u,v} = \bar{G}'^{\uparrow}_{(1,1)-u,(1,1)-v} = \bar{G}'_{(1,1)-u,(1,1)-v} + \text{another term},$$

where "another term" refers to the term for the zipper being deformed across u or v.

Next we move the location of the start of the zipper, translating it by $v+u-(1,1)$, by adding a finite number of new zipper edges. Then we deform the zipper again, making it go straight down; we have to add another term if the zipper gets deformed across either $(1,1)-u$ or $(1,1)-v$. Because the lattice $\mathbb{Z}^2$ is invariant under such translations, translating the starting face of the zipper is equivalent to translating the vertices in the opposite direction. Thus we have

$$\bar{G}'_{u,v} = \bar{G}'_{v,u} + \text{finite number of terms}.$$

Finally we use the antisymmetry of $\bar{G}'_{u,v}$:

$$\bar{G}'_{u,v} = \frac{\text{finite number of terms}}{2}.$$

This procedure is perhaps better explained by way of an example. We can write

$$\begin{aligned}
\bar{G}'_{(0,0),(2,1)} &= \bar{G}'^{\uparrow}_{(1,1),(-1,0)} \\
&= \bar{G}'_{(1,1),(-1,0)} + \bar{G}_{(1,1),(-1,0)} \\
&= \bar{G}'_{(2,1),(0,0)} + (\bar{G}_{(1,1),(0,0)}\bar{G}_{(0,1),(-1,0)} - \bar{G}_{(1,1),(0,1)}\bar{G}_{(0,0),(-1,0)}) + \bar{G}_{(1,1),(-1,0)} \\
&= \frac{(\bar{G}_{(1,1),(0,0)}\bar{G}_{(0,1),(-1,0)} - \bar{G}_{(1,1),(0,1)}\bar{G}_{(0,0),(-1,0)}) + \bar{G}_{(1,1),(-1,0)}}{2} \\
&= \frac{(\frac{1}{\pi}\frac{1}{\pi} - \frac{1}{4}\frac{1}{4}) - \frac{1}{4} + \frac{2}{\pi}}{2} \\
&= \frac{1}{2\pi^2} + \frac{1}{\pi} - \frac{5}{32}
\end{aligned}$$

In like manner we can compute $\bar{G}'_{u,v}$ for any pair of vertices u and v in $\mathbb{Z}^2$. The answer will always be in $\mathbb{Q} + \frac{1}{\pi}\mathbb{Q} + \frac{1}{\pi^2}\mathbb{Q}$.

7.2.4 Closed-form evaluation of $\bar{G}'$ on the triangular lattice

We can compute $\bar{G}'$ on the triangular lattice in essentially the same manner as for $\mathbb{Z}^2$. The key properties of the lattice that we used is that it is invariant under 180° rotations, and that for any pair of vertices there is a lattice-invariant translation that maps the first vertex to the second vertex.

7.2.5 Closed-form evaluation of $\bar{G}'$ on the hexagonal lattice

The hexagonal lattice is invariant under 180° rotations and is vertex-transitive. However, there are not lattice-invariant translations between any pair of vertices: we can partition the vertices into two color classes, black and white, such that any lattice-preserving translation will map the black vertices to the black vertices and the white vertices to the white vertices.

Suppose u is a black vertex and v is a white vertex. After a 180° rotation about a hexagon, u is mapped to a white vertex u' and v is mapped to a black vertex v'. We can then translate u' to v and v' to u while preserving the lattice. This allows us to compute $\bar{G}'_{u,v}$ when u is black and v is white (or vice versa).

Since $\bar{G}'_{u,v}$ is harmonic in both u and v (except along the zipper), when u and v have the same color, $\bar{G}'_{u,v}$ can be expressed as

$$\bar{G}'_{u,v} = \frac{1}{3}(\bar{G}'_{u,w_1} + \bar{G}'_{u,w_2} + \bar{G}'_{u,w_3})$$

(plus another term if one of the edges (v, w_i) crosses the zipper), where the w_i's are the neighbors of v, and the right-hand side we can compute by the above method.

7.3 Cutting edges

Suppose that in the vector bundle setting, we know the Green's function $\mathscr{G} = \mathscr{G}^{\mathcal{G}}$ for a graph $\mathcal{G}$, and we wish to know the Green's function for the graph $\mathcal{G} \setminus \{s, t\}$ obtained by deleting an edge $\{s, t\}$ of $\mathcal{G}$. Recall that $c_{s,t}$ denotes the conductance of edge (s, t), and let us denote by τ the parallel transport to s from t, so that $\Delta^{\mathcal{G}}_{s,t} = -c_{s,t}\tau$ and $\Delta^{\mathcal{G}}_{t,s} = -c_{s,t}\tau^*$. Then it is readily checked that

$$\mathscr{G}^{\mathcal{G}\setminus\{s,t\}}_{u,v} = \mathscr{G}_{u,v} - \frac{(\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*)(\mathscr{G}_{s,v} - \tau\mathscr{G}_{t,v})}{\alpha_{s,t}} \tag{17}$$

where

$$\alpha_{s,t} = \mathscr{G}_{s,s} + \mathscr{G}_{t,t} - \mathscr{G}_{s,t}\tau^* - \tau\mathscr{G}_{t,s} - 1/c_{s,t} \tag{18}$$

(which is a scalar). Indeed, if we let $f(u, v)$ denote the purported Green's function on the right-hand side of (17), then $f(u, v) = 0$ when either u or v is the boundary, and we have

$$\sum_v f(u, v)\Delta^{\mathcal{G}}_{v,w} = \delta_{u,w} - \frac{(\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*)(\delta_{s,w} - \tau\delta_{t,w})}{\alpha_{s,t}}.$$

If $w \neq s$ and $w \neq t$ then $\Delta^{\mathcal{G}\setminus\{s,t\}}_{v,w} = \Delta^{\mathcal{G}}_{v,w}$, so

$$\sum_v f(u, v)\Delta^{\mathcal{G}\setminus\{s,t\}}_{v,w} = \delta_{u,w} \qquad \text{(if } w \neq s \text{ and } w \neq t\text{)}.$$

Suppose now $w = s$. Then

$$\begin{aligned}
\sum_v f(u,v)\Delta^{\mathcal{G}\setminus\{s,t\}}_{v,s} &= \sum_v f(u,v)\Delta^{\mathcal{G}}_{v,s} + f(u,t)c_{t,s}\tau^* - f(u,s)c_{t,s} \\
&= \delta_{u,s} - \frac{\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*}{\alpha_{s,t}} + \left[\mathscr{G}_{u,t} - \frac{(\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*)(\mathscr{G}_{s,t} - \tau\mathscr{G}_{t,t})}{\alpha_{s,t}}\right] c_{t,s}\tau^* \\
&\qquad\qquad - \left[\mathscr{G}_{u,s} - \frac{(\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*)(\mathscr{G}_{s,s} - \tau\mathscr{G}_{t,s})}{\alpha_{s,t}}\right] c_{t,s} \\
&= \delta_{u,s} - \frac{\mathscr{G}_{u,s} - \mathscr{G}_{u,t}\tau^*}{\alpha_{s,t}} c_{t,s}\left[\frac{1}{c_{t,s}} + \alpha_{s,t} + (\mathscr{G}_{s,t}\tau^* - \mathscr{G}_{t,t} - \mathscr{G}_{s,s} + \tau\mathscr{G}_{t,s})\right] \\
&= \delta_{u,s}
\end{aligned}$$

by the choice of $\alpha_{s,t}$. The case $w = t$ is similar.

Let us return to the line bundle setting, with a zipper monodromy of z, that we are interested in near $z = 1$. If (s,t) is a zipper edge then $\tau = z$ or $\tau = 1/z$ (depending on the zipper direction), and otherwise $\tau = 1$. Let $\tau' = \partial_z\tau|_{z=1}$. Recall that $G = \mathscr{G}|_{z=1}$ and $G' = \partial_z\mathscr{G}|_{z=1}$. From (17) it is evident that

$$G^{\mathcal{G}\setminus\{s,t\}}_{u,v} = G_{u,v} - \frac{(G_{u,s} - G_{u,t})(G_{s,v} - G_{t,v})}{a_{s,t}} \tag{19}$$

where

$$a_{s,t} = G_{s,s} + G_{t,t} - 2G_{s,t} - 1/c_{s,t}. \tag{20}$$

We have

$$\begin{aligned}
\partial_z\alpha_{s,t} &= \partial_z\mathscr{G}_{s,s} + \partial_z\mathscr{G}_{t,t} - (\partial_z\mathscr{G}_{s,t})\tau^* - \mathscr{G}_{s,t}\partial_z\tau^* - (\partial_z\tau)\mathscr{G}_{t,s} - \tau\partial_z\mathscr{G}_{t,s} \\
\partial_z\alpha_{s,t}|_{z=1} &= 0.
\end{aligned}$$

Using this, we can differentiate (17) with respect to the zipper monodromy z and set $z = 1$ to obtain

$$(G^{\mathcal{G}\setminus\{s,t\}}_{u,v})' = G'_{u,v} - \frac{(G'_{u,s} - G'_{u,t} + G_{u,t}\tau')(G_{s,v} - G_{t,v}) + (G_{u,s} - G_{u,t})(G'_{s,v} - G'_{t,v} - \tau' G_{t,v})}{a_{s,t}} \tag{21}$$

One final edge-cutting formula is

$$\frac{Z^{\mathcal{G}\setminus\{s,t\}}}{Z^{\mathcal{G}}} = 1 - c_{s,t}(G_{s,s} + G_{t,t} - 2G_{s,t}). \tag{22}$$

This holds because directed edge (s,t) occurs in a uniform spanning tree of $\mathcal{G}$ with probability $c_{s,t}(G_{s,s}-G_{s,t})$, and likewise directed edge (t,s) may occur in the spanning tree.

