Exact solutions for small-amplitude capillary-gravity water waves

Delia IONESCU-KRUSE

Institute of Mathematics of the Romanian Academy,

P.O. Box 1-764, RO-014700, Bucharest, Romania

E-mail: Delia.Ionescu@imar.ro

Abstract

We present explicit solutions for the ordinary differential equations system describing the motion of the particles beneath small-amplitude capillary-gravity waves which propagate on the surface of an irrotational water flow with a flat bottom. The required computations involve elliptic integrals of first kind, the Legendre normal form and a solvable Abel differential equation of the second kind.

1 Introduction

We consider the problem of water waves in a domain of finite depth bounded above by a free surface and under the combined effects of gravity and surface tension. We suppose that the water flow is irrotational. Mathematically, the problem is formulated as a free boundary problem for incompressible Euler equations with the irrotational condition. After rewriting the equations in an appropriate non-dimensional form, we have two non-dimensional parameters δ and ϵ, the shallowness parameter and the amplitude parameter, respectively, and another non-dimensional parameter W_e called Weber number, which comes from the surface tension on the free surface. We simplify the governing equations with a linearization which is slightly different from the classical case in line with the Stokes condition for irrotational flows (see, for example, [5], [10]). By this linearization, we obtained a parameter c_0 by which we can describe different backward flows in the irrotational case: still water ($c_0 = 0$), favorable uniform current $c_0 > 0$, adverse uniform current $c_0 < 0$.
Further, we get the general solution of the linearized problem. Notice that there are only a few explicit solutions to the nonlinear governing equations: for gravity water waves, Gerstner's solution[1][16] and the edge wave solution related to it

[1] This solution was independently re-discovered later by Rankine [30]. Modern detailed descriptions of this wave are given in the recent papers [2] and [20].

(see [3]), for capillary water waves, Crapper's solution [13] and its generalization in the case of finite depth (see [26]).
After getting the general solution of the linearized problem we investigate the nonlinear equations of the motion of the fluid particles. In the case the constant c_0 equals the non-dimensional speed of propagation of the linear wave, the required computations involve elliptic integrals of first kind and their Legendre's normal form. The exact solutions obtained in this case contain in their expressions Jacobian elliptic functions. Only one solution is presented in detail, the others will be presented in a future paper. In the case the constant c_0 is different from the non-dimensional speed of propagation of the linear wave, the computations involve a solvable Abel differential equation of the second kind.
In the both cases we remark that the obtained solutions are not closed curves. This result is in the line with the recent results obtained for capillary-gravity water waves by using phase-plane considerations for the nonlinear system describing the particle motion (see [18], [19]). By the same method see also the results obtained for gravity water waves in [5], [10] and for constant vorticity gravity water waves in [14], [15]. Beside the phase-plane analysis, the exact solutions allow a better understanding of the dynamics (see [22], [23]). The same type of results are obtained for the governing equations without linearization, by analyzing a free boundary problem for harmonic functions in a planar domain (see [4] for Stokes waves, [9] for solitary waves and [17] for deep-water Stokes waves) or by applying local bifurcation theory (see [33] for small-amplitude waves with vorticity).
The existence of regular periodic travelling waves with vorticity was recently established (see [11], [32]). For steady periodic gravity waves the symmetry is known to be ubiquitous (see [6], [21]). The study of the symmetry of rotational water waves was initiated in the papers [7], [8]; for irrotational flows see also [28]. However, exact information about the flow beneath such waves, is not readily available even in the irrotational case. This paper addresses this issue.

