REFINING CASTELNUOVO-HALPHEN BOUNDS

VINCENZO DI GENNARO AND DAVIDE FRANCO

Abstract. Fix integers r, d, s, π with $r \geq 4$, $d \gg s$, $r-1 \leq s \leq 2r-4$, and $\pi \geq 0$. Refining classical results for the genus of a projective curve, we exhibit a sharp upper bound for the arithmetic genus $p_a(C)$ of an integral projective curve $C \subset \mathbb{P}^r$ of degree d, assuming that C is not contained in any surface of degree $< s$, and not contained in any surface of degree s with sectional genus $> \pi$. Next we discuss other types of bound for $p_a(C)$, involving conditions on the entire Hilbert polynomial of the integral surfaces on which C may lie.

Keywords and phrases: Castelnuovo-Halphen Theory, Hartshorne-Rao module, Hilbert polynomial, arithmetic genus.

MSC2010: Primary 14N15; Secondary 14H45, 14H99, 14M05, 14N05.

1. Introduction

A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree d and the dimension r of the space where they are embedded. In 1882 Halphen [10] and Noether [15] determined an upper bound $G(3, d)$ for the genus of an irreducible, non degenerate curve in $\mathbb{P}^3$, and in 1889 Castelnuovo [2] found the analogous bound $G(r, d)$ for the genus of irreducible, non degenerate curves in $\mathbb{P}^r$, $r \geq 3$.

Since curves of maximal genus $G(3, d)$ in $\mathbb{P}^3$ must lie on a quadric surface, it is natural to ask for the maximal genus $G(3, d, s)$ of space curves of degree d, not contained in surfaces of degree less than a fixed integer s. In fact Halphen gave such a refined bound. His argument was not complete, but in 1977 Gruson and Peskine [9] provided a complete proof in the range $d > s^2 - s$.

The same phenomenon occurs for curves of maximal genus $G(r, d)$ in $\mathbb{P}^r$, also called *Castelnuovo's curves*: at least when $d > 2r$, they must lie on surfaces of minimal degree $r - 1$. As before, one may refine Castelnuovo's bound, looking for the maximal genus $G(r, d, s)$ of curves of degree d in $\mathbb{P}^r$, not contained in surfaces of degree less than a fixed integer s. In 1982 Eisenbud and Harris ([5], Theorem (3.22), p. 117) determined such a bound for $r - 1 \leq s \leq 2r - 2$ and $d \gg s$. Next, in 1993, the bound $G(r, d, s)$ has been computed for any s and $d \gg s$ (see [3]).

A very special feature of the curves of maximal genus $G(r, d, s)$, which generalizes what we said about Castelnuovo's curves (i.e. when $s = r - 1$), is that they must lie on *Castelnuovo's surfaces* of degree s, i.e. on surfaces whose general hyperplane sections are themselves curves of maximal genus $G(s, r - 1)$ in $\mathbb{P}^{r-1}$ (see [3]).

Therefore, pushing further previous analysis, one may ask for *the maximal genus* $G(r,d,s,\pi)$ *of curves of degree* d*, not contained in surfaces of degree* $< s$*, neither in surfaces of degree* s *with sectional genus greater than a fixed integer* π (e.g. $\pi = G(r-1,s)-1$). Of course, one may assume $0 \leq \pi \leq G(r-1,s)$, and for $\pi = G(r-1,s)$ and $d \gg s$ we have $G(r,d,s,\pi) = G(r,d,s)$.

In the present paper we compute $G(r,d,s,\pi)$, in the range $r-1 \leq s \leq 2r-4$ and $d \gg s$ (Theorem 2.2)(except for the cases $s = 2r-3$ and $s = 2r-2$, it is the quoted Eisenbud-Harris range for s [5]). Next we discuss other types of bound for $p_a(C)$, involving conditions on the entire Hilbert polynomial of the integral surfaces on which C may lie (Proposition 2.3).

2. Notations and the statement of the main results

In order to state our results we need some preliminary notation, which we will use throughout the paper.

Notations 2.1. (i) Fix integers r, d, s, π and p, with $r \geq 3$ and $s \geq r-1$. Define m and ϵ by dividing $d-1 = ms+\epsilon$, $0 \leq \epsilon \leq s-1$. Set $\pi_0 := \pi_0(s, r-1) := s-r+1$. Notice that when $r-1 \leq s \leq 2r-3$ then $\pi_0 = G(r-1,s)$, i.e. π_0 is the Castelnuovo's bound for a curve of degree s in $\mathbb{P}^{r-1}$ [5]. Set

$$d_0(r) := \begin{cases} 16(r-2)(2r-3) & \text{if } 4 \leq r \leq 6 \\ 8(r-2)^3 & \text{if } 7 \leq r \leq 11 \\ 2^{r+1} & \text{if } r \geq 12. \end{cases}$$

(ii) When $r-1 \leq s \leq 2r-4$ define:

$$G^*(r,d,s,\pi,p) := \binom{m}{2}s + m(\epsilon+\pi) - p + \max\left(0, \left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right)$$

(square brackets indicate the integer part). Even if $G^*(r,d,s,\pi,p)$ does not depend on r, we prefer to use this notation in order to recall that $r-1 \leq s \leq 2r-4$. Observe that the number $G^*(r,d,s,\pi_0,0)$ is the quoted bound $G(r,d,s)$ determined in [5], Theorem (3.22), when $r-1 \leq s \leq 2r-3$ (in [5] these numbers are denoted by $\pi_\alpha(d,r)$, with $\alpha := s-r+2$).

(iii) We define the numerical function $h_{r,d,s,\pi}$ as follows:

$$h_{r,d,s,\pi}(i) := \begin{cases} 1-\pi+is-\max(0,\, \pi_0-\pi-i+1) & \text{if } 1 \leq i \leq m \\ d - \max\left(0, \left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right) & \text{if } i = m+1 \\ d & \text{if } i \geq m+2. \end{cases}$$

(iv) For a projective subscheme $X \subseteq \mathbb{P}^N$ we will denote by $\mathcal{I}_X$ its ideal sheaf in $\mathbb{P}^N$, and by $M(X) := \oplus_{i\in\mathbb{Z}} H^1(\mathbb{P}^N, \mathcal{I}_X(i))$ the Hartshorne-Rao Module. We will denote by h_X the Hilbert function of X [5], and by Δh_X the first difference of h_X, i.e. $\Delta h_X(i) := h_X(i) - h_X(i-1)$. We will say that X is a.C.M. if it is arithmetically Cohen-Macaulay.

(v) Given numerical functions $h_1 : \mathbb{Z} \to \mathbb{Z}$ and $h_2 : \mathbb{Z} \to \mathbb{Z}$, we say that $h_1 > h_2$ if $h_1(i) \geq h_2(i)$ for any $i \in \mathbb{Z}$, and if there exists some i such that $h_1(i) > h_2(i)$.

Our main result is the following:

Theorem 2.2. *Fix integers r, d, s, π with $r \geq 4$, $r-1 \leq s \leq 2r-4$, $0 \leq \pi \leq \pi_0$ and $d > d_0(r)$. Let $C \subset \mathbb{P}^r$ be an irreducible, reduced, nondegenerate, projective curve of degree d, and arithmetic genus $p_a(C)$. Let $\Gamma \subset \mathbb{P}^{r-1}$ be the general hyperplane section of C, and h_Γ its Hilbert function. Assume that C is not contained in any surface of degree $< s$, and not contained in any surface of degree s with sectional genus $> \pi$. Then one has:*

(a) $h_\Gamma(i) \geq h_{r,d,s,\pi}(i)$ for any $i \in \mathbb{Z}$;

(b) $p_a(C) \leq G^*\left(r, d, s, \pi, -\binom{\pi_0-\pi+1}{2}\right)$, and therefore

$$G(r, d, s, \pi) \leq G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right);$$

(c) the bound is sharp, and the curves with maximal genus are a. C.M. with $h_\Gamma = h_{r,d,s,\pi}$, and contained in surfaces S of degree s, sectional genus π and arithmetic genus $p_a(S) = -\binom{\pi_0-\pi+1}{2}$.