These formulas (19), (20), (21) and (22) of course apply to the modified graph $\bar{\mathcal{G}}$ simply by replacying G and G' with $\bar{G}$ and $\bar{G}'$. These formulas are well-suited to computer computations, so in later sections we shall use $\bar{G}$ and $\bar{G}'$ as needed on graphs with one more cut edges without showing the steps used to compute these quantities.

8 Loop-erased random walk

In this section we compute the probability that the LERW in $\mathbb{Z}^2$ from $(0,0)$ to ∞ passes through the vertex $(1,0)$, and the analogous probabilities for other points and for other lattices.

8.1 General remarks

We let $P_{v,w}$ denote the probability that the LERW started from $(0,0)$ to ∞ passes through edge (v,w), in the direction from v to w. Likewise we let P_w denote the probability that the LERW passes through vertex w. It is straightforward that

$$P_w = \sum_{v:v\sim w} P_{w,v} = \sum_{v:v\sim w} P_{v,w} + \delta_{w,0}.$$

Our strategy is compute these edge probabilities.

We find it conceptually convenient to work with finite graphs $\mathcal{G}$, set up our equations for spanning tree and grove event probabilities in terms of the finite-graph Green's function $G^{\mathcal{G}}$ and its derivative $(G^{\mathcal{G}})'$ using the formulas from section 6, and then afterwards take the limit $\mathcal{G}\to\mathbb{L}$ using the formulas from section 7. We are at liberty to use any convenient sequence $\mathcal{G}_k$ of finite graphs that converge to the lattice $\mathbb{L}$, since the limiting measure on spanning trees of $\mathbb{L}$ is independent of the choice of $\mathcal{G}_k$, and the event that the LERW from $(0,0)$ to ∞ uses a given edge is a measureable event in the limiting measure. So we choose a sequence $\mathcal{G}_k$ for which it is convenient to compute $(G^{\mathcal{G}_k})'$, as described in section § 7.2. These graphs $\mathcal{G}_k$ have a wired boundary vertex (see Figure 3) that we will label ∞, even though the graphs are finite. The other vertices of $\mathcal{G}_k$ we will label by their coordinates in $\mathbb{L}$. We define $P^{\mathcal{G}}_{v,w}$ and $P^{\mathcal{G}}_w$ in the same way as we defined $P_{v,w}$ and P_w; for fixed v and w, $\lim_{\mathcal{G}\to\mathbb{L}} P^{\mathcal{G}}_{v,w} = P_{v,w}$.

We remark that once an edge traversal probability $P^{\mathbb{L}}_{v,w}$ has been computed, finding the corresponding probability $P^{\mathbb{L}}_{w,v}$ for the reversed edge is straightforward:

Lemma 8.1 ([Ken00b]).

$$P^{\mathbb{L}}_{v,w} - P^{\mathbb{L}}_{w,v} = c_{v,w}(\bar{G}_{0,v}-\bar{G}_{0,w})$$

Proof. Let the origin 0 be at the center of a large box with wired boundary whose size we will send to infinity. For simple random walk started at 0, the expected number of traversals of (v, w) minus the expected number of traversals of (w, v) is the edge conductance $c_{v,w}$ times the difference in Green's functions. The same also holds for loop-erased random walk started at 0, since cycles are reversible. □

The intensity of the undirected edge $\{v, w\}$ is $P^{\mathbb{L}}_{\{v,w\}} = P^{\mathbb{L}}_{v,w} + P^{\mathbb{L}}_{w,v}$, and the undirected edge intensities turn out to be nicer numbers than the directed edge intensities. The vertex intensities are easily calculated from the undirected edge intensities, and given the potential kernel of $\mathbb{L}$, the directed edge intensities are easily recovered from the undirected intensities.

We show below how to calculate the specific edge intensities relevant to the $P_{(1,0)}$ computation before describing the general method that can be used for any edge.

8.2 Loop-erased random walk on $\mathbb{Z}^2$

We first recall the potential kernel for $\mathbb{Z}^2$ in Figure 10:

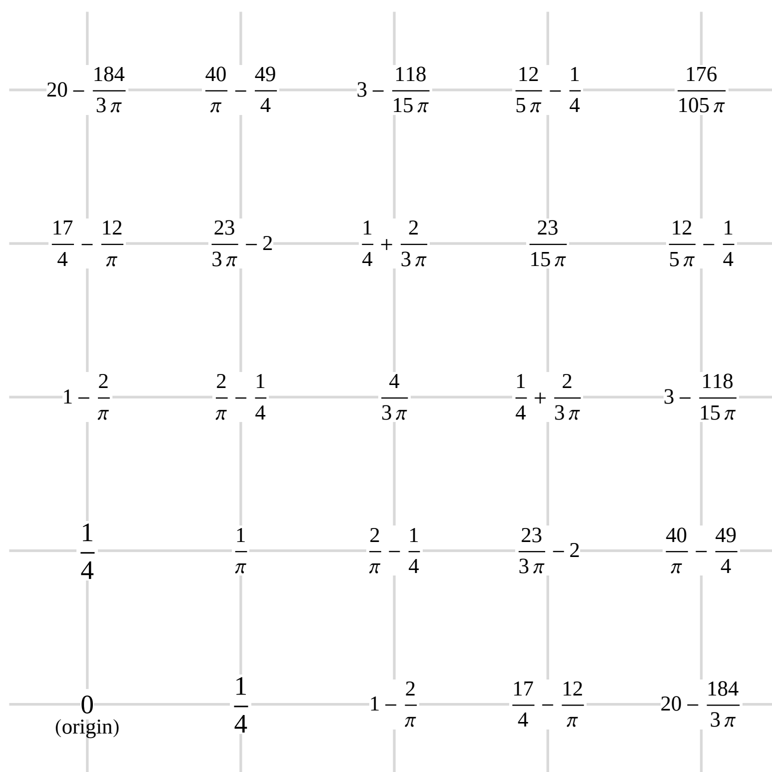
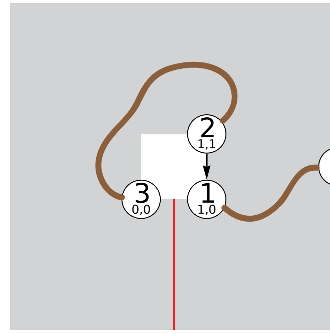


Figure 10: Square lattice potential kernel.

8.2.1 Probability LERW uses edge $(1,1)(1,0)$



To compute $P_{(1,1),(1,0)}$, the probability that the LERW uses edge $(1,1)(1,0)$, in fact we use the annular surface graph shown in Figure 3, and shown more schematically on the right. There are four nodes, whose coordinates are $(1,0)$, $(1,1)$, $(0,0)$, and ∞, and which we also refer to as nodes 1, 2, 3, and 4.

Consider a uniformly random spanning tree of this graph $\mathcal{G}$, and within it the path from $(0,0)$ to ∞, i.e., the path from node 3 to node 4. Those spanning trees in which the path from $(0,0)$ to ∞ contains the directed edge $(1,1)(1,0)$ are in bijective correspondence with groves with connection type $3,2|1,4$; the bijection is simply to delete the edge $(1,1)(1,0)$ from the spanning tree, or in the reverse direction, to add the edge to the grove. Therefore,

$$P^{\mathcal{G}}_{(1,1),(1,0)} = \frac{Z[3,2|1,4]}{Z[1,2,3,4]} = -\bar{G}'^{\bar{\mathcal{G}}}_{1,2} - G'^{\bar{\mathcal{G}}}_{2,3} - G'^{\bar{\mathcal{G}}}_{3,1} + G^{\bar{\mathcal{G}}}_{2,3} - G^{\bar{\mathcal{G}}}_{1,3}$$

by (10c) from section 6. Using the method described in section 7, we (or the computer) can compute

$$\begin{array}{c|ccc} \bar{G}^{\mathbb{Z}^2} & (1,0) & (1,1) & (0,0) \\ \hline (1,0) & 0 & -\frac{1}{4} & -\frac{1}{4} \\ (1,1) & -\frac{1}{4} & 0 & -\frac{1}{\pi} \\ (0,0) & -\frac{1}{4} & -\frac{1}{\pi} & 0 \end{array} \qquad \begin{array}{c|ccc} \bar{G}'^{\mathbb{Z}^2} & (1,0) & (1,1) & (0,0) \\ \hline (1,0) & 0 & -\frac{3}{32} & -\frac{5}{32} \\ (1,1) & \frac{3}{32} & 0 & -\frac{1}{2\pi} \\ (0,0) & \frac{5}{32} & \frac{1}{2\pi} & 0 \end{array}$$

We can now evaluate $\lim_{\mathcal{G}\to\mathbb{Z}^2} P^{\mathcal{G}}_{(1,1),(1,0)}$ by substituting $\bar{G}^{\mathbb{Z}^2}$ for $\bar{G}^{\mathcal{G}} = G^{\bar{\mathcal{G}}}$ and $\bar{G}'^{\mathbb{Z}^2}$ for $\bar{G}'^{\mathcal{G}} = (G^{\bar{\mathcal{G}}})'$:

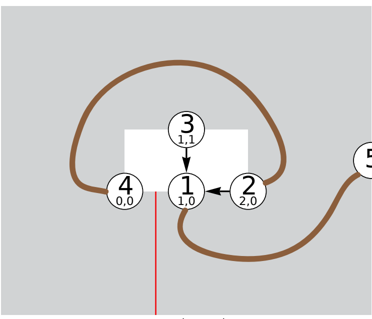
$$\begin{aligned} P^{\mathbb{Z}^2}_{(1,1),(1,0)} &= \frac{3}{32} + \frac{1}{2\pi} - \frac{5}{32} - \frac{1}{\pi} + \frac{1}{4} = +\frac{3}{16} - \frac{1}{2\pi} \\ P^{\mathbb{Z}^2}_{\{(1,0),(1,1)\}} &= \frac{1}{8}. \end{aligned} \tag{23}$$

With $P^{\mathbb{Z}^2}_{(1,1),(1,0)}$ in hand, it is possible to compute $P^{\mathbb{Z}^2}_{(1,0)}$ using an *ad hoc* argument and existing techniques [BP93, Ken97] for computing finite-dimensional marginals of the random spanning tree. However, here we take the systematic approach of computing the probabilities that various edges are on the LERW path.

