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BUNDLE GERBES AND MODULI SPACES

PETER BOUWKNEGT, VARGHESE MATHAI, AND SIYE WU

ABSTRACT. In this paper, we construct the *index bundle gerbe* of a family of self-adjoint Dirac-type operators, refining a construction of Segal. In a special case, we construct a geometric bundle gerbe called the *caloron bundle gerbe*, which comes with a natural connection and curving, and show that it is isomorphic to the analytically constructed index bundle gerbe. We apply these constructions to certain moduli spaces associated to compact Riemann surfaces, constructing on these moduli spaces, natural bundle gerbes with connection and curving, whose 3-curvature represent Dixmier-Douady classes that are generators of the third de Rham cohomology groups of these moduli spaces.

1. INTRODUCTION

Given a compact, simply-connected simple Lie group G, let $\mathcal{M}$ denote the moduli space of flat principal G-bundles on a compact Riemann surface Σ of genus greater than one. More precisely, we only consider the smooth, dense, open subset of $\mathcal{M}$ corresponding to irreducible homomorphisms from the fundamental group of Σ to G, which we will denote by the same symbol. Quantization of $\mathcal{M}$ was considered by Witten [36, 4] via the *determinant line bundle* $\mathcal{L}$ of the index bundle of the associated family of Cauchy-Riemann operators $\{\bar{\partial}_\rho : \rho \in \mathrm{Hom}(\pi_1(\Sigma), G)\}$. The fibre of $\mathcal{L}$ at ρ is

$$\mathcal{L}_\rho = \bigwedge^{\max} \ker(\bar{\partial}_\rho^*) \otimes \bigwedge^{\max} \mathrm{coker}(\bar{\partial}_\rho).$$

It carries the Quillen metric [32] and the curvature of the canonical hermitian connection coincides with the natural Kähler form on the moduli space $\mathcal{M}$. Moreover, the first Chern class of $\mathcal{L}$ generates the second de Rham cohomology group of $\mathcal{M}$. The moduli space $\mathcal{M}$ plays an important role in quantum field theory. In particular, the space of holomorphic sections of $\mathcal{L}$ can be naturally identified with what physicists call the space of *conformal blocks* in

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a standard conformal field theory, the WZW-model [7], on which there exists an extensive literature.

For a family of self-adjoint Dirac-type operators, an *index gerbe* was introduced by Carey-Mickelsson-Murray [12, 13] and by Lott [23]. The gerbe was described by local data, which can be viewed as the analogues of the transition functions of the determinant line bundle. The index gerbe is related to Hamiltonian anomalies. In [34], Segal constructs a projective Hilbert bundle; its Dixmier-Douady invariant is the obstruction to lifting it to a Hilbert bundle. In this paper, we construct the *index bundle gerbe* for such a family, which is a global version of prior constructions. Our construction of the index bundle gerbe is inspired by these papers and by the work on determinant line bundles [9, 17, 32]. Melrose-Rochon [25] have a completely different construction of an index bundle gerbe using pseudodifferential operators. In the special case of a circle fibration, we also construct the *caloron bundle gerbe* using geometric data and show that it is isomorphic to the analytically defined index bundle gerbe. We apply these constructions to the moduli space $\mathcal{M}$ and obtain on it natural bundle gerbes with connections and curvings. The 3-curvatures represent Dixmier-Douady classes that are generators of the third de Rham cohomology group of $\mathcal{M}$.

Applications of moduli spaces of Riemann surfaces are ubiquitous. The construction of explicit geometric realizations of the degree 3 classes on moduli spaces through bundle gerbes was motivated by the desire to generalize the geometric Langlands correspondence [18, 19, 21] by including background fluxes.

Here we outline the contents of the paper. In Section 2, we construct the *index bundle gerbe* associated to a set of geometric data using the spectrum of a family of Dirac-type operators. In Section 3, given a unitary representation of the structure group, we construct the *caloron bundle gerbe* using the projective Hilbert space bundle obtained from the caloron correspondence of Murray-Stevenson [29]. In Section 4, we establish an isomorphism (not just a stable isomorphism) between the geometrically constructed caloron bundle gerbe and the analytically defined index bundle gerbe for a circle fibration. Section 5 contains the application of these constructions to the moduli space $\mathcal{M}$, obtaining natural bundle gerbes with connections and curvings, whose Dixmier-Douady classes generate the third de Rham cohomology group of $\mathcal{M}$. We end with a section of conclusions, where we outline future work on constructing bundle gerbes on other moduli spaces such as the moduli space of anti-self-dual connections, on a compact four dimensional Riemannian manifold.

2. Index bundle gerbe

In this section we will construct an index bundle gerbe associated to the following set of geometric data. For an introduction to bundle gerbes, see [27].

Basic setup:
Let $Z \to X$ be a smooth fibre bundle whose typical fibre is a compact odd-dimensional manifold. Let $T(Z/X) \to Z$ denote the vertical tangent bundle, which is a sub-bundle of the tangent bundle TZ. We assume that there is a Riemannian metric on $T(Z/X)$. Suppose that $T(Z/X)$ has a spin structure and let $S \to Z$ denote the corresponding bundle of spinors. Let $E \to Z$ be a hermitian vector bundle with connection ∇^E.

In this case, there is a smooth family $\{D_x^E : x \in X\}$ of self-adjoint Dirac-type operators $D_x^E \colon \Gamma(Z_x, S \otimes E) \to \Gamma(Z_x, S \otimes E)$ acting on sections over the fibres Z_x $(x \in X)$ of $Z \to X$. The L^2-completions $\mathcal{H}_x$ of $\Gamma(Z_x, S \otimes E)$ form a Hilbert bundle $\mathcal{H}$ over X. We denote by the same symbol D_x^E the operator on $\mathcal{H}_x$. Recall that for any $x \in X$, $\mathrm{spec}(D_x^E) \subset \mathbb{R}$ is a closed countable set with accumulation point only at infinity. For $\lambda \in \mathbb{Q}$,[1] consider the open subset

$$U_\lambda = \{x \in X \big| \lambda \notin \mathrm{spec}(D_x^E)\} \tag{2.1}$$

of X. For $x \in U_\lambda$, let the Hilbert space $\mathcal{H}^+_{\lambda,x}$ ($\mathcal{H}^-_{\lambda,x}$, respectively) be the span of eigenspaces of D_x^E with eigenvalues greater than (less than, respectively) λ. They form Hilbert bundles $\mathcal{H}^\pm_\lambda$ over U_λ.

Suppose $\mu > \lambda$ and $\lambda, \mu \in \mathbb{Q}$. Observe that for $x \in U_\lambda \cap U_\mu$, we have

$$\mathcal{H}^+_{\lambda,x} = \mathcal{H}^+_{\mu,x} \oplus (\mathcal{H}^+_{\lambda,x} \cap \mathcal{H}^-_{\mu,x}), \quad \mathcal{H}^-_{\mu,x} = \mathcal{H}^-_{\lambda,x} \oplus (\mathcal{H}^+_{\lambda,x} \cap \mathcal{H}^-_{\mu,x}),$$

where $\mathcal{H}^+_{\lambda,x} \cap \mathcal{H}^-_{\mu,x}$ is a finite dimensional space. Let $\mathcal{L}_{\lambda\mu} = \det(\mathcal{H}^+_\lambda \cap \mathcal{H}^-_\mu)$, which is a line bundle over $U_\lambda \cap U_\mu$. Since

$$(\mathcal{H}^+_\lambda \cap \mathcal{H}^-_\mu) \oplus (\mathcal{H}^+_\mu \cap \mathcal{H}^-_\tau) = \mathcal{H}^+_\lambda \cap \mathcal{H}^-_\tau$$

on $U_\lambda \cap U_\mu \cap U_\mu$ for any $\lambda < \mu < \tau$ in $\mathbb{Q}$, we have a cocycle condition

$$\mathcal{L}_{\lambda\mu} \otimes \mathcal{L}_{\mu\tau} \cong \mathcal{L}_{\lambda\tau}. \tag{2.2}$$

The collection $\{\mathcal{L}_{\lambda\mu}\}$ defines the index gerbe [12, 13, 23] over X.

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[1] We restrict to $\lambda \in \mathbb{Q}$ to have a countable open cover of X.

