C^∞-HYPOELLIPTICITY AND EXTENSION OF *CR* FUNCTIONS

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Abstract. Let M be a *CR* submanifold of a complex manifold X. The main result of this article is to show that *CR*-hypoellipticity at $p_0 \in M$ is necessary and sufficient for holomorphic extension of all germs of *CR* functions to an ambient neighborhood in X. As an application, we obtain that *CR*-hypoellipticity implies the existence of generic embeddings and prove holomorphic extension for a large class of *CR* manifolds satisfying a higher order Levi pseudoconcavity condition.

1. Introduction

Let M be an abstract *CR* manifold, of arbitrary *CR* dimension m and *CR* codimension d. We say that M is *CR-hypoelliptic* at $p_0 \in M$ if every distribution satisfying the homogeneous tangential Cauchy-Riemann equations on a neighborhood of p_0 in M is C^∞-smooth on a neighborhood of p_0.

A *local CR-embedding* of M at p_0 is the datum of C^∞-smooth solutions $z_1, \dots, z_\nu$ to the homogeneous tangential Cauchy-Riemann equation on a neighborhood U of p_0 in M such that the map $p \mapsto (z_1(p), \dots, z_\nu(p))$ is a smooth embedding $U \hookrightarrow \mathbb{C}^\nu$. We have $\nu \geq m + d = n$, and when we have equality we say that the local *CR*-embedding is *generic*.

Note that from any local *CR*-embedding we can obtain a generic local *CR*-embedding of a smaller neighborhood of p_0, by choosing any subset $z_{i_1}, \dots, z_{i_n}$ of $z_1, \dots, z_\nu$ with $dz_{i_1}(p_0) \wedge \cdots \wedge dz_{i_n}(p_0) \neq 0$.

We say that M has the *holomorphic extension property* at p_0 if there is a generic local *CR*-embedding $\phi : U \hookrightarrow \mathbb{C}^n$ such that, for every distribution solution u of the homogeneous tangential Cauchy-Riemann equations on a neighborhood $U' \subset U$ of p_0 in M, there is a holomorphic function, defined on a neighborhood V of $\pi(p_0)$ in $\mathbb{C}^n$, such that $\phi^* \tilde{u}$ is defined and equal to u on a neighborhood of p_0 in U'.

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We can also consider weaker formulations of the holomorphic extension property, either by dropping the assumption that the local CR-embedding ϕ be generic, or allowing different embeddings for extending different CR-distributions, or keeping a same local CR-embedding but requiring local holomorphic extension only for smooth CR-functions.

The fact that the different formulations are in fact equivalent is a consequence of our main result:

Theorem 1.1. *Let M be a CR manifold, locally CR-embeddable at $p_0 \in M$. Then M has the holomorphic extension property at p_0 if and only if M is CR-hypoelliptic at p_0.*

An interesting consequence of Theorem 1.1 is a uniqueness result for the local CR-embedding of M at p_0:

Corollary 1.2. *Under the assumption of Theorem 1.1 we have:*

1. *If $p_0 \in U^{open} \subset M$ and $\phi : U \to \mathbb{C}^\nu$ is a local CR-embedding, then there is an n-dimensional complex submanifold X of an open neighborhood V of p_0 in $\mathbb{C}^\nu$ and ω, with $p_0 \in \omega^{open} \subset U$ such that $\phi(\omega) \subset X$.*
2. *If $p_0 \in U^{open} \subset M$ and $\phi_i : U \to \mathbb{C}^n$, for $i = 1, 2$, are two generic local CR-embeddings with $\phi_i(p_0) = 0$, then there are open neighborhoods V, W of 0 in $\mathbb{C}^n$ and a biholomorphic map $\psi : V \to W$ such that $\phi_2 = \psi \circ \phi_1$ on V.*

This corollary has the consequence that, when M is CR-hypoelliptic and locally embeddable at all points, its CR structure completely determines its hypo-analytic structure (see [24]). Moreover, the arguments of [5] also yield

Corollary 1.3. *Let M be a CR manifold of CR dimension m and CR codimension d, and $n = m + d$. Assume that M is locally CR-embeddable and CR-hypoelliptic at all points. Then M admits a smooth generic CR-embedding $M \hookrightarrow X$ into an n-dimensiona complex manifold X.*

If $\phi_i : M \to X_i$, $i = 1, 2$, are two generic CR-embeddings of M, then there are tubular neighborhoods Y_i of $\phi_i(M)$ in X_i, $i = 1, 2$, and a biholomorphic map $\psi : Y_1 \to Y_2$, such that $\psi(\phi_1(M)) = \phi_2(M)$.

Since holomorphic functions are real-analytic, holomorphic extendability trivially implies CR-hypoellipticity. Both CR-hypoellipticity and holomorphic extendability imply minimality. The main result of this note is that CR-hypoellipticity and holomorphic extendability are equivalent at minimal points.

Since real-analytic CR manifolds are locally CR-embeddable (see [5]), and holomorphic functions are real-analytic, we obtain

Corollary 1.4. *Assume that M is a real-analytic CR-manifold, and let $p_0 \in M$. Then the following are equivalent:*

1. *M is CR-hypoelliptic at p_0;*
2. *M is CR-analytic-hypoelliptic at p_0;*
3. *all smooth solutions of the homogeneous tangential Cauchy-Riemann equations on a neighborhood of p_0 are real-analytic at p_0.*

We also point out that our result applies to give concrete applications for the Siegel-type theorems proved in [11, 12] about the trancsendence degree of the fields of CR-meromorphic functions.

Despite of several contributions, the problem of finding a geometric characterization for the holomorphic extension property is still wide open, even for real analytic hypersurfaces. The interest of Theorem 1.1 is that it establishes a link between holomorphic extension and C^∞ regularity, a central and better understood topic in PDE theory. We illustrate this point of view by recalling in §6 the weak pseudoconcavity assumptions of [2], generalizing the *essential pseudoconcavity* of [10], which insure *CR*-hypoellipticity, and illustrating by some examples in §7 how this approach leads to the proof of the holomorphic extension property for manifolds with a highly degenerate Levi form. Extension theorems had been obtained before under stronger non-degeneracy assumptions on the Levi form (see e.g. [9, 19]), or for *CR* manifolds satisfying a third order pseudoconcavity condition (see [3]).

We notice that minimality is a necessary condition for *CR*-hypoellipticity by [6], and that some sort of pseudoconcavity is also necessary, as holomorphic extension does not hold e.g. when M lies in the boundary of a domain of holomorphy.