By property (c), combined with Corollary 3.6 below, we see that curves with *maximal* arithmetic genus lie in surfaces with *minimal* arithmetic genus. The proof of this fact relies on a general bound (see Proposition 3.9 below) which, as far as we know, although elementary, seems to have escaped explicit notice. We hope that Proposition 3.9 can be useful to obtain further information in the range $s \geq 2r-3$.

As for the other properties, the proof of Theorem 2.2 follows a now classic pattern in Castelnuovo-Halphen Theory (see [5]), taking into account [7] which allows us to estimate the Hartshorne-Rao module of the general hyperplane section of an integral surface $S \subset \mathbb{P}^r$ of degree $r-1 \leq s \leq 2r-4$ (compare also with [16]).

According to the above, previous Theorem 2.2 suggests a more refined analysis: *given integers r, d, s, π, p, find the maximal genus $G(r, d, s, \pi, p)$ for an integral curve in $\mathbb{P}^r$ of given degree $d \gg s$, not contained in any surface of degree $< s$, and not contained in any surface of degree s with sectional genus $> \pi$ and arithmetic genus $< p$.* By Theorem 2.2 we already know that when $r-1 \leq s \leq 2r-4$ then $G\left(r, d, s, \pi, -\binom{\pi_0-\pi+1}{2}\right) = G^*\left(r, d, s, \pi, -\binom{\pi_0-\pi+1}{2}\right)$. To this purpose we are able to prove the following partial result.

Proposition 2.3. *Fix integers r, d, s, π, p with $r \geq 4$, $r-1 \leq s \leq 2r-4$, $0 \leq \pi \leq \pi_0$, $-\binom{\pi_0-\pi+1}{2} \leq p \leq 0$, and $d > d_0(r)$. Let $C \subset \mathbb{P}^r$ be an irreducible, reduced,* *nondegenerate, projective curve of degree d, and arithmetic genus $p_a(C)$. Assume that C is not contained in any surface of degree $< s$, and not contained in any surface of degree s with sectional genus $> \pi$ and with arithmetic genus $< p$. Then one has:*

(a) $p_a(C) \leq G^*(r,d,s,\pi,p)$, i.e. $G(r,d,s,\pi,p) \leq G^*(r,d,s,\pi,p)$;

(b) if the bound is sharp, i.e. if $G(r,d,s,\pi,p) = G^*(r,d,s,\pi,p)$, then the curves with maximal genus are contained in surfaces of degree s, sectional genus π and arithmetic genus p;

(c) if there is a nondegenerate, irreducible, smooth curve $\Sigma \subset \mathbb{P}^{r-1}$ of degree s and genus π with the Hartshorne-Rao module of dimension $-p$ (in this case one has a fortiori $p \leq -(\pi_0 - \pi)$), then the bound is sharp, and there are extremal a.C.M. curves on the cone $S \subset \mathbb{P}^r$ over Σ (when $2\pi \geq s - 1 + \epsilon$ we must also assume that Σ is an isomorphic projection of a Castelnuovo curve $\Sigma' \subset \mathbb{P}^{r-1+\pi_0-\pi}$ contained in a smooth rational normal scroll surface);

(d) when $p = -\binom{\pi_0-\pi+1}{2}$ or $p = -(\pi_0 - \pi)$ or $p = 0$, then bound is sharp.

The line of the proof is similar to the proof of Theorem 2.2. However we are forced to slightly modify it because, in this more general setting, there is no a minimal Hilbert function for the general hyperplane section of C as in Theorem 2.2, (a) (and in fact there are extremal curves which are not a.C.M. (see Remark 4.2 below, (iii), (iv) and (v))). We are able to overcome this difficulty thanks to the quoted Proposition 3.9. As far as we know, the question of the existence of a curve Σ as in (c) of Proposition 2.3 (essentially of a curve with a prescribed Hartshorne-Rao module) is quite difficult. Therefore previous proposition appears as a partial result in this setting.

We refer to Remark 4.2 for other examples and comments on extremal curves in the sense of Theorem 2.3. Our assumption $d > d_0(r)$ is certainly not the best possible. It is only of the simplest form we were able to conceive. For $r \geq 12$ this assumption coincides with the one introduced in [5], p. 117, Theorem (3.22).

3. Preliminary results

In this section we collect some preliminary results which we need in order to prove the announced results. We start with the following consequence of Theorem 1 in [7] (compare also with Theorem 3.2 in [16]). We keep the notation introduced before.

Proposition 3.1. *Let $\Sigma \subset \mathbb{P}^{r-1}$ be a non degenerate integral curve of degree s with arithmetic genus π. Assume that $r - 1 \leq s \leq 2r - 4$. Then one has $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = 0$ for any $i \geq 1$, $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(1)) = \pi_0 - \pi$, and $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \leq$*

$\max\big(0,\, h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) - 1\big)$ *for any* $i \geq 2$. *In particular* $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \leq \max(0,\, \pi_0 - \pi + 1 - i)$ *for any* $i \geq 1$.

Proof. Since $\pi \leq \pi_0 = s - r + 1$ then $2\pi - 2 < s$. Therefore Σ is non special and so $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = 0$ for any $i \geq 1$. In particular $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(1)) = h^0(\Sigma, \mathcal{O}_\Sigma(1)) - h_\Sigma(1) = (1 - \pi + s) - r = \pi_0 - \pi$. It remains to prove that $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \leq \max\big(0, h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) - 1\big)$ for any $i \geq 2$.

To this purpose let H be the general hyperplane section of Σ. Since $r - 1 \leq s \leq 2r - 4$ then by Castelnuovo Theory [5] we know that $h_H(i) = s$ for any $i \geq 2$, and so $h^1(\mathbb{P}^{r-2}, \mathcal{I}_H(i)) = 0$ for any $i \geq 2$. Therefore for any $i \geq 2$ we have the following exact sequence:

$$(1)\ \ 0 \to H^0(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) \to H^0(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \to H^0(\mathbb{P}^{r-2}, \mathcal{I}_H(i)) \to$$

$$H^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) \to H^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \to 0.$$

In particular we have $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \leq h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1))$. Now suppose by contradiction that $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) = h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) > 0$ for some $i \geq 2$. Then the map $H^0(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \to H^0(\mathbb{P}^{r-2}, \mathcal{I}_H(i))$ should be surjective. But by [7] we know that the homogeneous ideal of H is generated by quadrics. It would follow that the map $H^0(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(j)) \to H^0(\mathbb{P}^{r-2}, \mathcal{I}_H(j))$ is onto for any $j \geq i$, which in turn would imply that $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(j)) = h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) > 0$ for any $j \geq i - 1$. This is absurd. □

Lemma 3.2. *With the same notation as above we have:*

(1) $\sum_{i=1}^{+\infty}(d - h_{r,d,s,\pi}(i)) = G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right)$;

(2) if $\pi' < \pi$ *then* $h_{r,d,s,\pi'} > h_{r,d,s,\pi}$, *therefore* $G^*\left(r, d, s, \pi', -\binom{\pi_0 - \pi' + 1}{2}\right) < G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right)$;

(3) if $d \geq (2s+1)(s+1)$ *then* $h_{r,d,s+1,\pi_0'} > h_{r,d,s,\pi}$, *therefore* $G^*(r, d, s+1, \pi_0', 0) < G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right)$ *(here we set* $\pi_0' := \pi_0(s+1, r-1) = (s+1) - r + 1$*).*

The proof is straightforward, and so we omit it.

