Nagata embedding and $\mathscr{A}$-schemes

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Abstract

We define the notion of normal $\mathscr{A}$-schemes, and approximable $\mathscr{A}$-schemes. Approximable $\mathscr{A}$-schemes inherit many good properties of ordinary schemes. As a consequence, we see that the Zariski-Riemann space can be regarded in two ways – either as the limit space of admissible blow ups, or as the universal compactification of the given non-proper scheme. We can prove Nagata embedding using Zariski-Riemann spaces.

Contents

0 Introduction

We introduced the concept of $\mathscr{A}$-schemes in [T]. In this paper, we will investigate further properties of $\mathscr{A}$-schemes, mainly focusing on Zariski-Riemann spaces.

First, we will show that there is a normalization of $\mathscr{A}$-schemes, just as for ordinary schemes. This is important, since we are aiming for an analog of Zariski's main theorem.

One of the advantage of introducing $\mathscr{A}$-schemes is that we can simplify the proof of Nagata embedding theorem; it can be proven intuitively, as in the original paper of Nagata [N]. Note that the essential part is already proven in Corollary 4.4.6 of [T]. Compare with the proof of Conrad [C], which only uses ordinary schemes, but is long (approximately 50 pages).

Also, we introduce the notion of approximable $\mathscr{A}$-schemes: an $\mathscr{A}$-scheme is approximable if it is a (filtered) projective limit of ordinary schemes. This notion is convenient, since locally free sheaves on approximable schemes always come from a pull back of a localy free sheaves on ordinary schemes. At the same time, we see that the Zariski-Riemann space defined in [T] is identified with the conventional one, namely the limit space of U-admissible blow ups along the exceptional locus $X \setminus U$ where U is an open subscheme of a scheme X. This shows that the conventional Zariski-Riemann space has the desirable universal property in the category of $\mathscr{A}$-schemes, not only with schemes.

This paper is organized as follows. In section 1, we quickly review the definitions and properties of $\mathscr{A}$-schemes, which plays the central role in this paper. In section 2, we construct the normalization functor of $\mathscr{A}$-schemes. In section 3, we give a notion of approximable $\mathscr{A}$-schemes. This actually determines the pro-category of the category ($\mathscr{Q}$-**Sch**) of ordinary schemes. In section 4, we will give a proof of the original version of Nagata embedding, which says that any separated scheme of finite type can be embedded as an open subscheme of a proper scheme.

Notation and conventions. In this paper, the algebraic type $\mathscr{A}$ is always that of rings. When we say ordinary schemes, we treat only coherent schemes and quasi-compact morphisms between them; to emphasize this assumption and to distinguish ordinary coherent schemes from $\mathscr{A}$-schemes, we will say $\mathscr{Q}$-schemes instead of coherent schemes.

For an $\mathscr{A}$-scheme X, the description $|X|$ stands for the underlying topological space, which is coherent.

An $\mathscr{A}$-scheme X will be called *integral* if it is irreducible and reduced. This condition is in fact, stronger than assuming any section ring $\mathscr{O}_X(U)$ is integral.

A morphism of $\mathscr{A}$-schemes is *proper*, if it is separated and universally closed. We *do not include the condition "of finite type"*.

1 A brief review of $\mathscr{A}$-schemes

In this section, we will recall some terminologies and definitions in [T]. A good reference for general lattice theories is [S].

A topological space X is *coherent*, if it is sober, quasi-compact, quasi-separated, and has a quasi-compact open basis.

The category (**Coh**) of coherent spaces and quasi-compact morphisms is isomorphic to the opposite category $(\mathbf{DLat})^{\mathrm{op}}$ of distributive lattices by the functor $C(-)_{\mathrm{cpt}}$. For a coherent space X, we may regard $C(X)_{\mathrm{cpt}}$ as the set of quasi-compact open subsets of X, or the set of their complements.

Therefore, we can regard an $\mathscr{A}$-valued sheaf on a coherent space X as a continuous covariant functor $C(X)_{\mathrm{cpt}} \to \mathscr{A}$.