Over U_λ, we construct the Fock bundle $\mathcal{F}_\lambda = \bigwedge(\mathcal{H}_\lambda^+ \oplus \overline{\mathcal{H}_\lambda^-})$. Hodge duality asserts that for any finite dimensional complex vector space V there is a canonical isomorphism

$$\bigwedge \overline{V} \otimes \det V \cong \bigwedge V.$$

Thus on the overlap $U_\lambda \cap U_\mu$, we get

$$\begin{aligned}
\mathcal{F}_\lambda &= \bigwedge \mathcal{H}_\lambda^+ \otimes \bigwedge \overline{\mathcal{H}_\lambda^-} \\
&\cong \bigwedge \mathcal{H}_\mu^+ \otimes \bigwedge (\mathcal{H}_\lambda^+ \cap \mathcal{H}_\mu^-) \otimes \bigwedge \overline{\mathcal{H}_\lambda^-} \\
&\cong \bigwedge \mathcal{H}_\mu^+ \otimes \bigwedge \overline{\mathcal{H}_\lambda^-} \otimes \bigwedge \overline{(\mathcal{H}_\lambda^+ \cap \mathcal{H}_\mu^-)} \otimes \mathcal{L}_{\lambda\mu} \\
&\cong \bigwedge \mathcal{H}_\mu^+ \otimes \bigwedge \overline{\mathcal{H}_\mu^-} \otimes \mathcal{L}_{\lambda\mu} \\
&= \mathcal{F}_\mu \otimes \mathcal{L}_{\lambda\mu}.
\end{aligned}$$

Therefore there is a canonical isomorphism of the projectivizations $\mathbb{P}(\mathcal{F}_\lambda) \cong \mathbb{P}(\mathcal{F}_\mu)$ on the overlap $U_\lambda \cap U_\mu$ and hence a well defined projectivized Fock bundle $\pi\colon \mathbb{P}(\mathcal{F}) \to X$.

To construct the index bundle gerbe, we now define a line bundle $\mathcal{L} \to \mathbb{P}(\mathcal{F})^{[2]}$ on the fibre product of $\mathbb{P}(\mathcal{F})$ with itself. Let $\mathcal{L}_\lambda$ be the universal line bundle over $\mathbb{P}(\mathcal{F}_\lambda) = \pi^{-1}(U_\lambda)$. That is, over any $\ell \in \mathbb{P}(\mathcal{F}_\lambda)$, the fibre is

$$(\mathcal{L}_\lambda)_\ell = \{(\ell, v) : v \in \ell\} \subset \{\ell\} \times (\mathcal{F}_\lambda)_{\pi(\ell)}.$$

$\mathcal{L}_\lambda$ is a hermitian line bundle, but does not glue properly on the overlaps in $\mathbb{P}(\mathcal{F})$ for the same reason that $\mathcal{F}_\lambda$ doesn't on X. In fact, on $\pi^{-1}(U_\lambda) \cap \pi^{-1}(U_\mu) = \pi^{-1}(U_\lambda \cap U_\mu)$, we have

$$\mathcal{L}_\lambda \cong \mathcal{L}_\mu \otimes \pi^* \mathcal{L}_{\lambda\mu}. \tag{2.3}$$

Now, $\delta(\mathcal{L}_\lambda) = \mathcal{L}_\lambda^* \boxtimes \mathcal{L}_\lambda$ is a line bundle on $\mathbb{P}(\mathcal{F}_\lambda) \times \mathbb{P}(\mathcal{F}_\lambda)$ which restricts to the total space of the bundle $\pi_\lambda^{[2]}\colon \mathbb{P}(\mathcal{F}_\lambda)^{[2]} \to U_\lambda$. However, this time, $\delta(\mathcal{L}_\lambda) = \delta(\mathcal{L}_\mu)$ on the overlap $(\pi_\lambda^{[2]})^{-1}(U_\lambda \cap U_\mu)$, since the fudge factor for each cancels, thereby defining a global hermitian line bundle $\mathcal{L} \to \mathbb{P}(\mathcal{F})^{[2]}$. There is a line bundle $\delta(\mathcal{L}) \to \mathbb{P}(\mathcal{F})^{[3]}$ defined by

$$\delta(\mathcal{L}) = \pi_{12}^* \mathcal{L} \otimes (\pi_{13}^* \mathcal{L})^{-1} \otimes \pi_{23}^* \mathcal{L},$$

where $\pi_{ij}\colon \mathbb{P}(\mathcal{F})^{[3]} \to \mathbb{P}(\mathcal{F})^{[2]}$ for $1 \le i < j \le 3$ is the projection onto the ith and jth factors. It is trivial precisely because of (2.2).

Therefore we have established the following.

Theorem 2.1. *Given the* **Basic setup***, there is a well-defined global fibre bundle of projective Hilbert spaces* $\pi\colon \mathbb{P}(\mathcal{F}) \to X$ *and a line bundle* $\mathcal{L} \to \mathbb{P}(\mathcal{F})^{[2]}$ *such that* $\delta(\mathcal{L}) \to \mathbb{P}(\mathcal{F})^{[3]}$ *is trivial.*

Thus we have a bundle gerbe over X, which we call the *index bundle gerbe*. Note that the line bundle $\mathcal{L}_\lambda \to \mathbb{P}(\mathcal{F}_\lambda)$ provides a local trivialization of the index bundle gerbe over $U_\lambda \subset X$.

We remark that the considerations in this section can straightforwardly be generalized to more general open coverings $\{U_\alpha\}$ of X so that for each U_α there exists a spectral cut $\lambda_\alpha \notin \operatorname{spec}(D^E_x)$, for all $x \in U_\alpha$. We restrict our discussion to the cover (2.1) for notational simplicity. Finally, we remark that the construction in this section works for a family of general elliptic self-adjoint operators on a compact manifold.

3. The caloron bundle gerbe

We begin by reviewing the caloron correspondence of Murray-Stevenson [29]. We then study the behavior of this construction under a unitary representation of the structure group. Finally, we obtain a fibre bundle whose typical fibre is a projective Hilbert space; we will call the associated bundle gerbe the caloron bundle gerbe.

3.1. The caloron correspondence.

Let X be a manifold and let S^1 denote the unit circle. Let $P \to S^1 \times X$ be a principal G-bundle, where G is a connected and simply-connected compact Lie group. Then there exists a canonical identification between equivalence classes of principal G-bundles over $S^1 \times X$ and principal LG-bundles over X,

$$\begin{pmatrix} G \longrightarrow & P \\ & \big\downarrow \\ & S^1 \times X \end{pmatrix} \Leftrightarrow \begin{pmatrix} LG \longrightarrow & Q \\ & \big\downarrow \\ & X \end{pmatrix}, \tag{3.1}$$

as follows. Given P, the bundle Q is the push-forward of P under the projection map $S^1 \times X \to X$. That is, the fibre of Q over $x \in X$ is $Q_x = \Gamma(S^1 \times \{x\}, P)$. Clearly, the free loop group LG acts freely and transitively on Q_x and hence Q is a principal LG-bundle. The total space Q can be constructed globally as follows. Consider the LG-bundle $LP \to L(S^1 \times X)$. Then $Q \to X$ is the pullback of LP by the map $\eta\colon X \to L(S^1 \times X)$ given by $\eta(x) = (\theta \mapsto (\theta, x))$. Conversely, given an LG-bundle $Q \to X$, we can define a principal G-bundle P over $S^1 \times X$ by $P = (Q \times G \times S^1)/LG$, where the right LG-action on $Q \times G \times S^1$ is given by $(p, g, \theta)\gamma = (p\gamma, \gamma(\theta)^{-1}g, \theta)$ and the G-action is by right multiplication in the second factor. Clearly these actions commute, establishing the canonical equivalence in (3.1). For a generalization of the caloron correspondence to principal G-bundles over non-trivial circle bundles or over more general fibrations, see [8, 11, 30, 20].