Lemma 3.3. *Fix integers* r, d, s *with* $r \geq 4$, $r - 1 \leq s \leq 2r - 4$, *and* $d \geq s(s-1)$. *Let* $C \subset \mathbb{P}^r$ *be an irreducible, reduced, non degenerate, projective curve of degree* d, *with general hyperplane section* Γ. *Assume that* C *is contained in an integral projective surface* $S \subset \mathbb{P}^r$ *of degree* s *and sectional genus* π. *Then one has* $h_\Gamma(i) \geq h_{r,d,s,\pi}(i)$ *for any* $i \geq 1$.

Proof. By Bezout Theorem we have $h_\Gamma(i) = h_\Sigma(i)$ for any $1 \leq i \leq m$, where Σ denotes the general hyperplane section of S. On the other hand, by Proposition 3.1, for $1 \leq i \leq m$ one has

$$h_\Sigma(i) = h^0(\Sigma, \mathcal{O}_\Sigma(i)) - h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) = 1 - \pi + is - h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i))$$
$$\geq 1 - \pi + is - \max(0, \pi_0 - \pi - i + 1) = h_{r,d,s,\pi}(i).$$

It remains to examine the range $i \geq m + 1$.

To this purpose first notice that if L is a general hyperplane such that $\Sigma = S \cap L$, then $Sing(\Sigma) = Sing(S) \cap L$, and so $\deg(Sing(S)) = \deg(Sing(\Sigma)) \leq \pi_0$. It follows that C is not contained in $Sing(S)$ because $d \gg s$. Hence Γ does not meet the singular locus of Σ, i.e. $\Gamma \subset \Sigma \backslash Sing(\Sigma)$, and so Γ defines an effective Cartier divisor on Σ. It follows the existence of the exact sequence:

$$0 \to \mathcal{O}_\Sigma(-\Gamma + (m + j)H) \to \mathcal{O}_\Sigma((m + j)H) \to \mathcal{O}_\Gamma \to 0,$$

where H denotes the general hyperplane section of Σ. Since $d \gg s$ then from [8] it follows that the natural map $H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(m + j)) \to H^0(\Sigma, \mathcal{O}_\Sigma(m + j))$ is surjective for any $j \geq 0$, and so from previous exact sequence we get:

$$h_\Gamma(m + j) = h^0(\Sigma, \mathcal{O}_\Sigma(m + j)) - h^0(\Sigma, \mathcal{O}_\Sigma(-\Gamma + (m + j)H)) \tag{2}$$
$$= 1 - \pi + (m + j)s - h^0(\Sigma, \mathcal{O}_\Sigma(-\Gamma + (m + j)H)).$$

If $h^1(\Sigma, \mathcal{O}_\Sigma(-\Gamma + (m+j)H)) = 0$ then $h^0(\Sigma, \mathcal{O}_\Sigma(-\Gamma + (m+j)H)) = 1 - \pi + (m+j)s - d$ and therefore $h_\Gamma(m+j) = d$. Otherwise $h^1(\Sigma, \mathcal{O}_\Sigma(-\Gamma+(m+j)H)) > 0$ and by Clifford's Theorem for possibly singular curves (see [5], p. 46, or [6], Proposition 1.5., and compare with [5], p.121) we know that

$$h^0(\Sigma, \mathcal{O}_\Sigma(-\Gamma + (m + j)H)) - 1 \leq \frac{(m + j)s - d}{2}$$

hence

$$h_\Gamma(m + j) \geq \frac{(m + j)s + d}{2} - \pi,$$

and so $h_\Gamma(i) \geq h_{r,d,s,\pi}(i)$ for any $i \geq m + 1$. □

Lemma 3.4. *Let $S \subset \mathbb{P}^r$ be an irreducible, reduced, non degenerate projective surface of degree s and arithmetic genus $p_a(S)$. Denote by Σ the general hyperplane section of S, and by H the general hyperplane section of Σ. For any integer i set $\delta_i := \Delta h_\Sigma(i) - h_H(i)$ and $\mu_i := \Delta h_S(i) - h_\Sigma(i)$. Then we have:*

$$p_a(S) = \sum_{i=1}^{+\infty}(i - 1)(s - h_H(i)) - \sum_{i=1}^{+\infty}(i - 1)\delta_i + \sum_{i=1}^{+\infty}\mu_i.$$

In particular, when $r - 1 \leq s \leq 2r - 4$, then

$$p_a(S) = -\dim_{\mathbb{C}} M(\Sigma) + \sum_{i=1}^{+\infty}\mu_i,$$

where $M(\Sigma)$ denotes the Hartshorne-Rao module of Σ.

Remark 3.5. By [4], p. 30, we know that

$$\delta_i = \dim_{\mathbb{C}} \left[Ker \left(H^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) \to H^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) \right) \right].$$

Similarly as in [4], p. 30 one may prove that

$$\mu_i = \dim_{\mathbb{C}} \left[Ker \left(H^1(\mathbb{P}^{r}, \mathcal{I}_S(i-1)) \to H^1(\mathbb{P}^{r}, \mathcal{I}_S(i)) \right) \right].$$

Proof of Lemma 3.4. Recall that when $t \gg 0$ then the Hilbert polynomial of S at level t coincides with the Hilbert function $h_S(t)$ of S. Therefore we have:

$$p_a(S) = h_S(t) - s\binom{t+1}{2} + t\pi - t - 1, \tag{3}$$

where π denotes the sectional genus of S. Now we may write:

$$h_S(t) = \sum_{j=0}^{t} \Delta h_S(j) = \sum_{j=0}^{t} h_\Sigma(j) + \mu_j = \sum_{j=0}^{t} \left(\sum_{i=0}^{j} \Delta h_\Sigma(i) \right) + \sum_{j=0}^{t} \mu_j$$

$$= \sum_{j=0}^{t} \left(\sum_{i=0}^{j} h_H(i) + \delta_i \right) + \sum_{j=0}^{t} \mu_j = \sum_{i=0}^{t} (t-i+1)(h_H(i) + \delta_i) + \sum_{j=0}^{t} \mu_j.$$

Taking into account that $\delta_0 = \mu_0 = 0$ and that $h_H(0) = 1$, inserting previous equality into (3) we obtain:

$$p_a(S) = t\left[\pi + \sum_{i=1}^{t} h_H(i) + \delta_i \right] - \sum_{i=1}^{t} (i-1)(h_H(i) + \delta_i) + \sum_{j=0}^{t} \mu_j - s\binom{t+1}{2}. \tag{4}$$

By [4], pg. 31, we have (recall that $t \gg 0$) $\pi = \sum_{i=1}^{t} (s - h_H(i) - \delta_i)$, therefore from (4) it follows that

$$p_a(S) = \left[t^2 - \binom{t+1}{2} \right] s - \sum_{i=1}^{t} (i-1)(h_H(i) + \delta_i) + \sum_{j=0}^{t} \mu_j$$

$$= \sum_{i=1}^{+\infty} (i-1)(s - h_H(i)) - \sum_{i=1}^{+\infty} (i-1)\delta_i + \sum_{i=1}^{+\infty} \mu_i.$$

As for the last claim, observe that when $r-1 \leq s \leq 2r-4$ we have $h_H(i) = s$ for any $i \geq 2$ by Castelnuovo Theory [5], and so $\sum_{i=1}^{+\infty} (i-1)(s - h_H(i)) = 0$. Moreover, by Remark 3.5 and (1) we see that $\delta_i = h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i-1)) - h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i))$, from which we get $\sum_{i=1}^{+\infty} (i-1)\delta_i = \dim_{\mathbb{C}} M(\Sigma)$. □

