ABELIANIZATIONS OF DERIVATION LIE ALGEBRAS OF FREE ASSOCIATIVE ALGEBRA AND FREE LIE ALGEBRA

SHIGEYUKI MORITA, TAKUYA SAKASAI, AND MASAAKI SUZUKI

ABSTRACT. We determine the abelianizations of the following three kinds of graded Lie algebras in a certain stable range: derivations of the free associative algebra, derivations of the free Lie algebra and symplectic derivations of the free associative algebra. As an application of the last case, and by making use of a theorem of Kontsevich, we obtain a new proof of the vanishing theorem of Harer concerning the top rational cohomology group of the mapping class group with respect to its virtual cohomological dimension.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

In this paper, we consider graded Lie algebras consisting of derivations of free associative or free Lie algebras generated by a finite dimensional vector space V over $\mathbb{Q}$. We also consider the cases where $\dim V$ is even and equipped with a non-degenerate skew-symmetric bilinear form. In this case, we consider the graded Lie algebras consisting of *symplectic* derivations. These Lie algebras appear naturally in various aspects of topology and it should be an important problem to analyze the structure of them.

To be more precise, let H_n denote an n-dimensional vector space over $\mathbb{Q}$ and let $T(H_n), \mathcal{L}_n$ be the free associative algebra without unit and the free Lie algebra generated by H_n, respectively. We denote by $\mathrm{Der}(T(H_n))$ and $\mathrm{Der}(\mathcal{L}_n)$ the graded Lie algebras consisting of derivations of $T(H_n)$ and $\mathcal{L}_n$. In the case where $n = 2g$ and H_{2g} is given a structure of a symplectic vector space as above, we denote by $\mathfrak{a}_g$ and $\mathfrak{h}_{g,1}$ the Lie subalgebras of $\mathrm{Der}(T(H_n))$ and $\mathrm{Der}(\mathcal{L}_n)$ consisting of *symplectic* derivations, respectively. See §3, 4, 5 for detailed definitions.

Our main result is the first and the third cases of the following theorem which determines the first stable homology groups of the three Lie algebras. The second statement is due to a beautiful work of Kassabov [11]. Here we give an alternative proof which we hope to be somewhat simpler than that of Kassabov.

Theorem 1.1. *The stable abelianizations of the three Lie algebras* $\mathrm{Der}(T(H_n)), \mathrm{Der}(\mathcal{L}_n), \mathfrak{a}_g$ *are given as follows.*

$$\text{(i)}\ \lim_{n\to\infty} H_1(\mathrm{Der}(T(H_n)) \cong \mathbb{Q}$$
$$\text{(ii)}\ \lim_{n\to\infty} H_1(\mathrm{Der}(\mathcal{L}_n)) \cong \mathbb{Q}$$
$$\text{(iii)}\ \lim_{g\to\infty} H_1(\mathfrak{a}_g) = 0.$$

2000 *Mathematics Subject Classification.* Primary 17B56; 32G15; 55R40, Secondary 17B65; 20J06.

Key words and phrases. moduli space of curves, Lie algebra homology, graph homology, derivation.

Let $\mathrm{Der}^+(T(H_n))$ and $\mathrm{Der}^+(\mathcal{L}_n)$ denote the ideals of $\mathrm{Der}(T(H_n))$ and $\mathrm{Der}(\mathcal{L}_n)$ consisting of derivations of *positive* degrees. Similarly we denote by $\mathfrak{a}_g^+ \subset \mathfrak{a}_g$ and $\mathfrak{h}_{g,1}^+ \subset \mathfrak{h}_{g,1}$ the ideals consisting of derivations of positive degrees. The proof of Theorem 1.1 is based on careful studies of the bracket operations in these ideals. We can summarize our results on the structures of these ideals as follows (see more precise statements in $\S 3, 4, 5$).

Theorem 1.2. *The Lie algebras* $\mathrm{Der}^+(T(H_n))$ *and* $\mathfrak{a}_g^+$ *are finitely generated in a certain stable range. More precisely we have the following.*

(i) *Up to degree* $n-1$, $\mathrm{Der}^+(T(H_n))$ *is generated by the degree* 1 *part* $H_n^* \otimes H_n^{\otimes 2}$ *together with a certain summand* $H_n^{\otimes 2}$ *of degree* 2

(ii) *Up to degree* g, $\mathfrak{a}_g^+$ *is generated by the degree* 1 *part* $S^3 H_{2g} \oplus \wedge^3 H_{2g}$ *together with a certain summand* $\wedge^2 H_{2g}/\langle\omega_0\rangle$ *of degree* 2

where $S^3 H_{2g}$ *and* $\wedge^3 H_{2g}$ *denote the third symmetric and exterior powers of* H_{2g} *respectively, and* $\langle\omega_0\rangle$ *denotes the submodule of the second exterior power* $\wedge^2 H_{2g}$ *spanned by the symplectic class.*

In a sharp contrast with the above result, the Lie algebras $\mathrm{Der}^+(\mathcal{L}_n)$ and $\mathfrak{h}_{g,1}^+$ are known to be *not* finitely generated. In fact, the degree 1 part and the trace maps introduced in [16] define *surjective* homomorphisms

$$\mathrm{Der}^+(\mathcal{L}_n) \longrightarrow (H_n^* \otimes \wedge^2 H_n) \oplus \bigoplus_{k=2}^{\infty} S^k H_n$$

$$\mathfrak{h}_{g,1}^+ \longrightarrow \wedge^3 H_{2g} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{2g}$$

of Lie algebras where the targets are understood to be *abelian* Lie algebras.

A theorem of Kassabov cited above implies that the upper homomorphism induces an isomorphism in the first homology group H_1 of Lie algebras in a certain stable range. The first author once conjectured that the lower homomorphism would also induce an isomorphism in H_1. However, very recently Conant, Kassabov and Vogtmann announced that this is not the case, indicating that the Lie algebra $\mathfrak{h}_{g,1}$ has a truly deep structure. Nevertheless, in view of known results together with numbers of explicit computations we have made so far, it seems still reasonable to make the following.

Conjecture 1.3. The stable abelianization of the Lie algebra $\mathfrak{h}_{g,1}$ vanishes. Namely

$$\lim_{g\to\infty} H_1(\mathfrak{h}_{g,1}) = 0.$$

The Lie algebra $\mathfrak{a}_g$ was introduced by Kontsevich in [13, 14]. It is one of the three Lie algebras considered in his theory of graph homology. One of the other Lie algebras, denoted ℓ_g by him, is the same as $\mathfrak{h}_{g,1}$ which appeared already in the theory of Johnson homomorphisms of the mapping class groups both in the contexts of topology and number theory.

Kontsevich proved a remarkable theorem which gives close relations between the stable homology of $\mathfrak{a}_g$ and $\mathfrak{h}_{g,1}$ with the totalities of the rational cohomology groups of the mapping class groups (see Theorem 6.2), and those of the outer automorphism groups $\mathrm{Out}\, F_n$ of free groups F_n $(n \geq 2)$, respectively.

If we combine Theorem 1.1 with the former case of this theorem of Kontsevich, we obtain a new proof of the following vanishing result of Harer for the top rational cohomology group of the mapping class group with respect to its virtual cohomological dimension which was also determined by Harer [8].

Theorem 1.4 (Harer [9])**.** *For any $g \geq 2$, the top degree rational cohomology group of the mapping class group $\mathcal{M}_g$, with respect to its virtual cohomological dimension, vanishes. Namely*

$$H^{4g-5}(\mathcal{M}_g; \mathbb{Q}) = 0 \quad (g \geq 2).$$

See Theorem 6.2 for details. We have heard that Church, Farb and Putman [1] have also proved the above vanishing theorem in their recent work.

Remark 1.5. We can deduce from the latter case of the theorem of Kontsevich mentioned above that Conjecture 1.3 is equivalent to the statement that the top rational cohomology group $H^{2n-3}(\mathrm{Out}\, F_n; \mathbb{Q})$ vanishes for any $n \geq 2$ with respect to its virtual cohomological dimension which was determined by Culler and Vogtmann [4].

Acknowledgement We would like to thank John Harer for informing us about his vanishing theorem and also for pointing out a possible relation to a recent work of Church, Farb and Putman.

2. Lie algebra and its homology

We begin by recalling a few basic facts from the theory of Lie algebras and their homology groups.

Definition 2.1. A vector space $\mathfrak{g}$ over $\mathbb{Q}$ is called a *Lie algebra* if it has a $\mathbb{Q}$-bilinear map

$$[\cdot\,,\,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g},$$

which is called the *bracket* map, satisfying the following two conditions:

- (anti-symmetry) $[x, y] = -[y, x]$ holds for any $x, y \in \mathfrak{g}$; and
- (Jacobi identity) $[x, [y, z]] + [y, [z, x] + [z, [x, y]] = 0$ holds for any $x, y, z \in \mathfrak{g}$.

The image $[\mathfrak{g}, \mathfrak{g}]$ of the bracket map is an ideal of $\mathfrak{g}$.

Definition 2.2. For a Lie algebra $\mathfrak{g}$, the quotient vector space

$$H_1(\mathfrak{g}) := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$$

considered as an abelian Lie algebra, is called the *abelianization* of $\mathfrak{g}$.

As the notation $H_1(\mathfrak{g})$ indicates, there is a general theory of (co)homology of Lie algebras due to Chevalley and Eilenberg, and the above can be interpreted as the first homology group of $\mathfrak{g}$.

Now suppose that the Lie algebra $\mathfrak{g}$ is graded. That is, there exists a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i\geq 0}^{\infty} \mathfrak{g}(i)$$

such that $[\mathfrak{g}(k), \mathfrak{g}(l)] \subset \mathfrak{g}(k+l)$ for any $k, l \geq 0$. Then the homology group $H_*(\mathfrak{g})$ becomes bigraded. In particular, the abelianization is decomposed as

$$H_1(\mathfrak{g}) \cong \bigoplus_{k\geq 0} H_1(\mathfrak{g})_k$$

where

$$H_1(\mathfrak{g})_k = \text{the quotient of } \mathfrak{g}(k) \text{ by } \sum_{\substack{i+j=k\\ i,j\geq 0}} [\mathfrak{g}(i), \mathfrak{g}(j)]$$

is called the *weight* k part of $H_1(\mathfrak{g})$.

If we set $\mathfrak{g}^+ = \bigoplus_{i\geq 1}^{\infty} \mathfrak{g}(i) \subset \mathfrak{g}$, then it becomes an ideal of $\mathfrak{g}$ and we have an extension

$$0 \longrightarrow \mathfrak{g}^+ \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}(0) \longrightarrow 0 \tag{1}$$

of Lie algebras, where the last map denotes the natural projection. It is easy to see that the above extension necessarily splits so that $\mathfrak{g}$ is isomorphic to the semi-direct product $\mathfrak{g}^+ \rtimes \mathfrak{g}(0)$. The abelianization of $\mathfrak{g}^+$ can be described by

$$H_1(\mathfrak{g}^+)_1 = \mathfrak{g}(1), \quad H_1(\mathfrak{g}^+)_k = \text{the quotient of } \mathfrak{g}(k) \text{ by } \sum_{\substack{i+j=k\\ i,j\geq 1}} [\mathfrak{g}(i), \mathfrak{g}(j)]$$

for $k \geq 2$. It follows that the computation of $H_1(\mathfrak{g}^+)$ is equivalent to the determination of a generating set of $\mathfrak{g}^+$ as a Lie algebra.

Finally the relation between the abelianizations of $\mathfrak{g}^+$ and $\mathfrak{g}$ is given by the following Hochschild-Serre exact sequence (see [10], here we use the homology version rather than the original cohomology version)

$$H_2(\mathfrak{g}) \longrightarrow H_2(\mathfrak{g}(0)) \longrightarrow H_1(\mathfrak{g}^+)_{\mathfrak{g}(0)} \longrightarrow H_1(\mathfrak{g}) \longrightarrow H_1(\mathfrak{g}(0)) \longrightarrow 0.$$

Here $H_1(\mathfrak{g}^+)_{\mathfrak{g}(0)}$ denotes the space of coinvariants of $H_1(\mathfrak{g}^+)$ with respect to the action of $\mathfrak{g}(0)$ on it. Since the extension (1) splits, the homomorphism $H_i(\mathfrak{g}) \to H_i(\mathfrak{g}(0))$ is *surjective* for any i so that we have a short exact sequence

$$0 \longrightarrow H_1(\mathfrak{g}^+)_{\mathfrak{g}(0)} \longrightarrow H_1(\mathfrak{g}) \longrightarrow H_1(\mathfrak{g}(0)) \longrightarrow 0.$$

3. Derivation Lie algebra of the free associative algebra

Let $H_n \cong \mathbb{Q}^n$ be an n-dimensional vector space over $\mathbb{Q}$ with a fixed ordered basis $\{x_1, x_2, \ldots, x_n\}$. We suppose that $n \geq 2$. The vector space H_n can be also seen as the first rational homology group of a free group of rank n. We write H_n^* for the dual space $\mathrm{Hom}(H_n, \mathbb{Q})$. The dual basis of H_n^* is denoted by $\{x_1^*, x_2^*, \ldots, x_n^*\}$.