2 Small-amplitude approximation of the water-wave problem

The water flow under consideration is two-dimensional, bounded by a rigid horizontal surface below at $z = 0$ and a free surface above at $z = h_0 + \eta(x, t)$, with $h_0 > 0$ a constant. The undisturbed water surface is $z = h_0$. Let $(u(x, z, t), v(x, z, t))$ be the velocity of the water and $p(x, z, t)$ be the pressure. Water can be assumed to be inviscid fluid, even though it is slightly viscous. In problems of water waves it is also reasonable to assume that the fluid is incompressible (constant density ρ) ([27]), which implies the equation of mass conservation (MC). A capillary-gravity wave is influenced by the effects of surface tension and gravity, as well as by the fluid inertia. The surface tension will play a role in the formulation of the boundary conditions but not in the equations of motion valid in the fluid domain. For the capillary-gravity water waves, the appropriate equations of motion are Euler's equations (EE)([24]). The boundary conditions for the water wave problem are the kinematic boundary conditions as well as the dynamic boundary condition. The kinematic boundary conditions (KBC) express the fact that the same particles always form the free water surface and that the fluid is assumed to be bounded below by a hard horizontal bed $z = 0$. The dynamic boundary condition (DBC) express the fact that the difference of pressure on the two sides of the surface η is balanced by the effects of surface tension. Thus, the boundary value problem for capillary-gravity water waves is:

$$\begin{array}{lr} u_t + uu_x + vu_z = -\frac{1}{\rho}p_x & \\ v_t + uv_x + vv_z = -\frac{1}{\rho}p_z - g & \text{(EE)} \\ u_x + v_z = 0 & \text{(MC)} \\ v = \eta_t + u\eta_x \ \text{ on } \ z = h_0 + \eta(x,t) & \\ v = 0 \ \text{ on } z = 0 & \text{(KBC)} \\ p = p_0 - \frac{\Gamma}{R}, \ \text{ on } z = h_0 + \eta(x,t) & \text{(DBC)} \end{array} \tag{1}$$

where g is the constant gravitational acceleration, p_0 is the constant atmospheric pressure, the parameter $\Gamma(> 0)$ is the coefficient of surface tension and $\frac{1}{R}$ is the mean curvature (up to a factor 1/2) of the surface. For the surface defined as a function $\eta(x,t)$, the mean curvature has the following expression

$$\frac{1}{R} = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \tag{2}$$

In respect of the well-posedness for the initial-value problem for (1) there has been significant recent progress, see [12] and the references therein.
A key quantity in fluid dynamics is the *curl* of the velocity field, called vorticity. For two-dimensional flows we denote the scalar vorticity of the flow by

$$\omega(x,z) = u_z - v_x \tag{3}$$

In what follows we consider a flow which is uniform with depth, that is, described by a zero vorticity (irrotational case).

We search for a linear approximation of the water-wave problem (1). First the system (1) is non-dimensionalized by making use of the following scales: the undisturbed depth of water h_0, as the vertical scale, a typical wavelength λ, as the horizontal scale, and $\sqrt{gh_0}$ as the scale of the horizontal component of the velocity. The surface wave itself leads to the introduction of a typical amplitude of the wave a. For more details see [24]. Thus, we define the set of non-dimensional variables

$$\begin{array}{c} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}}t, \\ u \mapsto \sqrt{gh_0}u, \quad v \mapsto h_0\frac{\sqrt{gh_0}}{\lambda}v \end{array} \tag{4}$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u, v, on the right-hand side.
We set the constant water density $\rho = 1$ and let us now define the non-dimensional pressure. If the water would be stationary, that is, $u \equiv v \equiv 0$,

from the equations (EE) and (DBC) with $\eta = 0$, $\Gamma = 0$, we get for a non-dimensionalised z, the hydrostatic pressure $p_0 + gh_0(1-z)$. Thus, the non-dimensional pressure is defined by

$$p \mapsto p_0 + gh_0(1-z) + gh_0 p \tag{5}$$

Taking into account (4) and (5) the two-dimensional capillary-gravity waves on water of finite depth are described, in non-dimensional variables, by the following boundary value problem

$$\begin{array}{cr} u_t + uu_x + vu_z = -p_x & \\ \delta^2(v_t + uv_x + vv_z) = -p_z & \\ u_x + v_z = 0 & \\ v = \epsilon(\eta_t + u\eta_x) & \text{on } z = 1 + \epsilon\eta(x,t) \\ p = \epsilon\left[\eta - \left(\frac{\Gamma}{g\lambda^2}\right)\frac{\eta_{xx}}{(1+\epsilon^2\delta^2\eta_x^2)^{3/2}}\right] & \text{on } z = 1 + \epsilon\eta(x,t) \\ v = 0 & \text{on } z = 0 \end{array} \tag{6}$$