On a coherent space X, there is a canonical (**DLat**)-valued sheaf τ_X on X, which is defined by $U \mapsto C(U)_{\mathrm{cpt}}$ for quasi-compact open U; this extends uniquely to the entire Zariski site of X.

We have a functor $\alpha_1 : (\mathbf{Rng}) \to (\mathbf{Dlat})$ from the category of commutative rings, which sends a ring R to the set of finitely generated ideals of R modulo the relation $I^2 = I$. Note that this gives the usual spectrum of rings, when combined with the previous isomorphism $C(-)_{\mathrm{cpt}}$.

Also, we have a natural homomorphism $R \to \alpha_1(R)$ of multiplicative monoids, sending $a \in R$ to the principal ideal generated by a. This homomorphism commutes with localizations.

An $\mathscr{A}$-scheme is a triple $X = (|X|, \mathscr{O}_X, \beta_X)$, where

(i) $|X|$ is a coherent space (the "underlying space"),

(ii) $\mathscr{O}_X$ is a ring-valued sheaf on $|X|$ (the "structure sheaf"), and

(iii) $\beta_X : \alpha_1\mathscr{O}_X \to \tau_X$ is a morphism of (**DLat**)-valued sheaves (the "support morphism"),

which satisfies the following condition: for an inclusion $V \hookrightarrow U$ of quasi-compact open subsets of $|X|$, the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ factors through $\mathscr{O}_X(U)_Z$, where $\mathscr{O}_X(U)_Z$ is the localization along the multiplicative system

$$\{a \in \mathscr{O}_X(U) \mid \beta_X(a) \subset Z\}.$$

By this property, $\mathscr{A}$-schemes are locally ringed spaces. A morphism of $\mathscr{A}$-schemes $f = (f, f^{\#}) : X \to Y$ is a morphism of ringed spaces, which commutes with the support morphism:

$$\begin{array}{ccc} \alpha_1\mathscr{O}_Y & \xrightarrow{\alpha_1 f^\#} & f_*\alpha_1\mathscr{O}_X \\ {\scriptstyle\beta_Y}\downarrow & & \downarrow{\scriptstyle\beta_X} \\ \tau_Y & \xrightarrow[f^{-1}]{} & f_*\tau_X \end{array}$$

The category ($\mathscr{A}$**-Sch**) of $\mathscr{A}$-schemes is complete, and co-complete.

We have a fully faithful functor ($\mathscr{Q}$**-Sch**) $\to$ ($\mathscr{A}$**-Sch**), which preserves pull backs and finite patchings by quasi-compact opens.

For the definition of a morphism of profinite type, we refer to [T].

2 Normalization

In this section, we fix an integral base $\mathscr{A}$-scheme S, and any $\mathscr{A}$-scheme is integral, of profinite type over S. We denote by (**Int.** $\mathscr{A}$**-Sch**) the category of integral $\mathscr{A}$-schemes of profinite type over S and dominant morphisms.

Definition 2.1. An $\mathscr{A}$-scheme X is *normal*, if the ring of every stalk $\mathscr{O}_{X,x}$ is integrally closed.

Remark 2.2. We do not assume Noetherian property on normal rings (or schemes) in this paper.

Theorem 2.3. Let (**N.** $\mathscr{A}$**-Sch**) be the full subcategory of (**Int.** $\mathscr{A}$**-Sch**), consisting of normal schemes, and U : (**N.** $\mathscr{A}$**-Sch**) $\to$ (**Int.** $\mathscr{A}$**-Sch**) be the underlying functor. Then, U has a right adjoint 'nor'. Moreover, the counit $\eta : U \circ \mathrm{nor} \Rightarrow \mathrm{Id}$ is proper dominant.

We will refer to this right adjoint as the *normalization functor*.

Proof. The proof is somewhat long, so we will divide it into several steps. The construction of the normalization functor is analogous to that of Zariski-Riemann spaces, described in detail in [T]. We will denote by R^{nor} the integral closure of a given integral domain R in the sequel.