Let $T(H_n) = \bigoplus_{i=1}^{\infty} H_n^{\otimes i}$ denote the tensor algebra without unit generated by H_n. A *derivation* of $T(H_n)$ is a self-linear map D of $T(H_n)$ satisfying

$$D(X \otimes Y) = D(X) \otimes Y + X \otimes D(Y) \tag{2}$$

for any $X, Y \in T(H_n)$. We denote the set of all derivations of $T(H_n)$ by $\mathrm{Der}(T(H_n))$, which has a natural structure of a vector space over $\mathbb{Q}$. Moreover we can endow $\mathrm{Der}(T(H_n))$ with a structure of a Lie algebra as follows. First, we note that a derivation is characterized by its action on the degree 1 part $T(H_n)(1) = H_n$ as the definition (2) implies. Conversely, any linear map in $\mathrm{Hom}(H_n, T(H_n))$ defines a derivation of $T(H_n)$. Therefore we have a natural decomposition

$$\mathrm{Der}(T(H_n)) \cong \mathrm{Hom}(H_n, T(H_n)) \cong \bigoplus_{k\geq 0} \mathrm{Der}(T(H_n))(k)$$

where

$$\mathrm{Der}(T(H_n))(k) := \mathrm{Hom}(H_n, H_n^{\otimes(k+1)}) = H_n^* \otimes H_n^{\otimes(k+1)}$$

denotes the degree k homogeneous part of $\mathrm{Hom}(H_n, T(H_n))$. Then for two elements

$$F = f \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_{p+1} \in \mathrm{Der}(T(H_n))(p) = H_n^* \otimes H_n^{\otimes(p+1)},$$
$$G = g \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{q+1} \in \mathrm{Der}(T(H_n))(q) = H_n^* \otimes H_n^{\otimes(q+1)},$$

where $f, g \in H_n^*$ and $u_1, \ldots, u_{p+1}, v_1, \ldots, v_{q+1} \in H_n$, their bracket $[F, G] \in \mathrm{Der}(T(H_n))(p+q) = H_n^* \otimes H_n^{\otimes(p+q+1)}$ is defined by

$$\begin{aligned}[F,G] = &\sum_{s=1}^{q+1} f(v_s)\, g \otimes v_1 \otimes \cdots v_{s-1} \otimes (u_1 \otimes \cdots \otimes u_{p+1}) \otimes v_{s+1} \otimes \cdots \otimes v_{q+1} \\ &- \sum_{t=1}^{p+1} g(u_t)\, f \otimes u_1 \otimes \cdots u_{t-1} \otimes (v_1 \otimes \cdots \otimes v_{q+1}) \otimes u_{t+1} \otimes \cdots \otimes u_{p+1}.\end{aligned} \tag{3}$$

We can check that this bracket operation gives a Lie algebra structure of $\mathrm{Der}(T(H_n))$. Note that $\mathrm{Der}(T(H_n))(0) = \mathrm{Hom}(H_n, H_n) \cong \mathfrak{gl}(n, \mathbb{Q})$, where $\mathfrak{gl}(n, \mathbb{Q})$ is the Lie algebra of all $(n \times n)$-matrices with entries in $\mathbb{Q}$.

Let

$$\mathrm{Der}^+(T(H_n)) = \bigoplus_{k\geq 1} \mathrm{Der}(T(H_n))(k)$$

be the Lie subalgebra of $\mathrm{Der}(T(H_n))$ consisting of all elements of positive degrees. We now compute $H_1(\mathrm{Der}^+(T(H_n)))$ in the stable range with respect to n.

In [17, Section 6], the first author showed that for $n \geq 2$ the homomorphism

$$C_{13} : \mathrm{Der}(T(H_n))(2) \longrightarrow H_n^{\otimes 2}$$

defined by

$$C_{13}(f \otimes u_1 \otimes u_2 \otimes u_3) = f(u_2)u_1 \otimes u_3,$$

where $f \in H_n^*$ and $u_1, u_2, u_3 \in H_n$, is surjective and that the composition

$$\wedge^2 \mathrm{Der}(T(H_n))(1) \xrightarrow{[\cdot,\cdot]} \mathrm{Der}(T(H_n))(2) \xrightarrow{C_{13}} H_n^{\otimes 2}$$

is trivial. Indeed, for $f, g \in H_n^*$ and $u_1, u_2, v_1, v_2 \in H_n$ we have

$$\begin{aligned}[f \otimes u_1 \otimes u_2, g \otimes v_1 \otimes v_2] &= f(v_1)g \otimes (u_1 \otimes u_2) \otimes v_2 + f(v_2)g \otimes v_1 \otimes (u_1 \otimes u_2) \\ &\quad - g(u_1)f \otimes (v_1 \otimes v_2) \otimes u_2 - g(u_2)f \otimes u_1 \otimes (v_1 \otimes v_2) \\ &\overset{C_{13}}{\longmapsto} f(v_1)g(u_2)u_1 \otimes v_2 + f(v_2)g(u_1)v_1 \otimes u_2 \\ &\quad - g(u_1)f(v_2)v_1 \otimes u_2 - g(u_2)f(v_1)u_1 \otimes v_2 = 0.\end{aligned}$$

Hence a surjective homomorphism $C_{13} : H_1(\mathrm{Der}^+(T(H_n)))_2 \twoheadrightarrow H_n^{\otimes 2}$ is induced.

Theorem 3.1. (1) *For $n \geq 2$, we have a direct sum decomposition*

$$\mathrm{Der}(T(H_n))(2) = H_n^{\otimes 2} \oplus \big[\,\mathrm{Der}(T(H_n))(1), \mathrm{Der}(T(H_n))(1)\big].$$

In particular, the homomorphism $C_{13} : H_1(\mathrm{Der}^+(T(H_n)))_2 \to H_n^{\otimes 2}$ is an isomorphism.
(2) *If $n \geq k \geq 3$, we have*

$$\begin{aligned}&\mathrm{Der}(T(H_n))(k) \\ &\quad = \big[\,\mathrm{Der}(T(H_n))(k-1), \mathrm{Der}(T(H_n))(1)\big] + \big[\,\mathrm{Der}(T(H_n))(k-2), \mathrm{Der}(T(H_n))(2)\big],\end{aligned}$$

namely $H_1(\mathrm{Der}^+(T(H_n)))_k = 0$ holds stably for any $k \geq 3$.

Remark 3.2. The formula (3) for the bracket operation in $\mathrm{Der}(T(H_n))$ looks slightly complicated. However, by using the following diagrammatic description, we can make it clear and intuitive. Generators of $\mathrm{Der}(T(H_n))(k) = H_n^* \otimes H_n^{\otimes(k+1)}$ are written in the form

$$x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}}$$

by using our basis. We associate to such a vector the diagram as in Figure 1:

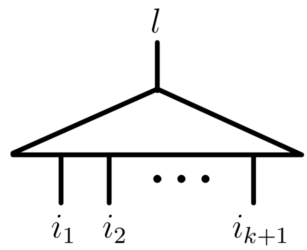

FIGURE 1. The diagram for the vector $x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}}$

Then the formula is diagrammatically written as in Figure 2, where we replace the diagrams in the right hand side under the rule shown in Figure 3.

Proof of Theorem 3.1. (1) We show that the image of the bracket map

$$[\cdot,\cdot] : \mathrm{Der}(T(H_n))(1) \otimes \mathrm{Der}(T(H_n))(1) \longrightarrow \mathrm{Der}(T(H_n))(2)$$

is a vector space of dimension $n^4 - n^2$.

First, the surjective homomorphism $C_{13} : \mathrm{Der}(T(H_n))(2) \longrightarrow H_n^{\otimes 2}$ has the kernel of dimension $n^4 - n^2$ and the kernel of C_{13} contains the image of the bracket map. Then we get

$$\dim \mathrm{Im}[\cdot,\cdot] \leq \dim \ker C_{13} = n^4 - n^2$$

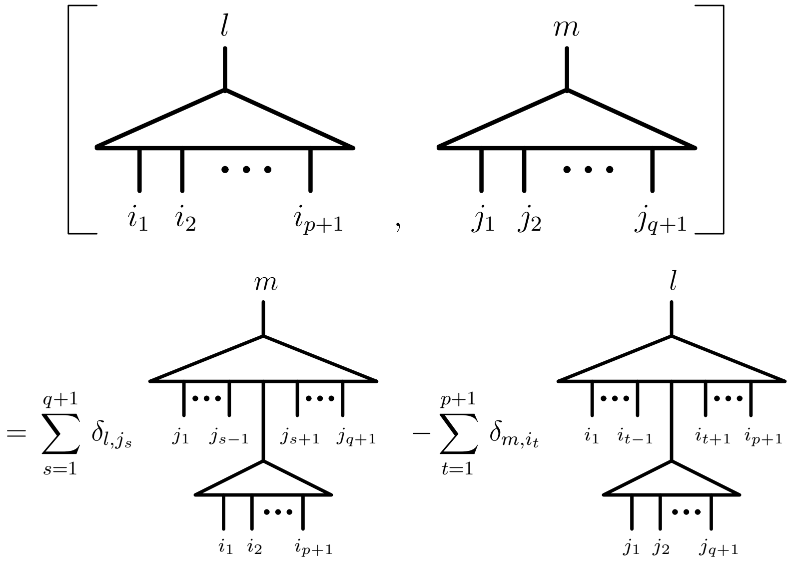

FIGURE 2. Diagrammatic description of the bracket operation

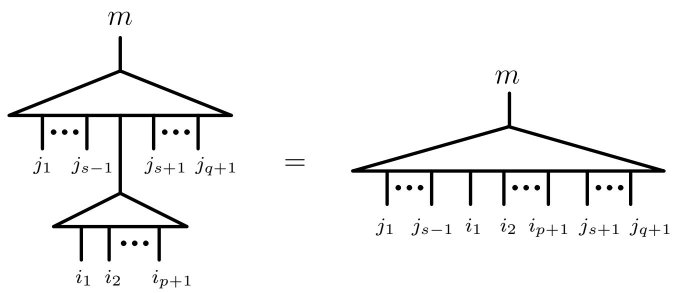

FIGURE 3. Replace the diagram in Figure 2 (similarly for the second one)

as vector spaces.

On the other hand, the image of the bracket map contains some types of elements as follows.

(Type 1) $x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes x_{i_3} = [x_l^* \otimes x_{i_1} \otimes x_{i_2}, x_l^* \otimes x_l \otimes x_{i_3}] \quad (l \neq i_1, i_2, i_3)$.
(Type 2) $x_l^* \otimes x_l \otimes x_{i_1} \otimes x_{i_2} = [x_{i_1}^* \otimes x_{i_1} \otimes x_{i_2}, x_l^* \otimes x_l \otimes x_{i_1}] \quad (l \neq i_1, i_2)$.
(Type 3) $x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes x_l = [x_{i_1}^* \otimes x_{i_1} \otimes x_{i_2}, x_l^* \otimes x_{i_1} \otimes x_l] \quad (l \neq i_1, i_2)$.
(Type 4) $x_l^* \otimes x_l \otimes x_{i_1} \otimes x_l = [x_l^* \otimes x_l \otimes x_{i_1}, x_l^* \otimes x_l \otimes x_l] \quad (l \neq i_1)$.