8.2.2 Probability LERW uses edge $(2,0)(1,0)$

Next we show here how to compute $P^{\mathbb{Z}^2}_{(2,0),(1,0)}$ using the methods developed in sections 6 and 7. The vertices of interest are $(0,0)$, $(2,0)$, $(1,0)$, and ∞, so we make them the nodes of an annular-one graph. Since vertices $(0,0)$ and $(2,0)$ are not on the boundary of the same face in $\mathbb{Z}^2$, we cut (undirected) edge $(1,0)(1,1)$ of the lattice, so that all the nodes (other than ∞) are on the boundary of the same face. The endpoints of any cut edge also become vertices of interest, so they are included as nodes in the resulting annular graph. Thus our annular graph has five nodes: $(1,0)$, $(2,0)$, $(1,1)$, $(0,0)$, and ∞, which we also refer to as nodes 1, 2, 3, 4, and 5. Nodes $1,2,3,4$ are arranged counterclockwise on the inner face with a zipper between nodes 1 and 4, as in section 6.

Next we relate the spanning trees of the original (uncut) graph to groves in the graph without (undirected) edge $\{(1,0),(1,1)\}$. Suppose the LERW from $(0,0)$ to ∞ passes through edge $(2,0)(1,0)$ (which we also call edge $2 \to 1$). There are three cases depending on whether or not edge $(1,1)(1,0)$ or $(1,0)(1,1)$ (edge $3 \to 1$ or $1 \to 3$) also occurs in the tree:



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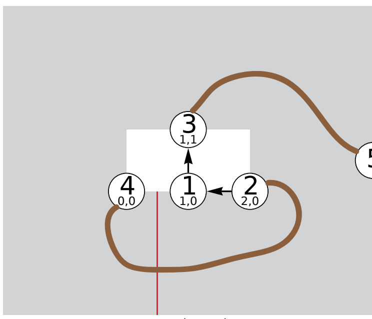

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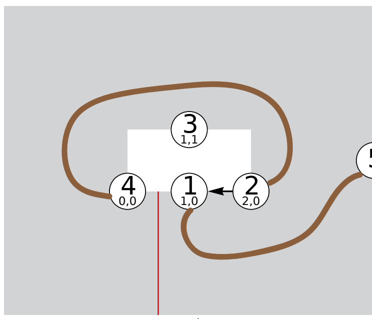

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1. Edge $3 \to 1$ occurs in the tree. Trees in this case are in bijective correspondence with groves of the cut graph of type $4,2|3|1,5$, where the bijection deletes edges $2 \to 1$ and $3 \to 1$ from the tree to get the grove, or adds them to the grove to get the tree.

2. Edge $1 \to 3$ occurs in the tree. Trees in this case are in bijective correspondence with groves of the cut graph of type $4,2|1|3,5$ (with the obvious bijection).

3. Neither $3 \to 1$ nor $1 \to 3$ occurs in the tree. Trees in this case are in bijective correspondence with groves of type $4,2|1,5$ (with the obvious bijection). (When a node is not listed in the partition, it is treated as internal, and could in principle occur in any of the parts, though in this case it would necessarily occur in the $4,2$ part.)

Putting these three cases together, we obtain

$$P^{\mathcal{G}}_{(2,0),(1,0)} = \frac{Z^{\tilde{\mathcal{G}}}[4,2|3|1,5] + Z^{\tilde{\mathcal{G}}}[4,2|1|3,5] + Z^{\tilde{\mathcal{G}}}[4,2|1,5]}{Z^{\mathcal{G}}}$$

$$P^{\mathcal{G}}_{(2,0),(1,0)} = \left[\frac{Z^{\tilde{\mathcal{G}}}[4,2|3|1,5]}{Z^{\tilde{\mathcal{G}}}} + \frac{Z^{\tilde{\mathcal{G}}}[4,2|1|3,5]}{Z^{\tilde{\mathcal{G}}}} + \frac{Z^{\tilde{\mathcal{G}}}[4,2|1,5]}{Z^{\tilde{\mathcal{G}}}}\right] \times \frac{Z^{\tilde{\mathcal{G}}}}{Z^{\mathcal{G}}}$$

Each of the expressions within the brackets is of a type which section 6 shows us how to compute in terms of $G^{\tilde{\mathcal{G}}}$ and $G'^{\tilde{\mathcal{G}}}$. These are

$$\frac{Z[4,2|3|1,5]}{Z} = \left\{\begin{matrix} -G_{1,3}G'_{2,3} + G'_{1,3}G_{2,3} - G'_{1,4}G_{2,3} + G_{1,4}G_{2,3} + G_{1,3}G'_{2,4} - G_{1,3}G_{2,4} \\ -G'_{1,2}G_{3,3} + G'_{1,4}G_{3,3} - G_{1,4}G_{3,3} - G'_{2,4}G_{3,3} + G_{2,4}G_{3,3} - G_{1,3}G'_{3,4} \\ +G_{2,3}G'_{3,4} + G'_{1,2}G_{3,4} - G'_{1,3}G_{3,4} + G_{1,3}G_{3,4} + G'_{2,3}G_{3,4} - G_{2,3}G_{3,4} \end{matrix}\right\}$$

$$\frac{Z[4,2|1|3,5]}{Z} = \left\{\begin{matrix} G_{1,2}G'_{1,3} - G'_{1,2}G_{1,3} - G_{1,2}G'_{1,4} + G_{1,3}G'_{1,4} + G'_{1,2}G_{1,4} - G'_{1,3}G_{1,4} \\ -G_{1,1}G'_{2,3} + G_{1,4}G'_{2,3} + G_{1,1}G'_{2,4} - G_{1,3}G'_{2,4} - G_{1,1}G'_{3,4} + G_{1,2}G'_{3,4} \end{matrix}\right\}$$

$$\frac{Z[4,2|1,5]}{Z} = -G'_{1,2} + G'_{1,4} - G_{1,4} - G'_{2,4} + G_{2,4}$$

From section 7 we know how to compute $\bar{G}^{\tilde{\mathcal{G}}}$ and $\bar{G}'^{\tilde{\mathcal{G}}}$ and also the ratio $Z^{\tilde{\mathcal{G}}}/Z^{\mathcal{G}}$ in the limit $\mathcal{G} \to \mathbb{Z}^2$. These values are

$$\lim_{\mathcal{G}\to\mathbb{Z}^2} \frac{Z^{\tilde{\mathcal{G}}}}{Z^{\mathcal{G}}} = \frac{1}{2},$$

and

$\bar{G}^{\mathbb{Z}^2\setminus\{(1,0)(1,1)\}}$	$(1,0)$	$(2,0)$	$(1,1)$	$(0,0)$
$(1,0)$	$\frac{1}{8}$	$\frac{1}{2\pi} - \frac{3}{8}$	$-\frac{3}{8}$	$\frac{1}{2\pi} - \frac{3}{8}$
$(2,0)$	$\frac{1}{2\pi} - \frac{3}{8}$	$\frac{1}{8} + \frac{2}{\pi^2} - \frac{1}{\pi}$	$\frac{1}{8} - \frac{3}{2\pi}$	$-\frac{7}{8} + \frac{2}{\pi^2} + \frac{1}{\pi}$
$(1,1)$	$-\frac{3}{8}$	$\frac{1}{8} - \frac{3}{2\pi}$	$\frac{1}{8}$	$\frac{1}{8} - \frac{3}{2\pi}$
$(0,0)$	$\frac{1}{2\pi} - \frac{3}{8}$	$-\frac{7}{8} + \frac{2}{\pi^2} + \frac{1}{\pi}$	$\frac{1}{8} - \frac{3}{2\pi}$	$\frac{1}{8} + \frac{2}{\pi^2} - \frac{1}{\pi}$

and

$\bar{G}'^{\mathbb{Z}^2\setminus\{(1,0)(1,1)\}}$	$(1,0)$	$(2,0)$	$(1,1)$	$(0,0)$
$(1,0)$	0	$\frac{1}{16\pi} - \frac{3}{32}$	$-\frac{3}{16}$	$\frac{7}{16\pi} - \frac{9}{32}$
$(2,0)$	$\frac{3}{32} - \frac{1}{16\pi}$	0	$\frac{1}{32} - \frac{9}{16\pi}$	$-\frac{9}{16} + \frac{3}{2\pi^2} + \frac{5}{8\pi}$
$(1,1)$	$\frac{3}{16}$	$\frac{9}{16\pi} - \frac{1}{32}$	0	$\frac{1}{32} - \frac{9}{16\pi}$
$(0,0)$	$\frac{9}{32} - \frac{7}{16\pi}$	$\frac{9}{16} - \frac{3}{2\pi^2} - \frac{5}{8\pi}$	$\frac{9}{16\pi} - \frac{1}{32}$	0

Upon substituting and simplifying we obtain

$$P^{\mathbb{Z}^2}_{(2,0),(1,0)} = -\frac{5}{16} + \frac{1}{\pi} \qquad\qquad P^{\mathbb{Z}^2}_{\{(1,0),(2,0)\}} = \frac{1}{8}. \tag{24}$$

8.2.3 Probability LERW passes through $(1,0)$ (sandpile value)

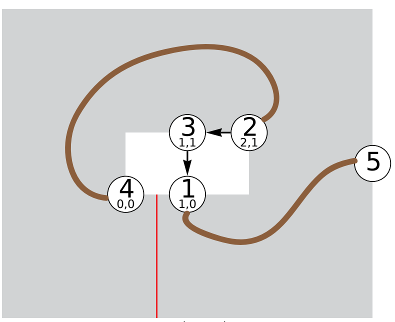
Combining (23) and (24) together with the symmetry relations $P^{\mathbb{Z}^2}_{(1,-1),(1,0)} = P^{\mathbb{Z}^2}_{(1,1),(1,0)}$ and $P^{\mathbb{Z}^2}_{(0,0),(1,0)} = 1/4$, we obtain

$$P^{\mathbb{Z}^2}_{(1,0)} = \frac{5}{16}. \tag{25}$$

8.2.4 Probability LERW uses edge $(2,1)(1,1)$

We delete edge $(1,0)(1,1)$ from $\mathbb{Z}^2$'s finite graph approximation $\mathcal{G}$, the same edge that we deleted in section 8.2, and again view $\tilde{\mathcal{G}} = \mathcal{G} \setminus (1,0)(1,1)$ as an annulus with 5 nodes, but this time nodes 1–5 are $(1,0)$, $(2,1)$, $(1,1)$, $(0,0)$, and ∞.