Step 1: First, we will construct the underlying space of the normalization of a given integral $\mathscr{A}$-scheme X. Let $\mathscr{N}_0^X$ be the set of finite sets of pairs (U, α), where

(a) U is a quasi-compact open subset of X, and

(b) $\alpha \in \mathscr{O}_X(U)^{\mathrm{nor}} \setminus \{0\}$.

Let $\mathfrak{a} = \{(U_i, \alpha_i)\}_i$, $\mathfrak{b} = \{(V_j, \beta_j)\}_j$ be two elements of $\mathscr{N}_0^X$. We define two operations $+, \cdot$ on $\mathscr{N}_0^X$ by

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{a} \cup \mathfrak{b}, \mathfrak{a} \cdot \mathfrak{b} = \{(U_i \cap V_j, \alpha_i \beta_j)\}_{ij}$$

For a pair (U, α), define $U[\alpha]$ as

$$U[\alpha] = \{x \in U \mid x \text{ is in the image of } \operatorname{Spec} \mathscr{O}_{X,x}[\alpha^{-1}] \to \operatorname{Spec} \mathscr{O}_{X,x}\},$$

where $\alpha^{-1} = \{a^{-1}\}_{a \in \alpha}$. For two elements $\mathfrak{a} = \{(U_i, \alpha_i)\}_i$, $\mathfrak{b} = \{(V_j, \beta_j)\}_j$, the relation $\mathfrak{a} \prec \mathfrak{b}$ holds if

(a) $U_i[\alpha_i] \subset \cup_j V_j[\beta_j]$ for any i, and

(b) For any $x \in U_i[\alpha_i]$, set $J_x = \{j \mid x \in V_j[\beta_j]\}$. Then $(\beta_j)_{j \in J_x}$ generates the unit ideal in $\mathscr{O}_{X,x}^{\mathrm{nor}}[\alpha_i^{-1}]$.

Let $\approx$ be the equivalence relation generated by $\prec$, and set $\mathscr{N}^X = \mathscr{N}_0^X / \approx$. The addition and multiplication of $\mathscr{N}_0^X$ descends to $\mathscr{N}^X$, which makes $\mathscr{N}^X$ into a distributive lattice. Set $|X^{\mathrm{nor}}| = \operatorname{Spec} \mathscr{N}^X$. This is the underlying space of the normalization X^{nor}.

Step 2: There is a natural homomorphism $C(X)_{\mathrm{cpt}} \to \mathscr{N}^X$ of distributive lattices, defined by $Z \mapsto \{(Z, 1)\}$. This defines a quasi-compact morphism $\pi : |X^{\mathrm{nor}}| \to |X|$ of coherent spaces.

Step 3: Let p be a point of $|X^{\mathrm{nor}}|$, and set $x = \pi(p)$. Then,

$$\mathfrak{p} = \{a \in \mathscr{O}_{X,x}^{\mathrm{nor}} \mid (X, a) \leq p\}$$

becomes a prime ideal of $\mathscr{O}_{X,x}^{\mathrm{nor}}$. Let R_p be the localization of $\mathscr{O}_{X,x}^{\mathrm{nor}}$ by $\mathfrak{p}$. Then, R_p dominates $\mathscr{O}_{X,x}$.

Step 4: The structure sheaf $\mathscr{O}_{X^{\mathrm{nor}}}$ is defined by

$$U \mapsto \{a \in K \mid a \in R_p \quad (p \in U)\},$$

where K is the function field of X. The support morphism $\beta_{X^{\mathrm{nor}}} : \alpha_1 \mathscr{O}_{X^{\mathrm{nor}}} \to \tau_{X^{\mathrm{nor}}}$ is defined by

$$\alpha_1 \mathscr{O}_{X^{\mathrm{nor}}}(U) \ni (a_1, \cdots, a_n) \mapsto \{(U, a_i)\}_i.$$

This defines an $\mathscr{A}$-scheme $X^{\mathrm{nor}} = (|X^{\mathrm{nor}}|, \mathscr{O}_{X^{\mathrm{nor}}}, \beta_{X^{\mathrm{nor}}})$.