(Type 5) $x_1^* \otimes x_{i_2} \otimes x_1 \otimes x_{i_3} - x_{i_1}^* \otimes x_{i_2} \otimes x_{i_1} \otimes x_{i_3}$ $(i_1 \neq 1)$ belongs to the image of the bracket map, since

$$\begin{aligned}
&x_1^* \otimes x_{i_2} \otimes x_1 \otimes x_{i_3} - x_{i_1}^* \otimes x_{i_2} \otimes x_{i_1} \otimes x_{i_3}\\
&= [x_{i_1}^* \otimes x_1 \otimes x_{i_3}, x_1^* \otimes x_{i_2} \otimes x_{i_1}] \quad (i_1 \neq i_2, i_3 \neq 1),\\
&x_1^* \otimes x_{i_2} \otimes x_1 \otimes x_{i_3} - x_{i_1}^* \otimes x_{i_2} \otimes x_{i_1} \otimes x_{i_3}\\
&= [x_{i_1}^* \otimes x_{i_2} \otimes x_1, x_1^* \otimes x_{i_1} \otimes x_{i_3}] \quad (i_1 \neq i_3, i_2 \neq 1),\\
&x_1^* \otimes x_{i_1} \otimes x_1 \otimes x_{i_1} - x_{i_1}^* \otimes x_{i_1} \otimes x_{i_1} \otimes x_{i_1}\\
&= [x_{i_1}^* \otimes x_{i_1} \otimes x_1, x_1^* \otimes x_{i_1} \otimes x_{i_1}] - [x_{i_1}^* \otimes x_{i_1} \otimes x_{i_1}, x_1^* \otimes x_{i_1} \otimes x_1],\\
&x_1^* \otimes x_1 \otimes x_1 \otimes x_1 - x_{i_1}^* \otimes x_1 \otimes x_{i_1} \otimes x_1\\
&= [x_1^* \otimes x_1 \otimes x_1, x_{i_1}^* \otimes x_1 \otimes x_{i_1}] - [x_1^* \otimes x_1 \otimes x_{i_1}, x_{i_1}^* \otimes x_1 \otimes x_1].
\end{aligned}$$

The above elements are linearly independent. Furthermore, the number of their elements is

$$n(n-1)^3 + n(n-1)^2 + n(n-1)^2 + n(n-1) + n^2(n-1) = n^4 - n^2.$$

Therefore we obtain that the dimension of $\mathrm{Im}[\cdot,\cdot]$ is $n^4 - n^2$.

(2) We now exhibit an algorithm to rewrite a generator $x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}}$ of $\mathrm{Der}(T(H_n))(k)$ as an element in $\big[\mathrm{Der}(T(H_n))(k-1), \mathrm{Der}(T(H_n))(1)\big] + \big[\mathrm{Der}(T(H_n))(k-2), \mathrm{Der}(T(H_n))(2)\big]$.

(Case 1) When $l \neq i_1, i_2, \ldots, i_{k+1}$, we have an equality

$$x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}} = [x_l^* \otimes x_{i_k} \otimes x_{i_{k+1}},\ x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_l]$$

as depicted in Figure 4 and we have done.

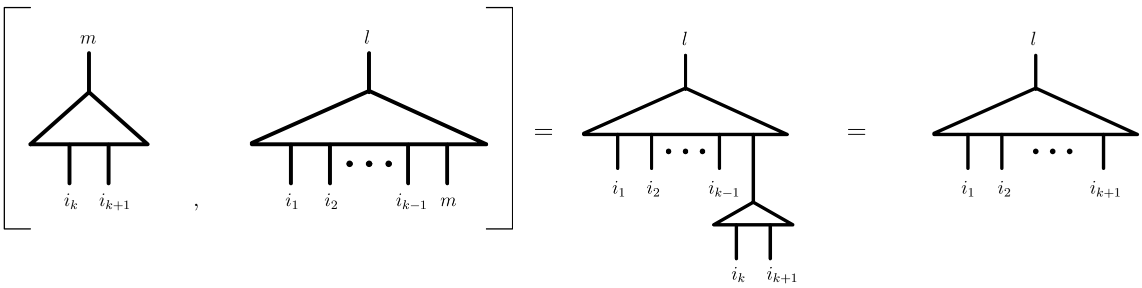

FIGURE 4. Case 1 of Proof of Theorem 3.1(2)

(Case 2) Suppose that l coincides with only one of $i_1, i_2, \ldots, i_{k+1}$ (say $l = i_j$). We rename $\{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k+1}\}$ by $\{j_1, j_2, \ldots, j_k\}$ so that $j_p \neq l$ for $1 \leq p \leq k$. By assumption, we have $k \geq 3$.

The equality

$$[x_{j_1}^* \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k},\ x_l^* \otimes x_l \otimes x_{j_1}] = x_l^* \otimes x_l \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k}$$

shows that the right hand side is in $\big[\mathrm{Der}(T(H_n))(k-1), \mathrm{Der}(T(H_n))(1)\big]$. Now we "slide" x_l to any other slot as follows.

If $q-p \geq 2$, then $x_l^* \otimes x_{j_{p+1}} \otimes x_{j_{p+2}} \otimes \cdots \otimes x_{j_q} \in \mathrm{Der}^+(T(H_n))$ and we have

$$\begin{aligned}&[x_l^* \otimes x_{j_{p+1}} \otimes x_{j_{p+2}} \otimes \cdots \otimes x_{j_q},\ x_l^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_p} \otimes x_l \otimes x_l \otimes x_{j_{q+1}} \otimes \cdots \otimes x_{j_k}] \\ &= x_l^* \otimes x_{j_1} \otimes \cdots x_{j_q} \otimes x_l \otimes x_{j_{q+1}} \otimes \cdots \otimes x_{j_k} + x_l^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_p} \otimes x_l \otimes x_{j_{p+1}} \otimes \cdots \otimes x_{j_k},\end{aligned}$$

which implies that modulo brackets and up to sign, we can slide x_l to the right by at least two slots, as depicted in Figure 5.

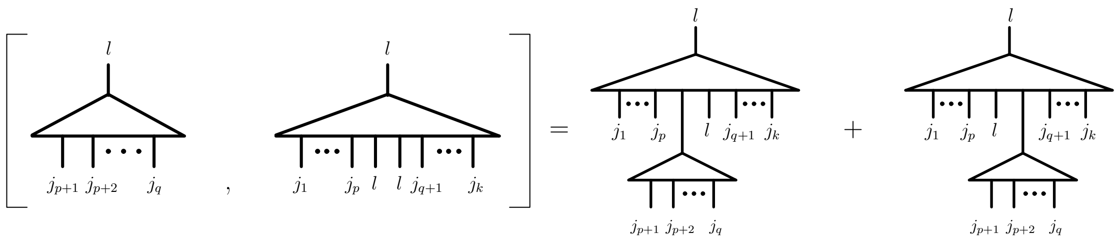

FIGURE 5. Slide x_l

By applying this observation to $x_l^* \otimes x_l \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k} \in \big[\mathrm{Der}(T(H_n))(k-1), \mathrm{Der}(T(H_n))(1)\big]$, we see that

$$\begin{aligned} x_l^* \otimes x_{j_1} \otimes x_l \otimes x_{j_2} \otimes x_{j_3} \otimes \cdots \otimes x_{j_k} &\equiv x_l^* \otimes x_{j_1} \otimes x_{j_2} \otimes x_{j_3} \otimes x_l \otimes x_{j_4} \otimes \cdots \otimes x_{j_k} \\ &\equiv x_l^* \otimes x_l \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k} \\ &\equiv 0 \end{aligned}$$

modulo $\big[\mathrm{Der}(T(H_n))(k-1), \mathrm{Der}(T(H_n))(1)\big] + \big[\mathrm{Der}(T(H_n))(k-2), \mathrm{Der}(T(H_n))(2)\big]$. Starting from $x_l^* \otimes x_l \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_k}$ and $x_l^* \otimes x_{j_1} \otimes x_l \otimes x_{j_2} \otimes x_{j_3} \otimes \cdots \otimes x_{j_k}$, we can slide x_l to any other slot modulo $\big[\mathrm{Der}(T(H_n))(k-2), \mathrm{Der}(T(H_n))(2)\big]$. Hence Case 2 was done.

(Case 3) Here we consider the general case. For $x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}}$, we take

$$m \in \{1,2,\ldots,n\} - \{i_1, i_2, \ldots, i_{k-1}\}$$

where $\{1,2,\ldots,n\} - \{i_1, i_2, \ldots, i_{k-1}\} \neq \emptyset$ by the assumption that $n \geq k$. Then we have

$$\begin{aligned} x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k+1}} &= [x_m^* \otimes x_{i_k} \otimes x_{i_{k+1}},\ x_l^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_m] \\ &\quad + \delta_{l,i_k} x_m^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_m \otimes x_{i_{k+1}} \\ &\quad + \delta_{l,i_{k+1}} x_m^* \otimes x_{i_k} \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_m \end{aligned}$$

as depicted in Figure 6.

If $m \neq i_{k+1}$, the second term of the right hand side is reduced to Case 2. Otherwise, we consider the equality

$$\begin{aligned}&[x_{i_1}^* \otimes x_{i_1} \otimes x_{i_2} \cdots \otimes x_{i_{k-1}},\ x_m^* \otimes x_{i_1} \otimes x_m \otimes x_m] \\ &= x_m^* \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_m \otimes x_m.\end{aligned}$$

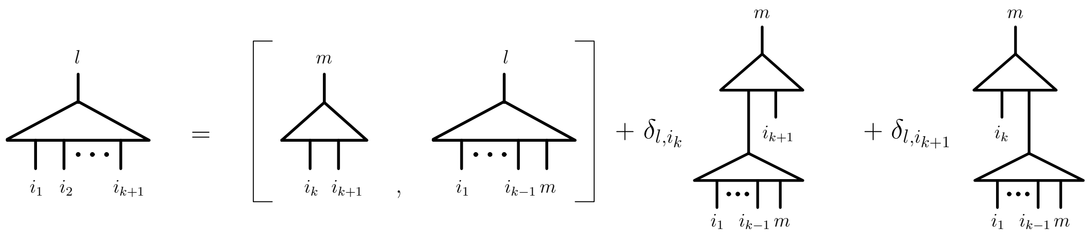

FIGURE 6. Case 3 of Proof of Theorem 3.1(2)

Then this term belongs to $\left[\operatorname{Der}(T(H_n))(k-2), \operatorname{Der}(T(H_n))(2)\right]$. Similarly, if $m \neq i_k$, the third term has already been considered in Case 2. In the other case $m = i_k$, we have

$$\begin{aligned}&[x_{i_1}^* \otimes x_{i_1} \otimes x_{i_2} \cdots \otimes x_{i_{k-1}},\ x_m^* \otimes x_m \otimes x_{i_1} \otimes x_m]\\ &= x_m^* \otimes x_m \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_m.\end{aligned}$$

Therefore this term also belongs to $\left[\operatorname{Der}(T(H_n))(k-2), \operatorname{Der}(T(H_n))(2)\right]$. This completes the proof. □

Theorem 1.1 (i) follows from the following.

Corollary 3.3. (1) *For any $n \geq 2$, The natural pairing $H_n^* \otimes H_n \to \mathbb{Q}$ induces an isomorphism $H_1(\operatorname{Der}(T(H_n)))_0 \cong \mathbb{Q}$.*
(2) *For any $n \geq 2$ we have $H_1(\operatorname{Der}(T(H_n)))_1 = 0$.*
(3) *If $n \geq k \geq 2$, we have $H_1(\operatorname{Der}(T(H_n)))_k = 0$.*

Proof. We have a split extension

$$0 \longrightarrow \operatorname{Der}^+(T(H_n)) \longrightarrow \operatorname{Der}(T(H_n)) \longrightarrow \operatorname{Der}(T(H_n))(0) \longrightarrow 0$$

of Lie algebras, which gives an exact sequence

$$0 \longrightarrow H_1(\operatorname{Der}^+(T(H_n)))_{\mathfrak{gl}} \longrightarrow H_1(\operatorname{Der}(T(H_n))) \longrightarrow H_1(\operatorname{Der}(T(H_n))(0)) \longrightarrow 0.$$

Here $H_1(\operatorname{Der}(T(H_n))(0)) = H_1(\mathfrak{gl}(n,\mathbb{Q}))$ as mentioned above and it is easy to see that the natural pairing $H_n^* \otimes H_n \to \mathbb{Q}$ gives an isomorphism $H_1(\mathfrak{gl}(n,\mathbb{Q})) \cong \mathbb{Q}$. By Theorem 3.1, we have

$$H_1(\operatorname{Der}^+(T(H_n)))_{\mathfrak{gl}} = ((H_n^* \otimes H_n^{\otimes 2}) \oplus H_n^{\otimes 2})_{\mathfrak{gl}} = 0$$

in the range $n \geq k \geq 2$. This completes the proof. □

Remark 3.4. The argument in this section works also for corresponding modules over $\mathbb{Z}$. That is, we may start from putting $H_n \cong \mathbb{Z}^n$. Then the derivation Lie algebra $\operatorname{Der}(T_1(H_n))$ is given as a Lie algebra over $\mathbb{Z}$ and the same conclusion (after replacing vector spaces with modules and linear maps with homomorphisms) is obtained.