Corollary 3.6. *Let $S \subset \mathbb{P}^r$ be an irreducible, reduced, non degenerate projective surface of degree s, sectional genus π, and arithmetic genus $p_a(S)$. Assume that $r-1 \leq s \leq 2r-4$. Then we have*

$$-\binom{\pi_0 - \pi + 1}{2} \leq p_a(S) \leq 0.$$

Proof. By previous Lemma 3.4 and Proposition 3.1 we deduce

$$p_a(S) \geq -\dim_{\mathbb{C}} M(\Sigma) \geq -\binom{\pi_0 - \pi + 1}{2}.$$

Therefore we only have to prove that $p_a(S) \leq 0$. To this aim first observe that $p_a(S) = -h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) \leq h^2(S, \mathcal{O}_S)$. Moreover by [14], Lemma 5, we know that $h^2(S, \mathcal{O}_S) \leq \sum_{i=1}^{+\infty}(i-1)(s-h_H(i))$. This number is 0 because $h_H(i) = s$ for any $i \geq 2$. Hence $p_a(S) \leq 0$. □

Remark 3.7. With the same assumption as in Corollary 3.6, previous argument proves that $p_a(S) = -\binom{\pi_0-\pi+1}{2}$ if and only if $M(S) = 0$, and $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) = \max(0, \pi_0 - \pi - i + 1)$ for any $i \geq 1$.

Next lemma, for which we did not succeed in finding an appropriate reference, states an explicit upper bound for Castelnuovo-Mumford regularity of an integral projective surface. We need it in order to make explicit the assumption $d \gg s$ appearing in Proposition 3.9 below (which in turn we will use, via Corollary 3.11, in the proof of Theorem 2.2, (c), and Proposition 2.3, (a)).

Lemma 3.8. *Let $S \subset \mathbb{P}^r$ be an irreducible, reduced, non degenerate projective surface of degree $s \geq r-1 \geq 2$ and Castelnuovo-Mumford regularity $reg(S)$. Then one has*

$$reg(S) \leq (s-r+2)\left(\frac{s^2}{2(r-2)} + 1\right) + 1.$$

Proof. Let Σ be the general hyperplane section of S. By [8] we know that:

$$reg(\Sigma) \leq s - r + 3. \tag{5}$$

Hence, by ([13], p. 102) we have

$$reg(S) \leq s - r + 3 + h^1(\mathbb{P}^r, \mathcal{I}_S(s-r+2)).$$

Therefore it suffices to prove that:

$$h^1(\mathbb{P}^r, \mathcal{I}_S(s-r+2)) \leq (s-r+2)\frac{s^2}{2(r-2)}. \tag{6}$$

To this purpose first notice that by (5) we know that $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) = 0$ for any $i \geq s-r+2$, so the natural map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) \to H^0(\Sigma, \mathcal{O}_\Sigma(i))$ is surjective for any $i \geq s-r+2$. A fortiori the natural map $H^0(\mathbb{P}^r, \mathcal{O}_S(i)) \to H^0(\mathbb{P}^r, \mathcal{O}_\Sigma(i))$ is surjective for any $i \geq s-r+2$. It follows that $H^1(S, \mathcal{O}_S(i-1)) \subseteq H^1(S, \mathcal{O}_S(i))$ for any $i \geq s-r+2$ in view of the exact sequence $0 \to \mathcal{O}_S(i-1) \to \mathcal{O}_S(i) \to \mathcal{O}_\Sigma(i) \to 0$, and from the vanishing $H^1(S, \mathcal{O}_S(i)) = 0$ for $i \gg 0$ we obtain $H^1(S, \mathcal{O}_S(i)) = 0$ for any $i \geq s-r+1$. Hence we have:

(7)

$$h^1(\mathbb{P}^r, \mathcal{I}_S(s-r+2)) = h^0(S, \mathcal{O}_S(s-r+2)) - h_S(s-r+2) \leq p_S(s-r+2) - h_S(s-r+2),$$

where $h_S(s-r+2)$ and $p_S(s-r+2)$ denote the Hilbert function and the Hilbert polynomial of S at level $s-r+2$. By [5], Lemma (3.1), we may estimate

$$h_S(s-r+2) \geq \sum_{i=0}^{s-r+2} h_\Sigma(i) \geq \sum_{i=0}^{s-r+2}\left[\sum_{j=0}^{i} h_H(j)\right] = \sum_{i=0}^{s-r+2}(s-r+3-i)h_H(i),$$

where h_H denotes the Hilbert function of the general hyperplane section H of Σ. Since

$$p_S(t) = s\binom{t+1}{2} + (1-\pi)t + 1 + p_a(S)$$

(π and $p_a(S)$ denote the sectional and the arithmetic genus of S) from (7) it follows that:

$$\text{(8)}\qquad h^1(\mathbb{P}^r, \mathcal{I}_S(s-r+2)) \leq \left[s\binom{s-r+3}{2} + (1-\pi)(s-r+2) + 1 + p_a(S)\right]$$

$$-\left[\sum_{i=0}^{s-r+2}(s-r+3-i)h_H(i)\right] = p_a(S) - \sum_{i=1}^{s-r+3}(i-1)(s-h_H(i))$$

$$+2s\binom{s-r+3}{2} - (s-r+2)(\pi + \sum_{i=1}^{s-r+3} h_H(i)).$$

From [14], Lemma 5, we know that:

$$p_a(S) = h^2(S,\mathcal{O}_S) - h^1(S,\mathcal{O}_S) \leq h^2(S,\mathcal{O}_S) \leq \sum_{i=1}^{+\infty}(i-1)(s-h_H(i)),$$

and from [5], Theorem (3.7), we have:

$$\sum_{i=1}^{+\infty}(i-1)(s-h_H(i)) = \sum_{i=1}^{s-r+3}(i-1)(s-h_H(i))$$

because $h_H(i) = s$ for $i \geq w+1$, and $w+1 \leq s-r+4$ (we define w by dividing $s-1 = w(r-2)+v$, $0 \leq v \leq r-3$). We deduce that:

$$p_a(S) - \sum_{i=1}^{s-r+3}(i-1)(s-h_H(i)) \leq 0,$$

and so from (8) we get:

$$h^1(\mathbb{P}^r, \mathcal{I}_S(s-r+2)) \leq 2s\binom{s-r+3}{2} - (s-r+2)(\pi + \sum_{i=1}^{s-r+3} h_H(i))$$

$$= (s-r+2)\left[\left(\sum_{i=1}^{s-r+3} s - h_H(i)\right) - \pi\right].$$

By [5], Corollary (3.3) and proof, and Theorem (3.7), the term $\sum_{i=1}^{s-r+3}(s-h_H(i))$ is bounded by Castelnuovo's bound $G(r-1,s) := \binom{w}{2}(r-2)+wv$ for the arithmetic genus of Σ. Since $G(r-1,s) \leq \frac{s^2}{2(r-2)}$ we get

$$\sum_{i=1}^{s-r+3}(s-h_H(i)) \leq \frac{s^2}{2(r-2)}.$$

Combining the last two estimates we obtain (6). $\square$

Proposition 3.9. *Let $S \subset \mathbb{P}^r$ be an irreducible, reduced, non degenerate projective surface of degree $s \geq r-1 \geq 2$, sectional genus π and arithmetic genus $p_a(S)$. Let $C \subset S$ be an irreducible, reduced, non degenerate projective curve of degree $d \geq \frac{s^4}{2(r-2)}$. Denote by $p_a(C)$, by $\mathcal{I}_C \subset \mathcal{O}_{\mathbb{P}^r}$ and by h_C the arithmetic genus, the ideal sheaf and the Hilbert function of C. Denote by Γ the general hyperplane section of C and by h_Γ its Hilbert function. Then one has:*

$$p_a(C) = \binom{m}{2}s + m(\epsilon+\pi) - p_a(S) + \sum_{i=m+1}^{+\infty} d - \Delta h_C(i). \tag{9}$$

In particular one has

$$p_a(C) \leq \binom{m}{2}s + m(\epsilon+\pi) - p_a(S) + \sum_{i=m+1}^{+\infty} d - h_\Gamma(i), \tag{10}$$

and $p_a(C)$ attains this bound if and only if $h^1(\mathbb{P}^r, \mathcal{I}_C(i)) = 0$ for any $i \geq m$.