Step 5: We have a canonical morphism of sheaves $\mathscr{O}_X \to \pi_* \mathscr{O}_{X^{\mathrm{nor}}}$, defined by the identity $a \mapsto a$. This yields a morphism $\pi : X^{\mathrm{nor}} \to X$ of $\mathscr{A}$-schemes. It is of profinite type, by the criterion 4.3.3 in [T].

Step 6: Let us show that π is proper.

We can see from the construction that we have a natural morphism $\mathrm{ZR}^f(K, X) \to X^{\mathrm{nor}}$: the morphism $|\,\mathrm{ZR}^f(K, X)| \to |X^{\mathrm{nor}}|$ of underlying spaces is defined by

$$\mathscr{N}^X \to \mathscr{M}^X \quad (\{(U_i, \alpha_i)\}_i \mapsto \{(X \setminus U_i), \{\alpha^{-1}\})\}_i,$$

where $\mathscr{M}^X = C(\mathrm{ZR}^f(K, X))_{\mathrm{cpt}}$, and the morphism between the structure sheaves is canonical. Note that $\mathrm{ZR}^f(K, X)$ is already normal. This shows that X^{nor} is proper over X by the valuative criterion.

Step 7: We will show that the normalization is a functor. Let $f : X \to Y$ be a dominant morphism of $\mathscr{A}$-schemes. $|f^{\mathrm{nor}}| : X^{\mathrm{nor}} \to Y^{\mathrm{nor}}$ is defined by

$$\mathscr{N}^Y \to \mathscr{N}^X : \{(U_i, \alpha_i)\}_i \mapsto \{(f^{-1}U_i, f^{\#}\alpha_i)\}_i.$$

The morphism $f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X$ extends canonically to $f^{\mathrm{nor},\#} : \mathscr{O}_{Y^{\mathrm{nor}}} \to f^{\mathrm{nor},*} \mathscr{O}_{X^{\mathrm{nor}}}$. This gives a functor $\mathrm{nor} : (\textbf{Int. } \mathscr{A}\textbf{-Sch}) \to (\textbf{N. } \mathscr{A}\textbf{-Sch})$.

Step 8: It remains to show that the normalization functor is indeed the right adjoint of the underlying functor. The unit $\epsilon : \mathrm{Id} \Rightarrow \mathrm{nor} \circ U$ is the identity, since the normalization of a normal $\mathscr{A}$-scheme is trivial. The counit $\eta : U \circ \mathrm{nor} \Rightarrow \mathrm{Id}$ is given by π defined above.

□

Lemma 2.4. Let X be a normal $\mathscr{A}$-scheme. Then. $\mathscr{O}_X(U)$ is normal for any open U.

Proof. Let $b \in K$ be an element which is integral over $\mathscr{O}_X(U)$, where K is the function field of X. Since b_x is integral over the stalk $\mathscr{O}_{X,x}$ for any $x \in U$ and $\mathscr{O}_{X,x}$ is integrally closed, we have $b_x \in \mathscr{O}_{X,x}$. Hence, $b \in \mathscr{O}_X(U)$. □

Proposition 2.5. The normalization functor coincides with the usual normalization, when restricted to $\mathscr{Q}$-schemes.

Proof. First, we will show for affine schemes $X = \operatorname{Spec} A$. The universality of the normalization functor gives a canonical morphism $f : \operatorname{Spec}(A^{\mathrm{nor}}) \to X^{\mathrm{nor}}$. Since $\Gamma(X^{\mathrm{nor}}, \mathscr{O}_{X,x})$ is normal, we have a canonical homomorphism $A^{\mathrm{nor}} \to \Gamma(X^{\mathrm{nor}}, \mathscr{O}_{X,x})$. This yields a morphism $g : X^{\mathrm{nor}} \to \operatorname{Spec}(A^{\mathrm{nor}})$. It is easy to check that these two morphisms f, g are inverse to each other.

It is obvious from the construction that normalization commutes with localizations. This shows that the normalization of any $\mathscr{Q}$-scheme coincides with the usual definition. □

3 Approximations by ordinary schemes

We fix an integral base $\mathscr{Q}$-scheme S in the sequel. The next proposition is pure category-theoretical and easy, so we will omit the proof.