4. Derivation Lie algebra of the free Lie algebra

Let $\mathcal{L}_n$ denote the free Lie algebra without unit generated by H_n. This Lie algebra is naturally graded and we have a direct some decomposition $\mathcal{L}_n = \bigoplus_{i=1}^{\infty} \mathcal{L}_n(i)$. For small degree i, the vector space $\mathcal{L}_n(i)$ is given by

$$\mathcal{L}_n(1) = H_n, \quad \mathcal{L}_n(2) \cong \wedge^2 H_n, \quad \mathcal{L}_n(3) \cong (H_n \otimes (\wedge^2 H_n))/\wedge^3 H_n, \quad \ldots$$

where $\wedge^2 H_n$ and $\wedge^3 H_n$ reflect the anti-symmetry and the Jacobi identity of the bracket operation of $\mathcal{L}_n$.

A *derivation* of $\mathcal{L}_n$ is a self-linear map D of $\mathcal{L}_n$ satisfying

$$D([X,Y]) = [D(X), Y] + [X, D(Y)]$$

for any $X, Y \in \mathcal{L}_n$. We denote by $\mathrm{Der}(\mathcal{L}_n)$ the set of all derivations of $\mathcal{L}_n$. By an argument similar to the case of $\mathrm{Der}(T(H_n))$, we have a natural decomposition

$$\mathrm{Der}(\mathcal{L}_n) \cong \mathrm{Hom}(H_n, \mathcal{L}_n) \cong \bigoplus_{k \geq 0} \mathrm{Der}(\mathcal{L}_n)(k)$$

where

$$\mathrm{Der}(\mathcal{L}_n)(k) := \mathrm{Hom}(H_n, \mathcal{L}_n(k+1)) = H_n^* \otimes \mathcal{L}_n(k+1)$$

denotes the degree k homogeneous part of $\mathrm{Hom}(H_n, \mathcal{L}_n)$. Also, we can endow $\mathrm{Der}(\mathcal{L}_n)$ with a graded Lie algebra structure with respect to the above decomposition by a formula, which is mentioned later by using diagrams, similar to $\mathrm{Der}(T(H_n))$. Again we have $\mathrm{Der}(\mathcal{L}_n)(0) = \mathrm{Hom}(H_n, H_n) \cong \mathfrak{gl}(n, \mathbb{Q})$.

It is easily checked that for each $k \geq 1$ the vector space $\mathcal{L}_n(k+1)$ is generated by elements of the form

$$[x_{i_1}, x_{i_2}, \ldots, x_{x_{i_{k+1}}}] := [[\cdots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots], x_{i_{k+1}}].$$

Therefore $\mathrm{Der}(\mathcal{L}_n)(k)$ is generated by elements of the form

$$x_l^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{x_{i_{k+1}}}].$$

Remark 4.1. As in the case of $\mathrm{Der}(T(H_n))$, the following diagrammatic description for $\mathrm{Der}(\mathcal{L}_n)$ is helpful and should be well-known. The vector space $\mathcal{L}_n$ is ly generated by rooted binary planer trees, each of whose univalent vertices has a cyclic order and is colored by an integer in $\{1, 2, \ldots, n\}$ corresponding to the basis of H_n, modulo anti-symmetry and IHX relations. For example, the element $[[[x_{i_1}, x_{i_2}], x_{i_3}], [x_{i_4}, x_{i_5}]] \in \mathcal{L}_n(5)$ is assigned to the left diagram of Figure 7. We can extend this description to a diagrammatic description for $\mathrm{Der}(\mathcal{L}_n)$ by labeling the root by an integer corresponding to the basis of H_n^*. The right diagram of Figure 7 represents $x_l^* \otimes [[[x_{i_1}, x_{i_2}], x_{i_3}], [x_{i_4}, x_{i_5}]] \in \mathrm{Der}(\mathcal{L}_n(4))$.
The bracket operation for generators is diagrammatically given as in Figure 8.

Let $\mathrm{Der}^+(\mathcal{L}_n) = \bigoplus_{k \geq 1} \mathrm{Der}(\mathcal{L}_n)(k)$ be the Lie subalgebra of $\mathrm{Der}(\mathcal{L}_n)$ consisting of all elements of positive degrees. The abelianization $H_1(\mathrm{Der}^+(\mathcal{L}_n))$ with n in the stable

FIGURE 7. The diagrams for $[[[x_{i_1}, x_{i_2}], x_{i_3}], [x_{i_4}, x_{i_5}]] \in \mathcal{L}_n(5)$ (left) and $x_l^* \otimes [[[x_{i_1}, x_{i_2}], x_{i_3}], [x_{i_4}, x_{i_5}]] \in \mathrm{Der}(\mathcal{L}_n)(4)$ (right)

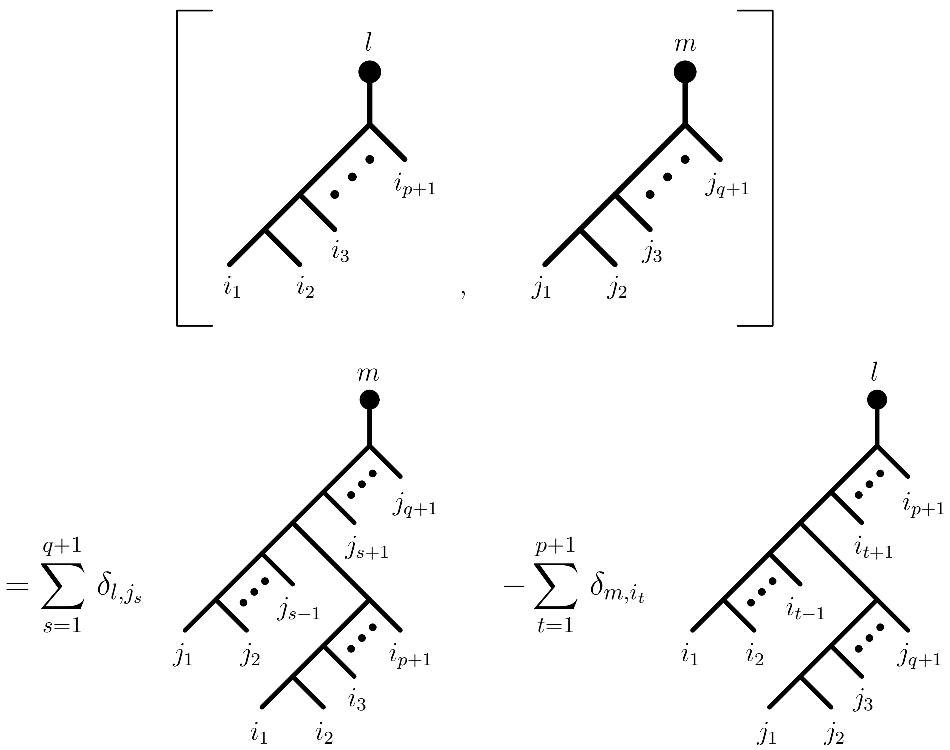

FIGURE 8. Diagrammatic description of the bracket operation

range was first computed by Kassabov [11, Theorem 1.4.11]. To explain the result, we recall the *trace map* introduced by the first author [16].

It is well known that the Lie algebra $\mathcal{L}_n$ can be embedded in $T(H_n)$ by replacing the bracket $[X, Y]$ with $X \otimes Y - Y \otimes X$ repeatedly. This operation keeps the degree. Then consider a sequence of homomorphisms

$$\mathrm{Der}(\mathcal{L}_n)(k) = H_n^* \otimes \mathcal{L}_n(k+1) \longrightarrow H_n^* \otimes H_n^{\otimes(k+1)} \xrightarrow{C_{12}} H_n^{\otimes k} \longrightarrow S^k H_n,$$

where the first map is the above mentioned embedding, the map C_{12} takes the pairing of H_n^* and the first component of $H_n^{\otimes(k+1)}$ and the last map is the symmetrization map to the k-th symmetric power of H_n. We put the composition by tr_k, namely

$$tr_k : \mathrm{Der}(\mathcal{L}_n)(k) \longrightarrow S^k H_n.$$

It was shown in [16] that tr_k vanishes on $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$. Now we show the following.

Theorem 4.2. *If $n \geq k+2 \geq 4$, we have a direct sum decomposition*

$$\mathrm{Der}(\mathcal{L}_n)(k) = S^k H_n \oplus \big[\,\mathrm{Der}(\mathcal{L}_n)(k-1), \mathrm{Der}(\mathcal{L}_n)(1)\big]$$

where the first projection is given by the trace map tr_k. In particular, the induced map $tr_k : H_1(\mathrm{Der}^+(\mathcal{L}_n))_k \to S^k H_n$ gives an isomorphism stably for any $k \geq 2$.

Remark 4.3. The assumption that $n \geq k+2 \geq 4$ makes our statement for the computation of $H_1(\mathrm{Der}^+(\mathcal{L}_n))$ *weaker* than that of Kassabov [11], where he showed (after reformulated by the trace maps) that $tr_k : H_1(\mathrm{Der}^+(\mathcal{L}_n))_k \to S^k H_n$ is an isomorphism if $n(n-1) \geq k \geq 2$. However, our statement has the following advantages at the cost of the stronger assumption:

- The proof is shorter and easily accessible (especially by using diagrams).
- We show that $\big[\,\mathrm{Der}(\mathcal{L}_n)(k-1), \mathrm{Der}(\mathcal{L}_n)(1)\big] = \sum_{\substack{i+j=k \\ i,j\geq 1}} \big[\,\mathrm{Der}(\mathcal{L}_n)(i), \mathrm{Der}(\mathcal{L}_n)(j)\big]$, which says more than the computation of the abelianization.
- The argument works also for corresponding modules over $\mathbb{Z}$.

Proof of Theorem 4.2. Let Γ be the generating set of $\mathrm{Der}(\mathcal{L}_n)(k)$ consisting of all elements of the form

$$x_l^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}].$$

First we make the set Γ smaller as a generating set of the quotient

$$Q := \mathrm{Der}(\mathcal{L}_n)(k)/\big[\,\mathrm{Der}(\mathcal{L}_n)(k-1), \mathrm{Der}(\mathcal{L}_n)(1)\big].$$

Suppose an element $x_l^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] \in \Gamma$ is given.

(Case 1) When $l \neq i_1, i_2$, we have an equality

$$\begin{aligned} x_l^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}] &= [x_l^* \otimes [x_{i_1}, x_{i_2}],\ x_l^* \otimes [x_l, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}]] \\ &\quad - \sum_{j=3}^{k+1} \delta_{i_j,l} x_l^* \otimes [x_l, x_{i_3}, \ldots, x_{i_{j-1}}, [x_{i_1}, x_{i_2}], x_{i_{j+1}}, \ldots, x_{i_{k+1}}]. \end{aligned}$$

The second term of the right hand side is rewritten as

$$\begin{aligned} -\sum_{j=3}^{k+1} \delta_{i_j,l} (x_l^* &\otimes [x_l, x_{i_3}, \ldots, x_{i_{j-1}}, x_{i_1}, x_{i_2}, x_{i_{j+1}}, \ldots, x_{i_{k+1}}] \\ &- x_l^* \otimes [x_l, x_{i_3}, \ldots, x_{i_{j-1}}, x_{i_2}, x_{i_1}, x_{i_{j+1}}, \ldots, x_{i_{k+1}}]) \end{aligned}$$

by applying the Jacobi identity

$$[X, [x_{i_1}, x_{i_2}]] = -[x_{i_1}, [x_{i_2}, X]] - [x_{i_2}, [X, x_{i_1}]] = -[[X, x_{i_2}], x_{i_1}] + [[X, x_{i_1}], x_{i_2}]$$

with $X = [x_l, x_{i_3}, \ldots, x_{i_{j-1}}]$. Therefore the quotient Q can be generated by the elements $x_l^* \otimes [x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}]$ in Γ with $l = i_1$ and $l \neq i_2$.

(Case 2) For an element $x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]$ with $l \neq i_2$, we take an integer m from the set $\{1, 2, \ldots, n\} - \{i_2, i_3, \ldots, i_{k+1}\}$ which is not empty. Then we have

$$\begin{aligned} x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}] &= [x_m^* \otimes [x_l, x_{i_2}],\ x_l^* \otimes [x_m, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}]] \\ &\quad + x_m^* \otimes [x_m, x_{i_3}, x_{i_4}, \ldots, x_{i_{k+1}}, x_{i_2}]. \end{aligned}$$

This shows that the quotient Q can be generated by the elements in Γ of the form $x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]$ with $l \neq i_2, i_3, \ldots, i_{k+1}$.