For spanning trees that use edge $(2,1)(1,1)$, there are again three cases depending on whether or not and in what direction the cut edge $\{(1,0),(1,1)\}$ is used. Spanning trees in each of these cases are as before in bijective correspondence with groves pairings on the annulus:



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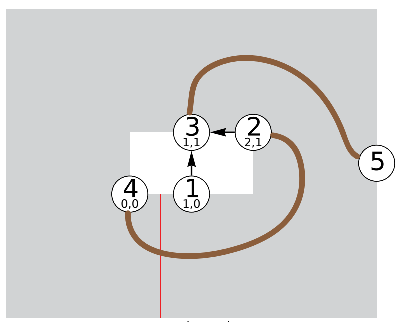

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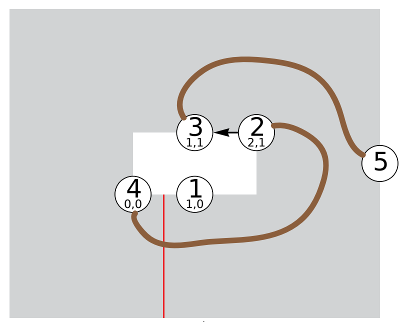


42|35

At this point it should be clear how to carry out the computation, which we did by computer, so we just state the answer:

$$P^{\mathbb{Z}^2}_{(2,1)(1,1)} = +\frac{3}{16} + \frac{1}{4\pi^2} - \frac{5}{8\pi} \qquad P^{\mathbb{Z}^2}_{\{(1,1),(2,1)\}} = \frac{1}{8} + \frac{1}{2\pi^2} - \frac{1}{4\pi}. \tag{26}$$

8.2.5 Probability LERW passes through $(1,1)$

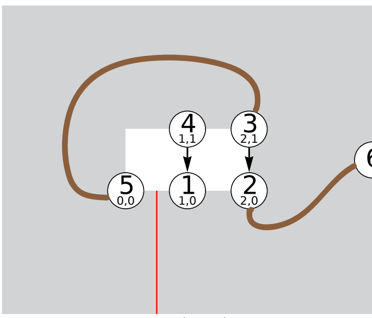
Since vertex $(1,1)$ lies on a symmetry axis, two edge probabilities suffice to determine $P^{\mathbb{Z}^2}_{(1,1)}$. Combining (26) and (23) we obtain

$$P^{\mathbb{Z}^2}_{(1,1)} = \frac{1}{4} + \frac{1}{2\pi^2} - \frac{1}{4\pi} = 0.221083\ldots \tag{27}$$

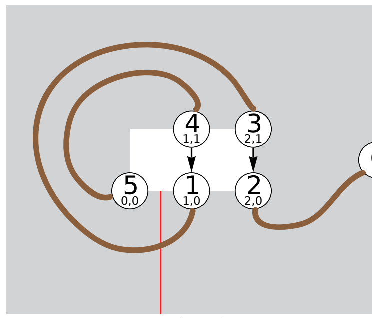
This compares well with the empirical probability of 0.221088 based on 10^7 LERW's from the center of a 1400×1400 grid with wired boundary.

8.2.6 Probability LERW uses edge $(2,1)(2,0)$

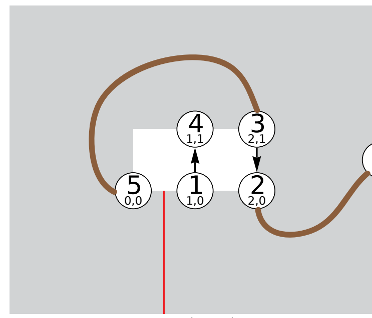
Again we delete edge $(1,0)(1,1)$ from $\mathbb{Z}^2$'s finite graph approximation $\mathcal{G}$. This time we need six nodes, which in order are $(1,0)$, $(2,0)$, $(2,1)$, $(1,1)$, $(0,0)$, and ∞. There are three main cases as before, depending on whether or not and in what direction the spanning tree of $\mathcal{G}$ uses edge $(1,0)(1,1)$ (which we now refer to as $1 \to 4$ or $4 \to 1$), but now there are several subcases. The edge we are studying is $3 \to 2$:



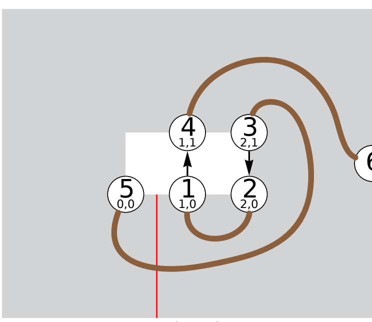
53|26|4



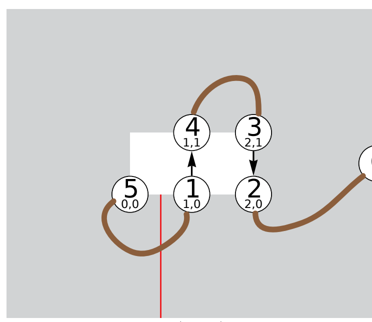
54|13|26



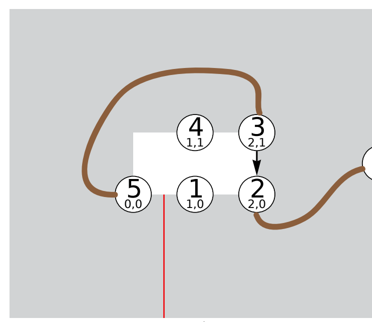
53|26|1



53|21|46



51|43|26



53|26

Once we have this list of cut edges, nodes, and grove types, we can carry out the computation and obtain

$$P^{\mathbb{Z}^2}_{(2,1)(2,0)} = \frac{5}{8} - \frac{3}{8\pi^2} - \frac{7}{4\pi} \qquad\qquad P^{\mathbb{Z}^2}_{\{(2,0),(2,1)\}} = -\frac{3}{4\pi^2} + \frac{1}{2\pi}. \tag{28}$$

8.3 General edge intensities

The calculations in section 8.2 illustrate the general method for computing the probability that LERW passes through an arbitrary given edge. We identify a set of edges to cut so as to place the starting point of the LERW and the endpoints of the edge on the same face. These three vertices, together with the endpoints of all of the cut edges, comprise the nodes on the inner boundary of the annulus. We number these nodes in counterclockwise order so

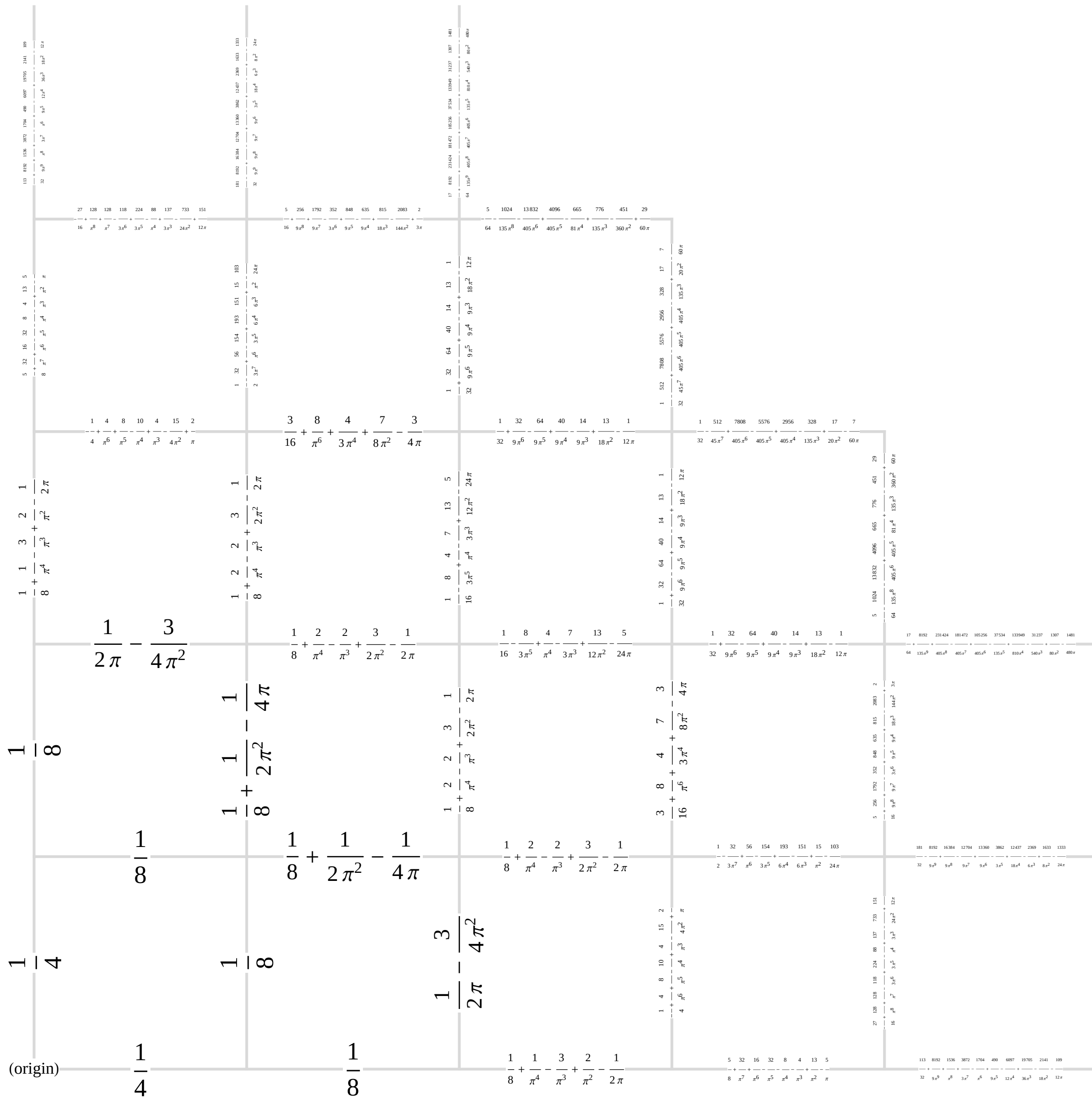


Figure 11: Undirected edge intensities of loop-erased random walk on $\mathbb{Z}^2$. The edge-intensities of $(x,x)(x,x-1)$ and $(x,x-1)(x+1,x-1)$ are identical for $x=1,2,3$ (and perhaps larger values), despite there being no lattice symmetry that would imply this.