Proposition 3.1. Let $\mathcal{B}, \mathcal{C}$ be two categories, with $\mathcal{B}$ finite complete and $\mathcal{C}$ small complete. Let $F : \mathcal{B} \to \mathcal{C}$ be a finite continuous functor, namely F preserves fiber products. For any object a of $\mathcal{C}$, The followings are equivalent:

(i) a is isomorphic to a limit of the objects in $\mathrm{Im}F$.

(ii) a is isomorphic to a filtered limit of the objects in $\mathrm{Im}F$.

Definition 3.2. Let X be an $\mathscr{A}$-scheme, and $\mathscr{P}$ be a class of $\mathscr{Q}$-schemes.

(1) X is *approximable by* $\mathscr{P}$, if X is isomorphic to a filtered limit of some objects of $\mathscr{P}$.

(2) X is *approximable*, if X is isomorphic to a filtered limit of some $\mathscr{Q}$-schemes.

Proposition 3.3. Any approximable $\mathscr{A}$-scheme is approximable by $\mathscr{Q}$-schemes of finite type.

Proof. It suffices to show that any $\mathscr{Q}$-scheme is approximable by $\mathscr{Q}$-schemes of finite type.

Let X be any $\mathscr{Q}$-scheme, and $\{U_{ijk} \to U_i\}$ be a finite affine covering of X. Since $U_{ijk} \to U_i$ is quasi-compact, U_{ijk} is of finite type over U_i. Thus, we have approximations $U_i = \varprojlim_\lambda U_i^\lambda$ and $U_{ijk} = \varprojlim_\lambda U_{ijk}^\lambda$ so that U_i^λ and U_{ijk}^λ are of finite type and $U_{ijk}^\lambda \to U_i^\lambda$ are open immersions. We may also assume that the above limits are filtered. Since filtered limits and finite colimits commute, we have

$$X = \varinjlim_i U_i = \varinjlim_i \varprojlim_\lambda U_i^\lambda = \varprojlim_\lambda \varinjlim_i U_i^\lambda$$

and $\varinjlim_i U_i^\lambda$ is a $\mathscr{Q}$-scheme of finite type. □

Definition 3.4. Let X, Y be two integral $\mathscr{A}$-schemes. A morphism $f : X \to Y$ is *birational*, if f induces an isomorphism $Q(X) \simeq Q(Y)$ between the rational function fields.

Remark 3.5. Note that, the morphism being birational does not imply the existence of an open dense subset U of X such that $U \simeq f(U)$.

Proposition 3.6. Let X be an approximable $\mathscr{A}$-scheme, say $X = \varprojlim_\lambda X^\lambda$ where X^λ's are $\mathscr{Q}$-schemes.

(1) If X is reduced, then X is approximable by reduced $\mathscr{Q}$-schemes.

(2) If X is integral, then X is approximable by integral $\mathscr{Q}$-schemes.

(3) Further, if the rational function field $Q(X)$ is finitely generated over an integral base $\mathscr{Q}$-scheme, then X is approximable by integral $\mathscr{Q}$-schemes birational to X.

(4) If X is normal, then X is approximable by normal $\mathscr{Q}$-schemes.

(5) If X is proper and approximable by separated $\mathscr{Q}$-schemes, then X is approximable by proper (and of finite type) $\mathscr{Q}$-schemes.

Proof. The proofs are all similar, so let us just see (1).

Since X is reduced, $X \to X^\lambda$ factors through the reduced $\mathscr{Q}$-scheme $(X^\lambda)_{\mathrm{red}}$. This shows that $X \simeq \varprojlim (X^\lambda)_{\mathrm{red}}$. □

Proposition 3.7. Let $f : X \to Y$ be a morphism of $\mathscr{A}$-schemes over S, and X approximable and Y a $\mathscr{Q}$-scheme, of finite type over S.

(1) Suppose X is a filtered projective limit $\varprojlim_\lambda X^\lambda$ of $\mathscr{Q}$-schemes. Then, f factors through $X \to X^\lambda$ for some λ.