(Case 3) Suppose an element $x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]$ of Γ with $l \neq i_2, i_3, \ldots, i_{k+1}$ is given. For every integer j with $2 \leq j \leq k$, we apply the Jacobi identity to $[[Y, x_{i_j}], x_{i_{j+1}}]$ with $Y = [x_l, x_{i_2}, \ldots, x_{i_{j-1}}]$. Then we have an equality

$$\begin{aligned} x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_j}, x_{i_{j+1}}, \ldots, x_{i_{k+1}}] &= x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{j-1}}, [x_{i_j}, x_{i_{j+1}}], x_{i_{j+2}}, \ldots, x_{i_{k+1}}] \\ &\quad + x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{j-1}}, x_{i_{j+1}}, x_{i_j}, x_{i_{j+2}}, \ldots, x_{i_{k+1}}]. \end{aligned}$$

As for the first term of the right hand side, we take an integer m from $\{1, 2, \ldots, n\} - \{l, i_2, i_3, \ldots, i_{k+1}\}$ which is not empty and consider the equality

$$\begin{aligned} &x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{j-1}}, [x_{i_j}, x_{i_{j+1}}], x_{i_{j+2}}, \ldots, x_{i_{k+1}}] \\ &= [x_m^* \otimes [x_{i_j}, x_{i_{j+1}}],\ x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{j-1}}, x_m, x_{i_{j+2}}, \ldots, x_{i_{k+1}}]]. \end{aligned}$$

Consequently the equality

$$x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_j}, x_{i_{j+1}}, \ldots, x_{i_{k+1}}] = x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{j-1}}, x_{i_{j+1}}, x_{i_j}, x_{i_{j+2}}, \ldots, x_{i_{k+1}}]$$

holds as an element of the quotient Q. In particular, we see that the element $x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]$ in Q with $l \neq i_2, i_3, \ldots, i_{k+1}$ is invariant under the permutation of the indices $x_{i_2}, x_{i_3}, \ldots, x_{i_{k+2}}$. Moreover the equality

$$\begin{aligned} x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}] &= [x_m^* \otimes [x_l, x_{i_2}],\ x_l^* \otimes [x_m, x_{i_3}, \ldots, x_{i_{k+1}}]] \\ &\quad + x_m^* \otimes [x_m, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] \end{aligned}$$

shows that as elements of the quotient Q, we have

$$x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}] = x_m^* \otimes [x_m, x_{i_3}, \ldots, x_{i_{k+1}}, x_{i_2}] = x_m^* \otimes [x_m, x_{i_2}, \ldots, x_{i_{k+1}}]$$

as long as $l, m \neq i_2, i_3, \ldots, i_{k+1}$.

For every $k \geq 2$, define a homomorphism $\Phi_k : S^k H_n \to Q$ by

$$\Phi_k(x_{i_2} x_{i_3} \cdots x_{i_{k+1}}) = x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]$$

where l is chosen for each generator $x_{i_2} x_{i_3} \cdots x_{i_{k+1}}$ of $S^k H_n$ so that $l \neq i_2, i_3, \ldots, i_{k+1}$. The argument in the previous paragraphs shows that Φ_k is well-defined, independent of the choices of l and surjective.

On the other hand, it follows from what was mentioned just before the statement of Theorem 4.2, we have a homomorphism

$$tr_k : Q \longrightarrow S^k H_n$$

and it is easily checked that

$$tr_k(x_l^* \otimes [x_l, x_{i_2}, \ldots, x_{i_{k+1}}]) = x_{i_2} x_{i_3} \cdots x_{i_{k+1}}$$

if $l \neq i_2, i_3, \ldots, i_{k+1}$. Therefore we have $tr_k \circ \Phi_k = \mathrm{id}_{S^k H_n}$ implying that $tr_k : Q \to S^k H_n$ is an isomorphism. This completes the proof. □

By an argument similar to the proof of Corollary 3.3, we have the following, which proves Theorem 1.1 (ii).

Corollary 4.4. (1) *For any $n \geq 2$, the natural pairing $H_n^* \otimes H_n \to \mathbb{Q}$ induces an isomorphism $H_1(\mathrm{Der}(\mathcal{L}_n))_0 \cong \mathbb{Q}$.*
(2) *For any $n \geq 2$, we have $H_1(\mathrm{Der}(\mathcal{L}_n))_1 = 0$.*
(3) *If $n \geq k+2 \geq 4$, we have $H_1(\mathrm{Der}(\mathcal{L}_n))_k = 0$.*

5. Symplectic derivation Lie algebra of the free associative algebra

Let $H_1(\Sigma_g;\mathbb{Q})$ be the first rational homology group of a closed oriented surface Σ_g of genus $g \geq 2$. We have $H_1(\Sigma_g;\mathbb{Q}) \cong H_{2g}$, a $2g$-dimensional vector space over $\mathbb{Q}$. This vector space has a natural intersection form

$$\mu : H_1(\Sigma_g;\mathbb{Q}) \otimes H_1(\Sigma_g;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

which is non-degenerate and skew symmetric. Let $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ be a symplectic basis of $H_1(\Sigma_g;\mathbb{Q})$ with respect to μ, namely

$$\mu(a_i,a_j)=0, \quad \mu(b_i,b_j)=0, \quad \mu(a_i,b_j)=\delta_{ij}.$$

The Poincaré duality gives a canonical isomorphism of $H_1(\Sigma_g;\mathbb{Q})$ with its dual vector space $H_1(\Sigma_g;\mathbb{Q})^* = H^1(\Sigma_g;\mathbb{Q})$, the first rational cohomology group of Σ_g. In this isomorphism, a_i (resp. b_i) $\in H_1(\Sigma_g;\mathbb{Q})$ corresponds to $-b_i^*$ (resp. a_i^*) $\in H^1(\Sigma_g;\mathbb{Q})$ where $\{a_1^*,\dots,a_g^*,b_1^*,\dots,b_g^*\}$ is the dual basis of $H^1(\Sigma_g;\mathbb{Q})$. We denote these canonically isomorphic vector spaces by H for simplicity. We write $\mathrm{Sp}(H)$ for the symplectic transformation group of H. It consists of all automorphisms of H preserving μ.

Denote the symplectic class by

$$\omega_0 = \sum_{i=1}^{g}(a_i \otimes b_i - b_i \otimes a_i) \in H \otimes H,$$

which is independent of the choice of a symplectic basis of H and is invariant under the action of $\mathrm{Sp}(H)$. A derivation $D \in \mathrm{Der}(T(H))$ is said to be *symplectic* if it satisfies $D(\omega_0)=0$. Let $\mathfrak{a}_g$ be the subset of $\mathrm{Der}(T(H))$ consisting of all symplectic derivations. It is easily checked that $\mathfrak{a}_g$ is a Lie subalgebra of $\mathrm{Der}(T(H))$. This Lie algebra was first studied by Kontsevich [13, 14] (see Section 6). A grading of $\mathfrak{a}_g$ is induced from $\mathrm{Der}(T(H))$ and define $\mathfrak{a}_g(k)$ to be its degree k homogeneous part. Namely we have

$$\mathfrak{a}_g = \bigoplus_{k\geq 0}\mathfrak{a}_g(k).$$

Note that $\mathfrak{a}_g(0) \cong \mathfrak{sp}(H) \cong S^2H$, the symplectic Lie algebra.

We also define a Lie subalgebra $\mathfrak{a}_g^+ := \bigoplus_{k\geq 1}\mathfrak{a}_g(k)$ consisting of all derivations of positive degrees. It is known (see [17, Proposition 2]) that under the identification

$$\mathrm{Hom}(H, H^{\otimes(k+1)}) = H \otimes H^{\otimes(k+1)} = H^{\otimes(k+2)},$$

the degree k part $\mathfrak{a}_g(k) \subset \mathrm{Hom}(H, H^{\otimes(k+1)})$ is rewritten as

$$\mathfrak{a}_g(k) = \left(H^{\otimes(k+2)}\right)^{\mathbb{Z}/(k+2)\mathbb{Z}},$$

where the right hand side is the invariant part of $H^{\otimes(k+2)}$ with respect to the action of the group $\mathbb{Z}/(k+2)\mathbb{Z}$ as cyclic permutations of elements in $H^{\otimes(k+2)}$. In particular,

$$\mathfrak{a}_g(1) = \left(H^{\otimes 3}\right)^{\mathbb{Z}/3\mathbb{Z}} \cong S^3 H \oplus \wedge^3 H.$$

In this section, we determine the abelianizations of $\mathfrak{a}_g$ and $\mathfrak{a}_g^+$. First we consider the latter. The weight 1 part $H_1(\mathfrak{a}_g^+)_1$ of $H_1(\mathfrak{a}_g^+)$ is given by $\mathfrak{a}_g(1)$. The weight 2 part $H_1(\mathfrak{a}_g^+)_2$ was calculated by the first author in [17, Theorem 6] and it is given by

$$H_1(\mathfrak{a}_g^+)_2 \cong \wedge^2 H/\langle \omega_0 \rangle$$

as $\mathrm{Sp}(H)$-modules, where $\langle \omega_0 \rangle$ denotes the submodule of $\wedge^2 H$ spanned by ω_0 as an element of $\wedge^2 H \subset H \otimes H$. We now prove the following:

Theorem 5.1. *If $g \geq k+3 \geq 6$, then $H_1(\mathfrak{a}_g^+)_k = 0$.*

For the proof of this theorem, we use more diagrammatic-minded argument than those in the previous cases. We introduce *spiders* and *chord diagrams* which play important roles in our proof.

The vector space $\mathfrak{a}_g(k) = \left(H^{\otimes(k+2)}\right)^{\mathbb{Z}/(k+2)\mathbb{Z}}$ is generated by vectors of the form

$$S(i_1, i_2, \ldots, i_{k+2}) := \sum_{j=1}^{k+2} \sigma_{k+2}^j (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_{k+2}}),$$

where $i_1, i_2, \ldots, i_{k+2} \in \{\pm 1, \pm 2, \ldots, \pm g\}$, $a_l := b_{-l}$ for $l < 0$ and σ_{k+2} is a generator of $\mathbb{Z}/(k+2)\mathbb{Z}$. We call such a vector $S(i_1, i_2, \ldots, i_{k+2})$ a *spider* (see also Conant-Vogtmann [3]). In a natural way, we can represent a spider in $\mathfrak{a}_g(k)$ by a graph with one $(k+2)$-valent vertex and $(k+2)$ univalent vertices, each of which is colored by an element in $\{\pm 1, \pm 2, \ldots, \pm g\}$ corresponding to the basis of H and is connected by an edge called a *leg* to the $(k+2)$-valent vertex. The edges (and hence vertices) are ordered cyclically. For example, the left of Figure 10 represents the spider $S(1, 4, -2, -1, 2, -1, 2, 1) = S(4, -2, -1, 2, -1, 2, 1, 1) = \cdots$.

For two spiders $S_1 = S(i_1, i_2, \ldots, i_{p+2}) \in \mathfrak{a}_g(p)$ and $S_2 = S(j_1, j_2, \ldots, j_{q+2}) \in \mathfrak{a}_g(q)$, their bracket $[S_1, S_2] \in \mathfrak{a}_g(p+q)$ is diagrammatically given by the formula shown in Figure 9.

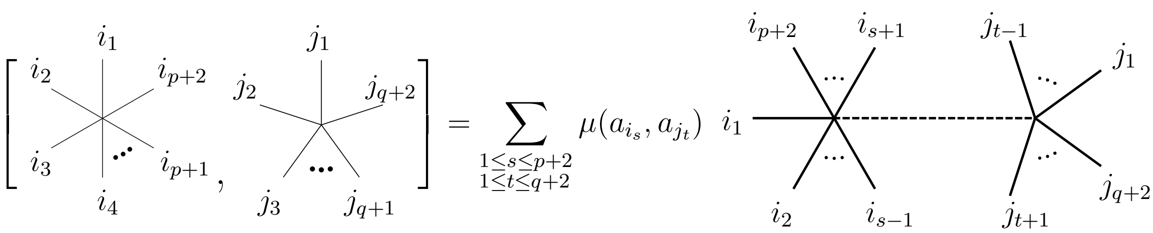

FIGURE 9. Bracket of spiders, where the dashed line in the right hand side is collapsed to a point to make a new spider

To a spider S, we associate a *chord diagram* $C(S)$ (in a generalized sense) so that the vertices of $C(S)$ are ordered and colored according to the legs of S and two vertices are connected by a chord if their colors differ by sign. (Two vertices with the same color are not connected.) We identify a spider with the corresponding chord diagram.