In general, germs of *CR* functions on a generically embedded *CR* manifold $M \hookrightarrow X$ may fail to holomorphically extend to a full neighborhood $\mathcal{U}$ of p_0 in X and one can consider instead open subsets $\mathcal{W}$ of X for which $M \cap \partial\mathcal{W}$ is a neighborhood of p_0 in M. A fundamental result of Tumanov [25] states that holomorphic local *wedge* extension is valid if M is minimal at p_0. By [6], this condition is also necessary. However, the known proofs of local holomorphic wedge extension merely yield existence, but no explicit information on its shape. The analytic or hypo-analytic wave front sets tautologically give the directions of holomorphic extension. We conjecture that, in analogy with Theorem 1.1, the union of the C^∞ wave front sets of all germs of *CR* distributions and that of their hypo-analytic wave front sets coincide. Theorem 5.1 in §5 is a first partial result in this direction.

Let us shortly describe the contents of the paper. In §2 we set notation and precise the notion of *CR*-hypoellipticity. §3 contains the proof of Theorem 1.1. In §4 we prove various equivalences of the extension property, easily implying Corollaries1.2, 1.3,1.4. Section §5 contains our result about wedge extension and the common C^∞ wave front set of germs of *CR* distributions. In §6 we rehearse the subellipticity result of [2] and in §7 we give some examples.

2. *CR*-HYPOELLIPTICITY

Let M be an abstract smooth *CR* manifold of *CR* dimension m and *CR* codimension d. The *CR* structure on M is defined by the datum of an m-dimensional subbundle $T^{0,1}M$ of the complexified tangent bundle $\mathbb{C}TM$ with

$$T^{0,1}M \cap \overline{T^{0,1}M} = \underline{0} \quad \text{and} \quad [\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M).$$

For $U^{\text{open}} \subset M$ we denote by $\mathcal{O}^\infty_M(U)$ the set of smooth solutions on U to the tangential Cauchy-Riemann equations:

$$\mathcal{O}^\infty_M(U) = \{u \in C^\infty(U, \mathbb{C}) \mid Zu = 0, \ \forall Z \in \Gamma(U, T^{0,1}M)\}.$$

Likewise, we denote by $\mathcal{O}^0_M(U)$ and $\mathcal{O}^{-\infty}_M(U)$ the spaces of complex valued continuous functions and of complex valued distributions, respectively, that weakly solve the homogeneous equations

$$Zu = 0, \ \forall Z \in \Gamma(U, T^{0,1}M) \text{ on } U \ ,$$

i.e. such that

$$\int u\, Z'\phi\, d\mu = 0, \quad \forall \phi \in C_0^\infty(U),\ \forall Z \in \Gamma(U, T^{0,1}M),$$

where μ is a positive measure with smooth density on M and the formal adjoint Z' of $Z \in \Gamma(U, T^{0,1}M)$ is defined by

$$\int Zv\,\phi\, d\mu = \int v\, Z'\phi\, d\mu, \quad \forall v, \phi \in C_0^\infty(U).$$

The assignments $U^{\text{open}} \to \mathcal{O}_M^a(U)$, for $a = -\infty, 0, \infty$, define sheaves of germs. We denote by $\mathcal{O}^a_{M,(p_0)}$ the stalk at $p_0 \in M$. When M is a complex manifold we drop the superscript a, because the three sheaves coincide by the regularity theorem for holomorphic functions.

Definition 2.1. We say that M is *CR-hypoelliptic* at $p_0 \in M$ if $\mathcal{O}^{-\infty}_{M,(p_0)} = \mathcal{O}^{\infty}_{M,(p_0)}$.

3. Proof of Theorem 1.1

By taking a generic CR-embedding, we can as well assume that $M \subset \mathbb{C}^n$, where $n = m + d$, and m is the CR dimension, d the CR codimension of M. We can also assume that $p_0 = 0$ and that the holomorphic coordinates of $\mathbb{C}^n$ have been chosen in such a way that M is the graph

$$y' = h(x', z'') \tag{3.1}$$

of a smooth map $h : V \to \mathbb{R}^d$, with $h(0) = 0$, $dh(0) = 0$, for an open neighborhood V of 0 in $\mathbb{R}^d \times \mathbb{C}^m$. Here $z = (z', z'') \in \mathbb{C}^d \times \mathbb{C}^m$, with $d + m = n$, and $z' = x' + iy'$, $z'' = x'' + iy''$ with $x', y' \in \mathbb{R}^d$, $x'', y'' \in \mathbb{R}^m$.

An *open wedge* $\mathcal{W}$ attached to M along an open set $E = \text{Edge}(\mathcal{W}) \subset M$ is, in the chosen coordinates, a set of the form

$$\mathcal{W} = \{z + (ix', 0) : z \in E, x' \in \mathrm{C}\}, \tag{3.2}$$

where $\mathrm{C} \subset \mathbb{R}^d$ is a truncated open cone with vertex at the origin. Note that $\mathcal{W}$ is foliated by the approach manifolds $E_{y'} = \{z + (iy', 0) : z \in E\}$, $y' \in \mathrm{C}$. Recall that $f \in \mathcal{O}(\mathcal{W})$ attains the weak boundary values $f^* \in \mathcal{D}'(E)$ along E if for every test function $\phi \in \mathcal{D}(E)$, we have

$$\lim_{h' \to 0, y' \in \mathrm{C}} \int f(x' + ih(x', z'') + iy', z'')\phi(x', z'')\, dm_{d+2m} = f^*[\phi]. \tag{3.3}$$

Here dm_{d+2m} denotes standard Lebesgue measure on $\mathbb{R}^d \times \mathbb{C}^m$. A function $f \in \mathcal{O}(\mathcal{W})$ has polynomial growth along E if for every compact $K \subset E$ there are an integer $N_K \geq 0$ and a constant $a_K > 0$ such that

$$|f(x' + ih(x', z'') + iy', z'')| \leq a_K |y'|^{-N_K},\ \forall (x', z'') \in K,\ \forall y' \in \mathrm{C}. \tag{3.4}$$

Holomorphic functions of polynomial growth attain unique distribution boundary values on E, which weakly satisfy the homogeneous tangential CR equations.

Proof of Theorem 1.1: Before going into the technical details of the proof, we sketch the main ideas involved. As already mentioned, we need only to show that CR-hypoellipticity implies holomorphic extension to full neighborhoods. First we observe that p_0 must be a minimal point of M. Otherwise, M contains a proper CR submanifold N through p_0, of the same CR dimension. Then a suitable distribution carried by N would define a non smooth CR-distribution on a neighborhood of p_0 (see [22, 6]). Thus CR-hypoellipticity implies minimality at p_0. Hence, all CR

distributions on a neighborhood $U \subset M$ of p_0 are boundary values of holomorphic functions defined on an open wedge $\mathcal{W} = \mathcal{W}_U$. Then we argue by contradiction, assuming that not all *CR* distributions holomorphically extend to a full neighborhood of p_0. We consider the envelope of holomorphy X of $\mathcal{W}$, and identify p_0 to a point of its abstract boundary bX. Then we construct a holomorphic function f on X with polynomial growth on bM, and whose modulus is unbounded in any neighborhood of p_0. Pushing down to $\mathcal{W}$, we obtain a function with polynomial growth along the edge with a *CR*-distribution boundary value which is unbounded, and hence discontinuous, at p_0.

