Generalized Patterson-Sullivan measures for products of Hadamard spaces

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Abstract

Let Γ be a discrete group acting by isometries on a product $X = X_1 \times X_2$ of Hadamard spaces. We further require that X_1, X_2 are locally compact and Γ contains two elements projecting to a pair of independent rank one isometries in each factor. Apart from discrete groups acting by isometries on a product of CAT(-1)-spaces, the probably most interesting examples of such groups are Kac-Moody groups over finite fields acting on the Davis complex of their associated twin building. In [Lin10] we showed that the regular geometric limit set splits as a product $F_\Gamma \times P_\Gamma$, where $F_\Gamma \subseteq \partial X_1 \times \partial X_2$ is the projection of the geometric limit set to $\partial X_1 \times \partial X_2$, and P_Γ encodes the ratios of the speed of divergence of orbit points in each factor. Our aim in this paper is a description of the limit set from a measure theoretical point of view. We first study the conformal density obtained from the classical Patterson-Sullivan construction, then generalize this construction to obtain measures supported in each Γ-invariant subset of the regular limit set and investigate their properties. Finally we show that the Hausdorff dimension of the radial limit set in each Γ-invariant subset of L_Γ is bounded above by the exponential growth rate introduced in [Lin10].

1 Introduction

Let (X_1, d_1), (X_2, d_2) be Hadamard spaces, i.e. complete simply connected metric spaces of non-positive Alexandrov curvature, and (X, d) the product $X_1 \times X_2$ endowed with the metric $d = \sqrt{d_1^2 + d_2^2}$. Assume moreover that X_1, X_2 are locally compact. Each metric space X, X_1, X_2 can be compactified by adding its geometric boundary ∂X, ∂X_1, ∂X_2 endowed with the cone topology (see [Bal95, Chapter II]). It is well-known that the regular geometric boundary ∂X^{reg} of X – which consists of the set of equivalence classes of geodesic rays which do not project to a point in one of the factors – is a dense open subset of ∂X homeomorphic to $\partial X_1 \times \partial X_2 \times (0, \pi/2)$. The last factor in this product is called the slope of a point in ∂X^{reg}. The singular geometric boundary $\partial X^{sing} = \partial X \setminus \partial X^{reg}$ consists of two strata homeomorphic to ∂X_1, ∂X_2 respectively. We assign slope 0 to the first and slope $\pi/2$ to the second one.

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For a group $\Gamma \subseteq \text{Is}(X_1) \times \text{Is}(X_2)$ acting properly discontinuously by isometries on X the limit set is defined by $L_\Gamma := \overline{\Gamma \cdot x} \cap \partial X$, where $x \in X$ is arbitrary. In order to relate the critical exponent of a Fuchsian group to the Hausdorff dimension of its limit set, S. J. Patterson ([Pat76]) and D. Sullivan ([Sul79]) developed a theory of conformal densities. It turned out that for higher rank symmetric spaces and Euclidean buildings these densities in general detect only a small part of the geometric limit set (see [Alb99]). In order to measure the limit set in each invariant subset of the limit set, a class of generalized conformal densities were independently introduced in [Qui02] and [Lin04]. One of the main goals in this paper is to adapt this construction to discrete groups $\Gamma \subseteq \text{Is}(X_1) \times \text{Is}(X_2)$ which contain a pair of isometries projecting to independent rank one elements in each factor. Related questions were considered by M. Burger ([Bur93]) for graphs of convex cocompact groups in a product of rank one symmetric spaces, and by F. Dal'bo and I. Kim ([DK]) for discrete isometry groups of a product of Hadamard manifolds of pinched negative curvature.

One important class of examples satisfying our conditions are Kac-Moody groups Γ over a finite field which act by isometries on a product $X = X_1 \times X_2$, the CAT(0)-realization of the associated twin building $\mathcal{B}_+ \times \mathcal{B}_-$. Indeed, there exists an element $h = (h_1, h_2)$ projecting to a rank one element in each factor by Remark 5.4 and the proof of Corollary 1.3 in [CF10]. Moreover, the action of the Weyl group produces many axial isometries $g = (g_1, g_2)$ with g_i rank one and independent from h_i for $i = 1, 2$. Notice that if the order of the ground field is sufficiently large, then $\Gamma \subseteq \text{Is}(X_1) \times \text{Is}(X_2)$ is an irreducible lattice (see e.g. [Rém99] and [CR09]).

A second type of examples are groups acting properly discontinuously on a product of locally compact Hadamard spaces of strictly negative Alexandrov curvature (compare [DK] in the manifold setting). In this special case every non-elliptic and non-parabolic isometry in one of the factors is already a rank one element. Prominent examples here which are already covered by the above mentioned results of J. F. Quint and the author are Hilbert modular groups acting as irreducible lattices on a product of hyperbolic planes and graphs of convex cocompact groups of rank one symmetric spaces (see also [Bur93]).

Before we can state our results we need some definitions. We fix a base point $o = (o_1, o_2) \in X$. For $\theta \in [0, \pi/2]$ we denote ∂X_θ the set of points in the geometric boundary of slope θ. Moreover, for $i \in \{1, 2\}$ and $\eta_i \in \partial X_i$ let $\mathcal{B}_{\eta_i}(\cdot, o_i)$ denote the Busemann function centered at η_i based at o_i.

A central role throughout the paper is played by the exponent of growth of Γ of given slope $\theta \in [0, \pi/2]$ introduced in [Lin10]. For $n \in \mathbb{N}$ and $\varepsilon > 0$ we put

$$N_\theta^\varepsilon(n) := \#\{\gamma = (\gamma_1, \gamma_2) \in \Gamma : n - 1 < d(o, \gamma o) \leq n, \, \Big| \arctan \frac{d_2(o_2, \gamma_2 o_2)}{d_1(o_1, \gamma_1 o_1)} - \theta \Big| < \varepsilon\} \, .$$

Definition 1.1 *The* exponent of growth *of* Γ *of slope* $\theta \in [0, \pi/2]$ *is defined by*

$$\delta_\theta(\Gamma) := \liminf_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log N_\theta^\varepsilon(n) \right) \, .$$

The quantity $\delta_\theta(\Gamma)$ can be thought of as a function of $\theta \in [0, \pi/2]$ which describes the exponential growth rate of orbit points converging to limit points of slope θ. It is an invariant of Γ which carries more information than the critical exponent $\delta(\Gamma)$. Moreover, Theorem 7.4 in [Lin10] implies that there exists a unique slope $\theta_* \in [0, \pi/2]$ such that the exponent of growth of Γ is maximal for this slope and equal to the critical exponent $\delta(\Gamma)$.

Our first result concerns the measures on the geometric boundary obtained by the classical Patterson-Sullivan construction. Similar to the case of higher rank symmetric spaces or Euclidean buildings we have the following result:

Theorem A *The Patterson-Sullivan construction produces a conformal density with support in a single Γ-invariant subset of the geometric limit set. Every point in its support has slope θ_* as above.*

Thus in order to measure the remaining Γ-invariant subsets of the limit set, we need a more sophisticated construction. Inspired by the paper [Bur93] of M. Burger we therefore consider densities with one more degree of freedom than the classical conformal density:

Definition 1.2 *Let $\mathcal{M}^+(\partial X)$ denote the cone of positive finite Borel measures on ∂X, $\theta \in [0, \pi/2]$ and $b = (b_1, b_2) \in \mathbb{R}^2$. A Γ-invariant (b, θ)-density is a continuous map*

$$\begin{array}{rcl} \mu : X & \to & \mathcal{M}^+(\partial X) \\ x & \mapsto & \mu_x \end{array}$$

such that for any $x \in X$ the following three properties hold:

(i) $\emptyset \neq supp(\mu_x) \subseteq L_\Gamma \cap \partial X_\theta$,

(ii) $\forall\, \gamma \in \Gamma \qquad \gamma_* \mu_x = \mu_{\gamma x}$[1],

(iii) *if* $\theta \in (0, \pi/2)$, *then* $\forall\, \tilde\eta = (\eta_1, \eta_2, \theta) \in supp(\mu_o)$

$$\frac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{b_1 \mathcal{B}_{\eta_1}(o_1, x_1) + b_2 \mathcal{B}_{\eta_2}(o_2, x_2)},$$

if $\theta = 0$, *then* $b_2 = 0$ *and* $\forall\, \tilde\eta = \eta_1 \in supp(\mu_o)$ $\quad \dfrac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{b_1 \mathcal{B}_{\eta_1}(o_1, x_1)}$,

if $\theta = \frac{\pi}{2}$, *then* $b_1 = 0$ *and* $\forall\, \tilde\eta = \eta_2 \in supp(\mu_o)$ $\quad \dfrac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{b_2 \mathcal{B}_{\eta_2}(o_2, x_2)}$.

Notice that the conformal density from Theorem A is a special case of such a density with support in ∂X_{θ_*} and parameters $b_1 = \delta(\Gamma)\cos\theta_*$, $b_2 = \delta(\Gamma)\sin\theta_*$.

We next give a criterion for the existence of a (b, θ)-density.

Theorem B *If $\theta \in (0, \pi/2)$ is such that $\delta_\theta(\Gamma) > 0$, then there exists a (b, θ)-density for some parameters $b = (b_1, b_2) \in \mathbb{R}^2$.*

In Section 6 we will explicitly describe the construction of such a (b, θ)-density. Notice that our method does not cover the cases $\theta = 0$ and $\theta = \pi/2$ in general. However, if $\delta_\theta(\Gamma) = \delta(\Gamma)$, then by Theorem A the classical Patterson-Sullivan construction

[1] Here $\gamma_* \mu_x$ denotes the measure defined by $\gamma_* \mu_x(E) = \mu_x(\gamma^{-1} E)$ for any Borel set $E \subseteq \partial X$

provides a (b,θ)-density, whether θ belongs to $(0,\pi/2)$ or not. Unfortunately, we do not know of an example with $\delta_\theta(\Gamma)=\delta(\Gamma)$ for $\theta=0$ or $\theta=\pi/2$.

The following results about (b,θ)-densities in particular apply to any conformal density supported in a single Γ-invariant subset of the geometric limit set, not only the one obtained by the classical Patterson-Sullivan construction. Our main tool is a so-called shadow lemma for (b,θ)-densities, which is a generalization of the well-known shadow lemma for conformal densities. It first gives a condition for the parameters of a (b,θ)-density in terms of the exponent of growth.

Theorem C *If a Γ-invariant (b,θ)-density exists for some $\theta\in(0,\pi/2)$, then*

$$\delta_\theta(\Gamma)\le b_1\cos\theta+b_2\sin\theta\,.$$

The following subsets of the geometric limit set will play an important role in the sequel.

DEFINITION 1.3 *A point $\tilde\xi\in\partial X$ is called a* radial limit point *of Γ if there exists a sequence $(\gamma_n)=\big((\gamma_{n,1},\gamma_{n,2})\big)\subset\Gamma$ such that $\gamma_n o$ converges to $\tilde\xi$, and the following condition holds:*

If $\tilde\xi=(\xi_1,\xi_2,\theta)\in\partial X^{reg}$, then for $i\in\{1,2\}$ $\gamma_{n,i}o_i$ stays at bounded distance of one (and hence any) geodesic ray in the class of ξ_i, if $i=1,2$ and $\tilde\xi=\xi_i\in\partial X_i\subset\partial X^{sing}$, then $\gamma_{n,i}o_i$ stays at bounded distance of one (and hence any) geodesic ray in the class of ξ_i.

We will denote the set of radial limit points of Γ by L_Γ^{rad}.

Notice that in general, a radial limit point $\tilde\xi$ is not approached by a sequence $\gamma_n o$, $\gamma_n\in\Gamma$, at bounded distance of a geodesic ray in the class of $\tilde\xi$.

Our next statement shows that for certain (b,θ)-densities the corresponding exponent of growth $\delta_\theta(\Gamma)$ is completely determined by the parameters $\theta\in(0,\pi/2)$ and $b=(b_1,b_2)\in\mathbb{R}^2$.

Theorem D *If $\theta\in(0,\pi/2)$ and μ is a Γ-invariant (b,θ)-density which gives positive measure to the radial limit set, then $\delta_\theta(\Gamma)=b_1\cos\theta+b_2\sin\theta$.*

The following theorem gives a restriction for the atomic part of our measures.

Theorem E *If $\theta\in(0,\pi/2)$ such that $\delta_\theta(\Gamma)>0$, and μ is a Γ-invariant (b,θ)-density, then a radial limit point is not a point mass for μ.*

Moreover, using a Hausdorff measure on the geometric boundary as proposed by G. Knieper ([Kni97, Section 4]), we have the following

Theorem F *For any $\theta\in[0,\pi/2]$ with $L_\Gamma\cap\partial X_\theta\neq\emptyset$ we have*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad}\cap\partial X_\theta)\le\delta_\theta(\Gamma)\,.$$

Unfortunately, a precise estimate for the Hausdorff dimension can be given only for a particular class of groups which we choose to call radially cocompact. Examples of such groups are uniform lattices and products of convex cocompact groups acting on a product of Hadamard manifolds of pinched negative curvature. Since Kac-Moody groups over finite fields are never cocompact, we do not know whether a similar result holds for them.

Theorem G *If* Γ *is radially cocompact and* $\theta \in (0, \pi/2)$ *such that* $\delta_\theta(\Gamma) > 0$*, then*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap \partial X_\theta) = \delta_\theta(\Gamma)\,.$$

The paper is organized as follows: In Section 2 we recall basic facts about Hadamard spaces and rank one isometries. Section 3 deals with the product case and provides some tools for the proof of the so-called shadow lemma in Section 7. In Section 4 we introduce and study the properties of the exponent of growth. Section 5 recalls the classical Patterson-Sullivan construction in our setting. The main new result here is Theorem A. In Section 6 we introduce a generalized Poincaré series that allows to construct (b, θ)-densities, and therefore proves Theorem B. Using the shadow lemma, in Section 7 we deduce properties of (b, θ)-densities and prove Theorems C, D and E. Section 8 finally is concerned with the Hausdorff dimension of the limit set and the proofs of Theorems F and G.

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2 Preliminaries

The purpose of this section is to introduce terminology and notation and to summarize basic results about Hadamard spaces and rank one isometries. The main references here are [BH99] and [Bal95] (see also [BB95], and [BGS85], [Bal82] in the case of Hadamard manifolds).

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map σ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\sigma(0) = x$, $\sigma(l) = y$ and $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. We will denote such a geodesic path $\sigma_{x,y}$. X is called geodesic if any two points in X can be connected by a geodesic path, if this path is unique we say that X is uniquely geodesic. In this text X will be a Hadamard space, i.e. a complete geodesic metric space in which all triangles satisfy the CAT(0)-inequality. This implies in particular that X is simply connected and uniquely geodesic. A geodesic or geodesic line in X is a map $\sigma : \mathbb{R} \to X$ such that $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$, a geodesic ray is a map $\sigma : [0, \infty) \to X$ such that $d(\sigma(t), \sigma(t')) = |t-t'|$ for all $t, t' \in [0, \infty)$. Notice that in the non-Riemannian setting completeness of X does not imply geodesically completeness, i.e. not every geodesic path or ray can be extended to a geodesic.

From here on we will assume that X is a locally compact Hadamard space. The geometric boundary ∂X of X is the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology (see e.g. [Bal95, chapter II]). The action of the isometry group $\mathrm{Is}(X)$ on X naturally extends to an action by homeomorphisms on the geometric boundary. Moreover, since X is locally compact, this boundary ∂X is compact and the space X is a dense and open subset of the compact space $\overline{X} := X \cup \partial X$. For $x \in X$ and $\xi \in \partial X$ arbitrary there exists a geodesic ray emanating from x which belongs to the class of ξ. We will denote such a ray $\sigma_{x,\xi}$.

We say that two points ξ, $\eta \in \partial X$ can be joined by a geodesic if there exists a geodesic $\sigma : \mathbb{R} \to X$ such that $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta$. It is well-known that if X is CAT(-1), i.e. of negative Alexandrov curvature bounded above by -1, then every pair of distinct points in the geometric boundary can be joined by a geodesic. This is not true in the general CAT(0)-case.

Let $x, y \in X$, $\xi \in \partial X$ and σ a geodesic ray in the class of ξ. We put

$$\mathcal{B}_\xi(x,y) := \lim_{s\to\infty} \big(d(x,\sigma(s)) - d(y,\sigma(s)) \big) . \tag{1}$$

This number is independent of the chosen ray σ, and the function

$$\begin{array}{rccc} \mathcal{B}_\xi(\cdot, y) : & X & \to & \mathbb{R} \\ & x & \mapsto & \mathcal{B}_\xi(x,y) \end{array}$$

is called the Busemann function centered at ξ based at y (see also [Bal95], chapter II). From the definition one immediately obtains the following properties of the Busemann function:

$$\begin{aligned} \mathcal{B}_\xi(x,y) &= -\mathcal{B}_\xi(y,x) && (2)\\ |\mathcal{B}_\xi(x,y)| &\leq d(x,y) && (3)\\ \mathcal{B}_\xi(x,z) &= \mathcal{B}_\xi(x,y) + \mathcal{B}_\xi(y,z) && (4)\\ \mathcal{B}_{g\cdot\xi}(g\cdot x, g\cdot y) &= \mathcal{B}_\xi(x,y) \end{aligned}$$

for all $x, y, z \in X$, $\xi \in \partial X$ and $g \in \mathrm{Is}(X)$. Moreover, $\mathcal{B}_\xi(x,y) = d(x,y)$ if and only if y is a point on the geodesic ray $\sigma_{x,\xi}$, and we have the following easy

Lemma 2.1 *Let $c > 0$, $x, y \in X$, $x \neq z$, and $\xi \in \partial X$ such that $d(z, \sigma_{x,\xi}) < c$. Then*

$$0 \leq d(x,z) - \mathcal{B}_\xi(x,z) < 2c .$$

Proof. The first inequality is (3). For the second one let $y \in X$ be a point on the geodesic ray $\sigma_{x,\xi}$ such that $d(z,y) < c$. Then for all $s > d(x,y)$ we have by the triangle inequality

$$\begin{aligned} d(x,\sigma_{x,\xi}(s)) - d(z,\sigma_{x,\xi}(s)) &\geq d(x,\sigma_{x,\xi}(s)) - d(z,y) - d(y,\sigma_{x,\xi}(s)) \\ &= d(x,y) - d(z,y) > d(x,y) - c , \end{aligned}$$

hence $d(x,z) - \mathcal{B}_\xi(x,z) \leq d(x,y) + c - d(x,y) + c = 2c$. □

A geodesic $\sigma : \mathbb{R} \to X$ is said to bound a flat half-plane if there exists a closed convex subset $i([0,\infty) \times \mathbb{R})$ in X isometric to $[0,\infty) \times \mathbb{R}$ such that $\sigma(t) = i(0,t)$ for all $t \in \mathbb{R}$. Similarly, a geodesic $\sigma : \mathbb{R} \to X$ bounds a flat strip of width $c > 0$ if there exists a closed convex subset $i([0,c] \times \mathbb{R})$ in X isometric to $[0,c] \times \mathbb{R}$ such that $\sigma(t) = i(0,t)$

for all $t \in \mathbb{R}$. We call a geodesic $\sigma : \mathbb{R} \to X$ a rank one geodesic if σ does not bound a flat half-plane.