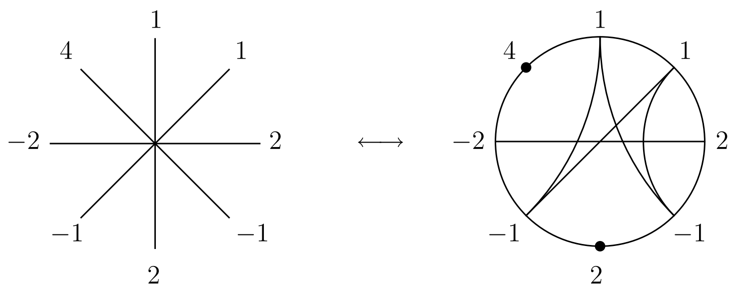

FIGURE 10. A spider and a chord diagram

Definition 5.2. A vertex v of a chord diagram is said to be

(a) *unpaired* if it is not connected to any other vertex by a chord.
(b) *single paired* if it is connected to only one other vertex, say w, by a chord and w is connected to only v.
(c) *multiple paired* if it is neither unpaired nor single paired.

By abuse of notation, we also say "a color i is *unpaired*", "a chord is *single paired*", etc.

Definition 5.3. For a chord diagram C, its *multiplicity* $m(C)$ is defined by

$$m(C) = 2(\text{number of chords}) - (\text{number of vertices having chords}).$$

For example, the multiplicity of the chord diagram in Figure 10 is 4. The multiplicity of a chord diagram without multiple paired vertices is zero by definition. Note that the multiplicity only depends on the set of colors of the diagram.

Definition 5.4. A chord diagram C is said to be *separable* if there exists an arc inside the outer circle of C connecting two points of the outer circle which are not vertices of C such that each region separated by the arc has at least two vertices and the arc does not intersect with the chords.

Lemma 5.5. *If $g \geq k+3 \geq 6$ and the chord diagram $C(S)$ of a spider S is separable, then $C(S)$ is in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$.*

Proof. Cut the chord diagram $C(S)$ by an arc separating it and for each region glue the two endpoints of the piece of the outer circle. We put vertices for the identified points and give them colors with opposite sign that are distinct from those possessed by $C(S)$, which is possible by the assumption $g \geq k+3$. The new chord diagrams C_1 and C_2 satisfy $[C_1, C_2] = C(S)$. □

Lemma 5.6. *If $g \geq k+3 \geq 6$, then the quotient $\mathfrak{a}_g(k)/\sum_{\substack{i+j=k\\ i,j\geq 1}}[\mathfrak{a}_g(i), \mathfrak{a}_g(j)]$ is generated by spiders corresponding to chord diagrams of the form, which we call the standard form, shown in Figure 11.*

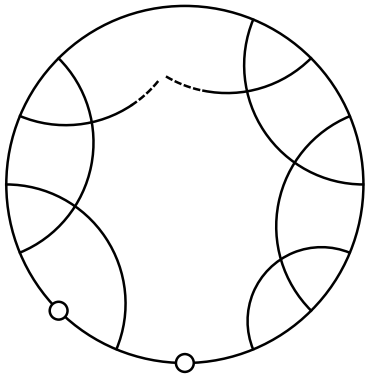

FIGURE 11. The standard form, where *each of white vertices may not exist*

Proof. It suffices to exhibit an algorithm by which a given chord diagram $C(S)$ corresponding to a spider S is rewritten modulo brackets as a linear combination of chord diagrams of the standard form.

Suppose we are given a chord diagram C corresponding to a spider with multiplicity $m(C)$. We may assume that C is not separable.

If C does not have a single paired chord, take two adjacent vertices. By using the colors i, j of these vertices, we can write $C = C(S(i,j,X))$ for some word X of colors with length bigger than 2. Then we have

$$\begin{aligned} S(i,j,X) &= [S(X,n),\ S(-n,i,j)] \\ &\quad + \sum_{\substack{\text{color } -i \\ \text{in } X}} \pm S(n,Z_1,j,-n,Z_2) + \sum_{\substack{\text{color } -j \\ \text{in } X}} \pm S(n,Z_1,-n,i,Z_2) \end{aligned}$$

where $n > 0$ and $-n$ are colors not possessed by C and Z_1, Z_2 are some words. While the words Z_1, Z_2 differ in each term of the summation, precisely speaking, we use the same letters here for simplicity. In the right hand side, each of $S(n,Z_1,j,-n,Z_2)$ and $S(n,Z_1,j,-n,Z_2)$ has a single paired color n and has multiplicity not bigger than $m(C)$.

Now we inductively show that a chord diagram C having the configuration $\mathcal{F}_l$ ($l = 1,2,\ldots$) shown in Figure 12, where the vertices colored by $\pm c_1, \pm c_2, \ldots, \pm c_l$ (some c_i may be negative) are all single paired, is written as a linear combination of chord diagrams having the configuration $\mathcal{F}_{l+1}$ and having multiplicities not bigger than $m(C)$ modulo brackets unless it is already of the standard form.

(The first step) By the argument in the third paragraph of this proof, we may assume that the chord diagram C has at least one single paired chord colored by $\pm c_1$. Let X and Y be the regions separated by the single paired chord so that the diagram C corresponds to the spider $S(c_1, X, -c_1, Y)$.

If X or Y has no vertices, then C is separable and we are done.

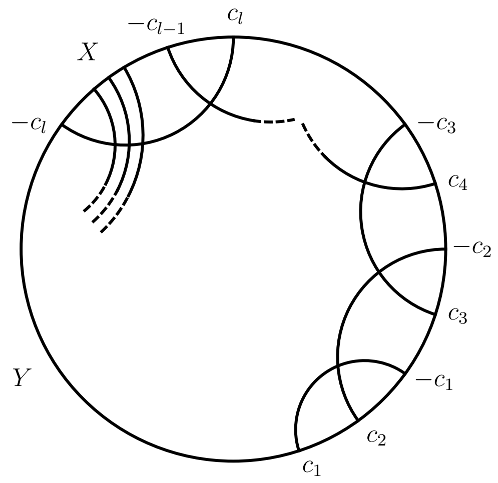

FIGURE 12. The configuration $\mathcal{F}_l$

If X has at least two vertices, we have

$$S(c_1, X, -c_1, Y) = [S(c_1, n, -c_1, Y),\ S(-n, X)] + \sum_{\substack{\text{pairing of} \\ X \text{ and } Y}} \pm S(c_1, n, -c_1, Z_1, -n, Z_2)$$

where $n > 0$ and $-n$ are new colors as before (hereafter we omit these words about the new color n). Each of the spiders $S(c_1, n, -c_1, Z_1, -n, Z_2)$ has the configuration $\mathcal{F}_2$ with $c_2 = n$ and multiplicity not bigger than $m(C)$.

If X has only one vertex v, then Y has at least two vertices since $k \geq 3$ and the diagram C corresponds to the spider $S(c_1, c_v, -c_1, Y)$, where c_v is the color of v. In this case, there are three possibilities:

(a) If v is unpaired, then it is separable.
(b) If v is single paired, then C has the configuration $\mathcal{F}_2$ with $c_2 = c_v$.
(c) If v is multiple paired, consider the equality

$$S(c_1, c_v, -c_1, Y) = [S(c_1, c_v, -c_1, n),\ S(-n, Y)] + \sum_{\substack{\text{color } -c_v \\ \text{in } Y}} \pm S(-c_1, n, c_1, Z_1, -n, Z_2).$$

Each of the spiders $S(-c_1, n, c_1, Z_1, -n, Z_2)$ has the configuration $\mathcal{F}_2$ and multiplicity less than $m(C)$ since a pair of multiple paired vertices colored by $\pm c_v$ was exchanged for single paired vertices colored by $\pm n$.

In any case, we have checked that we can proceed to the next step.

(The inductive step) Suppose that any chord diagram having the configuration $\mathcal{F}_i$ ($i = 1, 2, 3, \ldots, l-1$) is written as a linear combination of chord diagrams having the configuration $\mathcal{F}_{i+1}$ and having multiplicities not bigger than $m(C)$. Let C be a chord diagram having the configuration $\mathcal{F}_l$ as in Figure 11, where X is the region between the vertices colored by $-c_{l-1}$ and $-c_l$ and Y is the region between the vertices colored by $-c_l$ and c_1.

(I) Suppose that X has no vertices. If Y has at most one vertex, the diagram C is of standard form. Otherwise, Y has at least two vertices. Therefore C is separable.

(II) Suppose that X has at least two vertices. Then C corresponds to the spider $S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, X, -c_l, Y)$. Consider the equality

$$\begin{aligned} &S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, X, -c_l, Y) \\ &\quad = [S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, n, -c_l, Y), S(-n, X)] \\ &\qquad + \sum_{\substack{\text{pairing of} \\ X \text{ and } Y}} \pm S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, n, -c_l, Z_1, -n, Z_2). \end{aligned}$$

Each of the spiders $S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, n, -c_l, Z_1, -n, Z_2)$ has the configuration $\mathcal{F}_{l+1}$ and multiplicity not bigger than $m(C)$.

(III) Suppose that X has only one vertex v. Let c_v be the color of v.

III-a Suppose that v is unpaired. If Y has at most one vertex, then C is of standard form. Otherwise, C is separable.

III-b If v is single paired, C has the configuration of $\mathcal{F}_{l+1}$.

III-c If v is multiple paired, then Y has at least two vertices. Consider the equality

$$\begin{aligned} &S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, c_v, -c_l, Y) \\ &\quad = [S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, c_v, -c_l, n), S(-n, Y)] \\ &\qquad + \sum_{\substack{\text{color } -c_v \\ \text{in } Y}} \pm S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, Z_1, -n, Z_2, -c_l, n). \end{aligned}$$

Each of the spiders $S(c_1, c_2, -c_1, c_3, \dots, c_l, -c_{l-1}, Z_1, -n, Z_2, -c_l, n)$ has the configuration $\mathcal{F}_{l-1}$, namely we have stepped backward. However their multiplicity are less than $m(C)$ since a pair of multiple paired vertices colored by $\pm c_v$ was exchanged for single paired vertices colored by $\pm n$. Hence in repeating this rewriting process, we meet this case at most $m(C)$ times and we can finally go to the next step.

Therefore the induction works and we finish the proof. □

Next we introduce *chord slides* for chord diagrams to show that a chord diagram of the standard form is in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$. Hereafter we assume that all chord diagrams have no multiple paired vertices. Consider spiders having two adjacent vertices colored by i and j, which may be negative. Suppose first that both i and j are single paired. Then we have the following equalities:

$$\begin{aligned} S(X,i,j,Y,-j,Z,-i) &= \mathrm{sign}(n)[S(X,n,Y,-j,Z,-i), S(i,j,-n)] \\ &\quad + \mathrm{sign}(ni)S(X,n,Y,-j,Z,j,-n) \\ &\quad + \mathrm{sign}(nj)S(X,n,Y,-n,i,Z,-i), \\ S(X,i,j,Y,-i,Z,-j) &= \mathrm{sign}(n)[S(X,n,Y,-i,Z,-j), S(i,j,-n)] \\ &\quad + \mathrm{sign}(ni)S(X,n,Y,-i,Z,-n,i) \\ &\quad + \mathrm{sign}(nj)S(X,n,Y,j,-n,Z,-j), \end{aligned}$$

where $\mathrm{sign}(m) \in \{\pm 1\}$ denotes the sign of an integer $m \neq 0$. These equalities are diagrammatically expressed (up to sign) as in the first two equalities of Figure 13, which looks like "chord slides to two directions". Next suppose that i is single paired and j is unpaired. Then we have

$$\begin{aligned} S(X,i,j,Y,-i) &= \mathrm{sign}(n)[S(X,n,Y,-i), S(i,j,-n)] + \mathrm{sign}(ni)S(X,n,Y,j,-n), \\ S(X,j,i,Y,-i) &= \mathrm{sign}(n)[S(X,n,Y,-i), S(j,i,-n)] + \mathrm{sign}(ni)S(X,n,Y,-n,j), \end{aligned}$$

where we again assume that $n > 0$. These equalities are diagrammatically expressed (up to sign) as in the last two equalities of Figure 13. Note that in every case of the above, the color of the edge on which another chord slides changes after a chord slide.