Proof. Since for $t \gg 0$ we have $h_C(t) = 1 - p_a(C) + dt$ then we may write

$$p_a(C) = dt + 1 - h_C(t) = \sum_{i=1}^{t} d - \Delta h_C(i) = \sum_{i=1}^{+\infty} d - \Delta h_C(i) \tag{11}$$

$$= \sum_{i=1}^{m} d - \Delta h_C(i) + \sum_{i=m+1}^{+\infty} d - \Delta h_C(i).$$

On the other hand by Bezout's Theorem we have $h_C(i) = h_S(i)$ for any $i \leq m$, and therefore we have

$$\sum_{i=1}^{m} d - \Delta h_C(i) = md + 1 - h_C(m) = md + 1 - h_S(m).$$

By Lemma 3.8 we deduce that $h_S(m)$ coincides with the Hilbert polynomial $p_S(m)$ of S at level m, i.e.

$$h_S(m) = p_S(m) = \binom{m+1}{2}s + m(1-\pi) + 1 + p_a(S).$$

It follows that

$$\sum_{i=1}^{m} d - \Delta h_C(i) = md + 1 - h_S(m)$$

$$= md + 1 - \left[\binom{m+1}{2}s + m(1-\pi) + 1 + p_a(S)\right] = \binom{m}{2}s + m(\epsilon+\pi) - p_a(S).$$

Inserting this into (11) we obtain (9).

As for (10), we observe that

$$\sum_{i=m+1}^{+\infty} d - \Delta h_C(i) = \sum_{i=m+1}^{+\infty} d - h_\Gamma(i) - \sum_{i=m+1}^{+\infty} \Delta h_C(i) - h_\Gamma(i).$$

Hence (9) implies (10) because $\Delta h_C(i) - h_\Gamma(i) \geq 0$ for any i ([5], Lemma (3.1)). Moreover we deduce that $p_a(C)$ attains the bound appearing in (10) if and only if $\sum_{i=m+1}^{+\infty} \Delta h_C(i) - h_\Gamma(i) = 0$. And this is equivalent to say that $h^1(\mathbb{P}^r, \mathcal{I}_C(i)) = 0$

for any $i \geq m$ in view of Remark 3.5. This concludes the proof of Proposition 3.9. □

Remark 3.10. (i) From the proof it follows that *if there is an a.C.M. curve on S of degree $d \gg s$ then $\sum_{i=1}^{+\infty} \mu_i = 0$, and therefore the Hartshorne-Rao module of S vanishes.*

(ii) When S is smooth one knows that $reg(S) \leq s - r + 3$ [12], and so to prove Proposition 3.9 one may simply assume that $m \geq s-r+2$, or also $d \geq s(s-r+3)$. This last numerical assumption is enough also if one knows that $h^1(\mathbb{P}^r, \mathcal{I}_S(m)) = 0$, e.g. when S is a. C. M..

Combining (10) with Lemma 3.3 we get the following

Corollary 3.11. *Let $S \subset \mathbb{P}^r$ be an irreducible, reduced, non degenerate projective surface of degree s with $2 \leq r-1 \leq s \leq 2r-4$, sectional genus π and arithmetic genus $p_a(S)$. Let $C \subset S$ be an irreducible, reduced, non degenerate projective curve of arithmetic genus $p_a(C)$ and degree $d \geq \frac{s^4}{2(r-2)}$. Then one has:*

$$p_a(C) \leq G^*(r, d, s, \pi, p_a(S)). \tag{12}$$

4. Proof of Theorem 2.2 and of Proposition 2.3

We begin by proving Theorem 2.2.

(a) First assume C is not contained in any surface of degree s. Then C is not contained in any surface of degree $< s+1$. By [5] we know that $h_\Gamma(i) \geq h_{r,d,s+1,\pi_0'}(i)$ for any i, and by Lemma 3.2 we deduce $h_\Gamma(i) \geq h_{r,d,s,\pi}(i)$ for any i. Hence we may assume that C is contained in a surface of degree s, with sectional genus $\pi' \leq \pi$. By Lemma 3.2 and by Lemma 3.3 we get again $h_\Gamma(i) \geq h_{r,d,s,\pi}(i)$ for any i.

(b) Since in general we have $p_a(C) \leq \sum_{i=1}^{+\infty}(d - h_\Gamma(i))$ ([5], Corollary (3.2)) then by (a) and Lemma 3.2 we deduce

$$p_a(C) \leq \sum_{i=1}^{+\infty}(d - h_\Gamma(i)) \leq \sum_{i=1}^{+\infty}(d - h_{r,d,s,\pi}(i)) = G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right).$$

(c) If the bound is sharp, i.e. if $p_a(C) = G^*\left(r, d, s, \pi, -\binom{\pi_0-\pi+1}{2}\right)$, then previous inequality shows that $p_a(C) = \sum_{i=1}^{+\infty}(d - h_\Gamma(i))$, i.e. C is a.C.M., and $h_\Gamma = h_{r,d,s,\pi}$. Moreover, the same argument developed in (a) and (b), combined with Lemma 3.2, proves also that if C reaches the bound then C must be contained in a surface S of degree s and sectional genus π. As for $p_a(S)$, observe that, by Corollary 3.11, we have

$$p_a(C) = G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right) \leq G^*\left(r, d, s, \pi, p_a(S)\right).$$

It follows $p_a(S) \leq -\binom{\pi_0-\pi+1}{2}$, and by Corollary 3.6 we get $p_a(S) = -\binom{\pi_0-\pi+1}{2}$.

Now, to conclude the proof of Theorem 2.2, we only have to prove that the upper bound is sharp.

To this purpose, fix integers $r \geq 4$, $r-1 \leq s \leq 2r-4$, $0 \leq \pi < \pi_0 := s-r+1$. Let $\Sigma' \subset \mathbb{P}^{r-1+\pi_0-\pi}$ be a smooth Castelnuovo curve of degree s and genus π (which we may find on a smooth rational normal scroll surface in $\mathbb{P}^{r-1+\pi_0-\pi}$ (use [11], Corollary 2.18 and 2.19)).

Choose general $\pi_0 - \pi + 2$ points on Σ' (compare with [16], p. 13, Example 3.7). Denote by $\mathbb{P}^{\pi_0-\pi+1}$ the linear space generated by these points. A general subspace $\mathbb{P}^{\pi_0-\pi-1} \subset \mathbb{P}^{\pi_0-\pi+1}$ defines a projection $\varphi : \mathbb{P}^{r-1+\pi_0-\pi} \backslash \mathbb{P}^{\pi_0-\pi-1} \to \mathbb{P}^r$ which maps isomorphically Σ' to a curve $\Sigma \subset \mathbb{P}^{r-1}$. Since $\varphi(\mathbb{P}^{\pi_0-\pi+1} \backslash \mathbb{P}^{\pi_0-\pi-1})$ is a $(\pi_0 - \pi + 2)-$secant line to Σ then Castelnuovo-Mumford regularity of Σ is at least $\pi_0 - \pi + 2$. By Lemma 3.1 it follows that $h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(i)) = \max(0, \pi_0 - \pi + 1 - i)$ for any $i \geq 1$ and so $h_\Sigma(i) = 1 - \pi + si - \max(0, \pi_0 - \pi + 1 - i)$ for any $i \geq 1$. In particular, once fixed an integer $d \gg s$, we have $h_\Sigma(i) = h_{r,d,s,\pi}(i)$ for $1 \leq i \leq m$ (with $d - 1 = ms + \epsilon$, $0 \leq \epsilon \leq s - 1$).