where we have introduced the amplitude parameter $\epsilon = \frac{a}{h_0}$ and the shallowness parameter $\delta = \frac{h_0}{\lambda}$.
For irrotational flows the vorticity equation (3) writes in non-dimensional variables (4) as

$$u_z = \delta^2 v_x \tag{7}$$

We observe now that, on $z = 1 + \epsilon\eta$, both v and p are proportional to ϵ. Thus, with the following scaling of the non-dimensional variables, (avoiding again the introduction of a new notation),

$$p \mapsto \epsilon p, \quad (u,v) \mapsto \epsilon(u,v) \tag{8}$$

the problem (6) becomes

$$\begin{array}{cr} u_t + \epsilon(uu_x + vu_z) = -p_x & \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z & \\ u_x + v_z = 0 & \\ v = \eta_t + \epsilon u\eta_x & \text{on } z = 1 + \epsilon\eta(x,t) \\ p = \eta - \left(\frac{\Gamma}{g\lambda^2}\right)\frac{\eta_{xx}}{(1+\epsilon^2\delta^2\eta_x^2)^{3/2}} & \text{on } z = 1 + \epsilon\eta(x,t) \\ v = 0 & \text{on } z = 0 \end{array} \tag{9}$$

and the equation (7) keeps the same form. Therefore, the system which describes the full problem in the irrotational case is given by (9)+(7).
It is conventional to write $\frac{\Gamma}{\rho g\lambda^2} = \delta^2 W_e$, with $W_e = \frac{\Gamma}{\rho g h_0^2}$ a Weber number. This parameter is used to measure the size of the surface tension contribution.

By letting $\epsilon \to 0$, δ and W_e being fixed, we obtain a linear approximation of the scaled version (9)+(7) of our problem, that is,

$$\begin{aligned} u_t + p_x &= 0 \\ \delta^2 v_t + p_z &= 0 \\ u_x + v_z &= 0 \\ u_z - \delta^2 v_x &= 0 \\ v &= \eta_t \qquad \text{on } z = 1 \\ p &= \eta - \delta^2 W_e \eta_{xx} \qquad \text{on } z = 1 \\ v &= 0 \qquad \text{on } z = 0 \end{aligned} \tag{10}$$

From the first three equations in (10), we get that

$$v_{zzt} = -u_{xzt} = p_{xxz} = -\delta^2 v_{xxt} \tag{11}$$

Therefore,

$$v_{zzt} + \delta^2 v_{xxt} = 0 \tag{12}$$

and thus

$$v_{zz} + \delta^2 v_{xx} = f(x, z) \tag{13}$$

where f is an arbitrary function. Taking into account the forth equation in (10), we obtain that

$$\delta^2 v_{xx} = u_{zx} \tag{14}$$

Introducing (14) into (13), we have

$$(v_z + u_x)_z = f(x, z) \tag{15}$$

and in view of the third equation in (10), we get that

$$f(x, z) = 0 \tag{16}$$

The equation (13) becomes

$$v_{zz} + \delta^2 v_{xx} = 0 \tag{17}$$

We apply the method of separation of variables, seeking a solution of this equation in the form

$$v(x, z, t) = F(x, t)G(z, t) \tag{18}$$

Substituting (18) into the equation (17), we find

$$F\frac{\partial^2 G}{\partial z^2} + \delta^2 G\frac{\partial^2 F}{\partial x^2} = 0 \tag{19}$$

thus,

$$\frac{1}{G}\frac{\partial^2 G}{\partial z^2} = -\delta^2 \frac{1}{F}\frac{\partial^2 F}{\partial x^2} \tag{20}$$

We observe in the above equation that the left hand side does not depend on z and the right hand side does not depend on x. Therefore, each side must be a constant, say