Given a connection $\tilde{A}$ on P, we define a connection $A = \eta^* L\tilde{A}$ on Q by pulling back the connection $L\tilde{A}$ on LP. In addition, $\tilde{A}$ determines a Higgs field $\Phi = \eta^* \iota_{\Xi}(L\tilde{A})$, where Ξ is a canonical vector field on LP generating the translation on S^1. The map $\Phi\colon Q \to L\mathfrak{g}$ is smooth and satisfies

$$\Phi(p\gamma) = \mathrm{ad}(\gamma^{-1})\,\Phi(p) + \gamma^{-1}\partial_\theta \gamma \tag{3.2}$$

for $\gamma \in LG$. Conversely, given a connection A on Q and a Higgs field Φ for Q satisfying Eqn. (3.2), we have a 1-form

$$\tilde{A} = \mathrm{ad}(g^{-1})A(\theta) + \Theta + \mathrm{ad}(g^{-1})\,\Phi\, d\theta$$

on $Q \times G \times S^1$ which descends to a connection 1-form on P. Here Θ denotes the Cartan-Maurer 1-form on G. We summerize the above in the following

Proposition 3.1 (caloron correspondence [29])**.** *The canonical equivalence in* (3.1) *determines a bijection between isomorphism classes of G-bundles with a connection over $S^1 \times X$ and isomorphism classes of LG-bundles with a connection and a Higgs field over X.*

Recall that for a simple group G, there is a basic central extension (cf. [31])

$$1 \longrightarrow U(1) \longrightarrow \widehat{LG} \xrightarrow{\;p\;} LG \longrightarrow 1$$

corresponding to a cocycle that generates the second cohomology of LG. Denote by $\xi\colon Q^{[2]} \to LG$ the map given by $(q_1, q_2) \mapsto \gamma$, where $\gamma \in LG$ is the unique element such that $q_2 = q_1\gamma$. Let $\mathcal{P}^Q \to Q^{[2]}$ be the pullback of the $U(1)$-bundle $\widehat{LG} \to LG$ via ξ and let $\mathcal{L}^Q \to Q^{[2]}$ be the line bundle associated to $\mathcal{P}^Q$. Equivalently, if $\mathcal{L}^{LG} \to LG$ is the line bundle associated to $\widehat{LG}$, then $\mathcal{L}^Q = \xi^*(\mathcal{L}^{LG})$. Since ξ is a homomorphism of groupoids, the line bundle $\mathcal{L}^Q$ determines a bundle gerbe called the *lifting bundle gerbe* of Q.

Theorem 5.1 of [29] asserts that a connection A on Q and a Higgs field for Q determine a curving (or B-field)

$$B = \frac{\sqrt{-1}}{2\pi} \int_{S^1} \left(\frac{1}{2}\langle A, \partial_\theta A\rangle - \langle F_A, \nabla\Phi\rangle \right) d\theta,$$

where F_A is the curvature of the connection A and $\nabla\Phi = d\Phi + [A, \Phi] - \partial_\theta A$. Here and below, the inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ is chosen such that the length of a long root is $\sqrt{2}$. It induces an inner product (with the same notation) on $L\mathfrak{g}$. The 3-curvature

$H = dB \in \Omega^3(X)$, associated to B, represents the Dixmier-Douady class of the bundle gerbe $(Q, \mathcal{L}^Q)$ in de Rham cohomology. It pulls back to $-\frac{\sqrt{-1}}{4\pi^2}\int_{S^1}\langle F_A, \nabla\Phi\rangle\, d\theta$ on Q. Let $F_{\tilde{A}}$ denote the curvature of the connection $\tilde{A}$ on P. Then the first Pontryagin form of P is $\frac{1}{8\pi^2}\langle F_{\tilde{A}}, F_{\tilde{A}}\rangle$, and Theorem 6.1 in [29] asserts that

$$-\frac{1}{8\pi^2}\int_{S^1}\langle F_{\tilde{A}}, F_{\tilde{A}}\rangle = H. \tag{3.3}$$

If $\rho\colon G \to SU(V)$ is a unitary representation of G on a finite dimensional vector space V, we can study the behavior of the caloron correspondence under ρ. First, we have a principal $SU(V)$-bundle $P^\rho = P \times_G SU(V) \to S^1 \times X$; let $r_P\colon P \to P^\rho$ be the map induced by ρ that changes the structure group. Correspondingly, there is a principal $LSU(V)$-bundle $Q^\rho = \eta^*(LP^\rho) = Q \times_{LG} LSU(V)$ over X. Let $r_Q\colon Q \to Q^\rho$ be the map induced by $L\rho\colon LG \to LSU(V)$. Using the map $\xi_\rho\colon (Q^\rho)^{[2]} \to LSU(V)$, we have a line bundle $\mathcal{L}^\rho = \xi_\rho^*(\mathcal{L}^{LSU(V)})$ and hence a bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$. We recall that the line bundles $\mathcal{L}^{LG} \to LG$ and $\mathcal{L}^{LSU(V)} \to LSU(V)$ satisfy the relation $(L\rho)^*(\mathcal{L}^{LSU(V)}) \cong (\mathcal{L}^{LG})^{\otimes \iota_\rho}$, where ι_ρ is the Dynkin index of the representation ρ [5]. Using the commutative diagram

$$\begin{array}{ccc} Q^{[2]} & \xrightarrow{\ \xi\ } & LG \\ {\scriptstyle r_Q^{[2]}}\downarrow & & \downarrow{\scriptstyle L\rho} \\ (Q^\rho)^{[2]} & \xrightarrow{\ \xi_\rho\ } & LSU(V), \end{array}$$

we get $(r_Q^{[2]})^*(\mathcal{L}^\rho) \cong (\mathcal{L}^Q)^{\otimes \iota_\rho}$. Therefore, the bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$ on X is the ι_ρ-th power of $(Q, \mathcal{L}^Q)$.

Given a connection $\tilde{A}$ of $P \to S^1 \times X$, the induced connection $\tilde{A}_\rho$ on P^ρ satisfies $r_P^*(\tilde{A}_\rho) = \dot{\rho}(\tilde{A})$, where $\dot{\rho}\colon \mathfrak{g} \to \mathfrak{su}(V)$ is the representation of the Lie algebra $\mathfrak{g}$. The connection on Q^ρ is $A_\rho = \eta^*(L\tilde{A}_\rho)$; it satisfies $r_Q^*(A_\rho) = L\dot{\rho}(A)$, and therefore we have $r_Q^*(F_{A_\rho}) = L\dot{\rho}(F_A)$ for the corresponding curvatures. Similarly, the Higgs field $\Phi_\rho = \eta^*\iota_\Xi(L\tilde{A}_\rho)\colon Q^\rho \to L\mathfrak{su}(V)$ satisfies $r_Q^*(\Phi_\rho) = L\dot{\rho}(\Phi)$. Since the pull-back by $\dot{\rho}$ of the primitive bilinear form on $\mathfrak{su}(V)$ is ι_ρ times the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, the curving (B-field) B_ρ on Q^ρ is related to B on Q by $r_Q^*(B_\rho) = \iota_\rho B$. Therefore, the 3-curvature H_ρ on X associated to B_ρ is $H_\rho = \iota_\rho H$. This is compatible with an earlier result that the bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$ is the ι_ρ-th power $(Q, \mathcal{L}^Q)$.

We summarize the results in the following

Proposition 3.2. *Given a representation $\rho\colon G \to SU(V)$ in the above setting, the induced bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$ on X is the ι_ρ-th power of $(Q, \mathcal{L}^Q)$. Furthermore, its B-field and the*

3-*curvature are, respectively,*

$$r_Q^* B_\rho = \iota_\rho B, \qquad H_\rho = \iota_\rho H.$$

3.2. **Projective Hilbert bundle and the caloron bundle gerbe.** As before, G is a simple, connected and simply-connected compact Lie group and $\rho\colon G \to SU(V)$ is a unitary representation of G. Let $\mathbf{H} = L^2(S^1, V)$. Then there is an induced unitary representation $\tilde{\rho}\colon LG \to U(\mathbf{H})$ of LG defined by $\tilde{\rho}(\gamma)(\theta) = \rho(\gamma(\theta))$, where $\gamma \in LG$ and $\theta \in S^1$. For any $\lambda \in \mathbb{R}$, we have a decomposition $\mathbf{H} = \mathbf{H}_\lambda^+ \oplus \mathbf{H}_\lambda^-$, where $\mathbf{H}_\lambda^+$ ($\mathbf{H}_\lambda^-$, respectively) denotes the Hilbert space span of the Fourier modes not less than (less than, respectively) λ. Actually, we have $\tilde{\rho}\colon LG \to U_{\mathrm{res}}(\mathbf{H})$, where the *restricted unitary group* is

$$U_{\mathrm{res}}(\mathbf{H}) = \{S \in U(\mathbf{H}) : [S, I_\lambda] \text{ is a Hilbert-Schmidt operator}\}.$$

Here I_λ is an involution on $\mathbf{H}$ such that $I_\lambda = \mathrm{id}$ on $\mathbf{H}_\lambda^+$ and $I_\lambda = -\mathrm{id}$ on $\mathbf{H}_\lambda^-$ (cf. [31]). It is clear that $U_{\mathrm{res}}(\mathbf{H})$ does not depend on the choice of λ.