The following important lemma states that even though we cannot join any two distinct points in the geometric boundary of X, given a rank one geodesic we can at least join points in a neighborhood of its extremities. More precisely, we have the following well-known

LEMMA 2.2 *([Bal95], Lemma III.3.1) Let $\sigma : \mathbb{R} \to X$ be a rank one geodesic. Then there exist $c > 0$ and neighborhoods U of $\sigma(-\infty)$ and V of $\sigma(\infty)$ in $\overline{X}$ such that for any $\xi \in U$ and $\eta \in V$ there exists a rank one geodesic joining ξ and η. Moreover, any such geodesic σ' satisfies $d(\sigma', \sigma(0)) \leq c$..*

The following kind of isometries will play a central role in the sequel.

DEFINITION 2.3 *An isometry h of X is called* axial*, if there exists a constant $l = l(h) > 0$ and a geodesic σ such that $h(\sigma(t)) = \sigma(t + l)$ for all $t \in \mathbb{R}$. We call $l(h)$ the* translation length *of h, and σ an* axis *of h. The boundary point $h^+ := \sigma(\infty)$ is called the* attractive fixed point*, and $h^- := \sigma(-\infty)$ the* repulsive fixed point *of h. We further put $Ax(h) := \{x \in X \mid d(x, hx) = l(h)\}$.*

We remark that $\mathrm{Ax}(h)$ consists of the union of parallel geodesics translated by h, and $\overline{\mathrm{Ax}(h)} \cap \partial X$ is exactly the set of fixed points of h.

DEFINITION 2.4 *An axial isometry is called* rank one *if it possesses a rank one axis. Two rank one isometries are called* independent*, if their fixed point sets are disjoint.*

Notice that if h is rank one, then h^+ and h^- are the only fixed points of h. Let us recall the north-south dynamics of rank one isometries.

LEMMA 2.5 *([Bal95], Lemma III.3.3) Let h be a rank one isometry. Then*

(a) Any $\xi \in \partial X \setminus \{h^+\}$ can be joined to h^+ by a geodesic, and every geodesic joining ξ to h^+ is rank one,

(b) given neighborhoods U of h^- and V of h^+ in $\overline{X}$ there exists $N_0 \in \mathbb{N}$ such that $h^{-n}(\overline{X} \setminus V) \subset U$ and $h^n(\overline{X} \setminus U) \subset V$ for all $n \geq N_0$.

If Γ is a group acting by isometries on a locally compact Hadamard space X we define its geometric limit set by $L_\Gamma := \overline{\Gamma \cdot x} \cap \partial X$, where $x \in X$ is arbitrary.

3 Products of Hadamard spaces

Now let (X_1, d_1), (X_2, d_2) be locally compact Hadamard spaces, and $X = X_1 \times X_2$ the product space endowed with the product distance $d = \sqrt{d_1^2 + d_2^2}$. Notice that such a product is again a locally compact Hadamard space. To any pair of points $x = (x_1, x_2)$, $z = (z_1, z_2) \in X$ we associate the vector

$$H(x, z) := \begin{pmatrix} d_1(x_1, z_1) \\ d_2(x_2, z_2) \end{pmatrix} \in \mathbb{R}^2 \,, \tag{5}$$

which we call the distance vector of the pair (x, z). If $z \neq x$ we further define the direction of z with respect to x by

$$\theta(x, z) := \arctan \frac{d_2(x_2, z_2)}{d_1(x_1, z_1)} . \tag{6}$$

Moreover, for convenience we set $\theta(x, x) = 0$ for $x \in X$.

Clearly we have $H(z, x) = H(x, z)$ and $\theta(z, x) = \theta(x, z)$. Notice that we can also write

$$H(x, z) = d(x, z) \begin{pmatrix} \cos\theta(x, z) \\ \sin\theta(x, z) \end{pmatrix} ,$$

hence in particular $\|H(x, z)\| = d(x, z)$, where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$. The following easy lemma states that distance vectors and directions are invariant by $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$.

LEMMA 3.1 *If $g = (g_1, g_2) \in Is(X_1) \times Is(X_2)$, $x = (x_1, x_2)$, $z = (z_1, z_2) \in X$, then*

$$H(gx, gz) = H(x, z) \quad \textit{and} \qquad \theta(gx, gz) = \theta(x, z) .$$

Proof. Since $g_1 \in \mathrm{Is}(X_1)$ and $g_2 \in \mathrm{Is}(X_2)$ we have

$$d_1(g_1x_1, g_1z_1) = d_1(x_1, z_1) \quad \text{and} \qquad d_2(g_2x_2, g_2z_2) = d_2(x_2, z_2) .$$

Hence by (5) and (6)

$$\begin{aligned} H(gx, gz) &= \begin{pmatrix} d_1(g_1x_1, g_1z_1) \\ d_2(g_2x_2, g_2z_2) \end{pmatrix} = \begin{pmatrix} d_1(x_1, z_1) \\ d_2(x_2, z_2) \end{pmatrix} = H(x, z) , \\ \theta(gx, gz) &= \arctan \frac{d_2(g_2x_2, g_2z_2)}{d_1(g_1x_1, g_1z_1)} = \arctan \frac{d_2(x_2, z_2)}{d_1(x_1, z_1)} = \theta(x, z) . \end{aligned}$$

□

Denote $p_i : X \to X_i$, $i = 1, 2$, the natural projections. Every geodesic path $\sigma : [0, l] \to X$ can be written as a product $\sigma(t) = (\sigma_1(t\cos\theta), \sigma_2(t\sin\theta))$, where $\theta \in [0, \pi/2]$ and $\sigma_1 : [0, l\cos\theta] \to X_1$, $\sigma_2 : [0, l\sin\theta] \to X_2$ are geodesic paths in X_1, X_2. θ equals the direction of $\sigma(l)$ with respect to $\sigma(0)$ and is called the slope of σ. We say that a geodesic path σ is regular if its slope is contained in the open interval $(0, \pi/2)$. In other words, σ is regular if neither $p_1(\sigma([0, l]))$ nor $p_2(\sigma([0, l]))$ is a point.

If $x \in X$ and $\sigma : [0, \infty) \to X$ is an arbitrary geodesic ray, then by elementary geometric estimates one has the relation

$$\theta = \lim_{t \to \infty} \theta(x, \sigma(t)) \tag{7}$$

between the slope θ of σ and the directions of $\sigma(t)$, $t > 0$, with respect to x. Similarly, one can easily show that any two geodesic rays representing the same (possibly singular) point in the geometric boundary necessarily have the same slope. So we may define the slope $\theta(\tilde{\xi})$ of a point $\tilde{\xi} \in \partial X$ as the slope of an arbitrary geodesic ray representing $\tilde{\xi}$.

It is easy to see that a pair of distinct boundary points cannot be joined by a geodesic if they do not have the same slope. Moreover, two regular geodesic rays σ, σ' of the same slope represent the same point in the geometric boundary if and only if $\sigma_1(\infty) = \sigma_1'(\infty)$ and $\sigma_2(\infty) = \sigma_2'(\infty)$. The regular geometric boundary ∂X^{reg} of X is defined as the set of equivalence classes of regular geodesic rays and hence is homeomorphic to $\partial X_1 \times \partial X_2 \times (0, \pi/2)$.

If $\gamma \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$, then the slope of $\gamma{\cdot}\tilde{\xi}$ equals the slope of $\tilde{\xi}$. In other words, if ∂X_θ denotes the set of points in the geometric boundary of slope $\theta \in [0, \pi/2]$, then ∂X_θ is invariant by the action of $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$. Notice that points in $\partial X^{sing} := (\partial X)_0 \sqcup (\partial X)_{\pi/2}$ are equivalence classes of geodesic rays which project to a point in one of the factors of X. Hence $(\partial X)_0$ is homeomorphic to ∂X_1 and $(\partial X)_{\pi/2}$ is homeomorphic to ∂X_2. If $\theta \in (0, \pi/2)$, then the set $\partial X_\theta \subset \partial X^{reg}$ is homeomorphic to the product $\partial X_1 \times \partial X_2$. In the sequel we will often use the identification $\partial X = \partial X_1 \sqcup \partial X_2 \sqcup \big(\partial X_1 \times \partial X_2 \times (0, \pi/2)\big)$.

We remark that if $\theta \in (0, \pi/2)$, then a sequence $(y_n) = \big((y_{n,1}, y_{n,2})\big) \subset X$ converges to $\tilde{\eta} = (\eta_1, \eta_2, \theta)$ if and only if $y_{n,1} \to \eta_1$, $y_{n,2} \to \eta_2$ and $\theta(o, y_n) \to \theta$ as $n \to \infty$. Similarly, $(y_n) = \big((y_{n,1}, y_{n,2})\big) \subset X$ converges to $\tilde{\eta} = \eta_1 \in (\partial X)_0 \cong \partial X_1$ if and only if $y_{n,1} \to \eta_1$ and $\theta(o, y_n) \to 0$ as $n \to \infty$, and $(y_n) = \big((y_{n,1}, y_{n,2})\big) \subset X$ converges to $\tilde{\eta} = \eta_2 \in (\partial X)_{\pi/2} \cong \partial X_2$ if and only if $y_{n,2} \to \eta_2$ and $\theta(o, y_n) \to \pi/2$ as $n \to \infty$.

For higher rank symmetric spaces and Bruhat-Tits buildings there is a well-known notion of Furstenberg boundary which – for a product of rank one spaces – coincides with the product of the geometric boundaries. In our setting we choose to call the product $\partial X_1 \times \partial X_2$ endowed with the product topology the Furstenberg boundary $\partial^F X$ of X. Using the above parametrization of ∂X^{reg} we have a natural projection

$$\begin{array}{rrcl} \pi^F: & \partial X^{reg} & \to & \partial^F X \\ & (\xi_1, \xi_2, \theta) & \mapsto & (\xi_1, \xi_2) \end{array}$$

and a natural action of the group $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$ by homeomorphisms on the Furstenberg boundary of $X = X_1 \times X_2$.

We say that two points $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \partial^F X$ are opposite if ξ_1 and η_1 can be joined by a geodesic in X_1, and ξ_2, η_2 can be joined by a geodesic in X_2. Notice that if $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \partial^F X$ are opposite, then for any $\theta \in (0, \pi/2)$ the pair of points $\tilde{\xi} = (\xi_1, \xi_2, \theta)$, $\tilde{\eta} = (\eta_1, \eta_2, \theta) \in \partial X_\theta$ provides a pair of boundary points which can be joined by a geodesic in X. The same holds for the pairs $\tilde{\xi} = \xi_1$, $\tilde{\eta} = \eta_1 \in (\partial X)_0$ and $\tilde{\xi} = \xi_2$, $\tilde{\eta} = \eta_2 \in (\partial X)_{\pi/2}$.

Let σ_1, σ_2 be geodesics in X_1, X_2. We will call a set F in X of the form

$$F = \{(\sigma_1(t_1), \sigma_2(t_2)) : t_1, t_2 \in \mathbb{R}\}$$

a flat in X. Notice that a flat defined in this way is a particular case of a geometric 2-flat in X, i.e. a closed convex subset of X isometric to Euclidean 2-space. If X_1 and X_2 are CAT(-1), then every geometric 2-flat in X is a flat according to our definition. In general, however, this is not the case, because there may exist geometric 2-flats in each factor. In particular, X may contain geometric n-flats of larger dimensions n.

The boundary ∂F of a flat F is topologically a circle, and $\pi^F(\partial F \cap \partial X^{reg})$ consists of 4 points. We say that a flat F joins ξ, $\eta \in \partial^F X$, $\xi \neq \eta$, if $\xi, \eta \in \pi^F(\partial F \cap \partial X^{reg})$. Notice that even if ξ, $\eta \in \partial^F X$ can be joined by a flat, ξ and η need not be opposite.

It is easy to see that if X_1, X_2 are CAT(-1), then any two distinct points ξ, η in the Furstenberg boundary can be joined by a unique flat, analogously to the situation in higher rank symmetric spaces. This is clearly not true in general.

For $x = (x_1, x_2)$, $z = (z_1, z_2) \in X$ such that $z_1 \neq x_1$ and $z_2 \neq x_2$, the set

$$\mathcal{C}_{x,z} := \{(\sigma_{x_1,z_1}(t_1), \sigma_{x_2,z_2}(t_2)) : 0 \leq t_1 \leq d_1(x_1, z_1),\ 0 \leq t_2 \leq d_2(x_2, z_2)\} \tag{8}$$

is called the Weyl chamber with apex x containing z. Notice that if $z_1 = x_1$ or $z_2 = x_2$, then $\sigma_{x,z}$ is not defined, so the assignment in (8) is not well-defined. In this case we define $\mathcal{C}_{x,z}$ as follows:

$$\mathcal{C}_{x,z} := \begin{cases} \{(y_1, \sigma_{x_2,z_2}(t)) \in X : 0 \leq t \leq d_2(x_2, z_2),\ y_1 \in X_1\} & \text{if} \quad x_1 = z_1 \\ \{(\sigma_{x_1,z_1}(t), y_2) \in X : 0 \leq t \leq d_1(x_1, z_1),\ y_2 \in X_2\} & \text{if} \quad x_2 = z_2\,. \end{cases}$$

Similarly, for $x = (x_1, x_2) \in X$ and $\tilde{\xi} = (x_1, x_2, \theta) \in \partial X^{reg}$ we call

$$\mathcal{C}_{x,\tilde{\xi}} := \{(\sigma_{x_1,\xi_1}(t_1), \sigma_{x_2,\xi_2}(t_2)) : t_1, t_2 \geq 0\}$$

the Weyl chamber with apex x in the class of $\tilde{\xi}$. If $\tilde{\xi} \in \partial X^{sing}$, we set

$$\mathcal{C}_{x,\tilde{\xi}} := \begin{cases} \{(\sigma_{x_1,\xi_1}(t), y_2) \in X : t \geq 0,\ y_2 \in X_2\} & \text{if} \quad \tilde{\xi} = \xi_1 \in (\partial X)_0 \cong \partial X_1 \\ \{(y_1, \sigma_{x_2,\xi_2}(t)) \in X : t \geq 0,\ y_1 \in X_1\} & \text{if} \quad \tilde{\xi} = \xi_2 \in (\partial X)_{\pi/2} \cong \partial X_2\,. \end{cases} \tag{9}$$

In this way we have defined $\mathcal{C}_{x,z}$ for any $x \in X$ and $z \in \overline{X} \setminus \{x\}$.

The Weyl chamber shadow of a set $B \subset X$ viewed from $x = (x_1, x_2) \in X \setminus B$ is defined by

$$\mathrm{Sh}(x : B) := \{z \in \overline{X} : p_1(z) \neq x_1,\ p_2(z) \neq x_2,\ \mathcal{C}_{x,z} \cap B \neq \emptyset\}\,. \tag{10}$$

It consists of all Weyl chambers with apex x which intersect B non-trivially. Notice that in view of (9) we have

$$\begin{aligned} \mathrm{Sh}(x : B) \cap (\partial X)_0 &= \{\tilde{\xi} = \xi_1 \in \partial X_1 : \sigma_{x_1,\xi_1}(t) \in p_1(B) \ \text{ for some } t \geq 0\}\,, \\ \mathrm{Sh}(x : B) \cap (\partial X)_{\pi/2} &= \{\tilde{\xi} = \xi_2 \in \partial X_2 : \sigma_{x_2,\xi_2}(t) \in p_2(B) \ \text{ for some } t \geq 0\}\,. \end{aligned} \tag{11}$$

We next fix a base point $o = (o_1, o_2) \in X$. For $x \in X$ and $r > 0$ we denote by $B_x(r)$ the open ball of radius r centered at x. If $h \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$ is such that both projections $h_1 \in \mathrm{Is}(X_1)$ and $h_2 \in \mathrm{Is}(X_2)$ are axial, we denote $\widetilde{h^+} \in \partial X$ its attractive fixed point, and $\widetilde{h^-} \in \partial X$ its repulsive fixed point. If for $i \in \{1, 2\}$ $h_i^\pm \in \partial X_i$ are the attractive and repulsive fixed points of h_i, then we get

$$\widetilde{h^\pm} = (h_1^\pm, h_2^\pm, \arctan\big(l(h_2)/l(h_1)\big) \in \partial X^{reg}$$

by applying the estimate (6) to a point $x \in \mathrm{Ax}(h)$. Hence if $h^\pm := \pi^F(\widetilde{h^\pm})$, we have $h^\pm = (h_1^\pm, h_2^\pm)$.

The following proposition states that all Weyl chamber shadows of sufficiently large balls contain a given open set. This will be crucial in the proof of the shadow lemma. Notice that our idea of proof also considerably simplifies the proof of the analogous statement for one factor (see [Kni97, Proposition 3.6] and [Lin07, Lemma 3.5]).

PROPOSITION 3.2 *Suppose $g = (g_1, g_2)$ and $h = (h_1, h_2) \in Is(X)$ are axial isometries such that g_i and h_i are independent rank one elements in $Is(X_i)$ for $i = 1, 2$. Then there exist open neighborhoods $U_1 \subset \partial X_1$, $U_2 \subset \partial X_2$ of h_1^+, h_2^+ respectively, a finite set $\Lambda \subset \Gamma$ and $c_0 > 0$ with the following properties:*

If $U := U_1 \sqcup U_2 \sqcup \{(\xi_1, \xi_2, \theta) \in \partial X^{reg} : \xi_1 \in U_1, \xi_2 \in U_2, \theta \in (0, \pi/2)\} \subset \partial X$, then for any $r \geq c_0$ and all $y \in X \setminus B_o(r)$ there exists $\alpha \in \Lambda$ such that

$$\alpha U \subseteq \mathrm{Sh}(y : B_o(r)) .$$

Proof. For $i = 1, 2$ and $\eta = (\eta_1, \eta_2) \in \{g^-, g^+, h^-, h^+\}$ let $U_i(\eta) \subset \overline{X}_i$ be an arbitrary neighborhood of $\eta_i \in \partial X_i$ with $o_i \notin U_i(\eta)$ such that all $U_i(\eta)$ are pairwise disjoint in $\overline{X}_i$. Upon taking smaller neighborhoods, Lemma 2.2 provides a constant $c > 0$ such that for $i \in \{1, 2\}$ any pair of points in distinct neighborhoods can be joined by a rank one geodesic $\sigma_i \subset X_i$ with $d(o_i, \sigma_i) \leq c$. Moreover, according to Lemma 2.5 (b) there exists a constant $N \in \mathbb{N}$ such that for all $\gamma = (\gamma_1, \gamma_2) \in \{g, g^{-1}, h, h^{-1}\}$ and $i \in \{1, 2\}$

$$\gamma_i^N \big(\overline{X}_i \setminus U_i(\gamma^-)\big) \subseteq U_i(\gamma^+) . \qquad (12)$$

Let $y \in X$ arbitrary. Then one of the following cases occurs:

1. Case: $y_1 \in \overline{X}_1 \setminus U_1(h^+)$ and $y_2 \in \overline{X}_2 \setminus U_2(h^+)$
   Then by (12) $h^{-N} y \in U_1(h^-) \times U_2(h^-)$.

2. Case: $y_1 \in U_1(h^+)$ and $y_2 \in U_2(h^+)$
   Since $U_i(h^+) \subset \overline{X}_i \setminus U_i(g^-)$, $i = 1, 2$, we have again by (12) $g^N y \in U_1(g^+) \times U_2(g^+)$. Hence we are in Case 1 for $g^N y$, so $h^{-N} g^N y \in U_1(h^-) \times U_2(h^-)$.