Let us choose holomorphic coordinates (z', z'') centered at p_0 as in (3.1), and let $U \subset M$ be an open neighborhood of 0 which carries a *CR* distribution which does not holomorphically extend to an ambient neighborhood of 0. Since M is minimal at 0 as noticed above, Tumanov's theorem yields an open wedge $\mathcal{W}$ as in (3.2) such that every *CR* distribution on U has a holomorphic extension to $\mathcal{W}$.

Let $\pi : X \to \mathbb{C}^n$ be the envelope of holomorphy of $\mathcal{W}$. Recall that X is a Stein manifold spread over $\mathbb{C}^n$ by a locally biholomorphic mapping π. Moreover there is a canonical injective holomorphic map $\alpha : \mathcal{W} \to X$ satisfying $\pi \circ \alpha = \mathrm{id}_{\mathcal{W}}$ such that for every $g \in \mathcal{O}(\mathcal{W})$ the pushforward $\alpha_* g$ to $\mathcal{W}' = \alpha(\mathcal{W})$ extends to X holomorphically, and such that X is a maximal Riemann domain with this property (see [14, 18] for detailed information).

We recall the construction, due to Grauert and Remmert, which yields a canonical abstract closure $\overline{\pi} : \overline{X} \to \mathbb{C}^n$ in the following way: A boundary point is a maximal filter[1] $\mathfrak{a}$ of connected open sets in X such that

- **(i)** $\mathfrak{a}$ has no accumulation point in X,
- **(ii)** for every $U \in \mathfrak{a}$, there is $V^{\text{open}} \subset \mathbb{C}^n$ such that U is a connected component of $\pi^{-1}(V)$,
- **(iii)** the image filter $\pi_* \mathfrak{a}$ converges to a point $z \in \mathbb{C}^n$, and
- **(iv)** for every open neighborhood $V \subset \mathbb{C}^n$ of z one of the components of $\pi^{-1}(V)$ is a member of $\mathfrak{a}$.

We will denote the abstract boundary of X by bX. Setting $\overline{\pi}(\mathfrak{a}) = z$ in the above situation, one obtains an extension of π to $\overline{X} = X \cup bX$, and there is a natural Hausdorff topology on $\overline{X}$ such that $\overline{\pi}$ is continuous (see [14] for the details). Note that the topological boundary ∂D of a domain $D \subset \mathbb{C}^n$ may not coincide with its abstract boundary bD.

Our assumption that holomorphic extension to a full neighborhood of 0 fails implies that the abstract boundary bX contains a point $0'$ with $\overline{\pi}(0') = 0$. We denote by $\delta_X(p)$ the *distance from the boundary* in X. It can be defined by

$$\delta_X(p) = \sup\{r > 0 \mid \{|z - \pi(p)| < r\} \subset \pi(X)\}.$$

For each integer $k \geq 0$, we define the space of holomorphic functions on X, with k-polynomial growth on bX, by

$$\mathcal{O}^{(k)}(X) = \{f \in \mathcal{O}(X) \mid \delta_X^k f \text{ is bounded on } X\}.$$

It is a Banach spaces with the norm $\|f\|_{\mathcal{O}^{(k)}(X)} = \sup_{p \in X} |\delta_X^k(p) f(p)|$.

Lemma 3.1. *There is a sequence* $\{p_j\}_{j=1,2,\ldots} \in \mathcal{W}' = \alpha(\mathcal{W})$, *satisfying* $\pi(p_j) \to 0$, *and a function* $f \in \mathcal{O}^{(2n+1)}(X)$ *such that* $|f(p_j)| \to \infty$.

[1] A filter is a family of subsets such that for each pair of members U_1, U_2, there is a third member U_3 with $U_3 \subset U_1 \cap U_2$.

For subdomains of $\mathbb{C}^n$, more precise results can be found in [20].

Proof. We will use the following result, which is a particular case of [14, Proposition 2.5.4]: *There is a constant* $C > 0$*, only depending on* X*, such that*

$$\forall p \in X \ \ \exists f_p \in \mathcal{O}^{(2n+1)}(X) \ \ with \ \ f(p) = 1, \ \ \|f\|_{\mathcal{O}^{(2n+1)}(X)} \leq C\delta_X(p).$$

We will prove by induction that there are points $p_j \in X$ and functions $f_j \in \mathcal{O}^{(2n+1)}(X)$, $j = 1, 2, \ldots$, satisfying

- **(a)** $p_j \in \alpha(\mathcal{W} \cap B_0(1/j))$,
- **(b)** $|f_j(p_j)| \geq j$,
- **(c)** $\|f_j - f_{j-1}\|_{\mathcal{O}^{(2n+1)}(X)} < 2^{-j}$, and
- **(d)** $\sup_{X_{\leq\delta_{j-1}}} |f_j - f_{j-1}| \leq 2^{-j}$,

where we have abbreviated $\delta_j = \delta_X(p_j)$, $X_{\leq d} = \{p \in X : \delta_X(p) \leq d\}$.

Take any $p_1 \in \alpha(\mathcal{W} \cap B_0(1))$ and set $f_1 \equiv 1$. Assume by recurrence that we already found $p_1, \ldots, p_{k-1}$ and $f_1, \ldots, f_{k-1} \in \mathcal{O}^{(2n+1)}(X)$ satisfying **(a)-(d)** for $j \leq k-1$. Choose $p_k \in \alpha(\mathcal{W} \cap B_0(1/k))$ such that $\delta_k \leq \frac{\delta_{k-1}^{2n+1}}{k2^kC}$. If $|f_{k-1}(p_k)| \geq k$ holds, $f_k = f_{k-1}$ obviously satisfies **(a)-(d)** for $j = k$. Otherwise, we pick a function f_{p_k} as in the above-cited result and set $f_k = f_{k-1} + k\alpha f_{p_k}$, with $\alpha = 1$ if $f_{k-1}(p_k) = 0$ and $\alpha = \frac{f_{k-1}(p_k)}{|f_{k-1}(p_k)|}$ otherwise. This implies **(b)** for $j = k$. We verify that

$$\|f_k - f_{k-1}\|_{\mathcal{O}^{(2n+1)}(X)} = k\|f_{p_k}\|_{\mathcal{O}^{(2n+1)}(X)} \leq kC\delta_k \leq 2^{-k},$$

and

$$\sup_{X_{\leq\delta_{k-1}}} |f_k - f_{k-1}| = k \sup_{X_{\leq\delta_{k-1}}} |f_{p_k}| \leq \frac{C}{\delta_{k-1}^{2n+1}} \sup_{X_{\leq\delta_{k-1}}} |\delta_X^{2n+1} f_{p_k}| \leq 2^{-k},$$

completing the inductive step.