that the zipper starts between nodes 1 and $n-1$. The vertex labeled ∞ becomes node n, which is on the other boundary of the annulus. For each possible set of the cut edges that might occur within a spanning tree of the original graph $\mathcal{G}$, we associate spanning trees containing precisely those cut edges with groves in the cut graph $\tilde{\mathcal{G}}$. We can enumerate all possible compatible grove partitions, and then rewrite these as a linear combination of partial pairings as described in section 6.4. Using the methods described in sections 6 and 7, we compute the probability that the resulting grove is of the appropriate type for the edge of interest to be on the path from $(0,0)$ to ∞.

8.4 Loop-erased random walk on the triangular lattice

In this section we sketch the computation showing that on the triangular lattice, a LERW from $(0,0)$ to ∞ passes through vertex $(1,0)$ with probability $5/18$. (We choose coordinates for which vertex (x,y) is the one located at position $(x+y/2, y\sqrt{3}/2)$ in the plane.) The starting point of the calculations is of course the potential kernel, which is shown in Figure 12. The undirected edge intensities are shown in Figure 13.

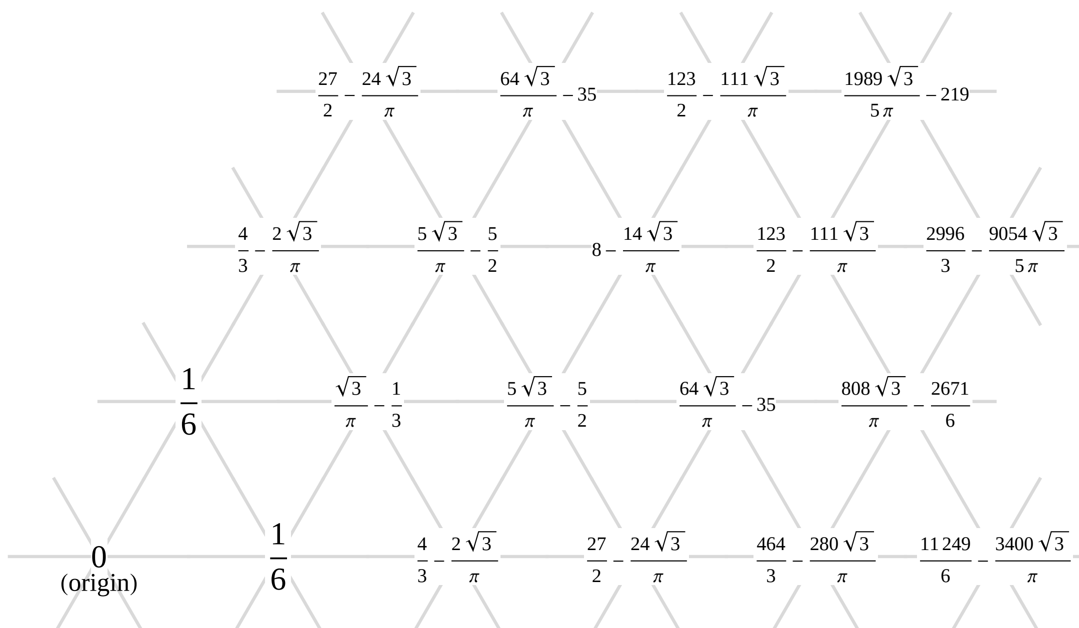
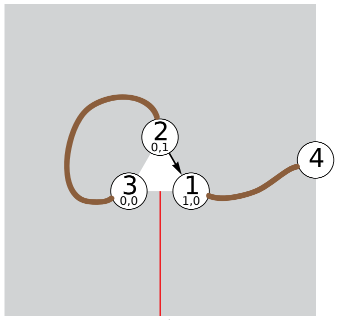


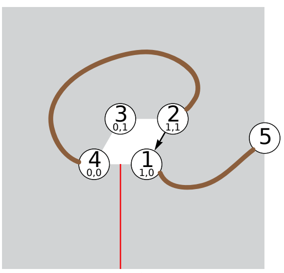
Figure 12: Triangular lattice potential kernel.

The following diagrams indicate how to compute the probability that, respectively, edge $(0,1)(1,0)$, edge $(1,1)(1,0)$ and edge $(2,0)(1,0)$ are on the LERW from $(0,0)$ to ∞. The methods of sections 7 and 8 automate these computations.

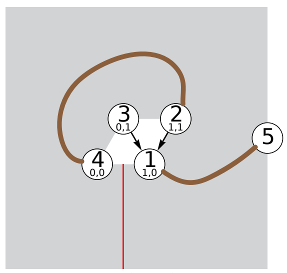


32|14

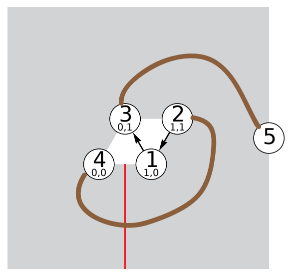
$$P^{\text{triangular}}_{(0,1),(1,0)} = \frac{1}{24}$$
$$P^{\text{triangular}}_{\{(1,0),(0,1)\}} = \frac{1}{12}.$$



42|15

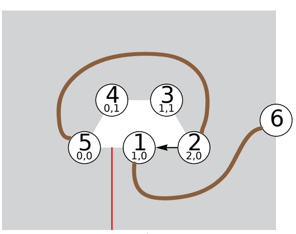


42|15|3

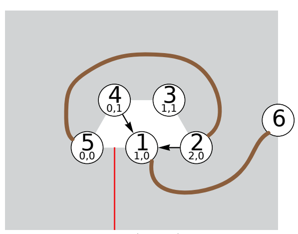


42|35|1

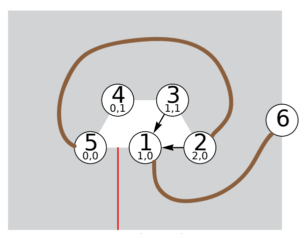
$$P^{\text{triangular}}_{(1,1),(1,0)} = \frac{31}{108} - \frac{\sqrt{3}}{2\pi}$$
$$P^{\text{triangular}}_{\{(1,0),(1,1)\}} = \frac{2}{27}.$$



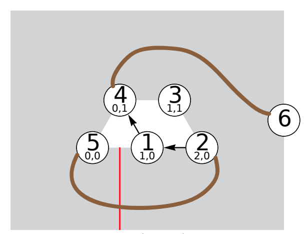
52|16



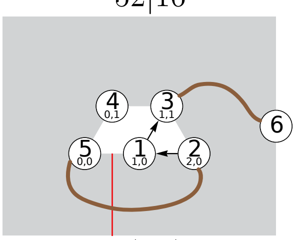
52|16|4



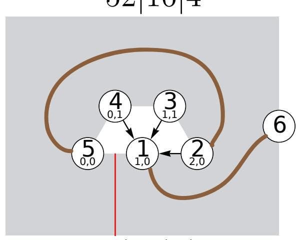
52|16|3



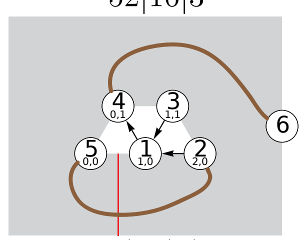
52|46|1



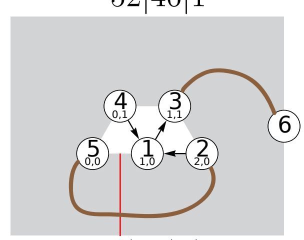
52|36|1



52|16|3|4



52|46|3|1



52|36|1|4

$$P^{\text{triangular}}_{(2,0),(1,0)} = \frac{\sqrt{3}}{\pi} - \frac{59}{108} \qquad P^{\text{triangular}}_{\{(1,0),(2,0)\}} = \frac{2}{27}.$$

Combining these probabilities we obtain the "sandpile value" of $P^{\text{triangular}}_{(1,0)} = \frac{5}{18}$.