3. Case: $y_1 \in U_1(h^+)$ and $y_2 \in \overline{X}_2 \setminus \big(U_2(h^+) \cup U_2(g^-)\big)$
   Then $g^N y \in U_1(g^+) \times U_2(g^+)$, which yields $h^{-N} g^N y \in U_1(h^-) \times U_2(h^-)$.

4. Case: $y_1 \in U_1(h^+)$ and $y_2 \in \overline{X}_2 \setminus \big(U_2(h^+) \cup U_2(g^+)\big)$
   Then $g^{-N} y \in U_1(g^-) \times U_2(g^-)$, which gives $h^{-N} g^{-N} y \in U_1(h^-) \times U_2(h^-)$.

5. Case: $y_1 \in \overline{X}_1 \setminus \big(U_1(h^+) \cup U_1(g^-)\big)$ and $y_2 \in U_2(h^+)$
   Similarly to case 3 we obtain $h^{-N} g^N y \in U_1(h^-) \times U_2(h^-)$.

6. Case: $y_1 \in \overline{X}_1 \setminus \big(U_1(h^+) \cup U_1(g^+)\big)$ and $y_2 \in U_2(h^+)$
   As in case 4 we get $h^{-N} g^{-N} y \in U_1(h^-) \times U_2(h^-)$.

So we have shown the existence of $\alpha = (\alpha_1, \alpha_2) \in \Lambda := \{h^N, g^N h^N, g^{-N} h^N\}$ such that $\alpha^{-1} y \in U_1(h^-) \times U_2(h^-)$. In particular, by our choice of the neighborhoods $U_i(h^\pm)$, $i = 1, 2$, we have for all $z = (z_1, z_2) \in U_1(h^+) \times U_2(h^+)$

$$d_i(\sigma_{\alpha_i^{-1} y_i, z_i}, o_i) \leq c , \qquad i \in \{1, 2\} .$$

We set $L := \max\{d_i(o_i, \lambda_i o_i) : i \in \{1,2\}, \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$. Then for $i = 1, 2$

$$\begin{aligned} d_i(\sigma_{y_i, \alpha_i z_i}, o_i) &\leq d_i(\alpha_i \sigma_{\alpha_i^{-1} y_i, z_i}, \alpha_i o_i) + d_i(\alpha_i o_i, o_i) \\ &< d_i(\sigma_{\alpha_i^{-1} y_i, z_i}, o_i) + L \leq c + L\,, \end{aligned}$$

which implies $\mathcal{C}_{y,\alpha z} \cap B_o(\sqrt{2}(c+L)) \neq \emptyset$. Hence the claim follows taking $U_1 := U_1(h^+) \cap \partial X_1$, $U_2 := U_2(h^+) \cap \partial X_2$ and $c_0 := \sqrt{2}(c+L)$. □

Recall the definition of the Busemann function from (1). The following easy lemma relates the Busemann function of the product to the Busemann functions on the factors. We include a proof for the convenience of the reader.

Lemma 3.3 *Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, $\tilde{\xi} = (\xi_1, \xi_2, \theta) \in \partial X^{reg}$. Then*

$$\mathcal{B}_{\tilde{\xi}}(x, y) = \cos\theta \cdot \mathcal{B}_{\xi_1}(x_1, y_1) + \sin\theta \cdot \mathcal{B}_{\xi_2}(x_2, y_2)\,. \tag{13}$$

Proof. Notice that from the definition of the Busemann functions in X_1, X_2 we have

$$\begin{aligned} \mathcal{B}_{\xi_1}(x_1, y_1) &= \lim_{s\to\infty} \big(s\cos\theta - d_1(y_1, \sigma_{x_1,\xi_1}(s\cos\theta))\big)\,, \\ \mathcal{B}_{\xi_2}(x_2, y_2) &= \lim_{s\to\infty} \big(s\sin\theta - d_2(y_2, \sigma_{x_2,\xi_2}(s\sin\theta))\big)\,. \end{aligned}$$

Now

$$\begin{aligned} s - d(y, \sigma_{x,\tilde{\xi}}(s)) &= \frac{s^2 - d(y, \sigma_{x,\tilde{\xi}}(s))^2}{s + d(y, \sigma_{x,\tilde{\xi}}(s))} \\ &= \frac{s^2\cos^2\theta - d_1(y_1, \sigma_{x_1,\xi_1}(s\cos\theta))^2}{s + d(y, \sigma_{x,\tilde{\xi}}(s))} + \frac{s^2\sin^2\theta - d_2(y_2, \sigma_{x_2,\xi_2}(s\sin\theta))}{s + d(y, \sigma_{x,\tilde{\xi}}(s))}\,, \end{aligned}$$

hence the assertion is proved if

$$\begin{aligned} &\lim_{s\to\infty} \frac{s\cos\theta + d_1(y_1, \sigma_{x_1,\xi_1}(s\cos\theta))}{s + d(y, \sigma_{x,\tilde{\xi}}(s))} = \cos\theta \qquad \text{and} \\ &\lim_{s\to\infty} \frac{s\sin\theta + d_2((y_2, \sigma_{x_2,\xi_2}(s\sin\theta))}{s + d(y, \sigma_{x,\tilde{\xi}}(s))} = \sin\theta\,. \end{aligned}$$

This claim follows immediately from the triangle inequalities

$$\begin{aligned} &s\cdot\cos\theta - d_1(y_1, x_1) \leq d_1(y_1, \sigma_{x_1,\xi_1}(s\cdot\cos\theta)) \leq s\cdot\cos\theta + d_1(y_1, x_1)\,, \\ &s\cdot\sin\theta - d_2(y_2, x_2) \leq d_2(y_2, \sigma_{x_2,\xi_2}(s\cdot\sin\theta)) \leq s\cdot\sin\theta + d_2(y_2, x_2)\,, \\ &s - d(y, x) \leq d(y, \sigma_{x,\tilde{\xi}}(s)) \leq s + d(y, x)\,. \end{aligned}$$

□

Recall that $\partial X_\theta \subset \partial X$ denotes the set of points of slope $\theta \in [0, \pi/2]$. Using similar arguments as in the proof above we get the following relation for singular boundary points.

LEMMA 3.4 *Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$. Then*

$$\mathcal{B}_{\tilde{\xi}}(x,y) = \begin{cases} \mathcal{B}_{\xi_1}(x_1, y_1) & \text{if } \tilde{\xi} = \xi_1 \in (\partial X)_0 \cong \partial X_1 \\ \mathcal{B}_{\xi_2}(x_2, y_2) & \text{if } \tilde{\xi} = \xi_2 \in (\partial X)_{\pi/2} \cong \partial X_2 \,. \end{cases}$$

To simplify notation in the sequel we further define for $b = (b_1, b_2) \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, $\tilde{\xi} \in \partial X$ the b-Busemann function $\mathcal{B}^b_{\tilde{\xi}}(x,y) \in \mathbb{R}$ via

$$\mathcal{B}^b_{\tilde{\xi}}(x,y) := \begin{cases} b_1\mathcal{B}_{\xi_1}(x_1, y_1) + b_2\mathcal{B}_{\xi_2}(x_2, y_2) & \text{if } \tilde{\xi} = (\xi_1, \xi_2, \theta) \in \partial X^{reg} \\ b_1\mathcal{B}_{\xi_1}(x_1, y_1) & \text{if } \tilde{\xi} = \xi_1 \in (\partial X_0) \cong \partial X_1 \\ b_2\mathcal{B}_{\xi_2}(x_2, y_2) & \text{if } \tilde{\xi} = \xi_2 \in (\partial X)_{\pi/2} \cong \partial X_2 \,. \end{cases} \tag{14}$$

For convenience we denote

$$H_\theta := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \in \mathbb{R}^2 \tag{15}$$

the unique unit vector of direction $\theta \in [0, \pi/2]$, and $\langle \cdot, \cdot \rangle$ the Euclidean inner product in $\mathbb{R}^2$. In the sequel we will need the following

DEFINITION 3.5 *The* directional distance *of the ordered pair $(x,y) \in X \times X$ with respect to the slope θ is defined by*

$$\begin{aligned} \mathcal{B}_\theta : \quad X \times X &\to \mathbb{R} \\ (x,y) &\mapsto \mathcal{B}_\theta(x,y) := \langle H_\theta, H(x,y) \rangle \,. \end{aligned}$$

In particular, if $\theta = 0$, then $\mathcal{B}_\theta(x,y) = d_1(p_1(x), p_1(y))$, if $\theta = \pi/2$, then $\mathcal{B}_\theta(x,y) = d_2(p_2(x), p_2(y))$.

By $\big(\mathrm{Is}(X_1) \times \mathrm{Is}(X_2)\big)$-invariance of the distance vector we immediately get that

$$\mathcal{B}_\theta(gx, gy) = \mathcal{B}_\theta(x,y)$$

for any $x, y \in X$ and $g \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$. Moreover, the symmetry and triangle inequality for the distances d_1 and d_2 directly imply the symmetry and triangle inequality for $\mathcal{B}_\theta$. The following important proposition states that for $\theta \in (0, \pi/2)$ the directional distance $\mathcal{B}_\theta$ is in fact a distance.

PROPOSITION 3.6 *For $\theta \in (0, \pi/2)$ the directional distance $\mathcal{B}_\theta$ is a distance.*

Proof. Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$. We clearly have

$$\mathcal{B}_\theta(x,y) = \cos\theta \cdot d_1(x_1, y_1) + \sin\theta \cdot d_2(x_2, y_2) \geq 0 \,,$$

because all terms involved are non-negative. Moreover, if $\mathcal{B}_\theta(x,y) = 0$, then $\cos\theta > 0$ and $\sin\theta > 0$ imply $d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$, hence $x = y$.

Finally, we have already noticed that the symmetry and triangle inequality follow directly from the symmetry and triangle inequality for the distances d_1 and d_2. □

The following easy facts will be convenient in the sequel.

Lemma 3.7 *Let $x \in X$ and $\tilde{\xi} \in \partial X_\theta$ for some $\theta \in [0, \pi/2]$. Then*

$$y \in \mathcal{C}_{x,\tilde{\xi}} \setminus \{x\} \quad \Longleftrightarrow \quad \mathcal{B}_\theta(x,y) = \mathcal{B}_{\tilde{\xi}}(x,y)\,.$$

Proof. Write $x = (x_1, x_2)$ and let $y = (y_1, y_2) \in X \setminus \{x\}$ arbitrary. Lemma 3.3, Lemma 3.4 and (3) imply

$$\mathcal{B}_{\tilde{\xi}}(x,y) \le \cos\theta \cdot d_1(x_1,y_1) + \sin\theta \cdot d_2(x_2,y_2) = \langle H_\theta, H(x,y)\rangle\,. \tag{16}$$

Assume first that $\theta \in (0, \pi/2)$ and write $\tilde{\xi} = (\xi_1, \xi_2, \theta)$. Then we have equality in (16) if and only if $\mathcal{B}_{\xi_i}(x_i, y_i) = d_i(x_i, y_i)$ for $i = 1, 2$. If $x_1 \ne y_1$ and $x_2 \ne y_2$, this is precisely the case if y_i is a point on the geodesic ray $\sigma_{x_i \xi_i}$ for $i = 1, 2$ which is equivalent to $y \in \mathcal{C}_{x,\tilde{\xi}}$. If $x_1 = y_1$, then $\mathcal{B}_{\tilde{\xi}}(x,y) = \cos\theta \cdot 0 + \sin\theta \cdot d_2(x_2, y_2)$ if and only if y_2 is a point on the geodesic ray σ_{x_2,ξ_2}. This again holds if and only if $y \in \mathcal{C}_{x,\tilde{\xi}}$. The case $x_2 = y_2$ is analogous.

If $\tilde{\xi} = \xi_1 \in (\partial X)_0 \cong \partial X_1$ we have equality in (16) if and only if y_1 is a point on the geodesic ray σ_{x_1,ξ_1}. This is equivalent to $y \in \mathcal{C}_{x,\tilde{\xi}}$.

Similarly, if $\tilde{\xi} = \xi_2 \in (\partial X)_{\pi/2} \cong \partial X_2$ we have equality in (16) if and only if $y \in \mathcal{C}_{x,\tilde{\xi}}$. □

If X is geodesically complete, this lemma allows to give the following nice geometric interpretation of the directional distance.

Corollary 3.8 *If $X = X_1 \times X_2$ is geodesically complete, $\theta \in [0, \pi/2]$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, then*

$$\mathcal{B}_\theta(x,y) = \max\{\mathcal{B}_{\tilde{\xi}}(x,y) : \tilde{\xi} \in \partial X_\theta\}\,.$$

If $\theta = 0$, then the conclusion holds under the weaker condition that X_1 is geodesically complete, if $\theta = \pi/2$, the conclusion holds under the condition that X_2 is geodesically complete.

Proof. If $y = x$, then $\mathcal{B}_\theta(x,y) = \langle H_\theta, H(x,y)\rangle = 0$ and $\mathcal{B}_{\tilde{\xi}}(x,y) \le d(x,y) = 0$ for all $\tilde{\xi} \in \partial X_\theta$. Hence we have $\max\{\mathcal{B}_{\tilde{\xi}}(x,y) : \tilde{\xi} \in \partial X_\theta\} = 0 = \mathcal{B}_\theta(x,y)$.

We next treat the case $y \ne x$. Since X is geodesically complete, every point $y \in X$ is contained in a Weyl chamber $\mathcal{C}_{x,\tilde{\xi}}$ for some $\tilde{\xi} \in \partial X_\theta$. Hence if $y \ne x$, the previous lemma implies $\mathcal{B}_\theta(x,y) = \mathcal{B}_{\tilde{\xi}}(x,y)$. Moreover, if $\tilde{\zeta} \in \partial X_\theta$ is arbitrary, then by Lemma 3.3, Lemma 3.4 and (3)

$$\mathcal{B}_{\tilde{\zeta}}(x,y) \le \langle H_\theta, H(x,y)\rangle\,.$$

Summarizing we conclude $\quad \mathcal{B}_\theta(x,y) = \mathcal{B}_{\tilde{\xi}}(x,y) = \max\{\mathcal{B}_{\tilde{\xi}}(x,y) : \tilde{\xi} \in \partial X_\theta\}\,.$

In order to prove the remaining assertions we write $x = (x_1, x_2)$, $y = (y_1, y_2)$ and assume that $y \ne x$. First assume that $\theta = 0$ and X_1 is geodesically complete. Then there exists $\xi_1 \in \partial X_1$ such that $y_1 = \sigma_{x_1,\xi_1}(t)$ for some $t \ge 0$. Hence if $\tilde{\xi} \in (\partial X)_0$ is the unique point identified with $\xi_1 \in \partial X_1$ we have $y \in \mathcal{C}_{x,\tilde{\xi}} \setminus \{x\}$. The claim now follows as before from the previous lemma, Lemma 3.4 and (3).

The case $\theta = \pi/2$ and X_2 geodesically complete is analogous. $\square$

Recall the definition of Weyl chamber shadows from (10). The following lemma will be needed in the proof of the shadow lemma Theorem 7.2.

Lemma 3.9 *Let $c > 0$, $z = (z_1, z_2) \in X$ with $d(o, z) > c$, and $\tilde\eta \in Sh\big(o : B_z(c)\big) \cap \partial X$. If $\tilde\eta = (\eta_1, \eta_2, \theta) \in \partial X^{reg}$, then*

$$0 \le d_1(o_1, z_1) - \mathcal{B}_{\eta_1}(o_1, z_1) < 2c \quad \text{ and } \quad 0 \le d_2(o_2, z_2) - \mathcal{B}_{\eta_2}(o_2, z_2) < 2c\,,$$

if $\tilde\eta = \eta_1 \in (\partial X)_0 \cong \partial X_1$, then $\quad 0 \le d_1(o_1, z_1) - \mathcal{B}_{\eta_1}(o_1, z_1) < 2c$, and
if $\tilde\eta = \eta_2 \in (\partial X)_{\pi/2} \cong \partial X_2$, then $\quad 0 \le d_2(o_2, z_2) - \mathcal{B}_{\eta_2}(o_2, z_2) < 2c$.

Proof. By definition $\tilde\eta \in \mathrm{Sh}\big(o : B_z(c)\big)$ translates to $\mathcal{C}_{o,\tilde\eta} \cap B_z(c) \neq \emptyset$. Hence if $\tilde\eta = (\eta_1, \eta_2, \theta) \in \partial X^{reg}$ there exist $t_1, t_2 \ge 0$ such that $d\big((z_1, z_2), (\sigma_{o_1,\eta_1}(t_1), \sigma_{o_2,\eta_2}(t_2))\big) < c$. Therefore we have $d_i(z_i, \sigma_{o_i,\eta_i}(t_i)) < c$ for $i \in \{1, 2\}$, so the claim follows from Lemma 2.1. The conclusion for $\tilde\eta \in \partial X^{sing}$ is clear in view of Lemma 3.4, (11) and Lemma 2.1. $\square$

4 The exponent of growth

For the remainder of the article X is a product of locally compact Hadamard spaces X_1, X_2, $o = (o_1, o_2)$ a fixed base point, and $\Gamma \subset \mathrm{Is}(X_1) \times \mathrm{Is}(X_2)$ a discrete group which contains two isometries $g = (g_1, g_2)$ and $h = (h_1, h_2)$ such that for $i = 1, 2$ g_i and h_i are independent rank one elements of Γ_i. Recall that the geometric limit set of a group Γ acting by isometries on a locally compact Hadamard space is defined by $L_\Gamma := \overline{\Gamma{\cdot}x} \cap \partial X$, where $x \in X$ is arbitrary. In this section we recall the notion of exponent of growth introduced in [Lin10] and give an important criterion for divergence or convergence of certain sums over Γ. This will play a central role in the construction of (generalized) Patterson-Sullivan measures in Sections 5 and 6.

We recall the notation introduced in Section 3 and put for $x, y \in X$, $\theta \in [0, \pi/2]$, $\varepsilon > 0$

$$\Gamma(x, y; \theta, \varepsilon) := \{\gamma \in \Gamma : \gamma y \neq x \quad \text{and} \quad |\theta(x, \gamma y) - \theta| < \varepsilon\}\,.$$

In order to define the exponent of growth of Γ of slope θ we put

$$\delta_\theta^\varepsilon(x, y) := \inf\{s > 0 : \sum_{\gamma \in \Gamma(x,y;\theta,\varepsilon)} e^{-sd(x,\gamma y)} \ \text{ converges}\}\,.$$

If $\delta(\Gamma)$ denotes the critical exponent of Γ defined by

$$\delta(\Gamma) := \inf\{s > 0 : \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} \text{ converges}\}\,, \tag{17}$$

we clearly have $\delta_\theta^\varepsilon(x, y) \le \delta(\Gamma)$ with equality if $\varepsilon > \pi/2$. Moreover, Lemma 6.1 in [Lin10] shows that $\delta_\theta^\varepsilon(x, y)$ is related to the numbers

$$\Delta N_\theta^\varepsilon(x, y; n) := \#\{\gamma \in \Gamma \ : \ n - 1 < d(x, \gamma y) \le n\,,\ |\theta(x, \gamma y) - \theta| < \varepsilon\}\,, \quad n \in \mathbb{N}\,,$$

via

$$\delta_\theta^\varepsilon(x,y) = \limsup_{n\to\infty} \frac{\log \Delta N_\theta^\varepsilon(x,y;n)}{n} \tag{18}$$

and thus can be interpreted as an exponential growth rate of the number of orbit points with slope ε-close to θ.