Now **(c)** implies that the $\mathcal{O}^{(2n+1)}(X)$-limit $f = \lim f_m$ exists, and **(b)**, **(d)** yield $|f(p_j)| \geq j - 1$ for all j. The proof is complete. □

The push forward $f \circ \alpha$ of the function f obtained in Lemma 3.1 is holomorphic on $\mathcal{W}$ and has polynomial growth while approaching the edge E of $\mathcal{W}$, because $E \subset \overline{\pi}(\overline{X})$. In particular, $f \circ \alpha$ has a boundary value, which is a *CR* distribution f^* on E. By [8, Lemma 7.2.6], f is continuous up to the edge near every point in E near which f^* happens to be continuous. Hence, by Lemma 3.1, f^* is not continuous at 0, because $f \circ \alpha$ is unbounded on a sequence in $\mathcal{W}$ which converges to 0. This completes the proof of Theorem 1.1. □

4. The holomorphic extension property

Let M be a *CR* submanifold, of *CR* dimension m, and *CR* codimension d, of a ν-dimensional complex manifold X. This means that M is a smooth real submanifold of X and $T^{0,1}M = T^{0,1}X \cap \mathbb{C}TM$.

Let $p_0 \in M$ and let $(W; z_1, \ldots, z_\nu)$ be any coordinate neighborhood in X centered at p_0. If $m + d = \nu$, then the embedding $M \hookrightarrow X$ is generic and the coordinate neighborhood $(W; z_1, \ldots, z_\nu)$ provides a generic *CR*-embedding of a neighborhood U of p_0 in $M \cap W$ into an open neighborhood of 0 in $\mathbb{C}^\nu$. If $m + d = n < \nu$, we can reorder the coordinates $z_1, \ldots, z_\nu$ in such a way that $dz_1(p_0), \ldots, dz_n(p_0)$ are linearly independent. Then the map $\phi : p \mapsto \phi(p) = (z_1(p), \ldots, z_n(p))$ yields again a generic *CR*-embedding of a neighborhood U of p_0 in $M \cap W$ into an open neighborhood of 0 in $\mathbb{C}^\nu$. We get

Theorem 4.1. *For each $a \in \{-\infty, 0, \infty\}$ the following are equivalent:*

(1) *the restriction map $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^a_{M,(p_0)}$ is onto;*
(2) *the map $\phi^* : \mathcal{O}_{\mathbb{C}^n,(p_0)} \to \mathcal{O}^a_{M,(p_0)}$ is an isomorphism.*

Proof. The equivalence is a consequence of Theorem 1.1. In fact (1) implies *CR*-hypoellipticity at p_0 and this, by Theorem 1.1, implies (2). The inference (2)⇒(1) is obvious. □

Moreover, we obtain

Theorem 4.2. *Assume that M is a CR submanifold of a complex manifold X and $p_0 \in M$. Then the following are equivalent:*

(1) *the restriction map $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^\infty_{M,(p_0)}$ is onto;*
(2) *the restriction map $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^0_{M,(p_0)}$ is onto;*
(3) *the restriction map $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^{-\infty}_{M,(p_0)}$ is onto.*

Proof. Since the statement is local, using Theorem 4.1, we can as well assume that M is a generic *CR* submanifold of an open ball in $\mathbb{C}^n$, centered at $p_0 = 0$.

To show that (1)⇒(2) it suffices to prove that, for every compact $K \subset M$ containing a neighborhood of 0 in M, the polynomial hull

$$\hat{K} = \{z \in \mathbb{C}^n \mid \ |f(z)| \le \sup_K |f|, \ \forall f \in \mathbb{C}[z_1, \dots, z_n]\}$$

of K in $\mathbb{C}^n$ contains a neighborhood of 0 in $\mathbb{C}^n$. The implication will indeed follow then by the approximation theorem in [7].

Let $\mathring{K}$ be the interior in M of an arbitrarily fixed compact neighborhood K of 0 in M.

For $r > 0$, set $B_r = \{z \in \mathbb{C}^n \mid \ |z| < r\}$. Fix $r > 0$ in such a way that $B_r \cap M$ is contained in some coordinate neighborhood $(U; t_1, \dots, t_{2m+d})$ in M, with $U \subset \mathring{K}$. Then for every integer k, the set

$$\mathbb{F}_k = \{(u, v) \in \mathcal{O}^\infty_M(\mathring{K}) \times \mathcal{O}(B_{r/2^k}) \mid v|_{M \cap B_{r/2^k}} = u|_{M \cap B_{r/2^k}}\}$$

is a closed subspace of the product $\mathcal{O}^\infty_M(\mathring{K}) \times \mathcal{O}(B_{r/2^k})$, endowed with its standard Fréchet topology, and hence a Fréchet space. The projection into the first coordinate defines continuous linear maps $\pi_k : \mathbb{F}_k \to \mathcal{O}^\infty_M(\mathring{K})$. By the assumption, $\bigcup_k \pi_k(\mathbb{F}_k) = \mathcal{O}^\infty_M(\mathring{K})$. Hence some $\pi_\nu(\mathbb{F}_\nu)$ is of the second Baire category. Then $\pi_\nu : \mathbb{F}_\nu \to \mathcal{O}^\infty_M(\mathring{K})$ is surjective and open by the Banach-Schauder theorem and we get:

$$\begin{cases} \exists C > 0, \ \ell \in \mathbb{Z}_+, \ K' \Subset K \quad \text{such that} \quad \forall u \in \mathcal{O}^\infty_M(\mathring{K}) \ \exists\, \tilde{u} \in \mathcal{O}_{\mathbb{C}^n}(B_{2^{-\nu}r}) \\ \text{with } \ \tilde{u}|_{M \cap B_{r/2^\nu}} = u|_{M \cap B_{r/2^\nu}}, \ \text{and} \ \sup_{B_{r/2^{\nu+1}}} |\tilde{u}| \le C\, \|u\|_{\ell, K'} = \sup_{K'} \sup_{|\alpha| \le \ell} \left| \frac{\partial^{|\alpha|} u}{\partial t^\alpha} \right|. \end{cases}$$