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -k^2, \quad \frac{1}{G}\frac{\partial^2 G}{\partial z^2} = k^2\delta^2 \tag{21}$$

where $k \geq 0$ is a constant that might depend on time. With the above choice, the solutions of the equations in (21) are

$$\begin{aligned} F(x,t) &= A\sin(kx) + B\cos(kx) \\ G(x,t) &= Ce^{k\delta z} + De^{-k\delta z}, \end{aligned} \tag{22}$$

where A, B, C, D are constants depending on time. We made this choice of the sign of the constant in the equations (21), in order to obtain this wave-like solution (22) propagating in the x-direction. On the bed $z = 0$, by the last equation in (10), we have $v = 0$, thus $C = -D$. Therefore,

$$v(x,z,t) = \sinh(k\delta z)\left(\mathcal{A}\sin(kx) + \mathcal{B}\cos(kx)\right) \tag{23}$$

where we introduced $C = -D$ into the constants $\mathcal{A}$ and $\mathcal{B}$.
Taking now into the account the fifth equation in (10), on $z = 1$ we get

$$\sinh(k\delta)\left(\mathcal{A}\sin(kx) + \mathcal{B}\cos(kx)\right) = \eta_t \tag{24}$$

which implies

$$\left(\mathcal{A}\sin(kx) + \mathcal{B}\cos(kx)\right) = \frac{\eta_t}{\sinh(k\delta)} \tag{25}$$

Hence,

$$v(x,z,t) = \frac{1}{\sinh(k\delta)}\sinh(k\delta z)\eta_t \tag{26}$$

For the component u of the velocity field, taking into account (26) and the fourth equation of the system (10), we obtain

$$u(x,z,t) = \frac{\delta}{k\sinh(k\delta)}\cosh(k\delta z)\eta_{tx} + \mathcal{F}(x,t) \tag{27}$$

where $\mathcal{F}(x,t)$ is an arbitrary function. The components u and v of the velocity have to fulfill also the third equation in (10), hence, in view of (26) and (27),

$$\frac{\delta}{k\sinh(k\delta)}\cosh(k\delta z)\eta_{txx} + \frac{\partial\mathcal{F}(x,t)}{\partial x} = -\frac{k\delta}{\sinh(k\delta)}\cosh(k\delta z)\eta_t \tag{28}$$

The above relation must hold for all values of $x \in \mathbf{R}$, and $0 \leq z \leq 1$. It follows

$$\frac{\partial\mathcal{F}(x,t)}{\partial x} = 0 \tag{29}$$

and

$$\eta_{txx} + k^2\eta_t = 0 \tag{30}$$

We seek periodic travelling wave solutions, thus, for the equation (30) with

$$k = 2\pi \tag{31}$$

we choose the following solution

$$\eta(x,t) = \cos(2\pi(x - ct)) \tag{32}$$

where c represents the non-dimensional speed of propagation of the linear wave and is to be determined.
From (29) the function $\mathcal{F}(x,t)$ is independent of x, therefore we will denote this function by $\mathcal{F}(t)$.
We return now to the systems (10) in order to find the the expressions of the pressure. Taking into account the first two equations in (10) and the expressions of the velocity field from above, we obtain

$$p(x,z,t) = \frac{2\pi\delta c^2}{\sinh(2\pi\delta)} \cosh(2\pi\delta z)\cos(2\pi(x-ct)) + x\mathcal{F}'(t) \tag{33}$$

On the free surface $z = 1$ the pressure (33) has to fulfill the sixth equation of the system (10). Hence, in view of (32), we get

$$2\pi\delta c^2 \coth(2\pi\delta)\cos(2\pi(x-ct)) + x\mathcal{F}'(t) = (1 + 4\pi^2\delta^2 W_e)\cos(2\pi(x-ct)) \tag{34}$$

The above relation must hold for all values $x \in \mathbf{R}$, therefore, we get

$$\mathcal{F}(t) = \text{constant} := c_0 \tag{35}$$

and we provide the non-dimensional speed of the linear wave

$$c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}(1 + 4\pi^2\delta^2 W_e) = \frac{\lambda}{2\pi h_0}\left(1 + \frac{4\pi^2\Gamma}{g\lambda^2}\right)\tanh\left(\frac{2\pi h_0}{\lambda}\right) \tag{36}$$

We observe thus, that the speed of propagation of the wave varies with the wavelength λ, with the undisturbed depth h_0 and with the coefficient of surface tension Γ.

Summing up, the solution of the linear system (10) is

$$\begin{array}{l} \eta(x,t) = \cos(2\pi(x-ct)) \\ p(x,z,t) = \frac{2\pi\delta c^2}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct)) \\ u(x,z,t) = \frac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct)) + c_0 \\ v(x,z,t) = \frac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin(2\pi(x-ct)) \end{array} \tag{37}$$

with c given by (36).