Let $\mathbf{F}_\lambda$ denote the associated Fock (Hilbert) space

$$\mathbf{F}_\lambda = \bigwedge(\mathbf{H}_\lambda^+ \oplus \overline{\mathbf{H}_\lambda^-}).$$

Then we have the Shale-Stinespring embedding $\sigma_\lambda\colon U_{\mathrm{res}}(\mathbf{H}) \hookrightarrow PU(\mathbf{F}_\lambda)$ (cf. [37]). Let $\hat{\rho}_\lambda = \sigma_\lambda \circ \tilde{\rho}\colon LG \to PU(\mathbf{F}_\lambda)$ be the composition. It is a positive energy representation of LG. In the next section, we will compare the representations for different values of λ, but for the remainder of this section we will only need $\lambda = 0$.

Given a principal G-bundle $P \to S^1 \times X$, we have the principal LG-bundle $Q \to X$ by the caloron correspondence. Together with a representation ρ of G on V, we form the associated fibre bundle

$$\mathbb{P}(\mathcal{F}^\rho) = Q \times_{LG} \mathbb{P}(\mathbf{F}_0) \to X,$$

where LG acts on the typical fibre $\mathbb{P}(\mathbf{F}_0)$ via $\hat{\rho}_0$. We will next define a line bundle $\mathcal{L}^\rho \to \mathbb{P}(\mathcal{F}^\rho)^{[2]}$ induced by $\mathcal{P}^Q$ or $\mathcal{L}^Q \to Q^{[2]}$. Since

$$\mathcal{P}^Q = \{(q_1, q_2, g) \in Q^{[2]} \times \widehat{LG} \mid q_2 = q_1 p(g)\},$$

there is a right action of $\widehat{LG} \times \widehat{LG}$ on $\mathcal{P}^Q$ given by

$$(g_1, g_2)\colon (q_1, q_2, g) \mapsto (q_1 p(g_1), q_2 p(g_2), g_1^{-1} g g_2).$$

On the other hand, if we denote the universal line bundle over $\mathbb{P}(\mathbf{F}_0)$ by $\mathbf{L}$, the left LG-action on $\mathbb{P}(\mathbf{F}_0)$ (via $\hat{\rho}_0$) lifts to an action of $\widehat{LG}$ on $\mathbf{L}$. We set $\mathcal{L}^\rho = \mathcal{P}^Q \times_{\widehat{LG} \times \widehat{LG}} (\mathbf{L} \boxtimes \mathbf{L})$; this is a line bundle over $Q^{[2]} \times_{(LG\times LG)} (\mathbb{P}(\mathbf{F}_0)\times\mathbb{P}(\mathbf{F}_0)) = \mathbb{P}(\mathcal{F}^\rho)^{[2]}$. The triviality of $\delta(\mathcal{L}^Q)\to Q^{[3]}$ implies that of $\delta(\mathcal{L}^\rho)\to\mathbb{P}(\mathcal{F}^\rho)^{[3]}$. Thus we have a bundle gerbe which we call the *caloron bundle gerbe*.

We explain the above construction using local data. Let $\{U_\alpha\}$ be an open cover of X which locally trivializes the principal LG-bundle Q. Then the restriction of Q to U_α, $Q_\alpha\to U_\alpha$, has a lift to a principal $\widehat{LG}$-bundle $\hat{Q}_\alpha\to U_\alpha$. On the overlap $U_\alpha\cap U_\beta$, the two bundles $\hat{Q}_\alpha$ and $\hat{Q}_\beta$ differ by a principal $U(1)$-bundle whose associated line bundle we denote by $\mathcal{L}^Q_{\alpha\beta}\to U_\alpha\cap U_\beta$. They satisfy $\mathcal{L}^Q_{\alpha\beta}\otimes\mathcal{L}^Q_{\beta\gamma}\cong\mathcal{L}^Q_{\alpha\gamma}$ on $U_\alpha\cap U_\beta\cap U_\gamma$ and the collection $\{\mathcal{L}^Q_{\alpha\beta}\}$ is a local description of the bundle gerbe $(Q,\mathcal{L}^Q)$. Given a unitary representation $\rho\colon G\to SU(V)$, the principal $LSU(V)$-bundle $Q^\rho_\alpha = Q_\alpha\times_{LG} LSU(V)$ lifts to a principal $\widehat{LSU(V)}$-bundle $\widehat{Q^\rho_\alpha} = \hat{Q}_\alpha\times_{\widehat{LG}}\widehat{LSU(V)}$ over U_α. On the overlap $U_\alpha\cap U_\beta$, the two lifts $\widehat{Q^\rho_\alpha}$ and $\widehat{Q^\rho_\beta}$ differ by a $U(1)$-bundle whose associated line bundle is $\mathcal{L}^\rho_{\alpha\beta} = (\mathcal{L}^Q_{\alpha\beta})^{\otimes\iota_\rho}$.

Let $\mathcal{F}^\rho_\alpha = \hat{Q}_\alpha\times_{\widehat{LG}}\mathbf{F}_0\to U_\alpha$ be the associated Fock bundles; the projectivizations $\mathbb{P}(\mathcal{F}^\rho_\alpha)$ glue properly to form the bundle $\pi_\rho\colon\mathbb{P}(\mathcal{F}^\rho)\to X$. On the overlap $U_\alpha\cap U_\beta$, the Fock bundles are related by $\mathcal{F}^\rho_\alpha\cong(\mathcal{L}^Q_{\alpha\beta})^{\otimes\iota_\rho}\otimes\mathcal{F}^\rho_\beta$. As in the construction of the index bundle gerbe, let $\mathcal{L}^\rho_\alpha$ be the universal bundle over $\mathbb{P}(\mathcal{F}^\rho_\alpha)$. Then $\mathcal{L}^\rho_\alpha = \mathcal{L}^\rho_\beta\otimes\pi^*_\rho(\mathcal{L}^\rho_{\alpha\beta})$. So the line bundles $(\mathcal{L}^\rho_\alpha)^{-1}\boxtimes\mathcal{L}^\rho_\alpha$ over $\mathbb{P}(\mathcal{F}^\rho_\alpha)^{[2]}\subset\mathbb{P}(\mathcal{F}^\rho_\alpha)\times\mathbb{P}(\mathcal{F}^\rho_\alpha)$ glue properly to define globally the line bundle $\mathcal{L}^\rho\to\mathbb{P}(\mathcal{F}^\rho)^{[2]}$. This defines the same caloron bundle gerbe. The line bundles $\mathcal{L}^\rho_\alpha$ are its local trivializations under which the bundle gerbe is described locally by a collection of line bundles $\mathcal{L}^\rho_{\alpha\beta} = (\mathcal{L}^Q_{\alpha\beta})^{\otimes\iota_\rho}$ over $U_\alpha\cap U_\beta$.

4. Comparison of the index bundle gerbe and the caloron bundle gerbe

Recall that P is a principal G-bundle over $S^1\times X$ and Q is the corresponding principal LG-bundle over X. The fibre of Q at $x\in X$ is $Q_x = \Gamma(S^1\times\{x\},P)$ with the obvious right LG-action. Given a finite dimensional unitary representation ρ of G on V, there is an associated hermitian vector bundle $E = P\times_G V\to S^1\times X$. Consider the Hilbert space bundle $\mathcal{H}\to X$ whose fibre $\mathcal{H}_x$ over $x\in X$ is the L^2-completion of the space $\Gamma(S^1\times\{x\},E)$. There is a family of Dirac operators $\{D_x\}$ on S^1 coupled to E acting on the fibres of $\mathcal{H}$. By Sect. 2, there is a projectivized Fock bundle $\mathbb{P}(\mathcal{F})\to X$ and a hermitian line bundle $\mathcal{L}\to\mathbb{P}(\mathcal{F})^{[2]}$ that defines the index bundle gerbe. By Sect. 3.2, there is a another projectivized Fock bundle $\mathbb{P}(\mathcal{F}^\rho)\to X$ constructed by using the data from caloron correspondence and the spectral cut at 0 and a hermitian line bundle $\mathcal{L}^\rho\to\mathbb{P}(\mathcal{F}^\rho)^{[2]}$ that defines the caloron bundle gerbe. The purpose of this section is to show that the two bundle gerbes are isomorphic.