For $\epsilon > 0$ set $K_\epsilon = \{z \in \mathbb{C}^n \mid \sup_{z' \in K} |z - z'| \le \epsilon\}$. By Cauchy's inequalities, there is a positive constant C_ϵ such that

$$\|f\|_{\ell, K'} \le C_\epsilon \sup_{K_\epsilon} |f|, \quad \forall f \in \mathbb{C}[z_1, \dots, z_n].$$

This implies that

$$\sup_{B_{r/2^{\nu+1}}} |f| \le C\, C_\epsilon \sup_{K_\epsilon} |f|, \quad \forall f \in \mathbb{C}[z_1, \dots, z_n].$$

An application of this inequality to the powers f^h of the holomorphic polynomials shows that in fact

$$\sup_{B_{r/2^{\nu+1}}}|f| \leq \sup_{K_\epsilon}|f|, \quad \forall f \in \mathbb{C}[z_1,\ldots,z_n],$$

i.e. that $B_{r/2^{\nu+1}}$ is contained in the polynomial hull $\hat{K}_\epsilon$ of K_ϵ. Since $\hat{K} = \bigcap_{\epsilon>0}\hat{K}_\epsilon$, the polynomial hull $\hat{K}$ contains $B_{r/2^{\nu+1}}$. This completes the proof of (1)⇒(2).

To prove the implication (2)⇒(3) we use the elliptic partial differential operator introduced in [7] (see also [24, Ch.II]). This is constructed in the following way. We can assume that $dz_1,\ldots,dz_n,d\bar{z}_1,\ldots,d\bar{z}_m$ define a maximal set of independent differentials on a neighborhood U of 0 in M. Then we uniquely define commuting smooth complex vector fields $L_1,\ldots,L_n,Z_1,\ldots,Z_m$ on U by requiring that

$$L_iz_j = \delta_{i,j},\ L_i\bar{z}_k = 0,\ Z_hz_j = 0,\ Z_h\bar{z}_k = \delta_{h,k}, \quad \text{for } 1 \leq i,j \leq n,\ 1 \leq h,k \leq m.$$

Then, for a large $c \in \mathbb{R}$,

$$\Delta_{L,cZ} = \sum\nolimits_{i=1}^{n} L_i^2 + c^2\sum\nolimits_{h=1}^{m} Z_h^2 \tag{4.1}$$

is elliptic on a neighborhood of 0 in M, that, after shrinking, we can take equal to U. If $f \in \mathcal{O}_{\mathbb{C}^n}(W)$ for an open neighborhood W of 0 in $\mathbb{C}^n$ and ν is a non negative integer, then

$$\Delta^k_{L,cZ} f|_{U\cap W} = \left(\left(\sum\nolimits_{i=1}^{n}\frac{\partial^2}{\partial z_i^2}\right)^k f\right)\Big|_{U\cap W}.$$

In [7] the following is proved

Lemma 4.3. *There is an open neighborhood U' of 0 in U such that for every $u \in \mathcal{O}^{-\infty}_M(U)$ there is $w \in \mathcal{O}^0_M(U')$ and an integer $k \geq 0$ such that*

$$u|_{U'} = \Delta^k_{L,cZ} w. \tag{4.2}$$ □

Let $u \in \mathcal{O}^{-\infty}_M(U)$. By Lemma 4.3 there is $w \in \mathcal{O}^0_M(U')$ satisfying (4.2). If (2) is valid, there is an open neighborhood W of 0 in $\mathbb{C}^n$ and a holomorphic function $\tilde{w} \in \mathcal{O}_{\mathbb{C}^n}(W)$ such that $\tilde{w}|_{U'\cap W} = w|_{U'\cap W}$. In view of (4.1), $\tilde{u} = (\sum_{i=1}^n \frac{\partial^2}{\partial z_i^2})^k\tilde{w}$ is a holomorphic function in W such that $\tilde{u}|_{U'\cap W} = u|_{U'\cap W}$. This shows that (2)⇒(3). Since the implication (3)⇒(1) is trivial, the proof is complete. □

As a corollary of Lemma 4.3, we also state the following regularity result, which will be useful to apply [2] to obtain holomorphic extension.

Corollary 4.4. *Let M be a CR submanifold of a complex manifold X, $p_0 \in M$ and assume that all germs $\alpha \in \mathcal{O}^{-\infty}_{M,(p_0)}$ which are in L^2_{loc} at p_0 are in $\mathcal{O}^{\infty}_{M,(p_0)}$. Then $\mathcal{O}^{-\infty}_{M,(p_0)} = \mathcal{O}^{\infty}_{M,(p_0)}$.* □

5. Wedge extension and the wave front set

Theorem 1.1 relates holomorphic extension to $\mathcal{C}^\infty$-regularity. Here we make a few remarks relating holomorphic wedge extension to the $\mathcal{C}^\infty$ wave front set. For extension to open wedges attached to M, it is known that the directions of extension are nicely reflected by the *analytic* wave front set, which provides information on the extension of any *individual CR* distribution. Below we will see that local properties for *simultaneous* extension are related to the $\mathcal{C}^\infty$-wave front sets of *all* the elements in $\mathcal{O}^{-\infty}_{M,(p_0)}$.

Let HM be the subbundle of the tangent bundle TM consisting of the real parts of vectors in $T^{0,1}M$.

For a point p of a smooth *CR* manifold M, we denote by $\mathfrak{O}_M(p)$ the *CR* orbit of p in M, i.e. the set of all points of M that can be linked with p by a piecewise smooth curve with velocity vectors in HM. A fundamental result of Sussmann ([21]) tells that each *CR* orbit $\mathfrak{O}_M(p)$ is a smooth *CR* submanifold, which turns out to have the same *CR* dimension of M. If U is an open neighborhood of p in M we can consider the orbit $\mathfrak{O}_U(p)$. Clearly, if $p \in V^{\text{open}} \subset U^{\text{open}} \subset M$, then $\mathfrak{O}_V(p) \subset \mathfrak{O}_U(p)$. The family of *CR* orbits $\mathfrak{O}_U(p)$, for $p \in U^{\text{open}} \subset M$, indexed by the filter of open neighborhoods of p, uniquely defines a *germ* of *CR* manifold $\mathfrak{O}_{M,loc}(p)$, which is called *the local CR orbit* of p. Tumanov's theorem in [25] yields local holomorphic extension to open wedges if $\mathfrak{O}_U(p_0)$ is open (see also [6, 15, 17, 18, 22]). More generally, the dimension of $\mathfrak{O}_U(p_0)$ can be related to the maximal number of independent directions of *CR* extension [26].