Recall that the exponent of growth of Γ of slope θ is defined by

$$\delta_\theta(\Gamma) := \liminf_{\varepsilon\to 0} \delta_\theta^\varepsilon(o,o)\,.$$

Notice that this number $\delta_\theta(\Gamma)$ does not depend on the choice of arguments of $\delta_\theta^\varepsilon$ by Lemma 6.3 in [Lin10], and $\delta_\theta(\Gamma) \le \delta(\Gamma)$ for all $\theta \in [0,\pi/2]$.

Furthermore, we recall the following properties from Section 6 in [Lin10]:

PROPERTIES:

(a) If $L_\Gamma \cap \partial X_\theta \neq \emptyset$, then $\delta_\theta(\Gamma) \ge 0$.

(b) If $(\theta_j) \subset [0,\pi/2]$ is a sequence converging to $\theta \in [0,\pi/2]$, then

$$\limsup_{j\to\infty} \delta_{\theta_j}(\Gamma) \le \delta_\theta(\Gamma)\,.$$

It will turn out useful to extend the exponent of growth to a homogeneous map $\Psi_\Gamma : \mathbb{R}^2_{\ge 0} \to \mathbb{R}$ as follows: If $x = (x_1,x_2) \in \mathbb{R}^2_{\ge 0}$ we put $\theta(x) := \arctan(x_2/x_1)$ and set

$$\Psi_\Gamma(x) := ||x|| \cdot \delta_{\theta(x)}\,. \tag{19}$$

In [Lin10] we showed that Ψ_Γ is concave. This implies in particular that there exists a unique $\theta_* \in [0,\pi/2]$ such that $\delta_{\theta_*}(\Gamma) = \max\{\delta_\theta(\Gamma) : \theta \in [0,\pi/2]\}$. The following important proposition will play a key role in the proof of Theorem A and for the construction of generalized Patterson-Sullivan measures. Recall the definition of the distance vector and of H_θ from (5) and (15) respectively.

PROPOSITION 4.1 *Let $D \subseteq [0,\pi/2]$ be a relatively open interval, set $\Gamma_D := \{\gamma \in \Gamma : \theta(o,\gamma o) \in D\}$, and let $f : \mathbb{R}^2_{\ge 0} \to \mathbb{R}$ be a continuous homogeneous function.*

(a) If there exists $\hat\theta \in D$ such that $f(H_{\hat\theta}) < \delta_{\hat\theta}(\Gamma)$, then the series $\sum_{\gamma\in\Gamma_D} e^{-f(H(o,\gamma o))}$ diverges.

(b) If $f(H_\theta) > \delta_\theta(\Gamma)$ for all $\theta \in \overline{D}$, then the series $\sum_{\gamma\in\Gamma_D} e^{-f(H(o,\gamma o))}$ converges.

Proof. For $\gamma \in \Gamma$ we set $H_\gamma := H(o,\gamma o)/d(o,\gamma o)$.

(a) Let $\hat\theta \in D$ such that $f(H_{\hat\theta}) < \delta_{\hat\theta}(\Gamma)$. Since $\delta_{\hat\theta}(\Gamma) = \liminf_{\varepsilon\to 0} \delta_{\hat\theta}^\varepsilon(o,o)$, there exists $\varepsilon \in (0,\pi/4)$ and $\hat s \in \mathbb{R}$ such that for $\gamma \in \Gamma_D$ with $\mid \theta(o,\gamma o) - \hat\theta \mid < \varepsilon$ we have

$$f(H_\gamma) < \hat s < \delta_{\hat\theta}^\varepsilon(o,o)\,.$$

Therefore

$$\sum_{\gamma\in\Gamma_D} e^{-f(H(o,\gamma o))} > \sum_{\substack{\gamma\in\Gamma \\ |\theta(o,\gamma o)-\hat\theta|<\varepsilon}} e^{-\hat s d(o,\gamma o)},$$

and the latter sum diverges since $\hat s < \delta^{\varepsilon}_{\hat\theta}(o,o)$.

(b) Let $\hat\theta \in \overline{D}$. Since $f(H_{\hat\theta}) > \delta_{\hat\theta}(\Gamma) = \liminf_{\varepsilon\to 0}\delta^{\varepsilon}_{\hat\theta}(o,o)$, there exists $\varepsilon' \in (0,\pi/4)$ and $\hat s < f(H_{\hat\theta})$ such that

$$\delta^{\varepsilon'}_{\hat\theta}(o,o) < \hat s < f(H_{\hat\theta})\,. \tag{20}$$

For $\theta\in[0,\pi/2]$ and $\varepsilon>0$ we put $B_\theta(\varepsilon) := \{\theta'\in[0,\pi/2] : |\theta'-\theta|<\varepsilon\}$. The continuity of the function f and inequality (20) imply the existence of $\hat\varepsilon < \varepsilon'$ such that for any $\theta\in B_{\hat\theta}(\hat\varepsilon)$ we have $\hat s < f(H_\theta)$. Hence for all $z\in X$ with $\theta(o,z)\in B_{\hat\theta}(\hat\varepsilon)$ we have

$$\frac{f(H(o,z))}{d(o,z)} > \hat s > \delta^{\varepsilon'}_{\hat\theta}(o,o) \ge \delta^{\hat\varepsilon}_{\hat\theta}(o,o)\,.$$

We now choose a sequence $(\theta_j)\subset\overline{D}$ and corresponding sequences $(s_j)\subset\mathbb{R}^+$ and $(\varepsilon_j)\subset\mathbb{R}^+$ such that for every $\theta\in B_{\theta_j}(\varepsilon_j)$ we have

$$\delta^{\varepsilon_j}_{\theta_j}(o,o) < s_j < f(H_\theta)\,, \quad\text{and}\quad \overline{D}\subseteq\bigcup_{j\in\mathbb{N}} B_{\theta_j}(\varepsilon_j)\,.$$

Since $\overline{D}$ is compact we may extract a finite covering $\bigcup_{j=1}^{l} B_{\theta_j}(\varepsilon_j)$, and conclude

$$\begin{aligned}\sum_{\gamma\in\Gamma_D} e^{-f(H(o,\gamma o))} &\le \sum_{j=1}^{l}\sum_{\substack{\gamma\in\Gamma\\ |\theta(o,\gamma o)-\theta_j|<\varepsilon_j}} e^{-f(H(o,\gamma o))}\\ &\le \sum_{j=1}^{l}\sum_{\substack{\gamma\in\Gamma\\ |\theta(o,\gamma o)-\theta_j|<\varepsilon_j}} e^{-s_j d(o,\gamma o)} < \infty\,,\end{aligned}$$

because $s_j > \delta^{\varepsilon_j}_{\theta_j}(o,o)$ for $1\le j\le l$. □

Taking $D=[0,\pi/2]$ and $f(H) = s\cdot\|H\|$ we obtain as a corollary that $\delta(\Gamma) = \max\{\delta_\theta(\Gamma) : \theta\in[0,\pi/2]\} = \delta_{\theta_*}(\Gamma)$. We conclude this section with two illustrative examples.

Example 1 (see [Lin10, Section 6]) If X is a product $X = X_1\times X_2$ of Hadamard manifolds with pinched negative curvature, and $\Gamma_1\subset \mathrm{Is}(X_1)$, $\Gamma_2\subset\mathrm{Is}(X_2)$ are convex cocompact groups with critical exponents δ_1,δ_2, then for $\Gamma := \Gamma_1\times\Gamma_2$ and for every $\theta\in[0,\pi/2]$

$$\delta_\theta(\Gamma) = \delta_1\cos\theta + \delta_2\sin\theta\,.$$

This number is maximal for $\theta_* = \arctan(\delta_2/\delta_1)$ and we have $\delta(\Gamma) = \delta_{\theta_*}(\Gamma) = \sqrt{\delta_1^2+\delta_2^2}$. The homogeneous function $\Psi_\Gamma : \mathbb{R}^2_{\ge 0}\to\mathbb{R}$ is simply the linear functional defined by

$\Psi_\Gamma = \langle \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}, \cdot \rangle.$

EXAMPLE 2 Consider a product of hyperbolic planes $X = H^2 \times H^2$ and a Hilbert modular group $\Gamma \subset \mathrm{Is}(X)$. Then Γ is an irreducible non-uniform lattice in a higher rank symmetric space, hence from Proposition 7.2 and 7.3 in [Alb99] we know that $\Psi_\Gamma = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \cdot \rangle$, i.e. $\delta_\theta(\Gamma) = \cos\theta + \sin\theta$. Here $\delta_\theta(\Gamma)$ is maximal for $\theta_* = \pi/4$ and we have $\delta(\Gamma) = \delta_{\theta_*}(\Gamma) = \sqrt{2}$.

5 The classical Patterson-Sullivan construction

In this section we will construct a conformal density for Γ using an idea originally due to S. J. Patterson ([Pat76]) in the context of Fuchsian groups. Taking advantage of Proposition 4.1 we will be able to describe precisely its support and hence prove Theorem A.

Recall that a Γ-invariant conformal density of dimension $\alpha \geq 0$ is a continuous map μ from X to the cone $\mathcal{M}^+(\partial X)$ of positive finite Borel measures on ∂X such that $\mathrm{supp}(\mu_o) \subseteq L_\Gamma$, $\gamma * \mu_x = \mu_{\gamma^{-1}x}$ for all $\gamma \in \Gamma$, $x \in X$ and

$$\frac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{\alpha \mathcal{B}_{\tilde\eta}(o,x)} \qquad \text{for } \tilde\eta \in \mathrm{supp}(\mu_o),\ x \in X\,.$$

In order to construct a Γ-invariant conformal density of dimension $\delta(\Gamma)$ we first suppose that we are given a map $b : \Gamma \to \mathbb{R}$, $\gamma \mapsto b_\gamma$, such that the sum

$$\sum_{\gamma\in\Gamma} e^{-sb_\gamma}$$

converges for $s > 1$ and diverges for $s < 1$. The following useful lemma states that if the above sum converges for $s = 1$, then we can slightly modify it to obtain a sum which diverges for $s \leq 1$ and converges for $s > 1$.

LEMMA 5.1 ((PATTERSON [PAT76])) *There exists a positive increasing function h on $[0,\infty)$ such that*

(i) $\sum_{\gamma\in\Gamma} e^{-sb_\gamma} h(b_\gamma)$ *has exponent of convergence $s = 1$ and diverges at $s = 1$;*

(ii) *for any $\varepsilon > 0$ there exists $r_0 > 0$ such that for $r \geq r_0$ and $t > 1$*

$$h(rt) \leq t^\varepsilon h(r)\,.$$

Recall the definition of the exponent of growth of Γ and its properties from Section 4. We have already noticed that there exists a unique $\theta_* \in [0, \pi/2]$ such that $\delta(\Gamma) = \delta_{\theta_*}(\Gamma)$.

Following the original idea of Patterson [Pat76], we apply the above lemma to the map
$$b : \Gamma \to \mathbb{R},\ \gamma \mapsto \delta(\Gamma) \cdot d(o, \gamma o)\,. \tag{21}$$
Then by definition (17) of the critical exponent $\delta(\Gamma)$ the series $\sum_{\gamma\in\Gamma} e^{-sb_\gamma}$ has exponent of convergence $s = 1$. If this sum converges at $s = 1$ we take the increasing function h from the previous lemma, otherwise we set $h \equiv 1$ and define
$$P^s := \sum_{\gamma\in\Gamma} e^{-sb_\gamma} h(b_\gamma)\,.$$
We then obtain a family of orbital measures on $\overline{X}$ as follows: If D denotes the unit Dirac point measure, then for $x \in X$ and $s > 1$ we set
$$\mu_x^s := \frac{1}{P^s} \sum_{\gamma\in\Gamma} e^{-s\delta(\Gamma)d(x,\gamma o)} h(b_\gamma) D(\gamma o)\,.$$
These measures are Γ-equivariant by construction and absolutely continuous with respect to each other.

Let $(\mathrm{C}^0(\overline{X}), \|\cdot\|_\infty)$ denote the space of real valued continuous functions on $\overline{X}$ with norm $\|f\|_\infty = \max\{|f(x)| : x \in \overline{X}\}$, $f \in \mathrm{C}^0(\overline{X})$. We endow the cone $\mathcal{M}^+(\overline{X})$ of positive finite Borel measures on $\overline{X}$ with the pseudo-metric
$$\rho(\mu_1, \mu_2) := \sup\left\{ \left| \int_{\overline{X}} f\, d\mu_1 - \int_{\overline{X}} f\, d\mu_2 \right| : f \in \mathrm{C}^0(\overline{X})\,,\ \|f\|_\infty = 1 \right\}, \tag{22}$$
$\mu_1, \mu_2 \in \mathcal{M}^+(\overline{X})$, and obtain the following:

LEMMA 5.2 *The family of maps $\mathcal{F} := \{x \mapsto \mu_x^s : 1 < s \le 2\}$ from X to $\mathcal{M}^+(\overline{X})$ is equicontinuous.*

Proof. Let $x, y \in X$. For $\gamma \in \Gamma$ we abbreviate
$$q_\gamma(y, x) := \delta(\Gamma)\big(d(y, \gamma o) - d(x, \gamma o)\big)$$
and notice that
$$|q_\gamma(y, x)| \le \delta(\Gamma) d(x, y)\,. \tag{23}$$
If $s \in (1, 2]$ and $f \in \mathrm{C}^0(\overline{X})$, the inequality $|1 - e^{-t}| \le e^{|t|} - 1$, $t \in \mathbb{R}$, then gives
$$\begin{aligned} \left| \int_{\overline{X}} f\, d\mu_x^s - \int_{\overline{X}} f\, d\mu_y^s \right| &\le \frac{1}{P^s} \sum_{\gamma\in\Gamma} e^{-s\delta(\Gamma)d(x,\gamma o)} \cdot h(b_\gamma) |f(\gamma o)|\, \big|1 - e^{-sq_\gamma(y,x)}\big| \\ &\le \frac{\|f\|_\infty}{P^s} \sum_{\gamma\in\Gamma} e^{-sb_\gamma} h(b_\gamma) e^{-sq_\gamma(x,o)} \big(e^{s|q_\gamma(y,x)|} - 1\big)\,. \end{aligned}$$
Since $f \in \mathrm{C}^0(\overline{X})$ was arbitrary, $s \le 2$ and $\sum_{\gamma\in\Gamma} e^{-sb_\gamma} h(b_\gamma) = P^s$, we conclude using (23)
$$\rho(\mu_x^s, \mu_y^s) \le e^{2\delta(\Gamma)d(o,x)} \cdot \big(e^{2\delta(\Gamma)d(x,y)} - 1\big)\,.$$
This proves that $\mathcal{F}$ is equicontinuous. $\square$

LEMMA 5.3 *For any $x \in X$ there exists a sequence $(s_n) \searrow 1$ such that the measures $\mu_x^{s_n} \subset \mathcal{M}^+(\overline{X})$ converge weakly to a measure $\mu_x := \mu_x(\theta, \tau, b)$ as $n \to \infty$.*

Proof. The compactness of the space $\overline{X}$ implies that every sequence of measures in $\mathcal{M}^+(\overline{X})$ possesses a weakly convergent subsequence. $\square$

The Theorem of Arzelà-Ascoli [Kel55, Theorem 7.17, p. 233] now allows to conclude that $\mathcal{F}$ is relatively compact in the space of continuous maps $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ endowed with the topology of uniform convergence on compact sets. From the definition of $(\mu_x^s)_{x\in X}$ it follows that every accumulation point $\mu = (\mu_x)_{x\in X}$ of $\mathcal{F}$ as $s \searrow 1$ takes its values in $\mathcal{M}^+(\partial X)$. Moreover, the following proposition shows that the family of measures obtained in this way are absolutely continuous with Radon-Nikodym derivative

$$\frac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{\delta(\Gamma)\mathcal{B}_{\tilde\eta}(o,x)}$$

for $\tilde\eta \in \mathrm{supp}(\mu_o)$, $x \in X$.

PROPOSITION 5.4 *Every accumulation point $\mu = (\mu_x)_{x\in X}$ of the family $\mathcal{F}$ in $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ is a $\delta(\Gamma)$-dimensional conformal density.*

Proof. Let $(\mu_x)_{x\in X}$ be an accumulation point of $\mathcal{F}$. By construction, the measures μ_x, $x \in X$, are Γ-equivariant and supported on the limit set L_Γ. It therefore remains to show

$$\frac{d\mu_x}{d\mu_o}(\tilde\eta) = e^{\delta(\Gamma)\mathcal{B}_{\tilde\eta}(o,x)} \quad \text{for any } x \in X\,,\ \tilde\eta \in \mathrm{supp}(\mu_o)\,.$$

Notice that if $(y_n) \subset X$ is a sequence converging to a point $\tilde\eta \in \partial X$, then $d(x, y_n) - d(\cdot, y_n) \to \mathcal{B}_{\tilde\eta}(x, \cdot)$ uniformly on compact sets in X. Hence it suffices to prove that for any $f \in \mathrm{C}^0(\overline{X})$

$$\lim_{s\searrow 1}\left| \int_{\overline{X}} f(z)d\mu_o^s(z) - \int_{\overline{X}} f(z)e^{-\delta(\Gamma)(d(o,z)-d(x,z))}d\mu_x^s(z)\right| = 0\,.$$

We choose $f \in \mathrm{C}^0(\overline{X})$ arbitrary, $s \in (1, 2]$ and estimate

$$\begin{aligned}
\Big| \int_{\overline{X}} f(z)d\mu_o^s(z) \;&-\; \int_{\overline{X}} f(z)e^{-\delta(\Gamma)(d(o,\gamma o)-d(x,\gamma o))}d\mu_x^s(z)\Big| \\
&= \frac{1}{P^s}\Big|\sum_{\gamma\in\Gamma} f(\gamma o)h(b_\gamma)\left(e^{-sb_\gamma} - e^{-\delta(\Gamma)(d(o,\gamma o)-d(x,\gamma o))}e^{-s\delta(\Gamma)d(x,\gamma o)}\right)\Big| \\
&\le \frac{1}{P^s}\sum_{\gamma\in\Gamma} |f(\gamma o)|\cdot e^{-sb_\gamma}h(b_\gamma)\cdot\left|1 - e^{\delta(\Gamma)(s-1)(d(o,\gamma o)-d(x,\gamma o))}\right| \\
&\le \frac{\|f\|_\infty}{P^s}\sum_{\gamma\in\Gamma} e^{-sb_\gamma}h(b_\gamma)\cdot\left(e^{\delta(\Gamma)(s-1)|d(o,\gamma o)-d(x,\gamma o)|} - 1\right) \\
&\le \|f\|_\infty\cdot\left(e^{\delta(\Gamma)(s-1)d(o,x)} - 1\right).
\end{aligned}$$

Since the last term tends to zero as s tends to 1 the claim follows. $\square$

Recall that $\theta_* \in [0, \pi/2]$ is the unique point such that $\delta_{\theta_*}(\Gamma) = \delta(\Gamma)$. In order to prove Theorem A it remains to show that the support of the conformal density μ constructed above is included in the unique Γ-invariant subset of the limit set which consists of all limit points with slope θ_*. This is the content of the following