3 Exact solutions to the nonlinear equations of the motion of fluid particles

Let $(x(t), z(t))$ be the path of a particle in the fluid domain, with location $(x(0), z(0)) := (x_0, z_0)$ at time $t = 0$. Taking into account (37), the motion of the particle is described by the following system of nonlinear differential equations

$$\left\{ \begin{array}{l} \frac{dx}{dt} = u(x,z,t) = \frac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct)) + c_0 \\ \frac{dz}{dt} = v(x,z,t) = \frac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin(2\pi(x-ct)) \end{array} \right. \tag{38}$$

The right-hand side of the differential system (38) is smooth and bounded, therefore, the unique solution of the Cauchy problem with initial data (x_0, z_0) is defined globally in time.
Notice that the constant c_0 is the average of the horizontal fluid velocity over any horizontal segment of length 1, that is,

$$c_0 = \frac{1}{1}\int_x^{x+1} u(s,z,t)ds, \tag{39}$$

representing therefore the strength of the underlying uniform current. Thus, $c_0 = 0$ will correspond to a region of still water with no underlying current, $c_0 > 0$ will characterize a favorable uniform current and $c_0 < 0$ will characterize an adverse uniform current.
To study the exact solution of the system (38) it is more convenient to re-write it in the following moving frame

$$X = 2\pi(x - ct), \quad Z = 2\pi\delta z \tag{40}$$

This transformation yields

$$\begin{cases} \frac{dX}{dt} = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\cosh(Z)\cos(X) + 2\pi(c_0 - c) \\ \frac{dZ}{dt} = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\sinh(Z)\sin(X) \end{cases} \tag{41}$$

I) $\mathbf{c_0 = c}$
In this case, differentiating with respect to t, the system (41) can be written in the following form:

$$\begin{cases} \frac{d^2X}{dt^2} = -\frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}\sin(2X) \\ \frac{d^2Z}{dt^2} = \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}\sinh(2Z) \end{cases} \tag{42}$$

This system integrates to

$$\begin{cases} \left(\frac{dX}{dt}\right)^2 = \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}\cos(2X) + c_1 \\ \left(\frac{dZ}{dt}\right)^2 = \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}\cosh(2Z) + c_2 \end{cases} \tag{43}$$

c_1, c_2 being the integration constants.
For the first equation in (43) we use the substitution

$$\tan(X) = y\,,\ \cos(2X) = \frac{1-y^2}{1+y^2}\,,\ \sin(2X) = \frac{2y}{1+y^2}\,,\ dX = \frac{1}{1+y^2}dy \tag{44}$$

In the new variable, the first equation in (43) takes the form

$$\left(\frac{dy}{dt}\right)^2 = \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}(1-y^4) + c_1(1+y^2)^2 \tag{45}$$

We denote by

$$a^2 := \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)} \tag{46}$$

The solution of the equation (45) involves an elliptic integral of first kind:

$$\pm \int \frac{dy}{\sqrt{(c_1 - a^2)y^4 + 2c_1 y^2 + c_1 + a^2}} = t \tag{47}$$

The elliptic integral of first kind from (47) may by reduced to Legendre's normal form. In order to do this we consider first the substitution

$$y^2 = s \tag{48}$$

Therefore, the left hand side in (47) becomes

$$\pm \int \frac{dy}{\sqrt{(c_1 - a^2)y^4 + 2c_1 y^2 + c_1 + a^2}} = \pm \int \frac{ds}{2\sqrt{(c_1 - a^2)s(s+1)\left(s + \frac{c_1 + a^2}{c_1 - a^2}\right)}} \tag{49}$$