(2) Furthermore, if X is proper over S and approximable by separated $\mathscr{Q}$-schemes, and Y is separated over S, then the above X^λ can be chosen to be a proper scheme over Y.

Proof. (1) We may assume that Y is affine. Since Y is of finite type and $\Gamma(\mathscr{O}_X)$ is a filtered colimit of $\Gamma(\mathscr{O}_{X^\lambda})$, f factors through X^λ for some λ.

(2) By the above proposition, we may assume that X^λ's are proper over the base scheme S. Since Y is separated, these morphisms are proper. □

Theorem 3.8. Let $f : X \to Y$ be a proper birational morphism, where X is an integral $\mathscr{A}$-scheme approximable by separated $\mathscr{Q}$-schemes, and Y a normal $\mathscr{Q}$-scheme separated and of finite type over S. Then, $f_*\mathscr{O}_X = \mathscr{O}_Y$.

Proof. The previous proposition shows that f factors through proper morphisms $f_\lambda : X^\lambda \to Y$, where $X = \varprojlim_\lambda X^\lambda$ and $\{X^\lambda\}$ is a filtered system of integral $\mathscr{Q}$-schemes, proper birational and of finite type over Y. Since Y is normal, the usual Zariski's main theorem tells that $\mathscr{O}_Y \to (f_\lambda)_*\mathscr{O}_{X^\lambda}$ is an isomorphism (Corollary III. 11.4 of [H]), and $f_*\mathscr{O}_X$ coincides with the right hand side, since it is a colimit of $(f_\lambda)_*\mathscr{O}_{X^\lambda}$'s. □

Since "of profinite type" morphisms are stable under taking limits, approximable $\mathscr{A}$-schemes are necessarily of profinite type over S.

Theorem 3.9. Let X be a normal $\mathscr{A}$-scheme, proper and profinite over the integral base $\mathscr{Q}$-scheme S. Assume that the rational function field $Q(X)$ is finitely generated over $Q(S)$. The followings are equivalent:

(i) X is approximable by separated $\mathscr{Q}$-schemes.

(ii) Let $\mathcal{U} = \{\amalg U_{ijk} \rightrightarrows \amalg U_i\}$ be any quasi-compact open covering of X. Then, there exists a refinement $\amalg V_{ijk} \rightrightarrows \amalg V_i$ of $\mathcal{U}$ such that $\operatorname{Spec} \mathscr{O}_X(V_{ijk}) \to \operatorname{Spec} \mathscr{O}_X(V_i)$ are open immersions.

Proof. (i)⇒(ii): X can be written as a filtered limit $X = \varprojlim_\lambda X^\lambda$, where X^λ's are normal $\mathscr{Q}$-schemes, proper and of finite type over S. Since the number of U_i's and U_{ijk}'s are finite, $U_i = \pi^{-1}\tilde{U}_i$, $U_{ijk} = \pi^{-1}\tilde{U}_{ijk}$ for some $\pi : X \to X^\lambda$, where $\tilde{U}_i$'s and $\tilde{U}_{ijk}$'s are quasi-compact open subsets of X^λ. Take any refinement $\{\amalg\tilde{V}_{ijk} \to \amalg\tilde{V}_i\}$ of $\tilde{\mathcal{U}} = \{\amalg\tilde{U}_{ijk} \to \amalg\tilde{U}_i\}$, by affine opens $\tilde{V}_{ijk}$ and $\tilde{V}_i$. Set $V_{ijk} = \pi^{-1}\tilde{V}_{ijk}$ and $V_i = \pi^{-1}\tilde{V}_i$. Since $V_i \to \tilde{V}_i$ is proper and $\tilde{V}_i$ is normal, of finite type, Theorem 3.8 implies that $\mathscr{O}_X(V_i) = \mathscr{O}_{X^\lambda}(\tilde{V}_i)$. This shows that $\operatorname{Spec} \mathscr{O}_X(V_{ijk}) \to \operatorname{Spec} \mathscr{O}_X(V_i)$ are open immersions.