PROPOSITION 5.5 *The support of the conformal density $\mu = (\mu_x)_{x\in X}$ is contained in $L_\Gamma \cap \partial X_{\theta_*}$.*

Proof. Recall the definition of b_γ from (21) and let h be the function from Lemma 5.1. Using Proposition 4.1 we will prove that for any $\varepsilon > 0$

$$\sum_{\substack{\gamma\in\Gamma \\ |\theta(o,\gamma o)-\theta_*|>\varepsilon}} e^{-b_\gamma} h(b_\gamma) < \infty\,.$$

Let $\varepsilon > 0$ arbitrary and set

$$s_\varepsilon := \max\{\delta_\theta(\Gamma) : \theta \in [0,\pi/2],\ |\theta - \theta_*| \geq \varepsilon\}\,.$$

Then by choice of θ_* we have $\delta(\Gamma) = \delta_{\theta_*}(\Gamma) > s_\varepsilon$. Fix $\alpha := \frac{1}{2}\big(\delta(\Gamma) - s_\varepsilon\big)$ and let $r_0 > 0$ such that for all $r \geq r_0$ and $t > 1$ we have $h(rt) \leq t^\alpha h(r)$. In particular, if $b_\gamma \geq r_0$, then

$$h(b_\gamma) = h(\frac{b_\gamma}{r_0}\cdot r_0) \leq \left(\frac{b_\gamma}{r_0}\right)^\alpha \cdot h(r_0) = \frac{\delta(\Gamma)^\alpha h(r_0)}{r_0^\alpha}\cdot e^{\alpha \log d(o,\gamma o)}\,.$$

Set $\Gamma_\varepsilon := \{\gamma \in \Gamma : |\theta(o,\gamma o) - \theta_*| > \varepsilon,\ b_\gamma \geq r_0\}$. Then

$$\begin{aligned}\sum_{\gamma\in\Gamma_\varepsilon} e^{-b_\gamma} h(b_\gamma) &\leq \sum_{\gamma\in\Gamma_\varepsilon} e^{-\delta(\Gamma) d(o,\gamma o)}\cdot \frac{\delta(\Gamma)^\alpha h(r_0)}{r_0^\alpha}\cdot e^{\alpha\log d(o,\gamma o)} \\ &= \frac{\delta(\Gamma)^\alpha h(r_0)}{r_0^\alpha}\cdot \sum_{\gamma\in\Gamma_\varepsilon} e^{-(\delta(\Gamma)-\alpha)d(o,\gamma o)}\,.\end{aligned}$$

Since $\delta(\Gamma) - \alpha = \frac{1}{2}\delta(\Gamma) + \frac{1}{2}s_\varepsilon > s_\varepsilon$, we conclude that

$$\sum_{\gamma\in\Gamma_\varepsilon} e^{-b_\gamma} h(b_\gamma) \quad \text{converges.}$$

The claim now follows from the fact that $\#\{\gamma \in \Gamma : d(o,\gamma o) < r_0/\delta(\Gamma)\}$ is finite. □

6 The generalized Patterson-Sullivan construction

According to the statement of Theorem A, the classical conformal density constructed in the previous section gives measure zero to the set of limit points of slope different from θ_*. In order to obtain measures on each Γ-invariant subset of the limit set we will use a variation of the classical Patterson-Sullivan construction with more degrees of freedom. The idea is to use a weighted version of the Poincaré series in order to get the main contribution from orbit points with slope close to the desired slope $\theta \in (0, \pi/2)$. At this point, properties of the exponent of growth and Proposition 4.1 will turn out to be central importance.

Recall that $\mathcal{B}_\theta$ denotes the directional distance introduced in Section 3. We observe that for any $b = (b_1, b_2) \in \mathbb{R}^2$, $\theta \in [0, \pi/2]$ and $\tau \geq 0$ fixed, the series

$$P_\theta^{s,b,\tau}(x, y) = \sum_{\gamma \in \Gamma} e^{-s(b_1 d_1(x_1, \gamma_1 y_1) + b_2 d_2(x_2, \gamma_2 y_2) + \tau(d(x, \gamma y) - \mathcal{B}_\theta(x, \gamma y)))}$$

possesses a critical exponent which is independent of $x, y \in X$ by the triangle inequalities for d, d_1, d_2 and $\mathcal{B}_\theta$. Notice that for $\tau = 0$, this is exactly the series considered by M. Burger in [Bur93]; here we will need to take τ large in order to make the contribution of the orbit points with slope far away from θ small.

For any $\theta \in [0, \pi/2]$ and $\tau \geq 0$, we define a region of convergence

$$\mathcal{R}_\theta^\tau := \{b{=}(b_1, b_2) : P_\theta^{s,b,\tau}(o, o) \text{ has critical exponent } s \leq 1\} \subseteq \mathbb{R}^2$$

and its boundary

$$\partial\mathcal{R}_\theta^\tau := \{b{=}(b_1, b_2) : P_\theta^{s,b,\tau}(o, o) \text{ has critical exponent } s \leq 1\} \subseteq \mathbb{R}^2.$$

We recall the definition of the distance vector from (5). In the sequel we will identify $b = (b_1, b_2)$ with the column vector b^t so that for $x = (x_1, x_2)^t \in \mathbb{R}^2$ we may write

$$\langle b, x\rangle = b_1 x_1 + b_2 x_2 \,.$$

The region of convergence possesses the following properties:

Lemma 6.1 *If $\tau \leq \tau'$, then $\mathcal{R}_\theta^\tau \subseteq \mathcal{R}_\theta^{\tau'}$.*

Proof. Let $\tau \leq \tau'$, $b \in \mathcal{R}_\theta^\tau$. Then for any $\gamma \in \Gamma$

$$e^{-s\big(\langle b, H(o,\gamma o)\rangle + \tau'(d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o))\big)} \leq e^{-s\big(\langle b, H(o,\gamma o)\rangle + \tau(d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o))\big)}$$

and therefore $P_\theta^{s,b,\tau'}(o, o) \leq P_\theta^{s,b,\tau}(o, o)$. Hence $P_\theta^{s,b,\tau'}(o, o)$ converges if $s > 1$. In particular, $P_\theta^{s,b,\tau'}(o, o)$ possesses a critical exponent less than or equal to 1. □

Lemma 6.2 *For any $\tau \geq 0$, the region $\mathcal{R}_\theta^\tau$ is convex.*

Proof. Let $\tau \geq 0$, $a, b \in \mathcal{R}_\theta^\tau$ and $t \in [0, 1]$. For $\gamma \in \Gamma$ we abbreviate

$$\big(ta + (1-t)b\big)_\gamma := \langle ta + (1-t)b, H(o, \gamma o)\rangle + \tau\big(d(o, \gamma o) - \mathcal{B}_\theta(o, \gamma o)\big).$$

Then by Hölder's inequality

$$\sum_{\gamma\in\Gamma} e^{-s(ta+(1-t)b)_\gamma} = \sum_{\gamma\in\Gamma} e^{-sta_\gamma} e^{-s(1-t)b_\gamma} \leq \Big(\sum_{\gamma\in\Gamma} e^{-sa_\gamma}\Big)^t \Big(\sum_{\gamma\in\Gamma} e^{-sb_\gamma}\Big)^{1-t}.$$

The latter sum converges if $s > 1$, hence $ta + (1-t)b \in \mathcal{R}_\theta^\tau$. □

With the help of Proposition 4.1 we can describe the region of convergence more precisely. The following result relates the region of convergence $\mathcal{R}_\theta^\tau$ to the exponent of growth of slope θ.

LEMMA 6.3 *Let* $\theta \in [0,\pi/2]$ *and* $\tau \geq 0$. *If* $(b_1,b_2) \in \mathcal{R}^\tau_\theta$, *then* $b_1\cos\theta + b_2\sin\theta \geq \delta_\theta(\Gamma)$.

Proof. Recall the definition of H_θ from (15) and suppose that $\langle b, H_\theta\rangle = b_1\cos\theta + b_2\sin\theta < \delta_\theta(\Gamma)$. Then there exists $s > 1$ such that $s\,\langle b, H_\theta\rangle < \delta_\theta(\Gamma)$. For $H \in \mathbb{R}^2_{\geq 0}$ we put

$$f(H) := s\big(\langle b, H\rangle + \tau(1 - \langle H_\theta, H\rangle)\big)\,.$$

Moreover, the continuous homogeneous function $f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ satisfies $f(H_\theta) = s\langle b, H_\theta\rangle < \delta_\theta(\Gamma)$, so according to Proposition 4.1 (a) applied to $D = [0,\pi/2]$, the series

$$\sum_{\gamma\in\Gamma} e^{-f(H(o,\gamma o))} \qquad \text{diverges}\,.$$

Since $f(H(o,\gamma o)) = s\big(b_1 d_1(o_1,\gamma_1 o_1) + b_2 d_2(o_2,\gamma_2 o_2) + \tau(d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o))\big)$ we obtain a contradiction to $(b_1,b_2) \in \mathcal{R}^\tau_\theta$. □

Using the above properties of the region of convergence and Patterson's Lemma 5.1 we are now going to construct (b,θ)-densities as defined in the introduction. Such densities are a natural generalization of Γ-invariant conformal densities if one wants to measure each Γ-invariant subset of the geometric limit set.

From here on we fix $\theta \in [0,\pi/2]$ such that $L_\Gamma \cap \partial X_\theta \neq \emptyset$, $\tau \geq 0$ and $b = (b_1,b_2) \in \partial\mathcal{R}^\tau_\theta$. For $\gamma = (\gamma_1,\gamma_2) \in \Gamma$ we abbreviate

$$b_\gamma := b_1 d_1(o_1,\gamma_1 o_1) + b_2 d_2(o_2,\gamma_2 o_2) + \tau\big(d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o)\big)\,. \tag{24}$$

Let h be a function as in Lemma 5.1 and recall the definition of the distance vector from (5). As in Section 5 we will construct a family of orbital measures on $\overline{X}$ in the following way: If D denotes the unit Dirac point measure, then for $x \in X$ and $s > 1$ we put

$$\mu^s_x := \frac{1}{P^s}\sum_{\gamma\in\Gamma} e^{-s\big(\langle b, H(x,\gamma o)\rangle + \tau(d(x,\gamma o) - \mathcal{B}_\theta(x,\gamma o))\big)} h(b_\gamma) D(\gamma o)\,.$$

As in the classical case, these measures are Γ-equivariant by construction, but they depend on the parameters $\theta \in [0,\pi/2]$, $\tau \geq 0$ and $b = (b_1,b_2) \in \partial\mathcal{R}^\tau_\theta$.

Recall from Section 5 that $(\mathrm{C}^0(\overline{X}), \|\cdot\|_\infty)$ is the space of real valued continuous functions on $\overline{X}$ with norm $\|f\|_\infty = \max\{|f(x)| : x \in \overline{X}\}$, $f \in \mathrm{C}^0(\overline{X})$, and ρ is the pseudo-metric on the cone $\mathcal{M}^+(\overline{X})$ of positive finite Borel measures on $\overline{X}$ defined in (22).

LEMMA 6.4 *Let* $\theta \in [0,\pi/2]$ *such that* $L_\Gamma \cap \partial X_\theta \neq \emptyset$, $\tau \geq 0$ *and* $b = (b_1,b_2) \in \partial\mathcal{R}^\tau_\theta$. *Then the family of maps* $\mathcal{F}(\theta,\tau,b) := \{x \mapsto \mu^s_x : 1 < s \leq 2\}$ *from* X *to* $\mathcal{M}^+(\overline{X})$ *is equicontinuous.*

Proof. Let $x, y \in X$. For $\gamma = (\gamma_1,\gamma_2) \in \Gamma$ we abbreviate

$$\begin{aligned} q_\gamma(y,x) := b_1\big(d_1(y_1,\gamma_1 o_1) - d_1(x_1,\gamma_1 o_1)\big) + b_2\big(d_2(y_2,\gamma_2 o_2) - d_2(x_2,\gamma_2 o_2)\big)\\ + \tau\big(d(y,\gamma o) - d(x,\gamma o) - \mathcal{B}_\theta(y,\gamma o) + \mathcal{B}_\theta(x,\gamma o)\big)\,, \end{aligned} \tag{25}$$

put $\|b\|_1 := |b_1| + |b_2|$ and estimate

$$\big|q_\gamma(y,x)\big| \le b_1 d_1(y_1,x_1) + b_2 d_2(y_2,x_2) + 2\tau d(y,x) \le d(x,y)\big(||b||_1 + 2\tau\big)\,. \qquad (26)$$

If $s \in (1,2]$ and $f \in \mathrm{C}^0(\overline{X})$, the inequality $|1 - e^{-t}| \le e^{|t|} - 1$, $t \in \mathbb{R}$, gives

$$\begin{aligned}\left|\int_{\overline{X}} f\, d\mu_x^s - \int_{\overline{X}} f\, d\mu_y^s\right| &\le \frac{1}{P^s}\sum_{\gamma\in\Gamma} e^{-s(b_1 d_1(x_1,\gamma_1 o_1)+b_2 d_2(x_2,\gamma_2 o_2)+\tau(d(x,\gamma o)-\mathcal{B}_\theta(x,\gamma o)))} \\ &\qquad\qquad \cdot h(b_\gamma)|f(\gamma o)|\,\big|1 - e^{-s q_\gamma(y,x)}\big| \\ &\le \frac{\|f\|_\infty}{P^s}\sum_{\gamma\in\Gamma} e^{-s b_\gamma} e^{-s q_\gamma(x,o)} h(b_\gamma)(e^{s|q_\gamma(y,x)|} - 1)\,.\end{aligned}$$

Since $f \in \mathrm{C}^0(\overline{X})$ was arbitrary, $s \le 2$ and $\sum_{\gamma\in\Gamma} e^{-s b_\gamma} h(b_\gamma) = P^s$, we conclude using (26)

$$\rho(\mu_x^s, \mu_y^s) \le e^{2d(o,x)(\|b\|_1+2\tau)}\cdot\big(e^{2d(x,y)(\|b\|_1+2\tau)} - 1\big)\,.$$

This proves that $\mathcal{F}(\theta,\tau,b)$ is equicontinuous. □

Lemma 6.5 *Let $\theta \in [0,\pi/2]$ such that $L_\Gamma \cap \partial X_\theta \ne \emptyset$, $\tau \ge 0$ and $b = (b_1,b_2) \in \partial\mathcal{R}_\theta^\tau$. Then for any $x \in X$ there exists a sequence $(s_n) \searrow 1$ such that the measures $\mu_x^{s_n} \subset \mathcal{M}^+(\overline{X})$ converge weakly to a measure $\mu_x := \mu_x(\theta,\tau,b)$ as $n \to \infty$.*

Proof. The compactness of the space $\overline{X}$ implies that every sequence of measures in $\mathcal{M}^+(\overline{X})$ possesses a weakly convergent subsequence. □

Hence as in the classical case, the family of maps $\mathcal{F}(\theta,\tau,b)$ is relatively compact in the space of continuous maps $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ endowed with the topology of uniform convergence on compact sets. From the definition of $(\mu_x^s)_{x\in X}$ it follows that every accumulation point $\mu = \mu(\theta,\tau,b) = (\mu_x)_{x\in X}$ of $\mathcal{F}(\theta,\tau,b)$ as $s \searrow 1$ takes its values in $\mathcal{M}^+(\partial X)$.

Unfortunately, the families of measures obtained in this way are not very useful, because in general they are not absolutely continuous with respect to each other and also depend on the parameter $\tau \ge 0$. However, we still have some freedom in choosing appropriate parameters $b = (b_1,b_2) \in \mathbb{R}^2$.

The following proposition paves the way towards the construction of orbital measures with support in a single Γ-invariant subset $L_\Gamma \cap \partial X_\theta \subseteq \partial X$.

Proposition 6.6 *Fix $\theta \in [0,\pi/2]$ such that $\delta_\theta(\Gamma) > 0$. Then there exist $b = (b_1,b_2) \in \mathbb{R}^2$ and $\tau_0 \ge 0$ such that for all $\tau \ge \tau_0$ and for all $\varepsilon > 0$*

$$\sum_{\substack{\gamma\in\Gamma \\ |\theta(o,\gamma o)-\theta|>\varepsilon}} e^{-b_\gamma} h(b_\gamma) < \infty\,,$$

where b_γ is defined in (24), and h is a function as in Lemma 5.1.

Proof. Recall the definition of H_θ from (15). Since the function Ψ_Γ defined in (19) is concave and upper semi-continuous, there exists a linear functional Φ on $\mathbb{R}^2$ such that

$$\Phi(H_\theta) = \Psi_\Gamma(H_\theta) \quad \text{and} \quad \Phi(H) \geq \Psi_\Gamma(H) \quad \forall\, H \in \mathbb{R}^2_{\geq 0}\,.$$

We choose $b = (b_1, b_2) \in \mathbb{R}^2$ such that $\Phi = \langle b, \cdot\rangle$, so

$$\langle b, H_\theta\rangle = \delta_\theta(\Gamma) \quad \text{and} \quad \langle b, H_{\hat\theta}\rangle \geq \delta_{\hat\theta}(\Gamma) \quad \forall\, \hat\theta \in [0, \pi/2]\,.$$

Since the map $\hat\theta \mapsto \langle b, H_{\hat\theta}\rangle$ is continuous and $\delta_\theta(\Gamma) > 0$, there exists $\hat\varepsilon \in (0,1)$ such that every $\hat\theta \in [0,\pi/2]$ with $|\hat\theta - \theta| < \hat\varepsilon$ satisfies $\langle b, H_{\hat\theta}\rangle > 0$. Notice that it suffices to prove the claim for $\varepsilon < \hat\varepsilon$, because the sum is non-increasing when ε gets bigger.

We now put $\|b\|_1 := |b_1| + |b_2|$, fix $\tau_0 := 12\max\{(2\delta(\Gamma) - \delta_\theta(\Gamma) + 2\|b\|_1)/\hat\varepsilon^2, 2\}$, and let $\varepsilon \in (0, \hat\varepsilon)$ be arbitrary. By property (ii) of the function h in Lemma 5.1 there exists $r_0 = r_0(\varepsilon) > 0$ such that for $r \geq r_0$ and $t > 1$ we have $h(rt) \leq (t)^{\varepsilon^2} h(r)$. Let $R = R(\varepsilon) > \max\{\frac{r_0}{2\varepsilon^2}, \frac{r_0}{2\delta(\Gamma)}\}$ such that $d(o,\gamma o) > R$ implies

$$d(o,\gamma o)\big(\|b\|_1 + 2\varepsilon^2 + 2\delta(\Gamma)\big) < \min\big\{e^{d(o,\gamma o)}, e^{\delta(\Gamma) d(o,\gamma o)}\big\}\,. \tag{27}$$

For $\gamma \in \Gamma$ we abbreviate $H_\gamma = H(o,\gamma o)/d(o,\gamma o)$, fix $\tau \geq \tau_0$ and set

$$\begin{aligned} \hat\Gamma &:= \{\gamma \in \Gamma : |\theta(o,\gamma o) - \theta| > \frac{\hat\varepsilon}{2}\},\\ \Gamma' &:= \{\gamma \in \Gamma : \frac{\varepsilon}{2} < |\theta(o,\gamma o) - \theta| < \hat\varepsilon\}; \end{aligned}$$

our goal is to show that $\sum_{\gamma \in \hat\Gamma} e^{-b_\gamma} h(b_\gamma) < \infty$ and $\sum_{\gamma\in\Gamma'} e^{-b_\gamma} h(b_\gamma) < \infty$.