Denote by $S \subset \mathbb{P}^r$ the projective cone on Σ. Fix an integer $k \gg s$ of type $k - 1 = \mu s + \epsilon$, $0 \leq \epsilon \leq s - 1$, and a set D of $s - 1 - \epsilon$ distinct points on Σ. Let $C(D) \subset S$ be the cone over D, and let $F \subset \mathbb{P}^r$ be a hypersurface of degree $\mu + 1$ containing $C(D)$, consisting of $\mu + 1$ sufficiently general hyperplanes. Let R be the residual curve to $C(D)$ in the complete intersection of F with S. Equipped with the reduced structure, R is a cone over k distinct points of Σ. In particular R is a (reducible) a.C.M. curve of degree k on S, and, if we denote by R' the general hyperplane section of R, we have $p_a(R) = \sum_{i=1}^{+\infty}(k - h_{R'}(i))$. We make the following claim. We will prove it in a while.

Claim. *For a suitable D one has $h_{R'}(i) = h_{r,k,s,\pi}(i)$ for any $i \geq 1$.*

It follows that

$$p_a(R) = \sum_{i=1}^{+\infty}(k - h_{R'}(i)) = \sum_{i=1}^{+\infty}(k - h_{r,k,s,\pi}(i)) = G^*\left(r, k, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right).$$

Now let $d \gg k$, with $d - 1 = ms + \epsilon$. Let $G \subset \mathbb{P}^r$ be a hypersurface of degree $m + 1$ containing $C(D)$ such that the residual curve C in the complete intersection of G with S, equipped with the reduced structure, is an integral curve of degree d, with a singular point of multiplicity k at the vertex p of S, and tangent cone at p equal to R. We are going to prove that C is the curve we are looking for, i.e.

$$p_a(C) = G^*\left(r, d, s, \pi, -\binom{\pi_0 - \pi + 1}{2}\right).$$

To this aim, let $\widetilde{S}$ be the blowing-up of S at the vertex. By [11], p. 374, we know that $\widetilde{S}$ is the ruled surface $\mathbb{P}(\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-1)) \to \Sigma$. Denote by E the exceptional divisor, by f the line of the ruling, and by L the pull-back of the hyperplane section. We have $L^2 = s$, $L\cdot f = 1$, $f^2 = 0$, $L \equiv E+sf$ and $K_{\widetilde{S}} \equiv -2L+(2\pi-2+s)f$. Let $\widetilde{C} \subset \widetilde{S}$ be the blowing-up of C at p, which is nothing but the normalization of C. Since C has degree d then $\widetilde{C}$ belongs to the numerical class of $(m+1+a)L+(1+\epsilon-(a+1)s)f$ for some integer a. Moreover $E\cdot\widetilde{C} = 1+\epsilon-(a+1)s = k$, so

$$a = -\frac{k+s-1-\epsilon}{s} = -(\mu+1).$$

By the adjunction formula we get

$$p_a(\widetilde{C}) = \binom{m}{2}s + m(\epsilon+\pi)+\pi-\frac{1}{2}a^2s+a(\pi+\epsilon-\frac{1}{2}s).$$

On the other hand we have

$$p_a(C) = p_a(\widetilde{C})+\delta_p$$

where δ_p is the delta invariant of the singularity (C,p). Since the tangent cone of C at p is R then the delta invariant is equal to the difference between the arithmetic genus of R and the arithmetic genus of k disjoint lines in the projective space, i.e.

$$\delta_p = p_a(R)-(1-k) = G^*\left(r,k,s,\pi,-\binom{\pi_0-\pi+1}{2}\right)-(1-k).$$

It follows that

$$p_a(C) = \binom{m}{2}s + m(\epsilon+\pi)+\pi-\frac{1}{2}a^2s+a(\pi+\epsilon-\frac{1}{2}s)$$
$$+G^*\left(r,k,s,\pi,-\binom{\pi_0-\pi+1}{2}\right)-(1-k).$$

Taking into account that $a = -\frac{k+s-1-\epsilon}{s}$, a direct computation proves that this number is exactly $G^*\left(r,d,s,\pi,-\binom{\pi_0-\pi+1}{2}\right)$.

It remains to prove the claim, i.e. that *for a suitable D one has $h_{R'}(i) = h_{r,k,s,\pi}(i)$ for any $i\geq 1$*. This certainly holds true for any D and any $1\leq i\leq\mu$ because in this range we have by construction $h_\Sigma(i) = h_{r,d,s,\pi}(i)$, and $h_{R'}(i) = h_\Sigma(i)$ by Bezout Theorem. This holds true also in the range $i\geq\mu+2$ by degree reasons (compare with the proof of Lemma 3.3). It remains to examine the case $i=\mu+1$. If $\max\left(0,\left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right) = 0$ then, as before, again by degree reasons we have $h_{R'}(\mu+1) = h_{r,k,s,\pi}(\mu+1) = k$. Otherwise $\max\left(0,\left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right) > 0$. In this case let $S'\subset\mathbb{P}^{r+\pi_0-\pi}$ be the cone over Σ'. By [3], Example 6.5 (here we need to choose Σ' on a smooth rational normal scroll surface), we know that for a suitable set D' (in [3] denoted by Z') of $s-1-\epsilon$ distinct points of Σ', a general curve C', obtained from the cone over D' through a linkage with S' and a hypersurface of degree $\mu+1$, is an integral curve of degree k and maximal arithmetic genus $p_a(C') = G(r+\pi_0-\pi,k,s) = G^*(r+\pi_0-\pi,k,s,\pi,0)$. Let Γ' and H' be the general hyperplane sections of C' and Σ'. We have $\mathcal{O}_{\Sigma'}(D')\cong\mathcal{O}_{\Sigma'}(-\Gamma'+(\mu+1)H')$. Since C' is maximal then by a similar computation as in (2) we see that $h^0(\Sigma',\mathcal{O}_{\Sigma'}(D')) = s-\epsilon-\pi+d-h_{r+\pi_0-\pi,k,s,\pi}(\mu+1)$. Since $h_{r+\pi_0-\pi,k,s,\pi}(\mu+1) = h_{r,k,s,\pi}(\mu+1)$ then

$h^0(\Sigma', \mathcal{O}_{\Sigma'}(D')) = s - \epsilon - \pi + d - h_{r,k,s,\pi}(\mu+1)$. Therefore if we choose D as the divisor on Σ corresponding to D' via the isomorphism $\Sigma' \cong \Sigma$, as in (2) we have

$$h_{R'}(\mu+1) = h^0(\Sigma, \mathcal{O}_\Sigma(\mu+1)) - h^0(\Sigma, \mathcal{O}_\Sigma(-R' + (\mu+1)H))$$

$$= 1-\pi+(\mu+1)s-h^0(\Sigma, \mathcal{O}_\Sigma(D)) = 1-\pi+(\mu+1)s-h^0(\Sigma', \mathcal{O}_\Sigma(D')) = h_{r,k,s,\pi}(\mu+1).$$

This concludes the proof of Theorem 2.2.

Remark 4.1. Constructing extremal curves as above, we need to choose Σ' on a smooth rational normal scroll surface only in the case $\max\left(0, \left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right) > 0$, i.e. when $2\pi \geq s - \epsilon + 1$.