Denote by $\mathcal{D}'(U)$, for $U^{\text{open}} \subset M$, the space of complex valued distributions in U, and by $\text{WF}(u) \subset T^*U$ the wave front set of $u \in \mathcal{D}'(U)$. For basic definitions and a thorough introduction to this topic we refer to [13]. Recall that $\text{WF}(u)$ is a closed conical subset of $\dot{T}^*U$, which is the cotangent bundle deprived of its zero section. It will be convenient to us to consider also $\overline{\text{WF}}(u) = \text{WF}(u) \cup \underline{0}$, where $\underline{0}$ is the zero section of T^*M.

If $U^{\text{open}} \subset M$, and $u \in \mathcal{O}^{-\infty}(U)$, then $\text{WF}(u) \subset H^0M$, where $H^0M = \{\xi \in T^*M \mid \xi(v) = 0,\ \forall v \in H_{\pi(\xi)}M\}$ is the *characteristic bundle* of the tangential *CR* system.

We prove the following

Theorem 5.1. *Let M be a CR submanifold, of CR dimension m and CR codimension d, of a complex manifold X, and $p_0 \in M$. Then the following are equivalent*

1. $\dim_{\mathbb{R}} \mathfrak{O}_{M,\text{loc}}(p_0) = 2m + k$ *($0 \le k \le d$);*
2. *there is a CR distribution u, defined on an open neighborhood U of p_0, such that $\overline{\text{WF}}(u) \cap T^*_{p_0}M$ contains a $(d-k)$-dimensional $\mathbb{R}$-linear subspace, and k is the smallest integer with this property.*

Assume that (1) *holds true and that $\mathfrak{O}_{M,\text{loc}}(p_0)$ does not have the holomorphic extension property at p_0. Then there exists a CR distribution u, defined on an open neighborhood U of p_0, such that $\overline{\text{WF}}(u) \cap T^*_{p_0}M$ properly contains a $(d-k)$-dimensional $\mathbb{R}$-linear subspace.*

Remark 5.2. Tumanov's theorem (see [25]) can be restated by saying that all *CR* functions defined on any fixed neighborhood of p_0 admit a holomorphic extension to an open wedge with edge containing p_0 if and only if no *CR* distribution u has a $\overline{\text{WF}}(u)$ which contains a real line of $T^*_{p_0}M$. Theorem 5.1 can be considered a generalization of that result to the non minimal case.

Proof. We can assume that M is a generic *CR* submanifold of $\mathbb{C}^n$.

Let $\dim_{\mathbb{R}} \mathfrak{O}_{M,loc}(p_0) = 2m + k$. Fix an open neighborhood U of p_0 in M. Then there are generic *CR* manifolds with boundary $M_1, \dots, M_k$ in $\mathbb{C}^n$, of dimension $2m+k+1$, attached to M along their boundaries near p_0, and such that every continuous *CR* function u on U uniquely extends to each M_j as a *CR* function, continuous up to the boundary. Moreover the M_j can be chosen so that there are linearly independent vectors $X_1, \dots, X_k \in T_{p_0}M \setminus T^c_{p_0}M$ such that JX_j points into M_j. Then a standard deformation argument shows that for any continuous *CR* function u on

U, $\mathrm{WF}(u)$ is contained in $\{\xi \in H^0M \mid \xi(X_j) \geq 0\}$ (this was observed in [23] for the larger analytic wave front set), so that $\overline{\mathrm{WF}}(u)$ cannot contain any $\mathbb{R}$-subspace of dimension larger than $d-k$.

To treat the case of a general CR distribution u, we utilize [7]. There it is shown that $u = (\Delta_{L+cZ})^q g$ on an open neighborhood U' of p_0 in M, where Δ_{L+cZ} is an appropriate second-order differential operator with smooth coefficients, q a sufficiently large positive integer and g a continuous CR function. Since $\mathrm{WF}(u) \subset \mathrm{WF}(g)$, the fact that $\overline{\mathrm{WF}}(u) \cap T^*_{p_0}M$ does not contain any $\mathbb{R}$-subspace of dimension larger than $d-k$ follows from the case of continuous CR functions.

On the other hand, assume that there is a CR submanifold N of an open neighborhood U of p_0 in M, with the same CR dimension m and $p_0 \in N$. By taking U small, we can find a CR distribution on U carried by N.

Indeed: When N is open, there is nothing to prove. If N has smaller dimension, we fix a positive measure μ with smooth density on N. A construction in [6] yields a function v which is C^∞-smooth in a neighborhood of p_0 in N, with $v(p_0) = 1$, and such that

$$T_N[\phi] = \int_N v\phi\, d\mu, \quad \phi \in \mathcal{D}(U), \tag{5.1}$$

is a CR distribution on a possibly smaller neighborhood U of p_0 in M. In this case we obtain $\mathrm{WF}(u) \cap T^*_{p_0}M = (T_{p_0}N)^\perp$. This completes the proof of the implication (1)⇒(2). The argument also shows that, if there is a CR distribution u, defined on a neighborhood U of p_0, such that $\overline{\mathrm{WF}}(u) \cap T^*_{p_0}M$ contains an ℓ-dimensional $\mathbb{R}$-subspace, then $\dim_{\mathbb{R}} \mathfrak{O}_{M,\mathrm{loc}}(p_0) \leq 2m+d-\ell$. Thus we obtain also the opposite implication (2)⇒(1).

Let us turn to the proof of the last statement. If $\mathfrak{O}_{M,loc}(p_0)$ is open, it is a consequence of Theorem 1.1, because a distribution u with $\mathrm{WF}(u) \cap T^*_{p_0}M = \emptyset$ is smooth near p_0. If $\mathfrak{O}_{M,loc}(p_0)$ is lower-dimensional, we fix a CR isomorphism $\pi : N \to N' \subset \mathbb{C}^{n'}$ from N to a generic CR manifold in some lower-dimensional space. As explained before Lemma 4.1, we may assume that π is induced by the projection of $\mathbb{C}^n$ onto the complex subspace $\mathbb{C}^{n'}$ of the first n' coordinates $z_1, \dots, z_{n'}$. The Baouendi-Treves approximation theorem says that there is a measure μ' on N', with a smooth density on N', such that any CR distribution S on N' can be approximated by polynomials $Q(z_1, \dots, z_{n'})$, in the sense that

$$\int_{N'} Q_j\phi\, d\mu' \to S[\phi], \quad \forall\phi \in \mathcal{D}(U'), \tag{5.2}$$

holds on an appropriate neighborhood $U' \subset N'$ of $0 = \pi(p_0)$. We can choose $\mu = \pi^*\mu'$ in (5.1). We have the following Lemma.