Further, we introduce a new variable φ. The definition of this variable depends on the sign of $c_1 - a^2$ and $c_1 + a^2$. There are three possibilities:
$c_1 - a^2 > 0$,
$c_1 - a^2 < 0$ and $c_1 + a^2 > 0$,
$c_1 - a^2 < 0$ and $c_1 + a^2 < 0$.
We present below only the second case, the investigation of the others will be presented in a future paper.
If

$$c_1 - a^2 < 0 \quad \text{and} \quad c_1 + a^2 > 0 \tag{50}$$

then, we introduce the variable φ by (see [31] Ch. VI, §4, page 602)

$$s = \frac{a^2 + c_1}{a^2 - c_1} \cos^2 \varphi \tag{51}$$

and we get

$$(c_1 - a^2)s(s+1)\left(s + \frac{c_1 + a^2}{c_1 - a^2}\right) = \frac{2a^2(a^2 + c_1)^2}{(a^2 - c_1)^2} \sin^2 \varphi \cos^2 \varphi \left[1 - k_1^2 \sin^2 \varphi\right]$$

$$ds = -\frac{2(a^2 + c_1)}{a^2 - c_1} \sin \varphi \cos \varphi d\varphi$$

where the constant $0 < k_1^2 < 1$ is given by

$$k_1^2 = \frac{a^2 + c_1}{2a^2} \tag{52}$$

Therefore we obtain the Legendre normal form of the integral in (47), that is,

$$\pm \frac{1}{\sqrt{2}\, a} \int \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}} = t \tag{53}$$

The inverse of the integral in (53) is the Jacobian elliptic function *sine amplitude* (see, for example, [1]), an odd periodic function of order two,

$$\operatorname{sn}\,\left(\pm\sqrt{2}a\,t;k_1\right) := \sin\varphi \tag{54}$$

In view of the notations (48), (51), we get that

$$y(t) = \pm\sqrt{\frac{a^2+c_1}{a^2-c_1}}\,\operatorname{cn}\,\left(\pm\sqrt{2}a\,t;k_1\right) = \pm\sqrt{\frac{a^2+c_1}{a^2-c_1}}\,\operatorname{cn}\,\left(\sqrt{2}a\,t;k_1\right) \tag{55}$$

cn being the Jacobian elliptic function *cosine amplitude*, an even periodic function of order two.
For the second equation in (43) we use the substitution

$$\tanh(Z) = w\,, \quad \cosh(2Z) = \frac{1+w^2}{1-w^2}\,, \quad dX = \frac{1}{1-w^2}dw \tag{56}$$

In the new variable, the second equation in (43) takes the form

$$\left(\frac{dw}{dt}\right)^2 = \frac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}(1-w^4) + c_2(1-w^2)^2 \tag{57}$$

The solution of the equation (57) involves an elliptic integral of first kind:

$$\pm\int\frac{dw}{\sqrt{(c_2-a^2)w^4-2c_2w^2+c_2+a^2}} = t \tag{58}$$

where a^2 is the constant from (46). The elliptic integral of first kind from (58) may by reduced to Legendre's normal form. In order to do this we consider first the substitution

$$w^2 = r \tag{59}$$

The left hand side in (58) becomes

$$\pm\int\frac{dw}{2\sqrt{(c_2-a^2)w^4-2c_2w^2+c_2+a^2}} = \pm\int\frac{ds}{2\sqrt{(c_2-a^2)r(r-1)(r-\frac{c_2+a^2}{c_2-a^2})}} \tag{60}$$

As in the case of the integral in (49), we introduce a new variable ϕ. The definition of ϕ depends on the sign of c_2-a^2 and c_2+a^2. There are three possibilities:
$c_2-a^2>0$,
$c_2-a^2<0$ and $c_2+a^2>0$,
$c_2-a^2<0$ and $c_2+a^2<0$.
We present below only the last case, the investigation of the others will be presented in detail in a future paper.
If

$$c_2-a^2<0 \quad \text{and} \quad c_2+a^2<0 \tag{61}$$

then, we introduce the variable ϕ by (see [31] Ch. VI, §4, page 602)

$$r = 1 + \frac{2a^2}{c_2 - a^2} \sin^2 \phi \tag{62}$$

and we get

$$(c_2 - a^2) r (r-1) \left(r - \frac{c_2 + a^2}{c_2 - a^2} \right) = \frac{4a^4}{(a^2 - c_2)} \sin^2 \phi \cos^2 \phi \left(1 - k_2^2 \sin^2 \phi\right)$$
$$ds = \frac{4a^2}{c_2 - a^2} \sin \phi \cos \phi d\phi$$

where the constant $0 < k_2^2 < 1$ is given by

$$k_2^2 = \frac{2a^2}{a^2 - c_2} \tag{63}$$

Therefore we obtain the Legendre normal form of the integral in (58), that is,

$$\pm \frac{1}{\sqrt{a^2 - c_2}} \int \frac{d\phi}{\sqrt{1 - k_2^2 \sin^2 \phi}} = t \tag{64}$$