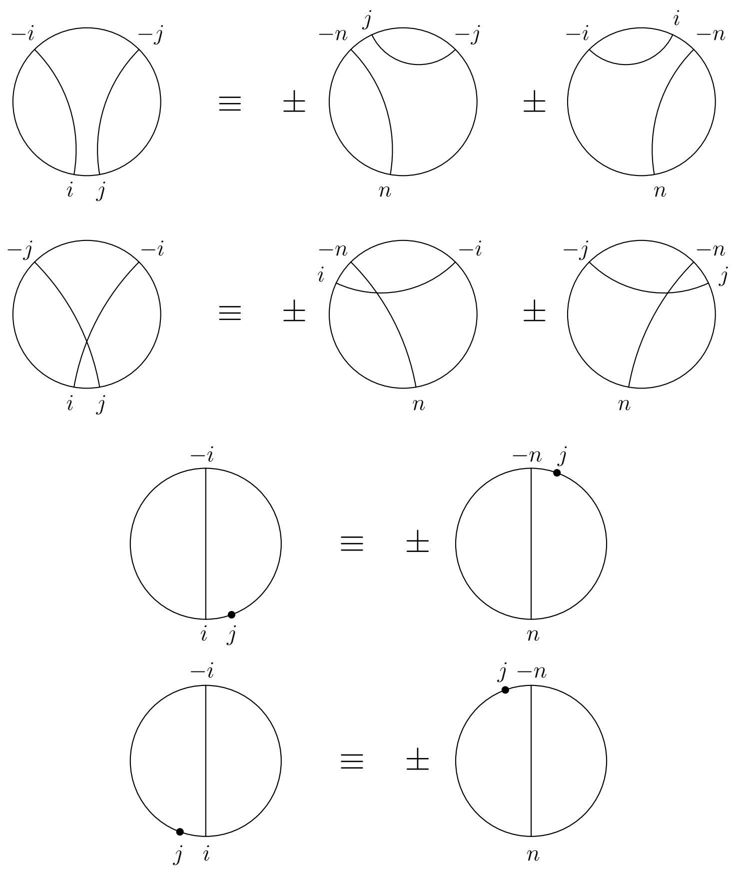

FIGURE 13. Chord slides

In using chord slides, the following observation is easy but important. Let Σ be a surface obtained from a chord diagram of the standard form with $l \geq 2$ single paired chords by fattening, where we ignore the crossings inside the outer circle. The boundary $\partial\Sigma$ of Σ contains the outer circle and we call the other components of $\partial\Sigma$ the *inner boundary*.

Lemma 5.7. *The inner boundary of Σ is connected when l is even and consists of two connected components if l is odd.*

Proof. It is easy to see that the statement holds for $l = 2$. Then we can inductively check that the statement holds for general cases by comparing the connection of the boundary before and after adding a new chord. □

Proof of Theorem 5.1 when $k \equiv 0 \pmod 4$. There are two patterns of the standard form. The first one consists of two unpaired vertices and an even number of single paired chords. In this case, we can slide the unpaired vertices so that they are adjacent, which is possible because the inner boundary of the fattened surface is connected. Then the chord diagram becomes separable.

The second pattern consists of no unpaired vertex and an odd number of single paired chords. To treat this pattern, we first consider a chord diagram having the configuration of the standard form with one more chord l intersecting with the others at one point as in the left hand side of Figure 14. For such a diagram, we can move the intersecting point by a chord slide as shown in the same figure, where the second diagram of the right hand side is separable.

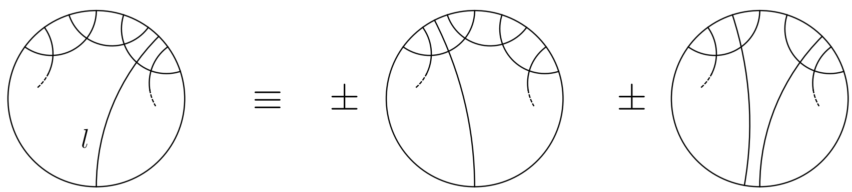

FIGURE 14. Sliding the intersection

Now take a chord diagram of the standard form consisting of $(k+2)/2$ chords as in the left hand side of Figure 15. We may consider it to be a diagram of the standard form consisting of $k/2$ chords with one more chord colored by $\pm c_1$. To this diagram, we apply the chord slide discussed above with regarding the chord colored by $\pm c_1$ as l. By iterating chord slides, we get to the chord diagram of the right hand side of Figure 15, which is shown to be in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$ by considering the result of the chord slide at $*$ (see also the first line of Figure 16). □

Hereafter we call the operation used in the second pattern of the above (i.e. moving the chord colored by $\pm c_1$ from right to left) a *chord cycling*.

Proof of Theorem 5.1 when $k \equiv 1 \pmod 4$. In this case, the standard form consists of a unique unpaired vertex and an odd number of single paired chords. Then by a chord

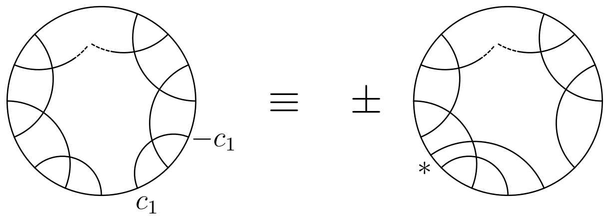

FIGURE 15. The right hand side is the final stage of a chord cycling

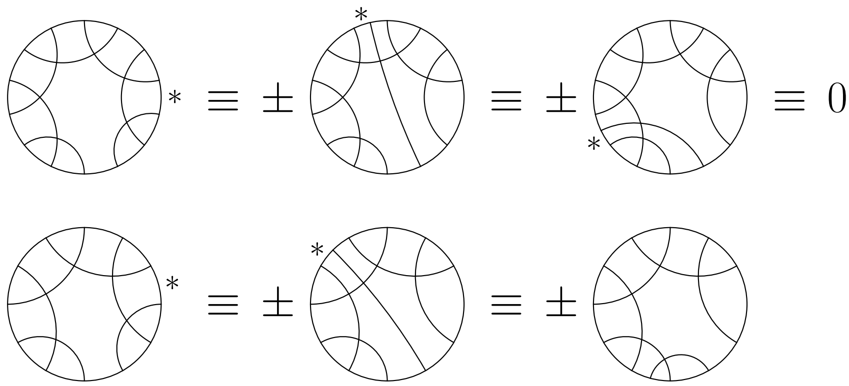

FIGURE 16. Chord cyclings for odd and even numbers of chords

cycling with ignoring the unpaired vertex, we can slide the diagram to a separable one. □

In the remaining two cases, we can apply the same argument as above only to chord diagrams of the standard form consisting of two unpaired vertices and an odd number of single paired chords, when $k \equiv 2 \pmod 4$. Therefore we can finish the proof of Theorem 5.1 by considering the following two types of chord diagrams (see Figure 17):

(a) chord diagrams of the standard form consisting of no unpaired vertices and $(2l+2)$ single paired chords, where $k = 4l+2$,
(b) chord diagrams of the standard form consisting of one unpaired vertex and $(2l+2)$ single paired chords, where $k = 4l+3$.

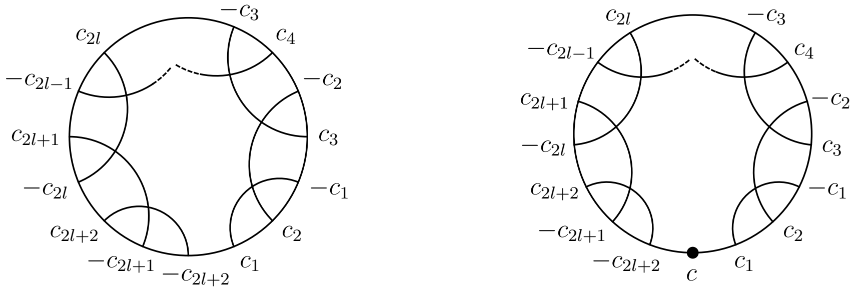

FIGURE 17. Type (a) and Type (b)

For each of them, a chord cycling results to another chord diagram of the standard form with distinct colors (see the second line of Figure 16).

Lemma 5.8. *Under the assumption $g \geq k+3 \geq 6$, we have the following.*

(1) *Every chord diagram of Type* (a) *shown in the left of Figure 17 is transformed up to sign to the one with $c_i = i$ for $i = 1, 2, \dots, 2l+2$ by chord slides.*

(2) *Let C be a chord diagram of Type* (b) *whose unique unpaired vertex is colored by c as shown in the right of Figure 17. Then for any fixed colors $\{d_1, d_2, \dots, d_{2l+2}\}$ consisting of positive integers and not including $\pm c$, the diagram C is transformed up to sign to the one with $c_i = d_i$ for $i = 1, 2, \dots, 2l+2$ by chord slides.*

Proof. (1) Let C be a chord diagram of Type (a) as in the left of Figure 17. By the assumption $g \geq 4l+5$, we have

$$\{1, 2, \dots, g\} - \{1, 2, \dots, 2l+2, |c_1|, |c_2|, \dots, |c_{2l+2}|\} \neq \emptyset.$$

This means that every time we apply a chord slide, we can choose an integer from this set as a new color, namely the integer n of the formulas in Figure 13.

For a chord diagram of the standard form, let $[e_1, e_2, \dots, e_{2l+2}]$ be the sequence of colors obtained by reading the color which appears for the first time for each of the chords while looking around the outer circle counter-clockwise. For example, the chord diagram C yields the sequence $[c_1, c_2, \dots, c_{2l+2}]$.

Taking account of the above discussion, we can apply a chord cycling to C so that the resulting chord diagram of the standard form is associated with the sequence

$$[c_2, n_3, c_4, n_5, \dots, n_{2l+1}, c_{2l+2}, c_1]$$

where $n_3, n_5, \dots, n_{2l+1} \in \{1, 2, \dots, g\} - \{1, 2, \dots, 2l+2\}$. By iterating chord cyclings, we obtain chord diagrams of the standard form associated with the sequences

$$\begin{aligned}
&[c_2, n_3, c_4, n_5, \dots, n_{2l+1}, c_{2l+2}, c_1] \\
\longrightarrow\ &[n_3, n_4, n_5, \dots, n_{2l+1}, n_{2l+2}, c_1, c_2] \\
\longrightarrow\ &[n_4, n'_5, \dots, n'_{2l+1}, n_{2l+2}, n_1, c_2, n_3] \\
\longrightarrow\ &[n'_5, n'_6, \dots, n'_{2l+1}, n'_{2l+2}, n_1, n_2, n_3, n_4] \\
\longrightarrow\ &[n'_6, 2l+1, n'_8, n''_9, \dots, n''_{2l+1}, n'_{2l+2}, n''_1, n_2, n''_3, n_4, n'_5] \\
\longrightarrow\ &[2l+1, 2l+2, n''_9, n''_{10}, \dots, n''_{2l+1}, n''_{2l+2}, n''_1, n''_2, n''_3, n''_4, n'_5, n'_6] \\
\longrightarrow\ &[2l+2, 1, n''_{10}, 3, n''_{12}, 5, \dots, 2l-4, n''_{2l+2}, 2l-3, n''_2, 2l-2, n''_4, 2l-1, n'_6, 2l+1] \\
\longrightarrow\ &[1, 2, 3, \dots, 2l+1, 2l+2]
\end{aligned}$$

where the positive integers n_i, n'_j, n''_k are taken from $\{1, 2, \dots, g\} - \{1, 2, \dots, 2l+2\}$. Our claim follows from this. Note that the above argument works also for small l.

(2) Take a chord diagram of Type (b) shown in the right of Figure 17. Since the inner boundary is connected, we can slide the unique unpaired vertex along all chords so that the colors of the other vertices are changed as indicated. This is possible because the assumption $g \geq 4l+6$ implies that

$$\{1, 2, \dots, g\} - \{|c|, |c_1|, |c_2|, \dots, |c_{2l+2}|, |d_1|, |d_2|, \dots, |d_{2l+2}|\} \neq \emptyset,$$

which enables us to use an argument similar to (1). □

To show that the chord diagrams specialized in Lemma 5.8 are in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$, we use the following mirror image argument. For a spider S, we define its *mirror* S^m as the spider obtained from S by sorting its legs in reverse order. In terms of chord diagrams, the chord diagram $C(S^m)$ is obtained from $C(S)$ by taking its mirror image. The following lemma is easily checked.

Lemma 5.9. *For spiders S_1 and S_2, their bracket $[S_1^m, S_2^m]$ is obtained from $[S_1, S_2]$ by taking the mirror for each spider in it.*

Proof of Theorem 5.1 when $k \equiv 2 \pmod 4$. There are two patterns of the standard form. The first one consists of two unpaired vertices and an odd number of single paired chords. In this case, we can use chord cyclings with ignoring the unpaired vertices to show that the chord diagram is in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$ as in the cases where $k \equiv 0, 1 \pmod 4$.