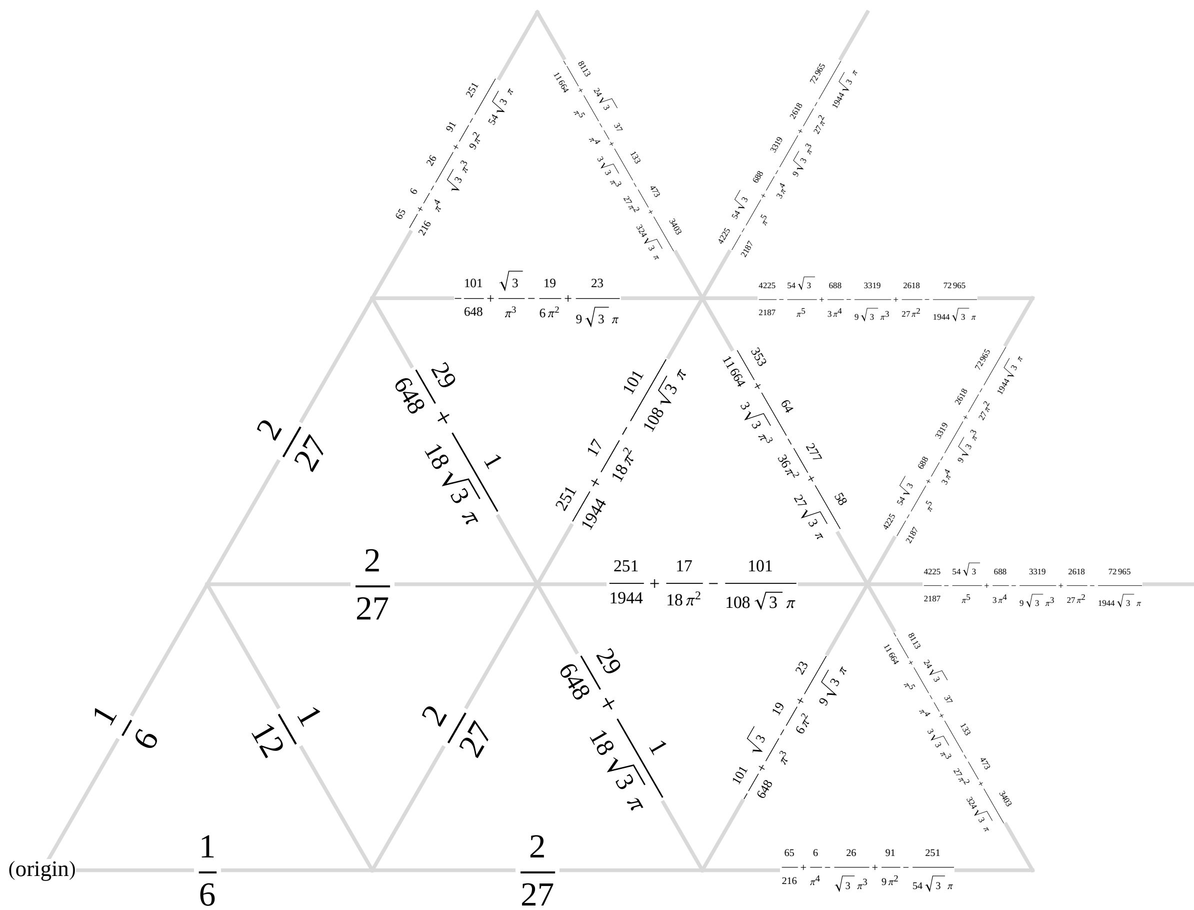


Figure 13: Undirected edge intensity of loop-erased random walk on the triangular lattice. The edge-intensities of $(x,x)(x,x-1)$ and $(x,x-1)(x+1,x-1)$ are identical for $x=1,2$ (and perhaps larger values), despite there being no lattice symmetry that would imply this.

8.5 Loop-erased random walk on the hexagonal lattice

The potential kernel of the hexagonal lattice is shown in Figure 14. Using the methods described above, we computed the undirected edge intensities for a LERW from $(0,0)$ to ∞, which are shown in Figure 15.

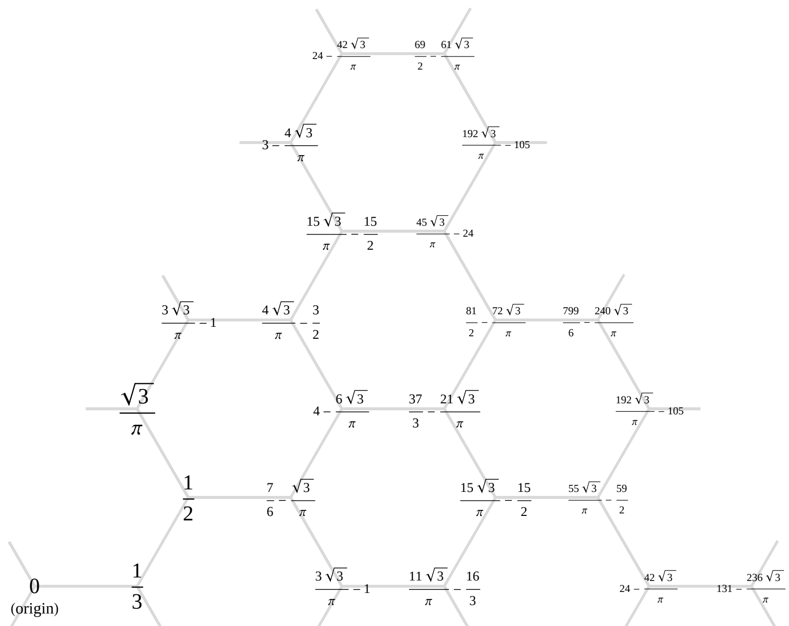


Figure 14: Hexagonal lattice potential kernel.

8.6 Loop-erased random walk on $\mathbb{Z}\times\mathbb{R}$

We consider next a weighted version of $\mathbb{Z}^2$, where each horizontal edge has weight c, and each vertical edge has weight $1/c$. This graph is isoradial, so we may compute the Green's function using [Ken02]. Because the lattice is symmetric under a 180° rotation and invariant under translations, we can also compute G'. This gives us all the necessary information we need to compute the probability that LERW from $(0,0)$ passes through $(1,0)$. It is convenient to let $c=\tan\theta$. After a computation similar to the ones above, we find that the LERW passes through vertex $(1,0)$ with probability

$$\frac{1}{4}+\frac{\theta}{2\pi}-\frac{\theta^2}{\pi^2\sin^2\theta}\left(1-\frac{2\theta}{\pi}\right).$$

When $\theta=\pi/4$, we have $c=1$, and this above probability reduces to $5/16$, in agreement with our earlier calculation for $\mathbb{Z}^2$.

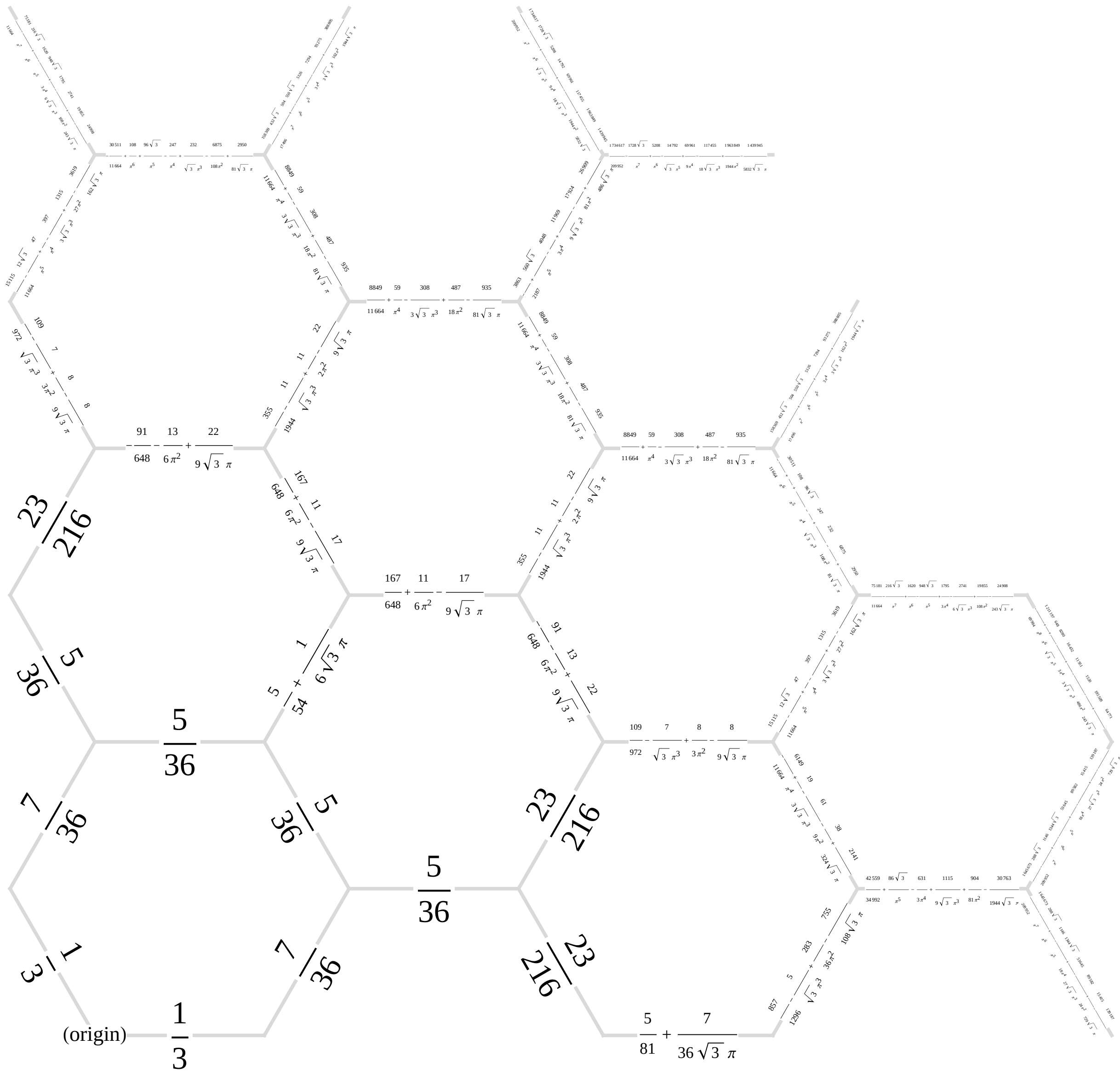


Figure 15: Undirected edge intensity of loop-erased random walk on the hexagonal lattice. Using coordinates for which vertex (x, y) is the one located at position $(x - y/2, y\sqrt{3}/2)$, the edge intensities $(3x-1, 3x-1)(3x-1, 3x-2)$ and $(3x-1, 3x-2)(3x, 3x-2)$ are identical, as are the intensities for edges $(3x, 1)(3x+1, 2)$ and $(3x, 1)(3x, 0)$ (for $x = 1, 2$ and perhaps larger values), despite there being no lattice symmetry that would imply this.