Lemma 5.3. *There is a neighborhood $U \subset M$ of p_0 such that for is any CR distribution u on N' the formula*

$$T_u[\phi] = u[(v\phi) \circ \pi^{-1}], \quad \forall\phi \in \mathcal{D}(U), \tag{5.3}$$

defines a CR distribution T_u on U with support contained in $N \cap U$.

Proof of Lemma 5.3. Let $\{Q_j = Q_j(z_1, \dots, z_{n'})\}$ be a sequence of polynomials approximating u on some neighborhood U' of 0 in N', as in (5.2). Since $\mu = \pi^*\mu'$, the distributions $Q_jT_N : \phi \mapsto \int_N Q_j v\phi\, d\mu$ approximate the distribution in (5.3), provided we take ϕ with support in an open $U \subset M$ with $p_0 \in U \cap N \subset \pi^{-1}(U')$. Being the products of a *CR* distribution by the restriction to U of holomorphic functions, the Q_jT_N are *CR* distributions on U, and therefore also their limit in the sense of distributions is a *CR* distribution on U. This completes the proof of the lemma. □

Since N' does not have the extension property, by Theorem 1.1 there is a *CR* distribution $\tilde{u}$ with $\mathrm{WF}_{N'}(\tilde{u}) \cap T^*_0 N' \neq \emptyset$. It remains to check that $\mathrm{WF}(T_{\tilde{u}})$ has the desired properties.

To this purpose, we introduce smooth coordinates $(s_1, \dots, s_{2m+k}, t_1, \dots, t_\ell)$, $\ell = d - k$, centered at p_0, such that $N = \{t_1 = 0, \dots, t_\ell = 0\}$. The distribution $T_{\tilde{u}}$ is a tensor product

$$T_{\tilde{u}} = (vg\tilde{u}) \otimes \delta_t,$$

where δ_t is the Dirac delta in the t-variables and g is a smooth nonvanishing function such that $d\mu' = g\, ds_1 \dots ds_{2m+k}$. Since $v(p_0) = 1$, we can assume after shrinking that $v \neq 0$ on U. Then $\mathrm{WF}(u^* vg) = \mathrm{WF}(u^*)$ and the general rule to compute the wave front set of a tensor product [13, Theorem 8.2.9] yields

$$\mathrm{WF}(T_{\tilde{u}}) \cap T^*_{p_0} M = (\,\overline{\mathrm{WF}}_N(u^*) \times \langle dt_1, \dots, dt_\ell \rangle) \setminus \{(0,0)\}, \tag{5.4}$$

The proof is complete. □

6. Some subellipticity conditions

In this section we recall some results of [2] that are relevant for our applications. In the following, M is an abstract *CR* manifold, $\mathscr{Z}(M) = \Gamma(M, T^{0,1}M)$ is the distribution of complex vector fields of type $(0,1)$ on M, and $\mathscr{H}(M) = \Gamma(M, HM)$ the distribution of the *real* vector fields which are real parts elements of $\mathscr{Z}(M)$.

6.1. The system $\Theta(M)$.

Definition 6.1. Set

$$\Theta(M) = \left\{ Z \in \mathscr{Z}(M) \;\middle|\; \begin{array}{l} \exists r \geq 0,\ \exists Z_1, \dots, Z_r \in \mathscr{Z}(M),\ \text{s.t.} \\ i[Z, \bar{Z}] + i \sum_{j=1}^r [Z_j, \bar{Z}_j] \in \mathscr{H}(M) \end{array} \right\}. \tag{6.1}$$

We denote by $\mathscr{A}(M)$ the Lie subalgebra of $\mathfrak{X}(M)$ generated by the real parts of vectors in $\Theta(M)$. If $\mathscr{H}'(M) = \{\mathrm{Re}\, Z \mid Z \in \Theta(M)\}$,

$$\mathscr{A}(M) = \mathscr{H}'(M) + [\mathscr{H}'(M), \mathscr{H}'(M)] + [\mathscr{H}'(M), [\mathscr{H}'(M), \mathscr{H}'(M)]] + \cdots$$

We showed in [2, Lemma 2.5] that:

Proposition 6.2. *With the notation introduced above, $\Theta(M)$ is a left $C^\infty(M)$-submodule of $\mathfrak{X}^{\mathbb{C}}(M)$. For every $Z \in \Theta(M)$ and every relatively compact open subset U of M there are a finite set $Z_1, \dots, Z_r$ of vector fields in $\mathscr{Z}(M)$ and a constant $C > 0$ such that*

$$\|\bar{Z}u\|_0^2 \leq C(\|u\|_0^2 + \textstyle\sum_{i=1}^r \|Z_i u\|_0^2), \quad \forall u \in C_0^\infty(U).$$

Hence, by [2, Corollary 1.15], we obtain

Theorem 6.3. *Let $\mathscr{M}(M)$ be the $\mathscr{A}(M)$-Lie submodule of $\mathfrak{X}(M)$ generated by $\mathscr{H}(M)$:*

$$\begin{aligned} \mathscr{M}(M) = \mathscr{H}(M) &+ [\mathscr{A}_{\mathscr{Z}}(M), \mathscr{H}(M)] \\ &+ [\mathscr{A}_{\mathscr{Z}}(M), [\mathscr{A}_{\mathscr{Z}}(M), \mathscr{H}(M)]] + \cdots \end{aligned} \tag{6.2}$$

If

$$\{X_{p_0} \mid X \in \mathscr{M}(M)\} = T_{p_0} M, \tag{6.3}$$

then the system $\mathscr{Z}(M)$ is subelliptic at p_0. This means that there exists an open neighborhood U of p_0 in M, vector fields $Z_1, \dots, Z_n \in \mathscr{Z}(M)$, and constants $C, \varepsilon > 0$ such that

$$\|u\|_{\varepsilon}^2 \leq C(\|u\|_0^2 + \sum\nolimits_{i=1}^{n} \|Z_i u\|_0^2), \quad \forall u \in C_0^\infty(U). \tag{6.4}$$

6.2. **The system $\mathscr{K}(M)$.** Under a certain constant rank assumption on $\mathscr{Z}(M)$, we can give a more explicit description of $\Theta(M)$.

Definition 6.4. The *characteristic bundle* H^0M of $\mathscr{Z}(M)$ is the set of *real* covectors ξ with $\langle Z, \xi \rangle = 0$ for all $Z \in \mathscr{Z}(M)$.