The inverse of the integral in (64) is

$$\text{sn } \left(\pm \sqrt{a^2 - c_2} t; k_2 \right) := \sin \phi \tag{65}$$

In view of the notations (59), (62), we get that

$$w(t) = \pm \sqrt{1 - \frac{2a^2}{a^2 - c_2} \text{ sn }^2 \left(\sqrt{a^2 - c_2} t; k_2 \right)} \tag{66}$$

Therefore, from (44) and (56), the solution of the system (43) has the following expression

$$\begin{array}{l} X(t) = \arctan \left[y(t) \right] \\ Z(t) = \text{arctanh } \left[w(t) \right] = \frac{1}{2} \ln \frac{1 + w(t)}{1 - w(t)} \end{array} \tag{67}$$

with $y(t)$ given by (55) and $w(t)$ given by (66). From (40) and (67), the solution of the system (38) with the constant c_0 equals the speed of propagation of the linear wave c, have the following expressions:

$$\begin{array}{l} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[\sqrt{\frac{a^2 + c_1}{a^2 - c_1}} \text{ cn } \left(\sqrt{2} a\, t; k_1 \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \text{arctanh } \left[\sqrt{1 - \frac{2a^2}{a^2 - c_2} \text{ sn }^2 \left(\sqrt{a^2 - c_2} t; k_2 \right)} \right] \end{array} \tag{68}$$

We remark that the curve in (68) is not a closed curve. This result is in the line with the results obtained in [4], [5], [10], [14], [15], [17], [18], [19], [22], [23].

II) $\mathbf{c_0 \neq c}$
Differentiating with respect to t the system (41) we get

$$\frac{d^2X}{dt^2} + b\tan(X)\frac{dX}{dt} + a^2\sin(2X) - b^2\tan(X) = 0 \tag{69}$$

where a^2 is the constant from (46) and

$$b := 2\pi(c_0 - c) \tag{70}$$

Using the substitution (44), the equation (69) takes the form

$$\frac{d^2y}{dt^2} - \frac{2y}{1+y^2}\left(\frac{dy}{dt}\right)^2 + by\frac{dy}{dt} + 2a^2y - b^2y(1+y^2) = 0 \tag{71}$$

For (see [25], 6.54, page 554)

$$p(y) = \frac{dy}{dt}, \tag{72}$$

the equation (71) becomes an Abel differential equation of the second kind

$$p\frac{dp}{dy} = \frac{2y}{1+y^2}p^2 - byp - 2a^2y + b^2y(1+y^2) \tag{73}$$

The substitution (see [25], 4.11, pages 26-27)

$$u(y) = p(y)E(y), \quad \text{where} \quad E(y) = \exp\left(-\int\frac{2y}{1+y^2}dy\right) = \frac{1}{1+y^2}, \tag{74}$$

brings this equation to the simpler form

$$u\frac{du}{dy} = -b\frac{y}{1+y^2}u - 2a^2\frac{y}{(1+y^2)^2} + b^2\frac{y}{1+y^2} \tag{75}$$

The equation (75) with the substitution

$$\xi = \int\left(-\frac{by}{1+y^2}\right)dy = -\frac{b}{2}\ln(1+y^2) \tag{76}$$

can be written in the canonical form:

$$u\frac{du}{d\xi} - u = \frac{2a^2}{b}\exp\left(\frac{2\xi}{b}\right) - b \tag{77}$$

The equation (77) is solvable (see [29], 8., page 111), its solution can be written out in the following parametric form

$$\begin{aligned} u(\tau) &= \tau\frac{C - b\ln|\tau + \sqrt{\tau^2 - 2a^2}|}{\sqrt{\tau^2 - 2a^2}} + b \\ \xi(\tau) &= -b\ln\left|\frac{\sqrt{\tau^2 - a^2}}{C - b\ln|\tau + \sqrt{\tau^2 - 2a^2}|}\right| \end{aligned} \tag{78}$$