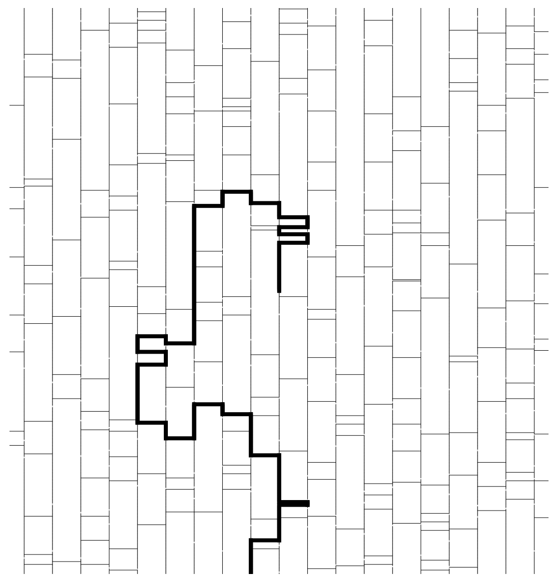


Figure 16: Portion of uniform spanning tree and LERW from $(0,0)$ to ∞ on $\mathbb{Z} \times (\mathbb{Z}/20)$, which approximates $\mathbb{Z} \times \mathbb{R}$. The probability that the LERW from $(0,0)$ passes through $(1,0)$ is $1/4 - 1/\pi^2$.

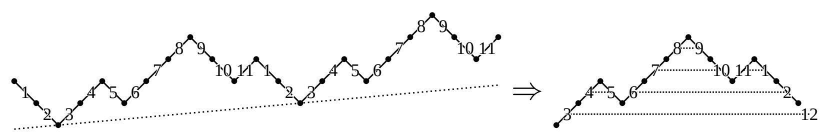
In the isoradial embedding of the lattice into the plane, if the horizonal edges have length 1, then the vertical edges have length c. An interesting special case is the limit $c \to 0$. Then random walk on this weighted graph converges to a standard Brownian motion in the vertical direction, except at a Poisson set of times with intensity 1, where the walk jumps left or right with equal probability. The random walk on this graph is then a continuous-time random walk on $\mathbb{Z}$ in the horizontal direction and a Brownian motion on $\mathbb{R}$ in the vertical direction. Figure 16 illustrates a portion of the uniform spanning tree on this graph, together with the path from $(0,0)$ to ∞. From the above formula, we see that in this limit, the probability that the LERW passes through $(1,0)$ converges to $1/4 - 1/\pi^2$.

A The annular-one matrix A_n

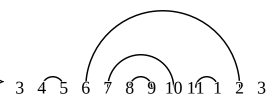
A.1 Association of rows and columns

Recall the matrix A_n for the annulus with $n-1$ nodes on one boundary and 1 node on the other boundary. We prove that A_n is invertible. More specifically, we prove that $\det \mathsf{A}_n$ is a polynomial of degree $2^{n-2} - \frac{1}{2}\binom{n}{n/2}$ with lead coefficient ± 1 and constant coefficient ± 1.

A key part of the proof is a natural correspondence between the rows and columns of the matrix A_n. This correspondence is based on the "cycle lemma" bijection of Dvoretzky and Motzkin [DM47] (see also [DZ90]). This bijection maps a cyclic sequences of $k+1$ $+$'s and k $-$'s to a Dyck path of length $2k$ together with an "offset", a number between 1 and $2k+1$. (There is a unique $+$ that can be deleted so that the subsequent sequence defines a Dyck path; the position of this $+$ is the "offset". Given the Dyck path and offset, the Dyck path is interpreted as a sequence of $+$'s and $-$'s, a $+$ is prepended to the this sequence, and the offset determines a cyclic shift.) In our setting, $k = n/2-1$, in $\det \mathscr{L}_R^S$, the indices in R give the locations of the $+$'s, and the indices in S give the locations of the $-$'s. The offset that the cycle-lemma bijection gives (a number from 1 to $n-1$) determines the index with which n is paired, and the chords under the Dyck path determine the rest of the pairing. The following example illustrates the bijection adapted to $(n-1,1)$-annular pairings:

$\begin{smallmatrix}\{1,2,5,9,10,12\}\\\{3,4,6,7,8,11\}\end{smallmatrix} \Rightarrow$  $\Rightarrow \begin{smallmatrix}5|2|10|9|1|12\\4|6|7|8|11|3\end{smallmatrix}$

and in the reverse direction,

$1,11|2,6|3,12|4,5|7,10|8,9 \Rightarrow$  $\Rightarrow \begin{smallmatrix}5|2|10|9|1|12\\4|6|7|8|11|3\end{smallmatrix} \Rightarrow \begin{smallmatrix}5,2,10,9,1,12\\4,6,7,8,11,3\end{smallmatrix}.$

A.2 Invertibility

We prove more than Theorem 6.1:

Theorem A.1. *The determinant of the annular-one matrix,* $\det A_n$*, as a polynomial in* ζ*, takes the value* 1 *when* $\zeta = 0$*, has degree* $2^{n-2} - \frac{1}{2}\binom{n}{n/2}$*, and has leading coefficient* ± 1*.*

Proof. It is convenient to work with a slightly different annular matrix A_n^* which corresponds to a symmetrized placement of the zipper, with a zipper starting from each interval of the first boundary between each pair of nodes. We do this by splitting the original zipper into $n-1$ zippers each with parallel transport $z^{1/(n-1)}$, and then deforming these zippers so that their endpoints lie in each of the $n-1$ intervals between the nodes. When we deform a zipper across node $i (\neq n)$ in the counterclockwise direction, then the parallel transport from i to any other node j is multiplied by $z^{1/(n-1)}$. For each column of the annular matrix, say indexed by directed pairing σ, the column is scaled by $z^{\mp 1/(n-1)}$ according to whether node i is a source or destination in σ. Likewise, each row of the annular matrix, say indexed by $\det \mathscr{L}_R^S$, is scaled by $z^{\pm 1/(n-1)}$ according to whether $i \in R$ or $i \in S$. Thus the effect of deforming these zippers is to conjugate the annular matrix by a diagonal matrix, so in particular $\det \mathsf{A}_n^* = \det \mathsf{A}_n$. We change variables to

$$w = z^{2/(n-1)} = \zeta^{1/(n-1)}$$

so that the nonzero entries of A_n^* are integral powers of w. For example, the first two rows of A_6^* are

$$
\begin{array}{r@{\;}c}
 & \begin{array}{cccccccccc}
\vdots\mathscr{L}\left[{}^{2}_{1}\middle|{}^{4}_{3}\middle|{}^{6}_{5}\right] &
\vdots\mathscr{L}\left[{}^{3}_{2}\middle|{}^{4}_{1}\middle|{}^{6}_{5}\right] &
\vdots\mathscr{L}\left[{}^{1}_{5}\middle|{}^{3}_{2}\middle|{}^{6}_{4}\right] &
\vdots\mathscr{L}\left[{}^{2}_{1}\middle|{}^{3}_{5}\middle|{}^{6}_{4}\right] &
\vdots\mathscr{L}\left[{}^{5}_{4}\middle|{}^{2}_{1}\middle|{}^{6}_{3}\right] &
\vdots\mathscr{L}\left[{}^{1}_{5}\middle|{}^{2}_{4}\middle|{}^{6}_{3}\right] &
\vdots\mathscr{L}\left[{}^{4}_{3}\middle|{}^{1}_{5}\middle|{}^{6}_{2}\right] &
\vdots\mathscr{L}\left[{}^{5}_{4}\middle|{}^{1}_{3}\middle|{}^{6}_{2}\right] &
\vdots\mathscr{L}\left[{}^{3}_{2}\middle|{}^{5}_{4}\middle|{}^{6}_{1}\right] &
\vdots\mathscr{L}\left[{}^{4}_{3}\middle|{}^{5}_{2}\middle|{}^{6}_{1}\right]
\end{array} \\
\begin{array}{r}
\det\mathscr{L}_{1,3,5}^{2,4,6} \\
\det\mathscr{L}_{2,1,5}^{3,4,6}
\end{array} &
\left[\begin{array}{cccccccccc}
1 & -w & 0 & 0 & -w & 0 & 0 & 0 & w^2 & -w^3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & w^4 & -w & 0
\end{array}\right]
\end{array}
$$

and the other rows are determined by cyclic rotations.

From the determinant formula, we see that the diagonal entries of A_n^* are all 1. Consider the column indexed by directed pairing σ. Since A_n^* is symmetric under cyclic rotations of the indices $1,\dots,n-1$, let us assume for convenience that σ pairs $n-1$ to n, so that we can write

$$\sigma = {}^{a_{1,1}}_{a_{1,0}}\big|\cdots\big|^{a_{n/2-1,1}}_{a_{n/2-1,0}}\big|^{\,n}_{n-1}.$$

Referring to the above bijection, since $n-1$ pairs to n, for each j we have $a_{j,0} < a_{j,1}$. Column σ contains $2^{n/2-1}$ nonzero entries, one for each sequence $f_1,\dots,f_{n/2-1}$ of $n/2-1$ 0's and 1's, where the row is indexed by

$$\det\mathscr{L}_{a_{1,f_1},\,\dots,\,a_{n/2-1,f_{n/2-1}},\,n-1}^{a_{1,1-f_1},\,\dots,\,a_{n/2-1,1-f_{n/2-1}},\,n}.$$

Since the pair $(a_{j,0},a_{j,1})$ crosses $a_{j,1}-a_{j,0}$ zippers, this pair contributes w^{f_j} to the matrix entry. In particular, all nonzero nondiagonal entries of A_n^* have positive powers of w. This implies

$$\det\mathsf{A}_n|_{\zeta=0} = \det\mathsf{A}_n^*|_{w=0} = 1,$$

which of course implies that the annular matrix is nonsingular.