Next we turn to the proof of Proposition 2.3.

(a) First assume C is not contained in any surface of degree s. Then C is not contained in any surface of degree $< s+1$. By [5] we know that $p_a(C) \leq G(r,d,s+1) = \frac{d^2}{2(s+1)} + O(d)$ which is strictly less than $G^*(r,d,s,\pi,p)$ because $d \gg s$ and $G^*(r,d,s,\pi,p) = \frac{d^2}{2s} + O(d)$. Hence we may assume that C is contained in a surface of degree s, with sectional genus $\pi' \leq \pi$. If $\pi' < \pi$ then by Theorem 2.2 we know that $p_a(C) \leq G^*(r,d,s,\pi', -\binom{\pi_0-\pi'+1}{2})$ which is strictly less than $G^*(r,d,s,\pi,p)$ because $\pi' < \pi$ and $d \gg s$. Therefore we may assume that C is contained in a surface S of degree s, with sectional genus π, and arithmetic genus $p_a(S) \geq p$. Then by Corollary 3.11 we know $p_a(C) \leq G^*(r,d,s,\pi,p_a(S))$ which is $\leq G^*(r,d,s,\pi,p)$ because $p_a(S) \geq p$. This establishes the upper bound.

(b) Previous argument also shows that if $p_a(C)$ reaches the upper bound then C is contained in a surface of degree s, sectional genus π, and arithmetic genus $p_a(S) \geq p$. Since $G^*(r,d,s,\pi,p_a(S)) = G^*(r,d,s,\pi,p)$ then $p_a(S) \leq p$, hence $p_a(S) = p$.

(c) Taking into account Remark 4.2 (i) below, one may construct a.C.M. extremal curves on the cone over Σ exactly as in the proof of Theorem 2.2. We omit the details.

(d) The bound is sharp when $p = -\binom{\pi_0-\pi+1}{2}$ by Theorem 2.2. Next let $\Sigma' \subset \mathbb{P}^{r-1+\pi_o-\pi}$ be a smooth Castelnuovo curve of degree $r-1 \leq s \leq 2r-4$. By [1], Theorem 2.6, p. 8, we know that a general projection $\Sigma \subset \mathbb{P}^r$ of Σ' remains 2-normal. By Proposition 3.1 it follows that Σ is k-normal for any $k \geq 2$. Therefore $\dim_{\mathbb{C}} M(\Sigma) = h^1(\mathbb{P}^{r-1}, \mathcal{I}_\Sigma(1)) = \pi_0 - \pi$. By property (c) this proves the sharpness of the bound in the case $p = -(\pi_0 - \pi)$. As for the case $p = 0$, let $S' \subset \mathbb{P}^{r+\pi_0-\pi}$ be a cone over a Castelnuovo curve of degree $r-1 \leq s \leq 2r-4$ as in [3], Example 6.4 and 6.5, and let $C' \subset S'$ be an extremal curve with arithmetic genus $p_a(C') = G(r+\pi_0-\pi, d, s)$. Projecting isomorphically in $\mathbb{P}^r$ we get extremal curves with genus $G^*(r,d,s,\pi,0) = G(r+\pi_0-\pi,d,s)$. Therefore the bound $G^*(r,d,s,\pi,p)$ is sharp also when $p = 0$.

This concludes the proof of Proposition 2.3.

Remark 4.2. (i) Let $S \subset \mathbb{P}^r$ be an integral nondegenerate surface of degree $r-1 \leq s \leq 2r-4$, with general hyperplane section Σ of arithmetic genus π. Fix an integer $d \gg s$ and consider the following numerical function

$$h_{d,\Sigma}(i) := \begin{cases} 1-\pi+is-h^1(\mathbb{P}^{r-1},\mathcal{I}_\Sigma(i)) & \text{if } 1 \leq i \leq m \\ d - \max\left(0, \left[\frac{2\pi-(s-1-\epsilon)}{2}\right]\right) & \text{if } i = m+1 \\ d \quad \text{if } i \geq m+2. \end{cases}$$

Observe that $h_{d,\Sigma}(i) = h_\Sigma(i)$ for $1 \leq i \leq m$. Using the same argument as in Lemma 3.3 we see that for any curve $C \subset S$ of degree d one has $h_\Gamma(i) \geq h_{d,\Sigma}(i)$ for any i, and so

$$(13) \qquad p_a(C) \leq \sum_{i=1}^{+\infty}(d-h_\Gamma(i)) \leq \sum_{i=1}^{+\infty}(d-h_{d,\Sigma}(i)) = G^*(r,d,s,\pi,-\dim_{\mathbb{C}} M(\Sigma))$$

where $M(\Sigma)$ denotes the Hartshorne-Rao module of Σ. This is another "natural" upper bound for $p_a(C)$. However notice that by Lemma 3.4 we know that $p_a(S) = -\dim_{\mathbb{C}} M(\Sigma) + \sum_{i=1}^{+\infty}\mu_i$, and therefore the bound appearing in Corollary 3.11 is more fine than this new bound (13), i.e.

$$G^*(r,d,s,\pi,p_a(S)) \leq G^*(r,d,s,\pi,-\dim_{\mathbb{C}} M(\Sigma)).$$

The inequality can be strict. For example, this is the case for a non linearly normal smooth surface S of arithmetic genus $p_a(S)=0$. In fact for such a surface we have $M(S) \neq 0$, and therefore $\sum_{i=1}^{+\infty}\mu_i > 0$.

(ii) Combining the examples in [16], p. 14, Table 1, with Proposition 2.3, (c), one may construct other examples of extremal curves with genus $G^*(r,d,s,\pi,p)$.

(iii) Let X be a ruled surface over a smooth curve R of genus π, defined by the normalized bundle $\mathcal{E} = \mathcal{O}_R \oplus \mathcal{O}_R(-\mathfrak{e})$, where $\mathfrak{e}$ is a fixed divisor on R of degree $-e \leq -2$ [11]. Let $\mathfrak{n}$ be a divisor on R of degree $n \geq 2\pi+1$. By ([11], Ex. 2.11, pg. 385), we know that $\Sigma := R_0 + \mathfrak{n}f$ is very ample on X (here R_0 denotes a section of X with $\mathcal{O}_X(R_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and f a fibre of the ruling $X \to R$). As in the proof of ([11], Theorem 2.17, pg. 379), we see that the complete linear system $|\,\Sigma\,|$ embeds X in $\mathbb{P}^{r+1}$ as a linearly normal surface S of degree s, sectional genus π and arithmetic genus $p_a(S) = -\pi = -(\pi_0(s,r)-\pi)$, with $s = 2n-e$ and $r+1 = s+1-2\pi$. In particular $r \leq s \leq 2r-4$. Now let C be any curve on S of degree d. For a suitable integer a and divisor $\mathfrak{b}_a$ on R of degree $b_a = 1+\epsilon-(a+1)s$ we have $C \in |\,(m+a+1)\Sigma + \mathfrak{b}_a f\,|$. Taking into account that the canonical divisor class of S is $|\,K_S\,| = |\,-2\Sigma + (s+2\pi-2)f\,|$, by the adjunction formula we may compute the arithmetic genus of C, which is equal to

$$g(a) := \binom{m}{2}s + m(\epsilon+\pi) + \pi - \frac{s}{2}a^2 + a(\pi+\epsilon-\frac{s}{2}).$$

Taking $a=0$, we deduce that, in the case $2\pi \leq s+1-\epsilon$, there are smooth curves C on S with maximal genus $g(0)=G^*(r+1,d,s,\pi,p)$, with $p=-\pi=-(\pi_0(s,r)-\pi)$. Projecting isomorphically S in $\mathbb{P}^r$, these examples show the existence of smooth extremal curves with genus $G^*(r,d,s,\pi,-(\pi_0(s,r-1)-\pi)+1)$ which are not a.C.M.. By contrast notice that in this range (i.e. $p=-(\pi_0-\pi)+1$) Proposition 2.3, (c), combined with the examples in [16], p. 14, Table 1, proves also the existence of a.C.M. extremal curves. So in certain range one can find both a.C.M. and not a.C.M. extremal curves. Therefore the classification of extremal curves appears somewhat complicated. Projecting in lower dimensional subspaces, this argument works well also for other values of $p \geq -(\pi_0-\pi)$.