4.1. **Spectral cuts, equivalent representations and Bogoliubov transformations.** In this section, we show that the projective loop group representation the projectivized Fock space $\mathbb{P}(\mathbf{F}_\lambda)$, corresponding to a spectral cut λ, is equivalent to $\mathbb{P}(\mathbf{F}_\mu)$, for $\mu \neq \lambda$. This is achieved by using a Bogoliubov transformation (see also [26], Sect. 12.3).

Let $\{\psi_n^i : n \in \mathbb{Z}, 1 \leq i \leq \dim V\}$ be a basis for $\mathbf{H}$. Then $\{\bar{\psi}_n^i : n \in \mathbb{Z}, 1 \leq i \leq \dim V\}$ is a basis for $\overline{\mathbf{H}}$, where $\bar{\psi}_n^i$ is the complex conjugation of ψ_{-n}^i. They are chosen so that $\mathbf{H}_\lambda^+$ is spanned by $\{\psi_n^i : n \in \mathbb{Z}, n \geq \lambda, 1 \leq i \leq \dim V\}$ and $\mathbf{H}_\lambda^-$ is spanned by $\{\psi_n^i : n \in \mathbb{Z}, n < \lambda, 1 \leq i \leq \dim V\}$. Similarly, $\overline{\mathbf{H}_\lambda^+}$ is spanned by $\{\bar{\psi}_n^i : n \in \mathbb{Z}, n \leq -\lambda, 1 \leq i \leq \dim V\}$ and $\overline{\mathbf{H}_\lambda^-}$ is spanned by $\{\bar{\psi}_n^i : n \in \mathbb{Z}, n > -\lambda, 1 \leq i \leq \dim V\}$. The Fock space $\mathbf{F}_\lambda$ is spanned by $\{\psi_{n_1}^{i_1} \dots \psi_{n_N}^{i_N} \bar{\psi}_{m_1}^{j_1} \dots \bar{\psi}_{m_M}^{j_M} |\lambda\rangle \ , \ n_1, \dots, n_N \geq \lambda, \ m_1, \dots, m_M > -\lambda\}$, where the 'vacuum' $|\lambda\rangle$ has the properties

$$\psi_n^i \, |\lambda\rangle = 0 \, , \quad n < \lambda \quad \text{and} \quad \bar{\psi}_n^i \, |\lambda\rangle = 0 \, , \quad n \leq -\lambda \, .$$

and ψ_m^i and $\bar{\psi}_n^j$ are operators act on $\mathbf{F}_\lambda$ satisfying

$$\{\psi_m^i, \psi_n^j\} = 0 = \{\bar{\psi}_m^i, \bar{\psi}_n^j\} \, ,$$
$$\{\psi_m^i, \bar{\psi}_n^j\} = \delta_{m+n,0} \delta^{i,j} \, .$$

If $\mu > \lambda$, then

$$|\mu\rangle = \left(\prod_{\lambda < n \leq \mu} \prod_i \psi_n^i \right) |\lambda\rangle \, , \qquad |\lambda\rangle = \left(\prod_{-\mu \leq n < -\lambda} \prod_i \bar{\psi}_n^i \right) |\mu\rangle \, .$$

The operator $\displaystyle\prod_{\lambda < n \leq \mu} \prod_i \psi_n^i$, corresponding to a base vector of the complex line $\det(\mathbf{H}^{\lambda +} \cap \mathbf{H}^{\mu -})$, relates the two vacua in $\mathbf{F}_\lambda$ and $\mathbf{F}_\mu$; this is the so called Bogoliubov transformation.

Just as $U_{\text{res}}(\mathbf{H})$, we have the restricted general linear group

$$GL_{\text{res}}(\mathbf{H}) = \{U \in GL(\mathbf{H}) : [U, I_\lambda] \text{ is a Hilbert-Schmidt operator}\} \, ,$$

which is also independent of λ. Define the bounded operator e_n^{ij} on $\mathbf{H}$ by $e_n^{ij}(\psi_m^k) = \delta^{j,k} \psi_{m+n}^i$. Then $\text{id} + e_n^{ij} \in GL_{\text{res}}(\mathbf{H})$. On the Fock space $\mathbf{F}_\lambda$ we define the operators $\sigma_\lambda(e_n^{ij}) = \sum_m : \psi_m^i \bar{\psi}_{n-m}^j :_\lambda$, where the λ-normal ordering is defined as

$$: \psi_m^i \bar{\psi}_n^j :_\lambda = \begin{cases} \psi_m^i \bar{\psi}_n^j, & \text{if } m \leq \lambda \, , \\ -\bar{\psi}_n^j \psi_m^i, & \text{if } m < \lambda \, . \end{cases}$$

A standard computation then gives

$$[\sigma_\lambda(e_m^{ij}), \sigma_\lambda(e_n^{kl})] = \delta^{j,k} \sigma_\lambda(e_{m+n}^{il}) - \delta^{i,l} \sigma_\lambda(e_{m+n}^{kj}) + \delta_{j,k} \delta^{i,l} m \delta_{m+n,0} \, ,$$

which shows that σ_λ is a representation on $\mathcal{F}_\lambda$ of the central extension $\widehat{\mathfrak{gl}_{\text{res}}(\mathbf{H})}$ of the Lie algebra $\mathfrak{gl}_{\text{res}}(\mathbf{H})$ of $GL_{\text{res}}(\mathbf{H})$.

Now let $\mu > \lambda$. Then one computes

$$\begin{aligned}\sigma_\mu(e_n^{ij}) &= \sigma_\lambda(e_n^{ij}) - \sum_{\lambda<m<\mu} \{\psi_m^i, \bar{\psi}_{n-m}^j\} \\ &= \sigma_\lambda(e_n^{ij}) - n_{\lambda\mu}\delta^{i,j}\delta_{n,0},\end{aligned}$$

where $n_{\lambda\mu} = \#\{m : \lambda < m \leq \mu\}$. Since e_n^{ij} generates $\widehat{\mathfrak{gl}_{\text{res}}(\mathbf{H})}$, we have

$$\sigma_\mu(K) = \sigma_\lambda(K) - n_{\lambda\mu}\text{Tr}(K)$$

for any $K \in \widehat{\mathfrak{gl}_{\text{res}}(\mathbf{H})}$ and hence

$$\sigma_\mu(U) = \sigma_\lambda(U)\det(U)^{-n_{\lambda\mu}}$$

for any $U \in \widehat{GL_{\text{res}}(\mathbf{H})}$. That is, the projective representations on $\mathbb{P}(\mathbf{F}_\lambda)$ and $\mathbb{P}(\mathbf{F}_\mu)$ of $U_{\text{res}}(\mathbf{H}) \subset GL_{\text{res}}(\mathbf{H})$, and hence those of LG are equal.

We note that in this particular case, not only do we have an action of a central extension of $GL_{\text{res}}(\mathbf{H})$ on $\mathbf{F}_\lambda$, but also an action of the Virasoro algebra (the central extension of the 2-dimensional conformal algebra). The calculations in this section are of course well-known to the experts in conformal field theory (see, e.g., [14] and references therein). It is an interesting question whether our construction of the caloron bundle gerbe has applications in the context of conformal field theory as well.

4.2. **Isomorphism between the index and caloron bundle gerbes.** In this section, we will show that the geometrically obtained caloron bundle gerbe is isomorphic (not just stably isomorphic) to the analytically defined index bundle gerbe in this context. For the notion of stable isomorphism of bundle gerbes, see [28].