First let $\gamma \in \hat\Gamma$. Using the Cauchy–Schwarz inequality, $\|H_\gamma - H_\theta\| \leq 2$, and the condition $\langle b, H_\theta\rangle = \delta_\theta(\Gamma)$, we have

$$\langle b, H_\gamma\rangle = \langle b, H_\theta\rangle + \langle b, H_\gamma - H_\theta\rangle \geq \delta_\theta(\Gamma) - \sqrt{b_1^2 + b_2^2}\cdot \|H_\gamma - H_\theta\| \geq \delta_\theta(\Gamma) - 2\|b\|_1\,.$$

Moreover, the estimate $\cos t < 1 - t^2/3$ for $t \in \mathbb{R}$, and the fact that $d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o) > d(o,\gamma o)\big(1 - \cos\frac{\hat\varepsilon}{2}\big)$ and $\tau \geq 12(2\delta(\Gamma) - \delta_\theta(\Gamma) + 2\|b\|_1)/\hat\varepsilon^2$ imply

$$\begin{aligned} b_\gamma &= \langle b, H(o,\gamma o)\rangle + \tau\big(d(o,\gamma o) - \mathcal{B}_\theta(o,\gamma o)\big)\\ &> d(o,\gamma o)\left(\delta_\theta(\Gamma) - 2\|b\|_1 + \frac{\tau\hat\varepsilon^2}{12}\right)\\ &\geq 2\delta(\Gamma) d(o,\gamma o)\,. \end{aligned}$$

Hence if $\gamma \in \hat\Gamma$ satisfies $d(o,\gamma o) > R$, then by choice of $R > \frac{r_0}{2\delta(\Gamma)}$ we have $b_\gamma > r_0$, hence

$$h(b_\gamma) = h\left(\frac{b_\gamma}{r_0} r_0\right) \leq \left(\frac{b_\gamma}{r_0}\right)^{\varepsilon^2} h(r_0) = \frac{h(r_0)}{(r_0)^{\varepsilon^2}} e^{\varepsilon^2 \log b_\gamma}\,. \tag{28}$$

Since the function $g(t) := t - \varepsilon^2 \log t$, $t > 0$, is monotone increasing, we conclude that for $\gamma \in \hat\Gamma$ with $d(o, \gamma o) > R$

$$\begin{aligned} g(b_\gamma) &> g\big(2\delta(\Gamma)d(o,\gamma o)\big) = 2\delta(\Gamma)d(o,\gamma o) - \varepsilon^2 \log\big(\underbrace{2\delta(\Gamma)d(o,\gamma o)}_{<e^{\delta(\Gamma)d(o,\gamma o)}}\big) \\ &> d(o,\gamma o)\big(2\delta(\Gamma) - \varepsilon^2\delta(\Gamma)\big)\,, \end{aligned}$$

where we used (27) in the last step. Summarizing, we estimate

$$\begin{aligned} \sum_{\substack{\gamma\in\hat\Gamma \\ d(o,\gamma o)>R}} e^{-b_\gamma} h(b_\gamma) &\le \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\substack{\gamma\in\hat\Gamma \\ d(o,\gamma o)>R}} e^{-b_\gamma + \varepsilon^2 \log b_\gamma} = \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\substack{\gamma\in\hat\Gamma \\ d(o,\gamma o)>R}} e^{-g(b_\gamma)} \\ &< \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\gamma\in\Gamma} e^{-(2-\varepsilon^2)\delta(\Gamma)d(o,\gamma o)}\,, \end{aligned}$$

which converges since $\varepsilon^2 \le \hat\varepsilon^2 < 1$.

Next let $\gamma \in \Gamma'$. As above, $|\theta(o,\gamma o) - \theta| > \varepsilon/2$ and $\tau \ge 24$ imply

$$b_\gamma > \langle b, H(o,\gamma o)\rangle + \tau d(o,\gamma o)\frac{\varepsilon^2}{12} > d(o,\gamma o)\big(\langle b, H_\gamma\rangle + 2\varepsilon^2\big)\,,$$

hence

$$g(b_\gamma) > d(o,\gamma o)\big(\langle b, H_\gamma\rangle + 2\varepsilon^2\big) - \varepsilon^2 \log\big(d(o,\gamma o)(\langle b, H_\gamma\rangle + 2\varepsilon^2)\big)\,.$$

By choice of $\hat\varepsilon$ we have $\langle b, H_\gamma\rangle > 0$, so $d(o,\gamma o) > R > \frac{r_0}{2\varepsilon^2}$ yields $b_\gamma > r_0$. Moreover, (27) implies $d(o,\gamma o)\big(\langle b, H_\gamma\rangle + 2\varepsilon^2\big) < e^{d(o,\gamma o)}$, which gives

$$g(b_\gamma) > d(o,\gamma o)\big(\langle b, H_\gamma\rangle + \varepsilon^2\big)\,.$$

Next we consider the continuous homogeneous function

$$f : \mathbb{R}^2_{\ge 0} \to \mathbb{R},\ H \mapsto \langle b, H\rangle + \varepsilon^2\|H\|.$$

Using inequality (28) we estimate

$$\begin{aligned} \sum_{\substack{\gamma\in\Gamma' \\ d(o,\gamma o)>R}} e^{-b_\gamma} h(b_\gamma) \le \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\substack{\gamma\in\Gamma' \\ d(o,\gamma o)>R}} e^{-g(b_\gamma)} &< \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\substack{\gamma\in\Gamma' \\ d(o,\gamma o)>R}} e^{-f(H(o,\gamma o))} \\ &< \frac{h(r_0)}{(r_0)^{\varepsilon^2}} \sum_{\gamma\in\Gamma'} e^{-f(H(o,\gamma o))}\,. \end{aligned}$$

This sum converges by Proposition 4.1 (b) applied to

$$D' := \{\hat\theta \in [0,\pi/2] : \varepsilon/2 < |\hat\theta - \theta| < \hat\varepsilon\},$$

because the continuous homogeneous function f satisfies $f(H_{\hat\theta}) > \delta_{\hat\theta}(\Gamma)$ for all $\hat\theta \in \overline{D'}$.

The claim now follows from the fact that the number of elements $\gamma \in \Gamma$ with $d(o, \gamma o) \leq R$ is finite. $\square$

Notice that the above proposition provides $b = (b_1, b_2) \in \mathbb{R}^2$ even if $\theta = 0$ or $\theta = \pi/2$. However, in general we do not have $b_2 = 0$ if $\theta = 0$, or $b_1 = 0$ if $\theta = \pi/2$. Hence for the construction of (b, θ)-densities we have to restrict ourselves to $\theta \in (0, \pi/2)$.

We therefore fix $\theta \in (0, \pi/2)$ such that $\delta_\theta(\Gamma) > 0$, hence in particular we have $L_\Gamma \cap \partial X_\theta \neq \emptyset$. Using the previous proposition we are finally able to construct a (b, θ)-density according to Definition 1.2.

To that end we choose $b = (b_1, b_2) \in \mathbb{R}^2$ and $\tau_0 > 0$ according to Proposition 6.6. For fixed $\tau > \tau_0$ we consider the corresponding family $\mathcal{F}(\theta, \tau, b)$ as in Lemma 6.4. Then by the Theorem of Arzelà-Ascoli, $\mathcal{F}(\theta, \tau, b)$ is relatively compact in the space of continuous maps $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ endowed with the topology of uniform convergence on compact sets. The following proposition characterizes the possible accumulation points.

Proposition 6.7 *Every accumulation point $\mu = \mu(\theta, \tau, b)$ of the family $\mathcal{F} = \mathcal{F}(\theta, \tau, b)$ in $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ is a (b, θ)-density.*

Proof. Let $(\mu_x)_{x \in X}$ be an accumulation point of $\mathcal{F}$. By construction, the measures μ_x, $x \in X$, are Γ-equivariant and supported on the limit set L_Γ. Proposition 6.6 further implies $\mathrm{supp}(\mu_o) \subseteq L_\Gamma \cap \partial X_\theta$. It therefore suffices to prove

$$\frac{d\mu_x}{d\mu_o}(\tilde{\eta}) = e^{b_1 \mathcal{B}_{\eta_1}(o_1, x_1) + b_2 \mathcal{B}_{\eta_2}(o_2, x_2)} \quad \text{for any } x \in X\,,\ \tilde{\eta} = (\eta_1, \eta_2, \theta) \in \mathrm{supp}(\mu_o)\,.$$

Notice that if $(y_n) = \big((y_{n,1}, y_{n,2})\big) \subset X$ is a sequence converging to a point $\tilde{\eta} = (\eta_1, \eta_2, \theta) \in \partial X_\theta$, then for $i \in \{1, 2\}$ $d_i(x_i, y_{n,i}) - d_i(\cdot, y_{n,i}) \to \mathcal{B}_{\eta_i}(x_i, \cdot)$ uniformly on compact sets in X_i, and $d(x, y_n) - d(\cdot, y_n) \to \mathcal{B}_{\tilde{\eta}}(x, \cdot)$ uniformly on compact sets in X. Using the definition and properties of Busemann functions and the directional distance $\mathcal{B}_\theta$ we conclude that for any constant $c \geq 0$ and $\varepsilon > 0$ arbitrary, there exist $R > 0$ and $\rho > 0$ with the following properties: If $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ satisfies $d(o, \gamma o) > R$ and $|\theta(o, \gamma o) - \theta| < \rho$, then for all $x = (x_1, x_2) \in X$ with $d(o, x) \leq c$

$$\big| d(o, \gamma o) - d(x, \gamma o) - \mathcal{B}_\theta(o, \gamma o) + \mathcal{B}_\theta(x, \gamma o) \big| < \varepsilon\,. \tag{29}$$

Let $\varepsilon > 0$ arbitrary, fix $x \in X$, put $c := d(o, x)$ and choose $R > 0$ and $\rho > 0$ as above. We set

$$\begin{aligned} \hat{\Gamma} &:= \big\{ \gamma \in \Gamma : \big|\theta(o, \gamma o) - \theta\big| > \rho/2 \big\}\,, \\ \Gamma' &:= \big\{ \gamma \in \Gamma : d(o, \gamma o) > R \ \text{ and } \ \big|\theta(o, \gamma o) - \theta\big| < \rho \big\}\,. \end{aligned}$$

Here, for $\gamma \in \Gamma$ we abbreviate

$$q_\gamma(o, x) := d(o, \gamma o) - d(x, \gamma o) - \mathcal{B}_\theta(o, \gamma o) + \mathcal{B}_\theta(x, \gamma o)$$

and recall that for $\gamma \in \Gamma'$ we have $|q_\gamma(o,x)| < \varepsilon$. Then for any $f \in \mathrm{C}^0(\overline{X})$, $s \in (1,2]$, we have

$$\begin{aligned}\Big| \int_{\overline{X}} f(z) d\mu_o^s(z) \;&-\; \int_{\overline{X}} f(z) e^{-b_1(d_1(o_1,z_1)-d_1(x_1,z_1))-b_2(d_2(o_2,z_2)-d_2(x_2,z_2))} d\mu_x^s(z)\Big| \\ &\le\; \frac{1}{P^s}\sum_{\gamma\in\Gamma} |f(\gamma o)|\cdot e^{-sb_\gamma} h(b_\gamma)\cdot \big|1 - e^{(s-1)\langle b, H(o,\gamma o)-H(x,\gamma o)\rangle + s\tau q_\gamma(o,x)}\big|\,.\end{aligned}$$

The triangle inequality and the estimate $|q_\gamma(o,x)| \le 2d(o,x)$ imply that for any $\gamma \in \Gamma$

$$\big|(s-1)\langle b, H(o,\gamma o) - H(x,\gamma o)\rangle + s\tau q_\gamma(o,x)\big| \le (s-1)\|b\|_1 d(o,x) + 2s\tau d(o,x)\,.$$

This proves that for $x \in X$ with $d(x,o) \le c$, and $s \le 2$, the term

$$\begin{aligned}\big|1 - e^{(s-1)\langle b,H(o,\gamma o)-H(x,\gamma o)\rangle + sq_\gamma(o,x)}\big| \;&\le\; e^{|(s-1)\langle b,H(o,\gamma o)-H(x,\gamma o)\rangle + sq_\gamma(o,x)|} - 1 \\ &\le\; e^{c(\|b\|_1+4\tau)} - 1\end{aligned}$$

is bounded above by a constant $A = A(c,b,\tau)$. If $\gamma \in \Gamma'$ then $|q_\gamma(o,x)| < \varepsilon$, hence

$$\lim_{s\searrow 1} \big|1 - e^{(s-1)\langle b,H(o,x)\rangle + sq_\gamma(o,x)}\big| \le \lim_{s\searrow 1} e^{(s-1)\|b\|_1 c + 2\tau\varepsilon} - 1 = e^{2\tau\varepsilon} - 1. \qquad (30)$$

We conclude

$$\begin{aligned}\Big| \int_{\overline{X}} f(z) d\mu_o^s(z) - &\int_{\overline{X}} f(z) e^{-b_1(d_1(o_1,z_1)-d_1(x_1,z_1))-b_2(d_2(o_2,z_2)-d_2(x_2,z_2))} d\mu_x^s(z)\Big| \\ \le \frac{\|f\|_\infty}{P^s}\Big(&\sum_{\substack{\gamma\in\Gamma \\ d(o,\gamma o)\le R}} e^{-sb_\gamma} h(b_\gamma) + \sum_{\gamma\in\hat\Gamma} e^{-sb_\gamma} h(b_\gamma) A \\ &+ \sum_{\gamma\in\Gamma'} e^{-sb_\gamma} h(b_\gamma)\big|1 - e^{(s-1)\langle b,H(o,\gamma o)-H(x,\gamma o)\rangle + s\tau q_\gamma(o,x)}\big|\Big)\,.\end{aligned}$$

Now the first term tends to zero as $s \searrow 1$ since the summation is over a finite number of $\gamma \in \Gamma$. By Proposition 6.6, $\sum_{\gamma\in\hat\Gamma} e^{-b_\gamma} h(b_\gamma)$ converges, hence the second term tends to zero as $s \searrow 1$. In the last term, we have $\sum_{\gamma\in\Gamma'} e^{-sb_\gamma} h(b_\gamma) \le P^s$ for any $s > 1$, and therefore by (30)

$$\begin{aligned}\lim_{s\searrow 1}\Big| \int_{\overline{X}} f(z) d\mu_o^s(z) - \int_{\overline{X}} f(z) e^{-b_1(d_1(o_1,z_1)-d_1(x_1,z_1))-b_2(d_2(o_2,z_2)-d_2(x_2,z_2))} d\mu_x^s(z)\Big| \\ \le \|f\|_\infty\big(e^{2\tau\varepsilon} - 1\big)\,.\end{aligned}$$

The claim follows taking the limit as $\varepsilon \searrow 0$. □

So in particular we can construct for any $\theta \in (0,\pi/2)$ with $\delta_\theta(\Gamma) > 0$ a (b,θ)-density with appropriate parameters $b = (b_1, b_2)$. This proves Theorem B from the introduction.

7 Properties of (b,θ)-densities

In this section we will study properties of (b,θ)-densities using the shadow lemma Theorem 7.2. If not otherwise specified we allow $\theta \in [0,\pi/2]$.

LEMMA 7.1 *Let μ be a (b,θ)-density, and $x \in X$. If $\tilde{U} \subset \partial X$ is an open neighborhood of a limit point $\tilde{\xi} \in \partial X_\theta$, then $\mu_x(\tilde{U}) > 0$.*

Proof. Let $\tilde{U} \subset \partial X$ be an open neighborhood of a limit point $\tilde{\xi} \in \partial X_\theta$ such that $\mu_x(\tilde{U}) = 0$. If $U := \tilde{U} \cap \partial X_\theta$, then by compactness and minimality of $L_\Gamma \cap \partial X_\theta$ (see Theorem A in [Lin10]) there exists a finite set $\Lambda \subset \Gamma$ such that

$$L_\Gamma \cap \partial X_\theta \subseteq \bigcup_{\gamma\in\Lambda} \gamma U\,.$$

Moreover, by Γ-equivariance

$$\mu_x(L_\Gamma \cap \partial X_\theta) \le \sum_{\gamma\in\Lambda} \mu_x(\gamma U) = \sum_{\gamma\in\Lambda} \mu_{\gamma^{-1}x}(U) \le \sum_{\gamma\in\Lambda} \mu_{\gamma^{-1}x}(\tilde{U}) = 0\,,$$

since $\mu_{\gamma^{-1}x}$, $\gamma \in \Lambda$, is absolutely continuous with respect to μ_x. □

Recall the definition of the distance vector (5) from Section 3.

Theorem 7.2 (Shadow lemma) *Let μ be a (b,θ)-density. Then there exists a constant $c_0 > 0$ such that for any $c > c_0$ there exists a constant $D(c) > 1$ with the property*

$$\frac{1}{D(c)} e^{-\langle b, H(o,\gamma o)\rangle} \le \mu_o\big(Sh(o : B_{\gamma o}(c))\big) \le D(c) e^{-\langle b, H(o,\gamma o)\rangle}.$$

Proof. Let $U_1 \subset \partial X_1$, $U_2 \subset \partial X_2$ be neighborhoods of h_1^+, h_2^+, $\Lambda \subset \Gamma$ finite, and $c_0 > 0$ as in Proposition 3.2. For $\theta \in [0,\pi/2]$ and $\alpha = (\alpha_1,\alpha_2) \in \Lambda$ the sets

$$U_\alpha := \begin{cases} \{(\eta_1,\eta_2,\theta) \in \partial X_\theta : \eta_1 \in \alpha_1 U_1, \eta_2 \in \alpha_2 U_2\} & \text{if } \theta \in (0,\pi/2) \\ \alpha_1 U_1 \subset \partial X_1 \cong (\partial X)_0 & \text{if } \theta = 0 \\ \alpha_2 U_2 \subset \partial X_2 \cong (\partial X)_{\pi/2} & \text{if } \theta = \pi/2 \end{cases}$$

are relatively open neighborhoods of a limit point in ∂X_θ, so by the previous lemma

$$q := \min\{\mu_o(U_\alpha) : \alpha \in \Lambda\}$$

is strictly positive. Moreover, if $c \ge c_0$ and $\gamma \in \Gamma$ such that $d(o,\gamma o) > c$ then by Proposition 3.2 there exists $\alpha \in \Lambda$ such that $U_\alpha \subseteq \mathrm{Sh}(\gamma^{-1}o : B_o(c))$. Hence for $c \ge c_0$ and $\gamma \in \Gamma$ with $d(o,\gamma o) > c$ we have

$$\mu_o(\partial X) \ge \mu_o\big(\mathrm{Sh}(\gamma^{-1}o : B_o(c))\big) \ge q > 0\,. \tag{31}$$

Put $S_\gamma := \mathrm{Sh}(o : B_{\gamma o}(c))$ and recall the definition of the b-Busemann function (14). The properties (ii) and (iii) of a (b,θ)-density imply

$$\mu_o\big(\mathrm{Sh}(\gamma^{-1}o : B_o(c))\big) = \mu_o(\gamma^{-1}S_\gamma) = \mu_{\gamma o}(S_\gamma) = \int_{S_\gamma} d\mu_{\gamma o}(\tilde\eta)$$
$$= \int_{\mathrm{Sh}(o:B_{\gamma o}(c))} e^{\mathcal{B}^b_{\tilde\eta}(o,\gamma o)} d\mu_o(\tilde\eta)\,.$$

By Lemma 3.9,

$$e^{-2c}e^{\langle b,H(o,\gamma o)\rangle}\mu_o(S_\gamma) < \mu_o\big(\mathrm{Sh}(\gamma^{-1}o : B_o(c))\big) \le e^{\langle b,H(o,\gamma o)\rangle}\mu_o(S_\gamma)\,,$$

so equation (31) allows us to conclude

$$e^{-\langle b,H(o,\gamma o)\rangle}q \le \mu_o(S_\gamma) \le e^{-\langle b,H(o,\gamma o)\rangle}e^{2c}\cdot\mu_o(\partial X)\,.$$ □

The following applications of Theorem 7.2 yield relations between the exponent of growth of a given slope $\theta \in [0,\pi/2]$ and the parameters of a (b,θ)-density. Recall the definition of H_θ from (15).