(ii)⇒(i): For any covering $\mathcal{U} = \{\amalg U_{ijk} \rightrightarrows \amalg U_i\}$ of X, the refinement $\amalg V_{ijk} \rightrightarrows \amalg V_i$ gives open immersions $\operatorname{Spec}\mathscr{O}_X(V_{ijk}) \to \operatorname{Spec}\mathscr{O}_X(V_i)$ which patches up to give a $\mathscr{Q}$-scheme $X(\mathcal{U})$ and the canonical morphism $\pi_{\mathcal{U}} : X \to X(\mathcal{U})$. The covering $\mathcal{U}$ is a pull back of a covering of $X(\mathcal{U})$, and ditto for the elements of $\mathscr{O}_X(U_i)$'s. From this observation, we see that the induced morphism $X \to \varprojlim_{\mathcal{U}} X(\mathcal{U})$ is an isomorphism. It is clear from the construction that $X(\mathcal{U})$ is proper. □

4 Another proof of Nagata embedding

In the sequel, any $\mathscr{A}$-schemes are integral.

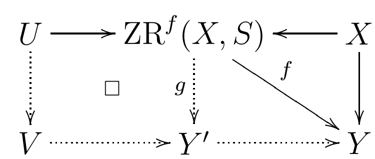
Definition 4.1. Let S be a $\mathscr{Q}$-scheme, and X be a $\mathscr{Q}$-scheme over S. We say that X is *compactifiable* over S, if there is an open immersion $X \to Y$ where Y is a $\mathscr{Q}$-scheme, proper, of finite type over S.

Proposition 4.2. Let S be a $\mathscr{Q}$-scheme, and X be a $\mathscr{Q}$-scheme over S. The followings are equivalent:

(i) X is compactifiable over S.

(ii) $\mathrm{ZR}^f(X, S)$ is approximable by separated $\mathscr{Q}$-schemes, and the natural map $X \to \mathrm{ZR}^f(X, S)$ is an open immersion.

Proof. (i)⇒(ii): There exists an open immersion $X \to Y$ into a $\mathscr{Q}$-scheme Y, proper of finite type over S. This morphism factors through $\mathrm{ZR}^f(X, S)$ by the universal property. We will show that for any quasi-compact open subset U of $\mathrm{ZR}^f(X, S)$, there exists a proper birational morphism $Y' \to Y$, such that $g^{-1}(V) = U$ for some quasi-compact open subset V of Y', where $g : \mathrm{ZR}^f(X, S) \to Y'$ is the canonical extension of $f : \mathrm{ZR}^f(X, S) \to Y$:



By the construction of $\mathrm{ZR}^f(X, S)$, we may assume U is of the form $U(W, \alpha)$, where W is a quasi-compact open subset of S and α is a finite subset of $K \setminus \{0\}$, and

$$U(W, \alpha) = \pi^{-1}(W) \cap \{p \in \mathrm{ZR}^f(X, S) \mid \alpha \subset \mathscr{O}_{\mathrm{ZR}^f(X,S),p}\}.$$

Note that $f(U \cap X)$ is open in Y, since $X \to \mathrm{ZR}^f(X,S)$ is an open immersion. Suppose $\alpha = \{a_i/b_i\}_i$, where $a_i, b_i \in \mathscr{O}_Y$ locally. Let $Y' \to Y$ be the blow up along $(Y \setminus X) \cap \mathrm{Supp}(a_i, b_i)$. Then, either a_i/b_i or b_i/a_i is in $\mathscr{O}_{Y'}$ locally, which shows that the domain of a_i/b_i is open in Y'. This shows that U is the pull back of some V by the morphism $g : \mathrm{ZR}^f(X,S) \to Y'$. Hence, $\mathrm{ZR}^f(X,S) \to \varprojlim_\lambda Y^\lambda$ becomes a homeomorphism on the underlying space, where $Y^\infty = \varprojlim_\lambda Y^\lambda$ is the filtered projective limit of X-admissible blow-ups of Y. A similar argument shows that $\mathscr{O}_{Y^\infty} \to \mathscr{O}_{\mathrm{ZR}^f(X,S)}$ also becomes an isomorphism. Note that Y^λ's are separated over S, since we only used blow-ups.