Consider the vector bundle $E = P \times_G V$ over $S^1 \times X$, which itself fibers over X with circle fibres. We first show that Hilbert bundle $\mathcal{H}$ defined in Sect. 2 is the associated bundle of Q by $\tilde{\rho}$, i.e., $\mathcal{H} \cong Q \times_{LG} \mathbf{H}$. Suppose X has an open covering $\{U_\alpha\}$ such that the bundle Q is trivial over each U_α. Then P is also trivial on $S^1 \times U_\alpha$; let $g_{\alpha\beta}\colon S^1 \times (U_\alpha \cap U_\beta) \to G$ be the transition functions. Then the transition functions $\tilde{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \to LG$ of Q are given by $\tilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(\cdot, x)$, $x \in U_\alpha \cap U_\beta$. Under the trivialisation over U_α, the fibre $\mathcal{H}_x$ ($x \in U_\alpha$) can be identified with $\mathbf{H}$. Using the local trivialisations of P, if $x \in U_\alpha \cap U_\beta$, the two identifications of $\mathcal{H}_x$ with $\mathbf{H}$ from U_α and U_β are related by $\tilde{\rho}(\tilde{g}_{\alpha\beta}(x)) \in U(\mathbf{H})$.

This shows the result that $\mathcal{H}$ is associated to Q and thus we have the bundle isomorphism $\mathcal{H} \cong Q \times_{LG} \mathbf{H}$.

We now show that $\mathbb{P}(\mathcal{F})$ defined in Sect. 2 is an associated bundle of Q by $\hat{\rho}_0$. We assume that on each U_α, we can choose $\lambda_\alpha \notin \mathrm{spec}(D_x)$ for all $x \in U_\alpha$. We then have a bundle of Fock spaces $\mathcal{F}_{\lambda_\alpha}$ over U_α. Since $\mathcal{H}$ is an associated bundle of Q by the representation $\tilde{\rho}$ of LG on the typical fibre $\mathbf{H}$, $\mathbb{P}(\mathcal{F}_{\lambda_\alpha})$ is an associated bundle of $Q_\alpha \to U_\alpha$ by the representation $\hat{\rho}_{\lambda_\alpha}\colon LG \to PU(\mathbf{F}_{\lambda_\alpha})$, where $\mathbf{F}_{\lambda_\alpha}$ is the Fock space of $\mathbf{H}$ with a polarization corresponding to λ_α. Under the natural identification of $\mathbb{P}(\mathbf{F}_{\lambda_\alpha})$ and $\mathbb{P}(\mathbf{F}_0)$, the representation $\hat{\rho}_{\lambda_\alpha}$ is the same as $\hat{\rho}_0$. This crucial step follows from the independence of representation of LG on the cuts, as explained in Sect 4.1. Therefore, $\mathbb{P}(\mathcal{F})$ is an associated bundle of Q; in fact it is isomorphic to $\mathbb{P}(\mathcal{F}^\rho)$.

Since the line bundles $\mathcal{L} \to \mathbb{P}(\mathcal{F})^{[2]}$ in Sect. 2 and $\mathcal{L}^Q \to \mathbb{P}(\mathcal{F}^\rho)^{[2]}$ in Sect. 3.2 are both constructed from the universal line bundle, we have the following results.

Theorem 4.1. *Let $P \to S^1 \times X$ be a principal G-bundle and let $Q \to X$ be the corresponding principal LG-bundle. Let ρ be a finite dimensional unitary representation of G on V and let $E = P \times_G V$. Let $\hat{\rho}_0$ be the projective representation of LG on $\mathbf{F}_0$, the Fock space of $\mathbf{H} = L^2(S^1, V)$ with polarization at 0. Let $\mathcal{H}$ be the Hilbert bundle whose fiber $\mathcal{H}_x$ at $x \in X$ is the L^2-completion of $\Gamma(S^1 \times \{x\}, E)$. Then the projectivized Fock bundle $\mathbb{P}(\mathcal{F})$, constructed from $\mathcal{H}$, is isomorphic to the associated bundle $Q \times_{\hat{\rho}_0} \mathbb{P}(\mathbf{F}_0) = \mathbb{P}(\mathcal{F}^\rho)$. Furthermore, the index bundle gerbe of the family of Dirac operators on S^1 coupled to E is isomorphic to the caloron bundle gerbe.*

An alternative, less explicit argument for the same result goes as follows. For simplicity, we will assume for the rest of the section that $H^3(X, \mathbb{Z})$ is torsion free, i.e., the torsion subgroup of $H^3(X, \mathbb{Z})$ is trivial. It is well known that fibre bundles over X whose typical fibre is a projective (infinite dimensional) Hilbert space are classified up to isomorphism by their Dixmier-Douady classes in $H^3(X, \mathbb{Z})$. Conversely, every class in $H^3(X, \mathbb{Z})$ is the Dixmier-Douady class of a fibre bundles over X whose typical fibre is a projective (infinite dimensional) Hilbert space. This is essentially contained in Proposition 2.1 of [2].

Given a unitary representation $\rho\colon G \to U(V)$, $E = P \times_G V$ is a hermitian vector bundle over $S^1 \times X$. A connection $\tilde{A}$ on P induces a hermitian connection ∇^E on E, whose curvature is F^E. For the family of self-adjoint Dirac operators on S^1 coupled to E, the index bundle gerbe $(\mathbb{P}(\mathcal{F}), \mathcal{L})$ is constructed in Sect. 2, and its Dixmier-Douady class in $H^3(X, \mathbb{Z})$

is represented by the 3-curvature form on X [12, 13, 23]

$$\frac{1}{8\pi^2}\int_{S^1} \mathrm{tr}_V\,(F^E \wedge F^E),$$

where F^E is the curvature of the induced connection ∇^E on E. This is equal to $H_\rho = \iota_\rho H$ according to Proposition 3.2. Therefore the Dixmier-Douady class of $\mathbb{P}(\mathcal{F})$ is equal to that of the bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$.

On the other hand, the homomorphism $LSU(V) \to U_{\mathrm{res}}(\mathbf{H})$ pulls back the basic central extension of $U_{\mathrm{res}}(\mathbf{H})$ to that of $LSU(V)$ (see [31], Sect. 6.7), whereas the Shale-Stinespring embedding $\sigma_0\colon U_{\mathrm{res}}(\mathbf{H}) \to PU(\mathbf{F}_0)$ pulls back the basic central extension $U(\mathbf{F}_0) \to PU(\mathbf{F}_0)$ of $PU(\mathbf{F}_0)$ to that of $U_{\mathrm{res}}(\mathbf{H})$. The latter can be argued from the literature as follows. One has the commutative diagram

$$\begin{array}{ccc} U_{\mathrm{res}}(\mathbf{H}) & \longrightarrow & PU(\mathbf{F}_0) \\ \downarrow & & \downarrow \\ Gr_{\mathrm{res}}(\mathbf{H}) & \longrightarrow & \mathbb{P}(\mathbf{F}_0), \end{array} \tag{4.1}$$

where $Gr_{\mathrm{res}}(\mathbf{H})$ denotes the restricted grassmannian, which is the quotient of $U_{\mathrm{res}}(\mathbf{H})$ by a contractible subgroup (cf. page 115 in [31]). The vertical arrows in equation (4.1) are the projection maps, which have degree one since the fibres are contractible. Similarly, since the fibre of $PU(\mathbf{F}_0) \to \mathbb{P}(\mathbf{F}_0)$ is also contractible, the Dixmier-Douady class of the $\mathbb{P}(\mathbf{F}_0)$ bundle is equal to that of the $PU(\mathbf{F}_0)$ bundle. The bottom horizontal arrow in (4.1) is the Plücker embedding. The sentence just after equation (7.7.4) on page 116 in [31] says that the pullback of the tautological line bundle on projective Fock space $\mathbb{P}(\mathbf{F}_0)$ (under the Plücker embedding) is the determinant line bundle on the restricted grassmannian $Gr_{\mathrm{res}}(\mathbf{H})$. Page 115 in [31] identifies the determinant line bundle on the restricted grassmannian $Gr_{\mathrm{res}}(\mathbf{H})$ with the central extension of restricted unitary group $U_{\mathrm{res}}(\mathbf{H})$. Therefore the Dixmier-Douady class of the bundle gerbe $(Q^\rho, \mathcal{L}^\rho)$ is equal to that of the induced $PU(\mathbf{F}_0)$ bundle and that of the projective Hilbert space bundle $\mathbb{P}(\mathcal{F}^\rho)$.