Theorem 7.3 *If for $\theta \in (0,\pi/2)$ a Γ-invariant (b,θ)-density exists, then*

$$\delta_\theta(\Gamma) \le \langle b, H_\theta\rangle.$$

Proof. Suppose μ is a (b,θ)-density. Let $c > c_0+1$, where $c_0 > 0$ is as in Theorem 7.2, $\varepsilon > 0$ and $R > 3c_0$ arbitrary. Let $\tilde\eta = (\eta_1,\eta_2,\theta) \in \mathrm{supp}(\mu_o)$. We only need $N(\varepsilon)R$ balls of radius 1 in X to cover the set

$$\{\big(\sigma_{o_1,\eta_1}(t\cos\hat\theta), \sigma_{o_2,\eta_2}(t\sin\hat\theta)\big) \in X : R-1 \le t < R,\ |\hat\theta-\theta| < \varepsilon\}\,,$$

and $N(\varepsilon)$ is independent of R. Since Γ is discrete, a $2c$-neighborhood of any of these balls contains a uniformly bounded number M_c of elements of $\Gamma\cdot o$.

The compactness of ∂X_θ implies the existence of a constant $A > 0$ such that every point in ∂X_θ is contained in at most $AM_cN(\varepsilon)R$ sets $\mathrm{Sh}(o : B_{\gamma o}(c))$, $\gamma \in \Gamma'$, where $\Gamma' := \{\gamma \in \Gamma : |\theta(o,\gamma o) - \theta| < \varepsilon,\ R-1 \le d(o,\gamma o) < R\}$. Therefore

$$\sum_{\gamma\in\Gamma'} \mu_o\big(\mathrm{Sh}(o : B_{\gamma o}(c))\big) \le AM_cN(\varepsilon)R\mu_o(\partial X_\theta) = AM_cN(\varepsilon)R\mu_o(\partial X)\,.$$

Furthermore, if $\gamma \in \Gamma'$ then $H_\gamma := H(o,\gamma o)/d(o,\gamma o)$ satisfies $\|H_\gamma - H_\theta\| \le \varepsilon$. Writing $\|b\|_1 := |b_1|+|b_2|$, using the Cauchy–Schwarz inequality and $\sqrt{b_1^2+b_2^2} \le \|b\|_1$ we obtain for $\gamma = (\gamma_1,\gamma_2) \in \Gamma'$

$$\langle b, H_\gamma\rangle = \langle b, H_\theta\rangle + \langle b, H_\gamma - H_\theta\rangle \le \langle b, H_\theta\rangle + \|b\|_1\varepsilon\,.$$

Using Theorem 7.2 and

$$\Delta N^\varepsilon_\theta(o,o;R) := \#\{\gamma \in \Gamma\ :\ R-1 \le d(o,\gamma o) < R\,,\ |\theta(o,\gamma o) - \theta| < \varepsilon\}\,,\quad R \gg 1\,,$$

we conclude

$$\Delta N_\theta^\varepsilon(o,o;R)\frac{1}{D(c)}e^{-\langle b,H_\theta\rangle R}\le \sum_{\gamma\in\Gamma'}\frac{1}{D(c)}e^{-\langle b,H(o,\gamma o)\rangle+\varepsilon\|b\|_1 d(o,\gamma o)}$$

$$\le e^{\varepsilon\|b\|_1 R}\sum_{\gamma\in\Gamma'}\mu_o\big(\mathrm{Sh}(o:B_{\gamma o}(c))\big)\le e^{\varepsilon\|b\|_1 R}AM_cN(\varepsilon)R\mu_o(\partial X)\,.$$

Hence

$$\begin{aligned}\delta_\theta^\varepsilon(o,o) &\le \limsup_{R\to\infty}\frac{1}{R}\log\Big(D(c)AM_cN(\varepsilon)\mu_o(\partial X)R\cdot\exp\big(\langle b,H_\theta\rangle R+\varepsilon\|b\|_1 R\big)\Big)\\ &= \langle b,H_\theta\rangle+\varepsilon\|b\|_1\end{aligned}$$

and the claim follows as $\varepsilon\searrow 0$. □

Unfortunately, the proof of the above proposition does not work for $\theta\in\{0,\pi/2\}$. So we do not know whether the estimate holds for $\delta_0(\Gamma)$ and $\delta_{\pi/2}(\Gamma)$.

We next recall the notion of the radial limit set from Definition 1.3 of the introduction. If $\theta\in[0,\pi/2]$, then the radial limit set in ∂X_θ is given by

$$L_\Gamma^{rad}\cap\partial X_\theta=\bigcup_{c>0}\bigcap_{R>c}\bigcap_{\varepsilon>0}\bigcup_{\substack{\gamma\in\Gamma\\ d(o,\gamma o)>R\\ |\theta(o,\gamma o)-\theta|<\varepsilon}}\mathrm{Sh}(o:B_{\gamma o}(c))\cap\partial X_\theta\,. \tag{32}$$

Together with the previous theorem the following implies that if a (b,θ)-density gives positive measure to the regular radial limit set, then the exponent of growth of Γ of slope θ is completely determined by its parameters.

Theorem 7.4 *If $\theta\in[0,\pi/2]$ and a (b,θ)-density gives positive measure to L_Γ^{rad}, then $\delta_\theta(\Gamma)\ge\langle b,H_\theta\rangle$.*

Proof. Suppose μ is a (b,θ)-density such that $\mu_o(L_\Gamma^{rad})>0$. Let $c>c_0$ with $c_0>0$ as in Theorem 7.2. Let $\varepsilon>0$ and $R>c$ arbitrary, and set

$$\Gamma':=\{\gamma\in\Gamma: d(o,\gamma o)>R,\ |\theta(o,\gamma o)-\theta|<\varepsilon\}.$$

Then by (32)

$$L_\Gamma^{rad}\cap\partial X_\theta\subseteq\bigcup_{\gamma\in\Gamma'}\mathrm{Sh}(o:B_{\gamma o}(c))\cap\partial X_\theta\,,$$

and we estimate

$$0<\mu_o(L_\Gamma^{rad})=\mu_o(L_\Gamma^{rad}\cap\partial X_\theta)\le\sum_{\gamma\in\Gamma'}\mu_o\big(\mathrm{Sh}(o:B_{\gamma o}(c))\big)\le D(c)\sum_{\gamma\in\Gamma'}e^{-\langle b,H(o,\gamma o)\rangle}.$$

This implies that for any $\varepsilon>0$ the tail of the series

$$\sum_{\substack{\gamma\in\Gamma\\ |\theta(o,\gamma o)-\theta|<\varepsilon}}e^{-\langle b,H(o,\gamma o)\rangle}$$

does not tend to zero. Therefore the sum above diverges, and by Proposition 4.1 (b) there exists $\hat\theta \in [0,\pi/2]$, $|\hat\theta - \theta| \le \varepsilon$ such that

$$\langle b, H_{\hat\theta}\rangle \le \delta_{\hat\theta}(\Gamma)\,.$$

Taking the limit as $\varepsilon \searrow 0$, we conclude $\langle b, H_\theta\rangle \le \delta_\theta(\Gamma)$. □

Recall the definition of the b-Busemann function (14) from Section 3. The following two lemmata hold for any $\theta \in [0,\pi/2]$ and will be important for the proof of Theorem 7.7.

LEMMA 7.5 *Let μ be a (b,θ)-density. If $\tilde\eta \in \partial X_\theta$ is a point mass for μ, and $\Gamma_{\tilde\eta}$ its stabilizer, then for any $\gamma \in \Gamma_{\tilde\eta}$ and $x \in X$ we have*

$$\mathcal{B}^b_{\tilde\eta}(x,\gamma x) = b_1 \mathcal{B}_{\eta_1}(x_1,\gamma_1 x_1) + b_2 \mathcal{B}_{\eta_2}(x_2,\gamma_2 x_2) = 0\,.$$

In particular, if $\gamma, \hat\gamma \in \Gamma$ are representatives of the same coset in $\Gamma/\Gamma_{\tilde\eta}$, then

$$\mathcal{B}^b_{\tilde\eta}(x,\gamma^{-1}x) = \mathcal{B}^b_{\tilde\eta}(x,\hat\gamma^{-1}x)\,.$$

Proof. If $\gamma \in \Gamma_{\tilde\eta}$, then for $x \in X$ we have by Γ-equivariance

$$\mu_x(\tilde\eta) = \mu_x(\gamma^{-1}\tilde\eta) = \mu_{\gamma x}(\tilde\eta)\,.$$

From the assumption that $\tilde\eta$ is a point mass and property (iii) in Definition 1.2 we get

$$1 = \frac{\mu_{\gamma x}(\tilde\eta)}{\mu_x(\tilde\eta)} = e^{\mathcal{B}^b_{\tilde\eta}(x,\gamma x)}\,,$$

hence $\mathcal{B}^b_{\tilde\eta}(x,\gamma x) = 0$ for any $\gamma \in \Gamma_{\tilde\eta}$.

Let $\gamma, \hat\gamma \in \Gamma$ such that $\gamma\Gamma_{\tilde\eta} = \hat\gamma\Gamma_{\tilde\eta} \in \Gamma/\Gamma_{\tilde\eta}$. Then $\hat\gamma^{-1}\gamma \in \Gamma_{\tilde\eta}$ and we obtain from the above, using the cocycle identity for the Busemann functions $\mathcal{B}_{\eta_1}$, $\mathcal{B}_{\eta_2}$,

$$\begin{aligned}\mathcal{B}^b_{\tilde\eta}(x,\gamma^{-1}x) &= \mathcal{B}^b_{\tilde\eta}(x,\gamma^{-1}x) + \mathcal{B}^b_{\tilde\eta}(\gamma^{-1}x,\hat\gamma^{-1}\gamma\gamma^{-1}x)\\ &= \mathcal{B}^b_{\tilde\eta}(x,\hat\gamma^{-1}\gamma\gamma^{-1}x) = \mathcal{B}^b_{\tilde\eta}(x,\hat\gamma^{-1}x)\,.\end{aligned}$$ □

LEMMA 7.6 *If $\tilde\eta \in \partial X_\theta$ is a point mass for a (b,θ)-density μ, then the sum*

$$\sum e^{\mathcal{B}^b_{\tilde\eta}(o,\gamma^{-1}o)}$$

taken over a system of coset representatives of $\Gamma/\Gamma_{\tilde\eta}$ converges.

Proof. If γ and $\hat\gamma$ are representatives of different cosets in $\Gamma/\Gamma_{\tilde\eta}$, then $\gamma\tilde\eta \ne \hat\gamma\tilde\eta$ and so, by Γ-equivariance, the sum $\sum \mu_{\gamma^{-1}o}(\tilde\eta) = \sum \mu_o(\gamma\tilde\eta)$ over a system of coset representatives of $\Gamma/\Gamma_{\tilde\eta}$ is bounded above by $\mu_o(\partial X)$. By property (iii) in Definition 1.2 and the assumption that $\tilde\eta$ is a point mass we conclude that the sum

$$\sum e^{\mathcal{B}^b_{\tilde\eta}(o,\gamma^{-1}o)} = \sum \frac{\mu_{\gamma^{-1}o}(\tilde\eta)}{\mu_o(\tilde\eta)} = \frac{1}{\mu_o(\tilde\eta)}\sum \mu_{\gamma^{-1}o}(\tilde\eta)$$

over a system of coset representatives of Γ/Γ_η is bounded above by $\mu_o(\partial X)/\mu_o(\tilde\eta)$. Since μ_o is a finite measure and $\mu_o(\tilde\eta) > 0$, the above sum converges. □

Theorem 7.7 *If $\delta_\theta(\Gamma) > 0$ then a regular radial limit point $\tilde\eta \in L_\Gamma^{rad} \cap \partial X^{reg}$ is not a point mass for any (b,θ)-density.*

Proof. Let μ be a (b,θ)-density. If $\tilde\eta \notin \partial X_\theta$, then $\tilde\eta \notin \mathrm{supp}(\mu_o)$, hence $\tilde\eta$ cannot be a point mass.

Suppose $\tilde\eta = (\eta_1, \eta_2, \theta) \in L_\Gamma^{rad} \cap \partial X_\theta$ is a point mass for μ. Then by Theorem 7.4 $\langle b, H_\theta\rangle = \delta_\theta(\Gamma) > 0$, hence by continuity of the map $\hat\theta \mapsto \langle b, H_{\hat\theta}\rangle$ there exists $\varepsilon > 0$ such that every $\hat\theta \in [0, \pi/2]$ with $|\hat\theta - \theta| < \varepsilon$ satisfies $\langle b, H_{\hat\theta}\rangle > q > 0$. Moreover, by definition of the radial limit set (32) there exists a constant $c > 0$ and a sequence $(\gamma_n) = \big((\gamma_{n,1}, \gamma_{n,2})\big) \subset \Gamma$ such that $|\theta(o, \gamma_n o) - \theta| < \varepsilon$ and $\tilde\eta \in S(o : B_{\gamma_n o}(c))$ for all $n \in \mathbb{N}$. Corollary 3.9 implies $\mathcal{B}_{\eta_i}(o_i, \gamma_{n,i} o_i) > d_i(o_i, \gamma_{n,i} o_i) - 2c$ for all $n \in \mathbb{N}$ and $i \in \{1,2\}$. We conclude

$$\mathcal{B}^b_{\tilde\eta}(o, \gamma_n o) > \langle b, H(o, \gamma_n o)\rangle - 2\|b\|_1 c \to \infty\,,$$

because $\langle b, H(o, \gamma_n o)\rangle > q \cdot d(o, \gamma_n o)$ and $d_i(o_i, \gamma_{n,i} o_i) \to \infty$ for all $i \in \{1,2\}$ as $n \to \infty$. Passing to a subsequence if necessary, we may therefore assume that $\mathcal{B}^b_{\tilde\eta}(o, \gamma_n o)\rangle$ is strictly increasing to infinity as $n \to \infty$.

Now suppose there exist $l, j \in \mathbb{N}$, $l \neq j$ such that $\gamma_l^{-1}\Gamma_{\tilde\eta} = \gamma_j^{-1}\Gamma_{\tilde\eta}$. Since $\tilde\eta$ is a point mass for μ Lemma 7.5 implies

$$\mathcal{B}^b_{\tilde\eta}(o, \gamma_j o) = \mathcal{B}^b_{\tilde\eta}(o, \gamma_l o)\,,$$

in contradiction to the choice of the subsequence (γ_n). Hence $\gamma_l^{-1}\Gamma_{\tilde\eta} \neq \gamma_j^{-1}\Gamma_{\tilde\eta}$ for all $l \neq j$, and the sum $\sum e^{\mathcal{B}^b_{\tilde\eta}(o,\gamma o)}$ over a system of coset representatives of $\Gamma/\Gamma_{\tilde\eta}$ is bounded below by

$$\sum_{n\in\mathbb{N}} e^{\mathcal{B}^b_{\tilde\eta}(o,\gamma_n o)},$$

and therefore diverges in contradiction to Lemma 7.6. We conclude that $\tilde\eta$ cannot be a point mass for μ_o. □

8 Hausdorff dimension

This final section introduces an appropriate notion of Hausdorff measure and Hausdorff dimension on the geometric boundary ∂X in order to estimate the size of the radial limit set in each Γ-invariant subset $L_\Gamma \cap \partial X_\theta$ of the geometric limit set. Our results are most precise for a class of groups which we call radially cocompact. In this case, the Hausdorff dimension of the radial limit set in a given subset $\partial X_\theta \subseteq \partial X^{reg}$ equals the exponent of growth of slope θ

We will follow the idea of G. Knieper ([Kni97, §4]) for a definition of Hausdorff measure on the geometric boundary. For $\tilde\xi \in \partial X$, $c > 0$ and $0 < r < e^{-c}$ we call the set

$$B^c_r(\tilde\xi) := \big\{\tilde\eta \in \partial X : d(\sigma_{o,\tilde\eta}(-\log r), \sigma_{o,\tilde\xi}(-\log r)) < c\big\}$$

a c-ball of radius r centered at $\tilde\xi$. Using this conformal structure, we define as in the case of metric spaces Hausdorff measure and Hausdorff dimension on the geometric boundary ∂X.

DEFINITION 8.1 *Let E be a Borel subset of ∂X, and*

$$\mathrm{Hd}^\alpha_\varepsilon(E) := \inf\Big\{\sum r_i^\alpha : |E \subseteq \bigcup B^c_{r_j}(\tilde\xi_j)\,,\ r_j < \varepsilon\Big\}\,.$$

The α-dimensional Hausdorff measure *of E is defined by $\mathrm{Hd}^\alpha(E) = \lim_{\varepsilon\to 0}\mathrm{Hd}^\alpha_\varepsilon(E)$, and the* Hausdorff dimension *of E is the number*

$$\dim_{\mathrm{Hd}}(E) = \inf\big\{\alpha \ge 0 \,\big|\, \mathrm{Hd}^\alpha(E) < \infty\big\}\,.$$

Recall the notion of Weyl chambers and Weyl chamber shadows from Section 3. The following lemma gives a relation between Weyl chamber shadows in ∂X_θ and c-balls.

LEMMA 8.2 *Let $c > 0$ and $\theta \in [0,\pi/2]$. We set $A := \max\{\sin\theta/\cos\theta, \cos\theta/\sin\theta\}$ if $\theta \in (0,\pi/2)$ and $A := 0$ otherwise. Let $\varepsilon \in (0,\pi/2)$ arbitrary with $\varepsilon \le \frac12\min\{\theta,\pi/2-\theta\}$ if $\theta\in(0,\pi/2)$, and $\tilde\eta \in \partial X_\theta$. If $y \in \mathcal{C}_{o,\tilde\eta}$ satisfies $d(o,y) > c$ and $|\theta(o,y)-\theta| < \varepsilon$, then with $r := \exp(-d(o,y)(\cos\varepsilon - A\sin\varepsilon))$ we have*

$$\mathrm{Sh}\big(o : B_y(c/2)\big) \cap \partial X_\theta \subseteq B^c_r(\tilde\eta)\,.$$

Proof. Fix $y = (y_1,y_2) \in \mathcal{C}_{o,\tilde\eta}$ with $d(o,y) > c$ and $|\theta(o,y)-\theta| < \varepsilon$, and set $t_i := d_i(o_i,y_i)$ for $i\in\{1,2\}$. We have to show that for $\tilde\zeta \in \mathrm{Sh}\big(o : B_y(c/2)\big)\cap\partial X_\theta$ arbitrary the inequality $d(\sigma_{o,\tilde\eta}(-\log r), \sigma_{o,\tilde\zeta}(-\log r)) < c$ holds. Notice that by definition of the Weyl chamber shadows we have $d(y,\mathcal{C}_{o,\tilde\zeta}) < c/2$.