The *scalar Levi form* at $\xi \in H_p^0 M$ is the Hermitian symmetric form

$$\mathfrak{L}_\xi(Z_1, \bar{Z}_2) = i\xi([Z_1, \bar{Z}_2]) \quad \text{for} \quad Z_1, Z_2 \in \mathscr{Z}(M). \tag{6.5}$$

The value of the right hand side of (6.5) only depends on the values $Z_1(p)$, $Z_2(p)$ of Z_1, Z_2 at the base point $p = \pi(\xi)$. Thus (6.5) is a Hermitian symmetric form on $T_p^{0,1}M$. Set:

$$H^\oplus M = \left\{ \xi \in H^0 M \mid \mathcal{L}_\xi \geq 0 \right\}, \tag{6.6}$$

$$\mathscr{K}(M) = \{ Z \in \mathscr{Z}(M) \mid \mathcal{L}_\xi(Z, \bar{Z}) = 0, \ \forall \xi \in H^\oplus M \}, \tag{6.7}$$

$$\mathrm{K}M = \dot{\bigcup}_{p \in M} \mathrm{K}_p M \text{ with } \mathrm{K}_p M = \{ Z_p \mid Z \in \mathbb{Z}(M) \}. \tag{6.8}$$

We have (see [2, Proposition 2.13])

Proposition 6.5. *$\mathscr{K}(M)$ is a left $C^\infty(M)$ submodule of $\Theta(M)$. Assume in addition that $H^\oplus M$ and $\mathrm{K}M$ are smooth vector bundles on M. Then*

$$\mathscr{K}(M) = \Theta(M). \tag{6.9}$$

6.3. **Hypoellipticity.** [1] Subelliptic estimates imply regularity. We have indeed (see [2, Theorem 4.1], [10, Theorem 4.3]):

Theorem 6.6. *Let M be an m-dimensional smooth manifold. Let U be an open subset of M, and $Z_1, \dots, Z_n$ complex vector fields on U such that, for some positive constants $C, \epsilon > 0$ (6.4) is valid. If $u \in L^2_{loc}$, $a_i \in L^\infty_{loc}(U)$, $f_i \in L^2_{loc}(U)$ for $i = 1, \dots, n$ satisfy*

$$Z_i u + a_i u = f_i, \quad \text{for } i = 1, \dots, n \quad \text{on } U, \tag{6.10}$$

then :

1. *$u \in W^\epsilon_{loc}(U)$;*
2. *if $0 < s \leq \frac{m}{2}$, $a_i \in C^s(U)$ and $f_i \in W^s_{loc}(U)$, then $u \in W^{s+\epsilon}_{loc}(U)$;*
3. *if $s > \frac{m}{2}$, $a_i \in W^s_{loc}(U)$ and $f_i \in W^s_{loc}(U)$, then $u \in W^{s+\epsilon}_{loc}(U)$;*
4. *in particular, if $a_i \in C^\infty(U)$, $f_i \in W^s_{loc}(U)$, then $u \in W^{s+\epsilon}_{loc}(U)$.*

Here we indicate by $W^s_{loc}(U)$ the L^2-Sobolev space of order s.

Then we obtain from Lemma 4.4:

Corollary 6.7. *If (6.3) holds true, then $\mathfrak{O}^{-\infty}_{M,(p_0)} = \mathfrak{O}^{\infty}_{M,(p_0)}$.*

7. Examples

A large class of examples of *CR* submanifolds of complex manifolds is provided by the orbits of the real forms in complex flag manifolds. We recall that a complex flag manifold is a compact homogeneous space X of a semisimple complex Lie group $\mathbf{G}$. The isotropy of a point of X is a *parabolic* subgroup $\mathbf{Q}$ of $\mathbf{G}$, i.e. a closed connected subgroup whose Lie algebra $\mathfrak{q}$ contains a maximal solvable Lie subalgebra $\mathfrak{b}$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. If $\mathbf{G}_0$ is a *real form* of $\mathbf{G}$, i.e. a connected real Lie subgroup of $\mathbf{G}_0$ with Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g}=\mathfrak{g}_0\oplus i\mathfrak{g}_0$, then $\mathbf{G}_0$ has finitely many orbits in X. In particular, there are open orbits and a minimal orbit M which is compact (see [27]). The structure of the orbits only depend on the Lie algebras involved, and are therefore completely determined by the pair $(\mathfrak{g}_0, \mathfrak{q})$, which is called a *CR algebra*, consisting of the Lie algebra of the real form $\mathbf{G}_0$ and of the Lie algebra of the parabolic subgroup $\mathbf{Q}$.

The embedding of M in X defines a *CR* structure on M. The minimal orbits are classified by their *cross-marked Satake diagrams*. A complete list of these diagrams is given e.g. in the appendix to [4]. Many properties of the minimal orbits are read off these diagrams: minimality is equivalent to the fact that the corresponding *CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and is described by [4, Theorem 9.3]. In [4, §13] all *essentially pseudoconcave* minimal orbits are classified in terms of their associated diagrams. Since essential pseudoconcavity (see [10]) implies (6.3), all these orbits are at every point *CR*-hypoelliptic and therefore have the holomorphic extension property by Theorem1.1. Globally defined *CR* functions on this class of *CR* manifolds and their properties were considered in [1].

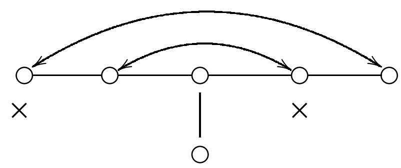
We give below some more explicit examples to illustrate this application.

Let X be the complex flag manifold consisting of the flags

$$\ell_1 \subset \ell_3 \subset \cdots \subset \ell_{2k-1} \subset \ell_{2k+2} \subset \cdots \subset \ell_{4k-2} \subset \mathbb{C}^{4k},$$

where k is a positive integer and ℓ_i is a $\mathbb{C}$-linear subspace of dimension i of $\mathbb{C}^{4k}$. Let M be the minimal orbit for the action of the group $\mathbf{SU}(2k, 2k)$ of complex $4k\times 4k$ matrices that leave invariant a Hermitian symmetric form of signature $(2k, 2k)$. Then M has *CR* dimension $2k$ and *CR* codimension $8k^2 - 6k - 1$ and we need $2k$ commutators of $\mathscr{H}(M)$ to span TM (these numbers were computed in [16]). However, M is minimal and essentially pseudoconcave and therefore is *CR*-hypoelliptic and has the holomorphic extension property at all points.

Another example is the minimal orbit of the special group $\mathbf{G}_0$ of type E_6III corresponding to the cross-marked Satake diagram



It corresponds to a *CR* manifold of *CR* dimension 4 and *CR* codimension 25, with 6 commutations needed to span TM from $\mathscr{H}(M)$. This is also essentially pseudoconcave and therefore is *CR*-hypoelliptic and has the holomorphic extension property at each point.

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