Thus, both the projective bundle $\mathbb{P}(\mathcal{F})$ on X associated to the index bundle gerbe and the projective bundle $\mathbb{P}(\mathcal{F}^\rho)$ on X associated to the caloron bundle gerbe have the same Dixmier-Douady class in de Rham cohomology. By our assumption on $H^3(X, \mathbb{Z})$ and by the classification theorem for projective Hilbert bundles over X, there is an isomorphism of projective Hilbert bundles over X, $\mathbb{P}(\mathcal{F}^\rho) \cong \mathbb{P}(\mathcal{F})$. In particular, there is an induced isomorphism of fibred products $\mathbb{P}(\mathcal{F}^\rho)^{[2]} \cong \mathbb{P}(\mathcal{F})^{[2]}$. Finally, we conclude that the caloron bundle gerbe $(\mathbb{P}(\mathcal{F}^\rho), \mathcal{L}^\rho)$ and the index bundle gerbe $(\mathbb{P}(\mathcal{F}), \mathcal{L})$ are isomorphic.

5. Bundle gerbes on moduli spaces associated to Riemann surfaces

In this section we apply the constructions of bundle gerbes in Sect. 2 and Sect. 3.2 to construct bundle gerbes on moduli spaces associated to Riemann surfaces whose Dixmier-Douady classes are generators of the third integral cohomology groups of these moduli spaces. We first review the construction of moduli spaces and universal bundles.

Let G be a compact, connected, semisimple real Lie group whose centre is $Z(G)$. Let G_{ad} be the quotient $G/Z(G)$, which has a trivial centre.

Let Σ be a compact, connected Riemann surface of genus $g > 1$, and let $\tilde{\Sigma}$ be its universal covering space, with a right action of the fundamental group $\pi_1\Sigma$. There is a left G-action on the set of homomorphisms $\mathrm{Hom}(\pi_1\Sigma, G)$ given by $G \ni h\colon \phi \mapsto \mathrm{Ad}_h \circ \phi$. An element ϕ is irreducible if the isotropy subgroup is $Z(G)$, that is, $\mathrm{Ad}_h \circ \phi = \phi$ for $h \in G$ implies $h \in Z(G)$. Let $\mathrm{Hom}^{\mathrm{irr}}(\pi_1\Sigma, G)$ be the subset of such. The quotient $\mathcal{M} = \mathrm{Hom}^{\mathrm{irr}}(\pi_1\Sigma, G)/G$ is the moduli space of flat G-connections on Σ. More generally, fix an element $z \in Z(G)$. Let Σ' be the surface Σ with a puncture. Its fundamental group $\pi_1\Sigma'$ is the central extension of $\pi_1\Sigma$ by $\mathbb{Z}$. Let $\mathrm{Hom}_z(\pi_1\Sigma', G)$ be the set of homomorphisms such that the generator of $\mathbb{Z}$ is mapped to z. Again there is a G-action on it and we have the subset $\mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G)$ of irreducible homomorphisms. The quotient $\mathcal{M}_z = \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G)/G$ is the moduli space of flat G-connections on Σ' whose holonomy around the puncture is z.

For any $z \in Z(G)$, there is a left G-action on $\tilde{\Sigma} \times \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G) \times G$ given by $G \ni h\colon (p, \phi, g) \mapsto (p, \mathrm{Ad}_h \circ \phi, hg)$. There is also a left $\pi_1\Sigma$-action on the same space given by $\pi_1\Sigma \ni \gamma\colon (p, \phi, g) \mapsto (p\gamma^{-1}, \phi, \phi(\gamma)g)$. It is easy to check that the two actions commute and hence there is a left action of $G \times \pi_1\Sigma$ on $\tilde{\Sigma} \times \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G) \times G$. Let $\mathcal{U} = (\tilde{\Sigma} \times \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G) \times G)/G \times \pi_1\Sigma$ be the quotient space and let $\pi\colon \mathcal{U} \to (\tilde{\Sigma} \times \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G))/G \times \pi_1\Sigma = \Sigma \times \mathcal{M}$ be the projection given by $\pi\colon [(p, \phi, g)] \mapsto [(p, \phi)] = ([p], [\phi])$, where $\phi \in \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G)$, $p \in \tilde{\Sigma}$ and $g \in G$. Then $\mathcal{U}$ is a principal G_{ad}-bundle over $\Sigma \times \mathcal{M}_z$. To see this, we note that there is a right G-action on each fibre $\pi^{-1}([p], [\phi]) = \{[(p, \phi, g)] \,|\, g \in G\}$ given by $G \ni h\colon [(p, \phi, g)] \mapsto [(p, \phi, gh)]$. The isotropic subgroup is $Z(G)$ because if $[(p, \phi, g)] = [(p, \phi, gh)] = [(p, \mathrm{Ad}_{ghg^{-1}} \circ \phi, g)]$, then $ghg^{-1} \in Z(G)$ or $h \in Z(G)$ by the irreducibility of ϕ. The right G-action thus descends to a free action of G_{ad} on $\mathcal{U}$.

The bundle $\pi\colon \mathcal{U} \to \Sigma \times \mathcal{M}_z$ is called the universal bundle [22] because for each $[\phi] \in \mathcal{M}_z$, the restriction of $\mathcal{U}$ to $\Sigma \times [\phi]$ is the flat G_{ad}-bundle over Σ defined by the homomorphism $\alpha \circ \phi \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1\Sigma, G_{\mathrm{ad}})$, where $\alpha\colon G \to G_{\mathrm{ad}}$ is the quotient map. In fact, there is a G_{ad}-equivariant 1-1 map from $\mathcal{U}|_{\Sigma\times[\phi]} = \pi^{-1}(\Sigma \times [\phi]) = \{[(p,\phi,g)] : p \in \tilde{\Sigma}, g \in G\}$ to $\tilde{\Sigma} \times_{\alpha\circ\phi} G_{\mathrm{ad}} = \{[(p,g)] : p \in \tilde{\Sigma}, g \in G_{\mathrm{ad}}\}$ given by $[(p,\phi,g)] \mapsto [(p,\alpha(g))]$. Moreover, for any $\sigma \in \Sigma$, the restriction of $\mathcal{U}$ to $\sigma\times\mathcal{M}_z$ is isomorphic to the G_{ad}-bundle $\mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G) \to \mathcal{M}_z$. To see this, we note that $\mathcal{U}|_{\sigma\times\mathcal{M}_z} = \pi^{-1}(\sigma \times \mathcal{M}_z) = \{[(p_0,\phi,g)] \mid \phi \in \mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G), g \in G\}$ for a fixed $p_0 \in \tilde{\Sigma}$ such that $[p_0] = \sigma$. Under the bijection $[(p_0,\phi,g)] = [(p_0, \mathrm{Ad}_g^{-1}\circ\phi, 1)] \mapsto \mathrm{Ad}_g^{-1}\circ\phi$, the right G-action $G \ni h\colon [(p_0,\phi,1)] \mapsto [(p_0,\phi,h)] = [(p_0,\mathrm{Ad}_h^{-1}\circ\phi,1)]$ on $\mathcal{U}|_{\sigma\times\mathcal{M}_z}$ corresponds to $G \ni h\colon \phi \mapsto \mathrm{Ad}_h^{-1}\circ\phi$ on $\mathrm{Hom}_z^{\mathrm{irr}}(\pi_1\Sigma', G)$.

We assume that the moduli space $\mathcal{M}_z$ is closed. When $G = SU(n)$, $\mathcal{M}_z$ is closed if and only if z generates the center $Z(SU(n)) \cong \mathbb{Z}_n$. Then the first Pontryagin class x of $\mathrm{Ad}\mathcal{U} \to \Sigma\times\mathcal{M}_z$ is a generator of $H^4(\Sigma \times \mathcal{M}_z, \mathbb{Q})$. For $G = SU(n)$, $H^2(\mathcal{M}, \mathbb{Q})$ is generated by the slant product $[\Sigma]\backslash x$ while $H^3(\mathcal{M}_z, \mathbb{Q})$ is generated by $[\gamma]\backslash x$, where $\gamma\colon S^1 \to \Sigma$ are smooth loops whose homology classes generate $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ when $g > 1$ [15]. There is no torsion for $H^3(\mathcal{M}_z, \mathbb{Z})$ if z generates $Z(SU(n))$ [6]. Given a Riemannian metric on Σ, the universal bundle has a canonical connection [3] whose curvature is denoted by $F^{\mathcal{U}} \in \Omega^2(\Sigma\times\mathcal{M}_z, \mathrm{Ad}\mathcal{U})$. In de Rham cohomology, x is represented by $-\frac{1}{8\pi^2}\langle F^{\mathcal{U}}, F^{\mathcal{U}}\rangle$. Therefore, the generators $[\Sigma]\backslash x$ and $[\gamma]\backslash x$ are represented by

$$-\frac{1}{8\pi^2}\int_\Sigma \langle F^{\mathcal{U}}, F^{\mathcal{U}}\rangle \quad \text{and} \quad H_{\rho,\gamma} = -\frac{1}{8\pi^2}\int_{S^1} (\gamma\times\mathrm{id})^*\langle F^{\mathcal{U}}, F^{\mathcal{U}}\rangle,$$

respectively.