Assume first that $\theta\in(0,\pi/2)$ and write $\tilde\eta = (\eta_1,\eta_2,\theta)$, $\tilde\zeta = (\zeta_1,\zeta_2,\theta)$. If for $i\in\{1,2\}$ we set $c_i := d_i(y_i,\sigma_{o_i,\zeta_i})$, then $d(y,\mathcal{C}_{o,\tilde\zeta}) < c/2$ implies $\sqrt{c_1^2+c_2^2} < c/2$. Since $y\in\mathcal{C}_{o,\tilde\eta}$ y_1 is a point on the geodesic ray σ_{o_1,η_1} and y_2 is a point on the geodesic ray σ_{o_2,η_2}. Moreover, by elementary geometric estimates we have for $i=1,2$

$$d_i(\sigma_{o_i,\eta_i}(t_i), \sigma_{o_i,\zeta_i}(t_i)) = d_i(y_i, \sigma_{o_i,\zeta_i}(t_i)) \le 2c_i\,.$$

Now $|\theta(o,y)-\theta| < \varepsilon$ and the choice of A imply

$$\begin{aligned}
t_1 &= d(o,y)\cos\theta(o,y) \ge d(o,y)\cos(\theta+\varepsilon) \ge d(o,y)\cos\theta\big(\cos\varepsilon - A\cdot\sin\varepsilon\big)\,,\\
t_2 &= d(o,y)\sin\theta(o,y) \ge d(o,y)\sin(\theta-\varepsilon) \ge d(o,y)\sin\theta\big(\cos\varepsilon - A\cdot\sin\varepsilon\big)\,.
\end{aligned}$$

Notice that $\cos\varepsilon - A\cdot\sin\varepsilon > 0$ because $\varepsilon \le \frac12\min\{\theta,\pi/2-\theta\}$. Using the definition of the constant r and the convexity of the distance function in X_1, X_2 we get

$$\begin{aligned}
d_1(\sigma_{o_1,\eta_1}(-\log r\cdot\cos\theta), \sigma_{o_1,\zeta_1}(-\log r\cdot\cos\theta)) &\le d_1(\sigma_{o_1,\eta_1}(t_1), \sigma_{o_1,\zeta_1}(t_1)) \le 2c_1\,,\\
d_2(\sigma_{o_2,\eta_2}(-\log r\cdot\sin\theta), \sigma_{o_2,\zeta_2}(-\log r\cdot\sin\theta)) &\le d_2(\sigma_{o_2,\eta_1}(t_2), \sigma_{o_2,\zeta_2}(t_2)) \le 2c_2\,.
\end{aligned}$$

Since $\sigma_{o,\tilde\eta}(t) = \big(\sigma_{o_1,\eta_1}(t\cos\theta), \sigma_{o_2,\eta_2}(t\sin\theta)\big)$ and $\sigma_{o,\tilde\zeta}(t) = \big(\sigma_{o_1,\zeta_1}(t\cos\theta), \sigma_{o_2,\zeta_2}(t\sin\theta)\big)$ for all $t > 0$, we conclude

$$d(\sigma_{o,\tilde\eta}(-\log r), \sigma_{o,\tilde\zeta}(-\log r)) \le \sqrt{(2c_1)^2 + (2c_2)^2} = 2\sqrt{c_1^2 + c_2^2} \le c\,.$$

If $\theta = 0$ we write $\tilde\eta = \eta_1$ and $\tilde\zeta = \zeta_1$. By (11) we have $d_1(y_1, \sigma_{o_1,\zeta_1}) < c/2$, hence by the same reasoning as before

$$d_1(\sigma_{o_1,\eta_1}(t_1), \sigma_{o_1,\zeta_1}(t_1)) = d_1(y_1, \sigma_{o_1,\zeta_1}(t_1)) \le c\,.$$

Moreover, $t_1 = d(o,y)\cos\theta(o,y) \ge d(o,y)\cos\varepsilon = d(o,y)\big(\cos\varepsilon - A\cdot\sin\varepsilon\big)$. The conclusion then follows from the convexity of the distance function as above.

The case $\theta = \pi/2$ is analogous; we only have to notice that

$$t_2 = d(o,y)\sin\theta(o,y) \ge d(o,y)\sin(\pi/2-\varepsilon) = d(o,y)\cos\varepsilon = d(o,y)\big(\cos\varepsilon - A\cdot\sin\varepsilon\big).$$

□

The inclusions from the previous lemma allow to give an upper bound for the Hausdorff dimension of the radial limit set.

Theorem 8.3 *If $\theta \in [0,\pi/2]$, then the Hausdorff dimension of the radial limit set in ∂X_θ is bounded above by $\delta_\theta(\Gamma)$.*

Proof. Let $\theta \in [0,\pi/2]$ and fix $c > 0$ sufficiently large. By definition of the radial limit set

$$L_\Gamma^{rad} \cap \partial X_\theta \subseteq \bigcap_{R>c}\bigcap_{\varepsilon>0} \bigcup_{\substack{\gamma\in\Gamma\\ d(o,\gamma o)>R\\ |\theta(o,\gamma o)-\theta|<\varepsilon}} \mathrm{Sh}(o : B_{\gamma o}(c/2))\,.$$

Fix $A \ge 0$ as in Lemma 8.2 and let $\varepsilon \in (0,\pi/2)$ arbitrary, $\varepsilon < \frac{1}{2}\min\{\theta, \pi/2-\theta\}$ if $\theta \in (0,\pi/2)$. Put $\hat\Gamma := \{\gamma\in\Gamma : \mathrm{Sh}(o : B_{\gamma o}(c/2)) \cap L_\Gamma^{rad}\cap\partial X_\theta \ne \emptyset,\ |\theta(o,\gamma o)-\theta| < \varepsilon\}$. For $\gamma\in\hat\Gamma$ we denote $\tilde\xi_\gamma$ a point in $\mathrm{Sh}(o : B_{\gamma o}(c/2)) \cap L_\Gamma^{rad}\cap\partial X_\theta$ and set

$$r_\gamma := \exp\big(-d(o,\gamma o)(\cos\varepsilon - A\sin\varepsilon)\big)\,.$$

Let $\rho < e^{-c}$ and set $\Gamma' := \{\gamma\in\hat\Gamma : r_\gamma < \rho\}$. By the previous lemma we have $\mathrm{Sh}(o : B_{\gamma o}(c/2)) \cap \partial X_\theta \subseteq B^c_{r_\gamma}(\tilde\xi_\gamma)$ for all $\gamma\in\Gamma'$, hence

$$L_\Gamma^{rad}\cap\partial X_\theta \subseteq \bigcup_{\gamma\in\Gamma'} B^c_{r_\gamma}(\tilde\xi_\gamma)\,.$$

Using the definition of Hd^α_ρ we estimate

$$\mathrm{Hd}^\alpha_\rho(L_\Gamma^{rad}\cap\partial X_\theta) \le \sum_{\gamma\in\Gamma'} r_\gamma^\alpha = \sum_{\gamma\in\Gamma'} e^{-\alpha(\cos\varepsilon - A\sin\varepsilon)d(o,\gamma o)}\,.$$

Recall from Section 4 that

$$Q_\theta^{s,\varepsilon}(o,o) := \sum_{\substack{\gamma\in\Gamma \\ |\theta(o,\gamma o)-\theta|<\varepsilon}} e^{-sd(o,\gamma o)}$$

converges for $s > \delta_\theta^\varepsilon(o,o)$. Hence if $\hat{s} := \alpha(\cos\varepsilon - A\sin\varepsilon) > \delta_\theta^\varepsilon(o,o)$, we have

$$\mathrm{Hd}_\rho^\alpha(L_\Gamma^{rad}\cap\partial X_\theta) \le Q_\theta^{\hat{s},\varepsilon}(o,o) < \infty\,.$$

This shows that for $\alpha > \delta_\theta^\varepsilon(o,o)/(\cos\varepsilon - A\sin\varepsilon)$, $\mathrm{Hd}_\rho^\alpha(L_\Gamma^{rad}\cap\partial X_\theta)$ is finite. Taking the limit as $\varepsilon\searrow 0$, we conclude that the same is true for $\alpha > \delta_\theta(\Gamma)$. Letting $\rho\searrow 0$, we obtain $\mathrm{Hd}^\alpha(L_\Gamma^{rad}\cap\partial X_\theta) < \infty$ if $\alpha > \delta_\theta(\Gamma)$, hence $\dim_{\mathrm{Hd}}(L_\Gamma^{rad}\cap\partial X_\theta) \le \delta_\theta(\Gamma)$. □

Notice that in the previous proof we only used the definition of $\delta_\theta(\Gamma)$ and not the existence of a (b,θ)-density. In particular, the claim also holds for slopes $\theta\in[0,\pi/2]$ for which $\delta_\theta(\Gamma) = 0$.

The notion of convex cocompact and geometrically finite groups plays an important role in the theory of Kleinian groups. For these groups the Hausdorff dimension of the limit set is equal to the critical exponent. We suggest here the following definition to replace convex cocompactness in a more general setting.

Definition 8.4 *If X is a locally compact Hadamard space, then a discrete group $\Gamma\subset$ Is(X) is called* radially cocompact *if there exists a constant $c_\Gamma > 0$ such that for any $\tilde\eta\in L_\Gamma^{rad}$ and for all $t>0$ there exists an element $\gamma\in\Gamma$ with*

$$d\big(\gamma o, \sigma_{o,\tilde\eta}(t)\big) < c_\Gamma\,.$$

The most familiar radially cocompact groups are convex cocompact isometry groups of rank one symmetric spaces and uniform lattices of higher rank symmetric spaces or Euclidean buildings. A further example is given by products of convex cocompact groups acting on the Riemannian product of two Hadamard spaces with pinched negative curvature.

For radially cocompact discrete groups $\Gamma\subset\mathrm{Is}(X_1)\times\mathrm{Is}(X_2)$, the existence of a (b,θ)-density μ together with Theorem 7.2 allows to obtain a lower bound for the Hausdorff dimension of the radial limit set in ∂X_θ. From here on, we fix $c > 2\max\{c_\Gamma, c_0\}$ with c_Γ as in Definition 8.4 and c_0 as in Theorem 7.2.

Theorem 8.5 *Let $\Gamma\subset$ Is(X) be radially cocompact, $\theta\in[0,\pi/2]$, and μ a (b,θ)-density. Then there exists a constant $C_0>0$ such that for any Borel subset $E\subseteq L_\Gamma^{rad}$*

$$\mathrm{Hd}^{\langle b,H_\theta\rangle}(E) \ge C_0\cdot\mu_o(E)\,.$$

Proof. Set $\alpha := \langle b, H_\theta\rangle$. Since $\mathrm{Hd}^\alpha(E)\ge\mathrm{Hd}^\alpha(E\cap\partial X_\theta)$ and $\mu_o(E) = \mu_o(E\cap\partial X_\theta)$, it suffices to prove the assertion for $E\subseteq L_\Gamma^{rad}\cap\partial X_\theta$. Let $\rho>0$, $q>0$ arbitrary, and choose a cover of E by balls $B_{r_n}^c(\tilde\eta_n)$, $r_n<\rho$, such that

$$\mathrm{Hd}_\rho^\alpha(E) \ge \sum_{n\in\mathbb{N}} r_n^\alpha - q\,.$$

If $B^c_{r_n}(\tilde\eta_n)\cap E=\emptyset$, we do not need $B^c_{r_n}(\tilde\eta_n)$ to cover $E\subseteq L_\Gamma^{rad}\cap\partial X_\theta$, otherwise we choose $\tilde\xi_n\in B^c_{r_n}(\tilde\eta_n)\cap E$. Since Γ is radially cocompact, there exists $\gamma_n=(\gamma_{n,1},\gamma_{n,2})\in\Gamma$ such that

$$d(\gamma_n o,\sigma_{o,\tilde\xi_n}(-\log r_n))\le c\,. \tag{33}$$

If $\theta\in(0,\pi/2)$ we write $\tilde\xi_n=(\xi_{n,1},\xi_{n,2},\theta)$, if $\theta=0$ we set $\xi_{n,1}=\tilde\xi_n\in\partial X_1$ and choose $\xi_{n,2}\in\partial X_2$ arbitrary, if $\theta=\pi/2$ we let $\xi_{n,1}\in\partial X_1$ arbitrary and set $\xi_{n,2}=\tilde\xi_n\in\partial X_2$. With this notation inequality (33) implies

$$d_1(\gamma_{n,1}o_1,\sigma_{o_1,\xi_{n,1}}(-\log r_n\cdot\cos\theta))\le c\quad\text{and}\qquad d_2(\gamma_{n,2}o_2,\sigma_{o_2,\xi_{n,2}}(-\log r_n\cdot\sin\theta))\le c\,,$$

so using the triangle inequalities we obtain

$$\begin{array}{rcll}\big|d_1(o_1,\gamma_{n,1}o_1)-d_1\big(o_1,\sigma_{o_1,\xi_{n,1}}(-\log r_n\cdot\cos\theta)\big)\big| &\le& c & \text{and}\\ \big|d_2(o_2,\gamma_{n,2}o_2)-d_2\big(o_2,\sigma_{o_2,\xi_{n,2}}(-\log r_n\cdot\sin\theta)\big)\big| &\le& c\,. &\end{array}$$

We therefore estimate

$$\begin{array}{rcl}\langle b,H(o,\gamma_n o)\rangle &\ge& -|b_1|c+b_1d_1\big(o_1,\sigma_{o_1,\xi_{n,1}}(-\log r_n\cdot\cos\theta)\big)-|b_2|c\\ && +b_2d_2\big(o_2,\sigma_{o_2,\xi_{n,2}}(-\log r_n\cdot\sin\theta)\big)=-\langle b,H_\theta\rangle\log r_n-\|b\|_1c,\end{array}$$

hence

$$-\langle b,H(o,\gamma_n o)\rangle\le\langle b,H_\theta\rangle\log r_n+c\|b\|_1\,.$$

Furthermore, we have $B^c_{r_n}(\tilde\eta_n)\subseteq\mathrm{Sh}(o:B_{\gamma_n o}(3c))$, hence $E\subseteq\bigcup_{n\in\mathbb N}\mathrm{Sh}(o:B_{\gamma_n o}(3c))$. We conclude

$$\begin{aligned}\mu_o(E)&\le\mu_o\Big(\bigcup_{n\in\mathbb N}\mathrm{Sh}\big(o:B_{\gamma_n o}(3c)\big)\Big)\le\sum_{n\in\mathbb N}\mu_o\big(\mathrm{Sh}(o:B_{\gamma_n o}(3c))\big)\\ &\le D(3c)\sum_{n\in\mathbb N}e^{-\langle b,H(o,\gamma_n o)\rangle}\le D(3c)\sum_{n\in\mathbb N}e^{\alpha\log r_n+c\|b\|_1}\\ &\le D(3c)e^{c\|b\|_1}\sum_{n\in\mathbb N}r_n^\alpha\le D(3c)e^{c\|b\|_1}\big(\mathrm{Hd}^\alpha_\rho(E)+q\big)\,.\end{aligned}$$

The claim now follows as $q\searrow 0$ and $\rho\searrow 0$. □

Theorem 8.6 *Let $\Gamma\subset Is(X_1)\times Is(X_2)$ be radially cocompact, and $\theta\in(0,\pi/2)$ such that $\delta_\theta(\Gamma)>0$. Then*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad}\cap\partial X_\theta)=\delta_\theta(\Gamma)\,.$$

Proof. From Section 6 we know that there exists a (b,θ)-density μ for appropriate parameters $b=(b_1,b_2)$. So the previous theorem implies that for $\alpha:=\langle b,H_\theta\rangle$

$$\mathrm{Hd}^\alpha(L_\Gamma^{rad}\cap\partial X_\theta)\ge C_0\mu_o(L_\Gamma^{rad})\ge 0,$$

hence

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad}\cap\partial X_\theta)\ge\alpha=\langle b,H_\theta\rangle\ge\delta_\theta(\Gamma)$$

by Theorem 7.3. The assertion now follows directly from Theorem 8.3. □

For Example 1 described in Section 4 we deduce the following

Corollary 8.7 *Let $X = X_1 \times X_2$ be a product of two Hadamard manifolds of pinched negative curvature, $\Gamma_1 \subset Is(X_1)$, $\Gamma_2 \subset Is(X_2)$ convex cocompact groups with critical exponents δ_1, δ_2, and $\Gamma = \Gamma_1 \times \Gamma_2 \subset Is(X)$. Then for any $\theta \in (0, \pi/2)$ we have*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap \partial X_\theta) = \delta_1 \cos\theta + \delta_2 \sin\theta \,.$$

References

[Alb99] P. Albuquerque, *Patterson-Sullivan theory in higher rank symmetric spaces*, Geom. Funct. Anal. **9** (1999), no. 1, 1–28.

[Bal82] Werner Ballmann, *Axial isometries of manifolds of nonpositive curvature*, Math. Ann. **259** (1982), no. 1, 131–144.

[Bal95] Werner Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995, With an appendix by Misha Brin.

[BB95] Werner Ballmann and Michael Brin, *Orbihedra of nonpositive curvature*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 169–209 (1996).

[BGS85] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[Bur93] Marc Burger, *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank* 2, Internat. Math. Res. Notices (1993), no. 7, 217–225.

[CF10] Pierre-Emmanuel Caprace and Koji Fujiwara, *Rank-one isometries of buildings and quasi-morphisms of Kac-Moody groups*, Geom. Funct. Anal. **19** (2010), no. 5, 1296–1319.

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009), no. 1, 169–221.

[DK] Françoise Dal'Bo and Inkang Kim, *Shadow lemma on the product of Hadamard manifolds and applications*, Séminaire de Théorie Spectrale et Géométrie. Vol. 25. Année 2006–2007, Univ. Grenoble I, pp. 105–119.

[Kel55] John L. Kelley, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[Kni97] G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, Geom. Funct. Anal. **7** (1997), no. 4, 755–782.

[Lin04] Gabriele Link, *Hausdorff dimension of limit sets of discrete subgroups of higher rank Lie groups*, Geom. Funct. Anal. **14** (2004), no. 2, 400–432.

[Lin07] Gabriele Link, *Asymptotic geometry and growth of conjugacy classes of non-positively curved manifolds*, Ann. Global Anal. Geom. **31** (2007), no. 1, 37–57.

[Lin10] Gabriele Link, *Asymptotic geometry in products of Hadamard spaces with rank one isometries*, Geom. Topol. **14** (2010), no. 2, 1063–1094.

[Pat76] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), no. 3-4, 241–273.

[Qui02] J.-F. Quint, *Mesures de Patterson-Sullivan en rang supérieur*, Geom. Funct. Anal. **12** (2002), no. 4, 776–809.

[Rém99] Bertrand Rémy, *Construction de réseaux en théorie de Kac-Moody*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 6, 475–478.

[Sul79] Dennis Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202.

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