(ii)$\Rightarrow$(i): The Zariski-Riemann space $\mathrm{ZR}^f(X,S)$ can be written as a form $\varprojlim_\lambda Y^\lambda$, where Y^λ's are proper, of finite type $\mathscr{Q}$-schemes. Since $X \to \mathrm{ZR}^f(X,S)$ is an open immersion and X is quasi-compact, $X \to \mathrm{ZR}^f(X,S) \to Y^\lambda$ becomes an open immersion for some λ. $\square$

Now, we are on the stage to give the proof of the Nagata embedding.

Theorem 4.3 (Nagata)**.** Let S be a $\mathscr{Q}$-scheme, and X be a $\mathscr{Q}$-scheme, separated and of finite type over S. Then, X is compactifiable.

In this section, we will prove this theorem for the essential case, namely when S and X are integral. This restriction is due to the fact that we simply haven't established the theorem of Zariski-Riemann spaces for non-integral schemes.

Since X is quasi-compact, and affine schemes of finite type over S is obviously compactifiable, it suffices to prove the following proposition:

Proposition 4.4. Let V_1 and V_2 be compactifiable open sub-$\mathscr{Q}$-schemes of a $\mathscr{Q}$-scheme X separated over S, with $X = V_1 \cup V_2$. Then X is also compactifiable.

Proof. Consider $\mathrm{ZR}^f(X,S)$. Since X is separated, of finite type over S, the morphism $X \to \mathrm{ZR}^f(X,S)$ is an open immersion by Corollary 4.4.6 of [T]. Let W_1 (resp. W_2) be the complement of the closure of $V_2 \setminus V_1$ (resp. $V_1 \setminus V_2$) in $\mathrm{ZR}^f(X,S)$.

We can see that $W_1 \cap W_2 = V_1 \cap V_2$, since the open kernel of the complement of $V_1 \cup V_2$ is empty. Next, we see that $W_1 \cup W_2 = \mathrm{ZR}^f(X,S)$. For this, it suffices to show that $\overline{V_2 \setminus V_1} \cap \overline{V_1 \setminus V_2} = \emptyset$. Suppose there is a point p in $\overline{V_2 \setminus V_1} \cap \overline{V_1 \setminus V_2}$. Since $V_2 \setminus V_1$ and $V_1 \setminus V_2$ are coherent subset of $\mathrm{ZR}^f(X,S)$,

p must be a specialization of some $x_1 \in V_2 \setminus V_1$ and $x_2 \in V_1 \setminus V_2$ by Corollary 1.2.8 of [T]. Since $\mathrm{ZR}^f(K,S) \to \mathrm{ZR}^f(X,S)$ is surjective, there are inverse images $y_i \in \mathrm{ZR}^f(K,S)$ of x_i such that y_i specializes to p. The points in $\mathrm{ZR}^f(K,S)$ are valuation rings, hence y_2 must be the specialization of y_1, or the converse. In either cases, this contradicts to the fact that x_1 and x_2 has no specialization-generalization relations. This also shows that W_1 and W_2 are quasi-compact. The morphism $p_1 : \mathrm{ZR}^f(V_1,S) \to \mathrm{ZR}^f(X,S)$ induces an isomorphism on W_1, hence W_1 is approximable by $\mathscr{Q}$-morphisms of finite type over S, ditto for W_2.

Take any $\mathscr{Q}$-model Y_i of W_i (namely, a morphism $\pi_i : W_i \to Y_i$ where Y_i is a $\mathscr{Q}$-scheme) such that the morphism π_i induces an isomorphism on V_i. Then, Y_1 and Y_2 can be patched along $\pi_1(W_1 \cap W_2) \simeq \pi_2(W_1 \cap W_2)$ to obtain a $\mathscr{Q}$-scheme Y of finite type, and a surjective morphism $\mathrm{ZR}^f(X,S) = W_1 \cup W_2 \to Y$. This shows that Y is proper. □

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