When $G = SU(n)$ and z generates $Z(G)$, we apply the constructions of the index and caloron bundle gerbes to obtain bundle gerbes on $\mathcal{M}_z$ whose Dixmier-Douady invariants are the generators of the third de Rham cohomology group. Given a smooth loop $\gamma\colon S^1 \to \Sigma$, let $P_\gamma = (\gamma\times\mathrm{id})^*\mathcal{U}$ be the principal G_{ad}-bundle over $S^1 \times \mathcal{M}_z$. With a finite dimensional unitary representation ρ of G_{ad} on V, we have a principal $SU(V)$-bundle $P_\gamma^\rho \to S^1 \times \mathcal{M}_z$ and hence an $LSU(V)$-bundle $Q_\gamma^\rho \to \mathcal{M}_z$ by caloron correspondence, with a line bundle $\mathcal{L}_\gamma^\rho$ over $(Q_\gamma^\rho)^{[2]}$. The universal connection on $\mathcal{U}$ pulls back to P_γ and induces one on P_γ^ρ. The latter determines a connection and a Higgs field for Q_γ^ρ. In particular, the Dixmier-Douady class of bundle gerbe $(Q_\gamma^\rho, \mathcal{L}_\gamma^\rho)$ is represented by $\iota_\rho H_{\rho,\gamma}$, which is one of the $2g$ generators of $H^3(\mathcal{M}, \mathbb{R})$ if $g > 1$.

Moreover, there are two ways to construct a projective Fock space bundles over $\mathcal{M}_z$ of the same Dixmier-Douady class as above. First, by considering the the family of self-adjoint Dirac operators on S^1 coupled to the vector bundle $E_\gamma = P_\gamma \times_G V$, we get the index bundle gerbe $(\mathbb{P}(\mathcal{F}_\gamma), \mathcal{L}_\gamma)$ from Sect. 2. Second, using the $LSU(V)$ bundle Q^ρ_γ, we have the caloron bundle gerbe $(\mathbb{P}(\mathcal{F}^\rho_\gamma), \mathcal{L}^\rho_\gamma)$ from Sect. 3.2. They are isomorphic by Sect. 4.2 and their Dixmier-Douady class is equal to that of $(Q^\rho_\gamma, \mathcal{L}^\rho_\gamma)$.

The construction of natural bundle gerbes also applies to other moduli spaces associated to Riemann surfaces, such as the Hitchin moduli space and the monopole moduli space.

6. Conclusions and outlook

In this paper, we have constructed bundle gerbes on moduli spaces $\mathcal{M}$ of flat G-bundles on a compact Riemann surface Σ of genus $g > 1$, where G is a compact connected simply-connected Lie group. These are the index bundle gerbe and the caloron bundle gerbe, which we show are isomorphic (not just stably isomorphic). We have constructed a bundle gerbe connection and curving (or B-field), and have computed the 3-curvature which represents the Dixmier-Douady class of the bundle gerbe in de Rham cohomology. The construction is such that it extends without change to other moduli spaces, such as the moduli space of principal $U(n)$-bundles with fixed determinant bundle and the moduli space of Higgs bundles.

In Sect. 2, given the basic setup, we have constructed the index bundle gerbe, refining a construction of Segal. It remains to define a natural bundle gerbe connection and curving on it, and to compute the 3-curvature form. This has been established in the paper in a special case using Sect. 4.2, which is the case that applies to the moduli spaces considered here.

In the setting of Sect. 3, let Q be a principal LG-bundle over X and $\hat{\rho}_0 \colon LG \to PU(\mathbf{F}_0)$ a positive energy representation of LG. Then we can form the algebra bundle

$$\mathcal{K}_Q = Q \times_{LG} \mathcal{K}(\mathbf{F}_0) \longrightarrow X$$

with fibre the algebra of compact operators on Fock space, $\mathcal{K}(\mathbf{F}_0)$. The space of continuous sections of $\mathcal{K}_{Q,\hat{\rho}_0}$ vanishing at infinity, $C_0(X, \mathcal{K}_{Q,\hat{\rho}_0})$, is a C^*-algebra, which is a continuous trace algebra with spectrum equal to X. This non-commutative algebra is only locally Morita equivalent to continuous functions on X. The operator K-theory $K_\bullet(C_0(X, \mathcal{K}_{Q,\hat{\rho}_0}))$ is the twisted K-theory $K^\bullet(X, H)$ (cf. [33]), where H is the Dixmier-Douady class of $\mathbb{P}(\mathcal{F}^\rho)$ (see [10] for a alternate description of twisted K-theory). In particular, for each of the $2g$ bundle gerbes that we have constructed on the moduli space $\mathcal{M}$, we can form the twisted K-theory, and it would be interesting to compute these.

Another interesting problem is to give natural constructions of holomorphic bundle gerbes on the moduli space stable holomorphic bundles over a Riemann surface Σ. If the degree and the rank are coprime, then the $2g$ generators of the third cohomology are of types $(2,1)$ and $(1,2)$ for $g>1$ [16]. According to section 7 in [24], there are hermitian holomorphic bundle gerbes that have these as Dixmier-Douady classes, but the challenge is to find natural constructions for them.

Next we outline a construction of bundle gerbes on other moduli spaces such as the moduli space of anti-self-dual (ASD) connections on a compact four dimensional Riemannian manifold M such that $\dim(H_1(M,\mathbb{R}))>0$. More precisely, let P denote a principal G-bundle over M, where G is a compact semisimple Lie group, and let $\mathcal{A}_P^-$ denote the space of all ASD connections on P. We will assume that $\mathcal{A}_P^-\neq\emptyset$, which occurs under various hypotheses on P and M, cf. the introduction in [35].

If $\mathcal{G}_P$ denotes the gauge group of P, then one has the universal principal G_{ad} bundle

$$G_{\mathrm{ad}}\longrightarrow(P\times\mathcal{A}_P^-)/\mathcal{G}_P\longrightarrow M\times\mathcal{A}_P^-/\mathcal{G}_P.$$

Using this, we may also construct our bundle gerbes, for instance as follows. For $\gamma\colon S^1\to M$ a generator of $H_1(M,\mathbb{Z})$, consider the restricted bundle

$$G_{\mathrm{ad}}\longrightarrow(\gamma^*P\times\mathcal{A}_P^-)/\mathcal{G}_P\longrightarrow S^1\times\mathcal{A}_P^-/\mathcal{G}_P, \tag{6.1}$$

which as before determines the caloron bundle gerbe over the ASD moduli space.

$$LG_{\mathrm{ad}}\longrightarrow Q_\gamma\longrightarrow\mathcal{A}_P^-/\mathcal{G}_P.$$

One can similarly define the index bundle gerbe and show that these bundle gerbes are isomorphic to each other and that their Dixmier-Douady is class given by

$$\int_{S^1}p_1\left(\gamma^*(P)\times\mathcal{A}_P^-)/\mathcal{G}_P\right)\in H^3(\mathcal{A}_P^-/\mathcal{G}_P,\mathbb{Z}).$$

Here $p_1\left(\gamma^*P\times\mathcal{A}_P^-)/\mathcal{G}_P\right)$ denotes the first Pontryagin class of the principal bundle in equation (6.1).

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(P Bouwknegt) Department of Theoretical Physics, Research School of Physics and Engineering and Department of Mathematics, Mathematical Sciences Institute, The Australian National University, Canberra, ACT 0200, Australia

E-mail address: peter.bouwknegt@anu.edu.au

(V Mathai) Department of Pure Mathematics, University of Adelaide, Adelaide, SA 5005, Australia

E-mail address: mathai.varghese@adelaide.edu.au

(S Wu) Department of Mathematics, University of Hong Kong, Pokfulam Road, Hong Kong, China

E-mail address: swu@maths.hku.hk