(iv) In the case $p=0$, any extremal curve C cannot be a.C.M.. In fact if C would a.C.M. then the surface S (of degree s, sectional genus π and arithmetic genus $p_a(S)=0$) on which it lies should be a.C.M. in view of Remark 3.10. This is impossible when $\pi < \pi_0$.

(v) Let $C \subset \mathbb{P}^r$ be an extremal curve in the case $p=-(\pi_0-\pi)$, contained in a cone over a curve $\Sigma \subset \mathbb{P}^{r-1}$ with $\dim_{\mathbb{C}} M(\Sigma)=\pi_0-\pi$. Then we have $h_\Gamma(2)=h_\Sigma(2)=1-\pi+2s$. On the other hand, the Hilbert function at level 2 of the general hyperplane section of an extremal curve with genus $G(r,d,s+1)$ is equal to $h_{r,d,s+1,\pi_0'}(2)=s+r+3$, which is strictly less than $h_\Gamma(2)$ as soon as $\pi_0-\pi>3$. Therefore we see that (at least in this case) there is no a minimal Hilbert function for the general hyperplane section of a curve satisfying the conditions in Proposition 2.3.

(vi) If S is smooth then $p_a(S) \geq -\pi$ and so inequality (12) implies $p_a(C) \leq G(r,d,s,\pi,-\pi)$.

(vii) From the proof of Corollary 3.11 we see that the bound

$$p_a(C) \leq \binom{m}{2}s+m(\epsilon+\pi)-p_a(S)$$

holds true for any s and $d \gg s$, if $2\pi \leq s+1-\epsilon$. So when $\pi=0$ then we have the bound

$$p_a(C) \leq \binom{m}{2}s+m\epsilon-p_a(S).$$

In certain cases it is sharp. In fact, let $S \subset \mathbb{P}^4$ be a general projection of a smooth rational normal scroll $S' \subset \mathbb{P}^{s+1}$, and let δ_S be the number of double points of S. From the double point formula we know that $\delta_S=\binom{s-2}{2}$. On the other hand we have $p_a(S)=-\delta_S$. So previous bound becomes

$$p_a(C) \leq \binom{m}{2}s+m\epsilon+\binom{s-2}{2}.$$

Now take a Castelnuovo's curve $C' \subset S'$ of degree $d \gg s$ passing through the double point set of S'. Then the projection C of C' acquires δ_S nodes and so

$$p_a(C)=p_a(C')+\delta_S=\binom{m}{2}s+m\epsilon+\binom{s-2}{2}.$$

(viii) The arithmetic genus of a curve C complete intersection of a surface S with a hypersurface of degree $m+1$ is $p_a(C) = \binom{m}{2}s + m(\epsilon + \pi) + \pi$, where s and π are the degree and sectional genus of S. On the other hand, in this range, i.e. when $\epsilon = s-1$, we have $G^*(r,d,s,\pi,p) = \binom{m}{2}s + m(\epsilon+\pi) - p + \pi$, which is strictly greater than $p_a(C)$ when $p < 0$. In other words, in contrast with the classical case, in our setting complete intersections are not extremal curves.

(ix) Let C be an extremal curve as in Theorem 2.2, and assume $\epsilon = s-1$. Let S be the surface of degree s, sectional genus π and arithmetic genus $p_a(S) = -\binom{\pi_0-\pi+1}{2}$ on which C lies. We remark that S cannot be locally Cohen-Macaulay. In fact, by the proof of Lemma 3.3 we see that since C is extremal then Γ is the complete intersection of Σ with a hypersurface of degree $m+1$. Since C is a. C. M. one may lift such a hypersurface to a hypersuface $F \subset \mathbb{P}^r$ of degree $m+1$ containing C and not containing S. If S would be locally Cohen-Macaulay then C, as a scheme, would be the complete intersection of S with F for degree reasons. This is absurd in view of previous remark (viii).

Aknowledgements

We would like to thank Ciro Ciliberto for valuable discussions and suggestions on the subject of this paper.

References

[1] Alzati, A. - Russo, F.: *On the k-normality of projected algebraic varieties*, Bull Braz Math Soc, New Series 33(1), 1-22, 2002, Sociedade Brasileira de Matemtica.

[2] Castelnuovo, G.: *Ricerche di geometria sulle curve algebriche*, Zanichelli, Bologna (1937).

[3] Chiantini, L. - Ciliberto, C. - Di Gennaro, V.: *The genus of projective curves*, Duke Math. J. **70**/2, 229-245 (1993).

[4] Ciliberto, C.: *Hilbert functions on finite sets of points and the genus of a curve in a projective space*, in *Space Curves: Proceedings, Rocca di Papa, 1985*, Lecture Notes in Math., Springer-Verlag, Berlin **1266**, 24-73.

[5] Eisenbud, D. - Harris, J.: *Curves in Projective Space*, Sém. Math. Sup. **85**, Les Presses du l'Université de Montréal, Montréal (1982).

[6] Fujita, T.: *Defining Equations for Certain Types of Polarized Varieties*, in Complex Analysis and Algebraic Geometry (edited by Baily and Shioda), Iwanami and Cambridge Univ. Press, 165-173 (1977).

[7] Green, M. - Lazarsfeld, R.: *Some results on the syzygies of finite sets and algebraic curves*, Compositio Mathematica, **67**, n^o 3, 301-314 (1988).

[8] Gruson, L. - Lazarsfeld, R. - Peskine, C.: *On a Theorem of Castelnuovo and the Equations Defining Space Curves*, Inven. Math. **72**, 491-506 (1983).

[9] Gruson, L. - Peskine, C.: *Genre des courbes dans l'espace projectif*, Algebraic Geometry: Proceedings, Norway, 1977, Lecture Notes in Math. **687**, Springer-Verlag, New York, 31-59 (1978).

[10] Halphen, G.: *Mémoire sur la classification des courbes gauches algébriques*, Oeuvres Complètes, vol. III; also J. École Polytechnique **52**, 1-200 (1882).

[11] Hartshorne, R.: *Algebraic Geometry*, GTM, 52, Springer-Verlag, 1983.

[12] Lazarsfeld, R.: *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. **55**, 423-438 (1987).

[13] Mumford, D.: *Varieties defined by quadratic equations*, In Questions on Algebraic Varieties, 29-100, C.I.M.E., Cremonese, Roma (1970).

[14] Nagel, U. - Vogel, W.: *Bounds for Castelnuovo's Regularity and Hilbert functions*, Topics in Algebra, Banach Center Publications, Volume **26**, Part 2, PWN-Polish Scientific Publishers, Warsaw, pp. 163-183 (1990).

[15] Noether, M.: *Zur Grundlegung der Theorie der Algebraischen Raumcurven*, Verlag der Königlichen Akademie der Wissenshaften, Berlin (1883).

[16] Park, E.: *On hypersurfaces containing projective varieties*, arxiv.org/abs/1104.5599, 1-28 (2011).

Università di Roma "Tor Vergata", Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.

E-mail address: `digennar@axp.mat.uniroma2.it`

Università di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", P.le Tecchio 80, 80125 Napoli, Italy.

E-mail address: `davide.franco@unina.it`