The second one is of Type (a), where $k = 4l + 2$. By Lemma 5.8, it suffices to show that the spider

$$\widetilde{S} = S(1, 2, -1, 3, -2, 4, \dots, -2l, 2l + 2, -(2l + 1), -(2l + 2))$$

is in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$. For that, we "divide" the corresponding chord diagram at the center of the chain of chords. That is, we consider the equality

$$\begin{aligned}\widetilde{S} = {} & [S(1, 2, \dots, \underline{l + 1}, -l, \underline{l + 2}, n),\ S(-n, \underline{-(l + 1)}, l + 3, \underline{-(l + 2)}, \dots, -(2l + 2))] \\ & - S(1, 2, \dots, -(l - 1), \underline{l + 3, -(l + 2), l + 4, \dots, -(2l + 1), -(2l + 2), -n}, -l, l + 2, n) \\ & - S(-n, -(l + 1), l + 3, \underline{n, 1, 2, \dots, l + 1, -l}, l + 4, \dots, -(2l + 1), -(2l + 2)).\end{aligned}$$

Here we remark that the third term of the right hand side is obtained up to sign from the second term by taking its mirror and applying the symplectic action

$$\begin{aligned}& a_i \longmapsto -b_{2l+3-i}, \quad b_i \longmapsto a_{2l+3-i} \quad (i = 1, 2, \dots, 2l + 2), \\ & a_n \longmapsto -b_n, \quad b_n \longmapsto a_n.\end{aligned}$$

We use Lemma 5.6 to rewrite the second term as the linear combination P of chord diagrams of the standard form. As for the third term, Lemma 5.9 and the fact that the bracket operation is equivariant with respect to the symplectic action show that we can rewrite it as the linear combination Q obtained from P by taking the mirror and applying the symplectic action to each chord diagram. It follows from Lemma 5.8 that the sum $P + Q$ is rewritten as $2m\widetilde{S}$ by some even number $2m$. Therefore we have $\widetilde{S} \equiv 2m\widetilde{S}$, which implies that $\widetilde{S} \equiv 0$ in $H_1(\mathfrak{a}_g^+)$. □

Proof of Theorem 5.1 when $k \equiv 3 \pmod 4$. The standard form consists of a unique unpaired vertex and $(2l + 2)$ single paired chords, where $k = 4l + 3$. By Lemma 5.8, it suffices to show that the spider

$$S(c, d_1, d_2, -d_1, d_3, -d_2, d_4, \dots, -d_{2l}, d_{2l+2}, -d_{2l+1}, -d_{2l+2})$$

is in $\mathrm{Im}[\,\cdot\,,\,\cdot\,]$. For that, we can use almost the same argument as the case where $k \equiv 2 \pmod 4$ by ignoring the unique unpaired vertex. Note that the algorithm of Lemma 5.6 keeps the color c of the unpaired vertex. □

Proof of Theorem 1.1 (iii). By an argument similar to the previous cases, we have

$$\lim_{g\to\infty} H_1(\mathfrak{a}_g) \cong \lim_{g\to\infty} H_1(\mathfrak{a}_g^+)_{\mathfrak{sp}}.$$

See also Conant-Vogtmann [3, Proposition 8]. Now the first author's computation [17, Theorem 6] and Theorem 5.1 show that

$$\begin{aligned}\lim_{g\to\infty} H_1(\mathfrak{a}_g^+)_{\mathfrak{sp}} &\cong \lim_{g\to\infty} (\mathfrak{a}_g(1) \oplus (\wedge^2 H/\langle\omega_0\rangle))_{\mathfrak{sp}} \\ &\cong \lim_{g\to\infty} (S^3 H \oplus \wedge^3 H \oplus (\wedge^2 H/\langle\omega_0\rangle))_{\mathfrak{sp}} = 0.\end{aligned}$$

This completes the proof. □

6. Application to cohomology of moduli spaces of curves

In this section, we apply one of our main theorems, Theorem 1.1, to obtain a new proof of the vanishing theorem of Harer (Theorem 1.4).

We denote by $\mathbf{M}_g^m$ the moduli space of curves of genus g with m distinct marked points where we assume $2g-2+m>0$. Then as is well known, there exists a canonical isomorphism

$$H^*(\mathbf{M}_g^m;\mathbb{Q}) \cong H^*(\mathcal{M}_g^m;\mathbb{Q})$$

where $\mathcal{M}_g^m$ denotes the mapping class group of Σ_g with m distinct marked points.

Harer proved the following foundational result.

Theorem 6.1 (Harer [8]). *The virtual cohomological dimension of $\mathcal{M}_g^m$ is given by*

$$\operatorname{vcd}\mathcal{M}_g^m = \begin{cases} 4g-5 & (g\geq 2, m=0) \\ 4g-4+m & (g>0, m>0) \\ m-3 & (g=0) \end{cases}$$

so that the rational cohomology group

$$H^k(\mathbf{M}_g^m;\mathbb{Q}) \cong H^k(\mathcal{M}_g^m;\mathbb{Q})$$

vanishes for any $k > \operatorname{vcd}\mathcal{M}_g^m$.

Now we prove the following result which gives an alternative proof of a theorem of Harer mentioned above.

Theorem 6.2. *For any $g\geq 2$, the top degree rational cohomology group of the moduli space $\mathbf{M}_g^m$ $(m=0,1)$ as well as the mapping class group $\mathcal{M}_g^m$ $(m=0,1)$, with respect to its virtual cohomological dimension, vanishes. More precisely, we have*

$$H^{4g-5}(\mathbf{M}_g;\mathbb{Q}) \cong H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0$$
$$H^{4g-3}(\mathbf{M}_g^1;\mathbb{Q}) \cong H^{4g-3}(\mathcal{M}_g^1;\mathbb{Q}) = 0$$

for any $g\geq 2$.

Remark 6.3. In contrast with the above result, the situation in the cases of genus 0 and 1 is completely different. According to Getzler [6], the Poincaré polynomial of $\mathbf{M}_0^m$ is given by

$$\sum_{i=0}^{n-2}(-t)^i\dim H^i(\mathbf{M}_0^m;\mathbb{Q}) = \prod_{i=2}^{n-2}(1-it).$$

In particular, the rational cohomology group of top degree $H^{m-3}(\mathbf{M}_0^m;\mathbb{Q})$ has dimension $(m-2)!$. In [7], Getzler also determined the $\mathfrak{S}_m$-equivariant Serre characteristic for $\mathbf{M}_1^m$. In particular, the top degree $\mathfrak{S}_m$-invariant rational cohomology group $H^m(\mathbf{M}_1^m;\mathbb{Q})^{\mathfrak{S}_m}$ is highly non-trivial for infinitely many m.

In [17], the first author combined the computational result of the weight 2 part $H_1(\mathfrak{a}_g^+)_2$ of the abelianization of $\mathfrak{a}_g^+$ with the theorem of Kontsevich cited below (Theorem 6.4), so that he constructed a series of cohomology classes in $H^{4m+1}(\mathbf{M}_1^{4m+1})^{\mathfrak{S}_{4m+1}}$ for $m=1,2,\cdots$. Then Conant [2] proved that these classes are all non-trivial. It would be an interesting problem to seek for possible special property of these classes among the whole classes which Getzler determined.

To prove Theorem 6.2, we recall a theorem of Kontsevich given in [13, 14] which is one of the three types of graph (co)homologies he presented, more precisely the *associative* version.

Theorem 6.4 (Kontsevich [13, 14]). *There exists an isomorphism*

$$PH_k\big(\lim_{g\to\infty}\mathfrak{a}_g\big)_{2n} \cong PH_k(\mathfrak{sp}(\infty)) \oplus \bigoplus_{\substack{2g-2+m=n\\ m>0}} H^{2n-k}(\mathbf{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m}.$$

Proof of Theorem 6.2. (1) First we prove the vanishing $H^{4g-3}(\mathcal{M}_g^1;\mathbb{Q})=0$ for any $g\geq 2$.

By Theorem 1.1, we know that $\lim_{g\to\infty}H_1(\mathfrak{a}_g)_{2n}=0$ for any n. If we substitute this in Theorem 6.4, then we obtain

$$H^{4g-5+2m}(\mathcal{M}_g^m;\mathbb{Q})^{\mathfrak{S}_m}=0 \quad \text{for any } m\geq 1.$$

If $g=0$, then $m\geq 3$ and hence $2m-5>m-3=\operatorname{vcd}\mathcal{M}_0^m$. It follows that the above vanishing is already known by Theorem 6.1. If $g\geq 1$, then $\operatorname{vcd}\mathcal{M}_g^m=4g-4+m$ again by Theorem 6.1. On the other hand, since we have $m\geq 1$ the inequality $4g-5+2m\geq 4g-4+m$ holds. Moreover the equality $4g-5+2m=4g-4+m$ holds if and only if $m=1$. We can now conclude that

$$H^{4g-3}(\mathcal{M}_g^1;\mathbb{Q})=0 \quad \text{for any } g\geq 2.$$

(2) Next, we deduce $H^{4g-5}(\mathcal{M}_g;\mathbb{Q})=0$ $(g\geq 2)$ from (1). For this, we recall the following result proved in [15]. Namely there exists an isomorphism

$$H^k(\mathcal{M}_g^1;\mathbb{Q})\cong H^{k-2}(\mathcal{M}_g;\mathbb{Q})\oplus H^{k-1}(\mathcal{M}_g;H)\oplus H^k(\mathcal{M}_g;\mathbb{Q})$$

which holds for any k. If we put $k=4g-3$ here, then the left hand side vanishes by (1). On the other hand, the last two terms on the right hand side vanish because their cohomology degrees, namely $4g-4$ and $4g-3$, exceed $\operatorname{vcd}\mathcal{M}_g=4g-5$. We can now conclude that $H^{4g-5}(\mathcal{M}_g;\mathbb{Q})=0$ as required. □

7. Concluding remarks

In this section, we make a few remarks concerning the ingredients of this paper.

Remark 7.1. We have been investigating not only the first homology groups of Lie algebras $\mathfrak{a}_g$ and $\mathfrak{h}_{g,1}$ but also higher homology groups as well. In particular, we had already a glimpse of considerable difference between the structures of $H_2(\mathfrak{a}_g)$ and $H_2(\mathfrak{h}_{g,1})$. We will discuss this in a forthcoming paper.

Remark 7.2. The *unstable* structure of the Lie algebras $\mathfrak{h}_{g,1}$ for small values of g, is very important in the study of the arithmetic mapping class group developed extensively by number theorists.

We are trying to find a route which would connect our topological study with the works of number theorists, in particular those of Hain and Matsumoto, and also those of Nakamura.

Remark 7.3. The Lie algebra $\mathfrak{a}_g$ appeared in a recent work of Enomoto and Satoh [5] and also in Kawazumi and Kuno [12], where they found certain new roles of this Lie algebra. We refer to the above cited papers for details.

References

1. T. Church, B. Farb, A. Putman, *private communication*.
2. J. Conant, *Ornate necklaces and the homology of the genus one mapping class group*, Bull. London. Math. Soc. 39 (2007) 881–891.
3. J. Conant, K. Vogtmann, *On a theorem of Kontsevich*, Algebr. Geom. Topol. 3 (2003) 1167–1224.
4. M. Culler, K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986) 91–119.
5. N. Enomoto, T. Satoh, *New series in the Johnson cokernels of the mapping class groups of surfaces*, preprint (2011).
6. E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, In "The moduli space of curves", Progr. Math. 129 (1995) 199–230.
7. E. Getzler, *Resolving mixed Hodge modules on configuration spaces*, Duke Math. J. 96 (1999) 175–203.
8. J. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. 84 (1986) 157–176.
9. J. Harer, *unpublished*.
10. G. Hochschild, J-P. Serre, *Cohomology of Lie algebras*, Ann. Math. 57 (1953) 591–603.
11. M. Kassabov, *On the automorphism tower of free nilpotent groups*, PhD thesis, Yale University (2003).
12. N. Kawazumi, Y. Kuno, *The logarithms of Dehn twists*, preprint (2010).
13. M. Kontsevich, *Formal (non)commutative symplectic geometry*, from: "The Gel'fand Mathematical Seminars, 1990–1992", Birkhäuser, Boston (1993) 173–187.
14. M. Kontsevich, *Feynman diagrams and low-dimensional topology*, from: "First European Congress of Mathematics, Vol. II (Paris, 1992)", Progr. Math. 120, Birkhäuser, Basel (1994) 97–121.
15. S. Morita, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987) 551–577.
16. S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J. 70 (1993), 699–726.
17. S. Morita, *Lie algebras of symplectic derivations and cycles on the moduli spaces*, Geom. Topol. Monogr. 13 (2008) 335–354.

E-mail address: morita@ms.u-tokyo.ac.jp

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OH-OKAYAMA, MEGURO-KU, TOKYO, 152-8551, JAPAN
E-mail address: sakasai@math.titech.ac.jp

DEPARTMENT OF MATHEMATICS, AKITA UNIVERSITY, 1-1 TEGATA-GAKUENMACHI, AKITA, 010-8502, JAPAN
E-mail address: macky@math.akita-u.ac.jp