We can also compute the degree of the polynomial $\det\mathsf{A}_n^*$. For a given column, the row that maximizes the power of w is the one for which $f_0,\dots,f_{n/2-1} = 1,\dots,1$, and the power is the area under the Dyck path. The mapping from a column σ to the row $\det\mathscr{L}_R^S$ which has the highest power of w is also a bijection, in fact it is a simple variant of the bijection described above. This implies that the leading coefficient of the polynomial $\det\mathsf{A}_n^*$ is ±1, and the degree is

$$\deg\det\mathsf{A}_n^* = (n-1)\times\sum_{\text{Dyck paths of length } n-2}\text{area under Dyck path.}$$

For Dyck paths of length $2k = n-2$, the above sum is known (see Sloane's A008549) to be

$$4^k - \binom{2k+1}{k} = 2^{n-2} - \binom{n-1}{n/2-1} = 2^{n-2} - \frac{1}{2}\binom{n}{n/2}.$$

Consequently, the degree of the polynomial $\det\mathsf{A}_n$ (a polynomial in ζ) is $2^{n-2}-\frac{1}{2}\binom{n}{n/2}$. $\square$

A.3 Cancellation of higher order derivatives

Next we restate and prove Theorem 6.3:

Theorem A.2. *For all positive even n and pairings σ of $\{1,\dots,n\}$, $Z_\sigma/Z_{1|2|\cdots|n}$ is a polynomial of degree $n/2$ in the quantities*

$$\{L_{i,j} : 1 \le i < j \le n\} \quad \textit{and} \quad \{L'_{i,j} : 1 \le i < j \le n-1\}.$$

Proof of Theorem 6.3. For a general finite graph with general parallel transports, define $\ddot{\mathscr{Z}}[\sigma]$ as in the case of a planar graph or annular-one graph. Let us then define D_R^S to be the result of multiplying row (R,S) of the matrix A_n by the vector of $\ddot{\mathscr{Z}}[\sigma]$'s. For annular graphs we have $D_R^S = \det \mathscr{L}_R^S$, but for general graphs these two quantities will be different. We can "recover" the $\ddot{\mathscr{Z}}[\sigma]$'s from these D_R^S's by multiplying by A_n^{-1}:

$$\frac{\mathscr{Z}[\sigma]}{\mathscr{Z}[1|2|\cdots|n]} = \frac{1}{(1-\zeta)^{m_\sigma}} \sum_{R,S} \alpha_{\sigma,R,S}(\zeta) D_R^S, \tag{29}$$

where

$$\alpha_{\sigma,R,S} = (1-\zeta)^{m_\sigma} (\mathsf{A}_n^{-1})_{\sigma,R,S}$$

and $m_\sigma \ge 0$ is the smallest integer such that for each (R,S), the coefficient $(1-\zeta)^{m_\sigma}(\mathsf{A}_n^{-1})_{\sigma,R,S}$ is nonsingular at $\zeta = 1$.

Let us consider now the complete graph on n nodes, so that $\mathscr{L} = -\Delta$ and $\mathscr{L}_{i,j}$ is the edge weight between nodes i and j, times the parallel transport to i from j. Note that in this case $\mathscr{L}$ is (except for the diagonal entries) a general Hermitian matrix. Each D_R^S is a polynomial in the entries of $\mathscr{L}$, with coefficients that involve powers of ζ. (For the complete graph, any matrix times the vector of $\ddot{\mathscr{Z}}[\sigma]$'s will yield polynomials in $\mathscr{L}$.)

Next we change variables by setting $\zeta = e^t$. This makes it more convenient to differentiate the annulus response matrix, since $\frac{d}{dt}\mathscr{L}_{j,i} = -\frac{d}{dt}\mathscr{L}_{i,j}$. The $\zeta \to 1$ limit is of course equivalent to $t \to 0$, and $(1-e^t)^m$ has a zero of order m at $t=0$, with $\frac{d^m}{dt^m}(1-e^t)^m = (-1)^m m!$.

For general nonzero edge weights and smooth (in t) parallel transports on the complete graph, $\mathscr{Z}[\sigma]$ is finite and $\mathscr{Z}[1|2|\cdots|n] = 1$, so for any $\ell < m_\sigma$ it must be that we get zero when we differentiate the numerator from (29), i.e., $\sum_{R,S} \alpha_{\sigma,R,S}(\zeta) D_R^S$, ℓ times with respect to t and then set t to 0:

$$0 = \sum_{k=0}^{\ell} \binom{\ell}{k} \sum_{R,S} \frac{d^{\ell-k}}{dt^{\ell-k}} \alpha_{\sigma,R,S}\Big|_{t=0} \times \frac{d^k}{dt^k} D_R^S\Big|_{t=0}.$$

Since $\mathscr{L}$ is generic, we can rescale the kth derivative of each $\mathscr{L}_{i,j}$ by a factor of β^k, and deduce that for each k,ℓ with $k \le \ell < m_\sigma$

$$0 = \sum_{R,S} \frac{d^{\ell-k}}{dt^{\ell-k}} \alpha_{\sigma,R,S}\Big|_{t=0} \times \frac{d^k}{dt^k} D_R^S\Big|_{t=0}. \tag{30}$$

Next we write $D_R^S()$ for the polynomial function of a Hermitian matrix, which, when evaluated on the response matrix $\mathscr{L}$ of the complete graph, gives $D_R^S = D_R^S(\mathscr{L})$. Let d_i denote the differential operator for which $d_i \mathscr{L}_{i,j} = \frac{1}{2}\frac{d}{dt}\mathscr{L}_{i,j}$ and $d_i \mathscr{L}_{j,i} = \frac{1}{2}\frac{d}{dt}\mathscr{L}_{j,i}$ but $d_i \mathscr{L}_{h,j} = 0$ for $h, j \neq i$. For the complete graph, each monomial of the polynomial D_R^S includes each index i exactly once. (This also holds for annular graphs, though of course the polynomials are different.) Using this property of these polynomials, we can write the kth derivative of $\mathscr{L} = \mathscr{L}(t)$ as follows:

$$\frac{d^k}{dt^k} D_R^S(\mathscr{L}) = \sum_{i_1, i_2, \dots \in \{1,\dots,n\}} d_{i_1} \cdots d_{i_k}\ D_R^S(\mathscr{L}). \tag{31}$$

Given a set T of k nodes, for each $i \in T$ and each j we rescale $\frac{d}{dt}\mathscr{L}_{i,j}\Big|_{t=0}$ by a factor of β, without changing any of the other derivatives at $t = 0$. If $i, j \in T$, then we rescale $\frac{d^2}{dt^2}\mathscr{L}_{i,j}\Big|_{t=0}$ by a factor of β^2. (Recall that $\mathscr{L}$ is generic Hermitian, so we can do this.) The coefficient of β^k within the kth derivative is obtained from (31) by including only those terms for which $i_1, \dots, i_k$ is a permutation of T. Then substituting (31) into (30) and taking the coefficient of β^k, we find

$$0 = \sum_{R,S} \frac{d^{\ell-k}}{dt^{\ell-k}} \alpha_{\sigma,R,S}\Big|_{t=0} \times D_R^S(\mathscr{L}^{(T)})\Big|_{t=0}, \tag{32}$$

where $\mathscr{L}^{(T)}$ where is the Hermitian matrix obtained from $\mathscr{L}$ by replacing $\mathscr{L}_{i,j}$ with $\frac{d}{dt}\mathscr{L}_{i,j}$ for each $i \in T$ and $j \notin T$ or $i \notin T$ and $j \in T$, and replacing $\mathscr{L}_{i,j}$ with $\frac{d^2}{dt^2}\mathscr{L}_{i,j}$ for each $i, j \in T$.

From our definition of D_R^S, for the complete graph we have

$$D_R^S(\mathscr{L}^{(T)}) = \sum_\tau (\mathsf{A}_n)_{R,S,\tau} \prod_{\substack{\{i,j\}\in\tau \\ i\in R, j\in S}} \mathscr{L}_{i,j}^{(T)},$$

where the sum is over directed pairings τ that happen to be annular. Next we take the pairing σ, and for each pair $\{i, j\}$ of σ, we rescale $\mathscr{L}_{i,j}^{(T)}$ by a factor β. Then the coefficient of $\beta^{n/2}$ in (32) only arises when $\tau = \sigma$ in the above sum, so

$$0 = \sum_{R,S} \frac{d^{\ell-k}}{dt^{\ell-k}} \alpha_{\sigma,R,S}\Big|_{t=0} \times (\mathsf{A}_n)_{R,S,\tau} \times \prod_{\substack{\{i,j\}\in\sigma \\ i\in R, j\in S}} \mathscr{L}_{i,j}^{(T)}\Big|_{t=0}, \tag{33}$$

The product term is the formula always takes the same (generically nonzero) value, except for a sign, which is given by the parity of the number of indices in T which are also in S.

So we cancel this factor (keeping the sign), rewrite α in terms of A_n, and obtain

$$0 = \sum_{R,S:\sigma \text{ pairs } R \text{ to } S} \frac{d^{\ell-|T|}}{dt^{\ell-|T|}} \left[(\mathsf{A}_n^{-1}(e^t))_{\sigma,R,S}(1-e^t)^{m_\sigma}\right]\Big|_{t=0} \times (-1)^{|S\cap T|}, \tag{34}$$

whenever $|T| \le \ell < m$. This is purely an identity about the matrix A_n^{-1}.

Next we return to annular graphs, and differentiate the numerator $m = m_\sigma$ times:

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{R,S} \frac{d^{m-k}}{dt^{m-k}} \alpha_{\sigma,R,S}\Big|_{t=0} \times \frac{d^k}{dt^k} \det \mathscr{L}_R^S\Big|_{t=0}.$$

We can rewrite the kth derivative of $\det \mathscr{L}_R^S$ using the differential operators d_i, as in (31). Let T be the set of nodes for which we applied d_i an odd number of times. If for some i we applied d_i more than once, then the size of the set T will be less than k, and then from (34) we see that the coefficient of such terms is 0.

Next, suppose the variable $\mathscr{L}_{i,n}$ is differentiated. Since n is in each set S, we may as well replace T with $T \setminus \{n\}$ and introduce a global sign, but then since $|T|$ is smaller, we see from (34) that the coefficient of such terms is 0. □

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