C being a constant. From (76) and (78) we get the expression of y

$$y(\tau) = \pm\sqrt{\frac{\tau^2 - 2a^2}{\left(C - b\ln|\tau + \sqrt{\tau^2 - 2a^2}|\right)^2} - 1}, \tag{79}$$

By (72) and (74), we have $\frac{dt}{d\tau} = \frac{1}{(1+y^2)u}\left(\frac{dy}{d\tau}\right)$ and therefore, from (78) and (79), the relation between t and τ is the following:

$$t = \int \frac{1}{\sqrt{\tau^2 - 2a^2}\sqrt{\tau^2 - 2a^2 - (C - b\ln|\tau + \sqrt{\tau^2 - 2a^2}|)^2}}\, d\tau \tag{80}$$

Thus, taking into account (44), we obtain

$$X(t) = \arctan\,[y(t)], \tag{81}$$

with $y(\tau)$ given by (79) and τ given implicitly by (80).
In order to determine $Z(t)$ from the system (41), with (81) in view, we write the second equation of this system in the form

$$\frac{dZ}{\sinh(Z)} = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\sin(\arctan\,[y(t)])\, dt = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\frac{y(t)}{\sqrt{1 + y^2(t)}}\, dt \tag{82}$$

Integrating, we get

$$\ln\left[\tanh\left(\frac{Z}{2}\right)\right] = \int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\frac{y(t)}{\sqrt{1 + y^2(t)}}\, dt + \text{const} \tag{83}$$

If

$$\int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\frac{y(t)}{\sqrt{1 + y^2(t)}}\, dt + \text{const} < 0 \tag{84}$$

then

$$Z(t) = 2\text{arctanh}\,\left[\exp\left(\int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)}\frac{y(t)}{\sqrt{1 + y^2(t)}}\, dt + \text{const}\right)\right] \tag{85}$$

From (40), (81) and (85), the solution of the system (38) is written now as

$$\begin{aligned} x(t) &= ct + \tfrac{1}{2\pi}\arctan\,\,[y(t)] \\ z(t) &= \tfrac{1}{\pi\delta}\text{arctanh}\,\left[\exp\left(\int \tfrac{4\pi^2\delta c}{\sinh(2\pi\delta)}\tfrac{y(t)}{\sqrt{1+y^2(t)}}\, dt + \text{const}\right)\right] \end{aligned} \tag{86}$$

with $y(\tau)$ given by (79) and τ given implicitly by (80).
We remark that the curve in (86) is not a closed curve. This result is in the line with the results obtained in [4], [5], [10], [14], [15], [17], [18], [19], [22], [23].

References

[1] Byrd P. F. and Friedman M. D., *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag Berlin Heidelberg New York, 1971.

[2] Constantin A., *On the deep water wave motion*, J. Phys. A, 34 (2001), pp. 1405–1417.

[3] Constantin A., *Edge waves along a sloping beach*, J. Phys. A, 34 (2001), pp. 9723–9731.

[4] Constantin A., *The trajectories of particles in Stokes waves*, Invent. Math., 166 (2006), pp. 523–535.

[5] Constantin A., Ehrnström M., and Villari G., *Particle trajectories in linear deep-water waves*, Nonlinear Anal. Real World Appl., 9 (2008), pp. 1336-1344.

[6] Constantin A, Ehrnström M. and E. Wahlen, *Symmetry of steady periodic gravity water waves with vorticity*, Duke Math. J., 140 (2007), pp. 591–603.

[7] Constantin A. and Escher J., *Symmetry of steady deep-water waves with vorticity*, European J. Appl. Math., 15 (2004), 755–768.

[8] Constantin A. and Escher J., *Symmetry of steady periodic surface water waves with vorticity*, J. Fluid Mech., 498 (2004), 171–181.

[9] Constantin A. and Escher J., *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc., 44 (2007), 423–431.

[10] Constantin A. and Villari G., *Particle trajectories in linear water waves*, J. Math. Fluid Mech., 10 (2008), pp. 1–18.

[11] Constantin A. and Strauss W., *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math., 57 (2004), pp. 481–527.

[12] Coutand D. and Shkoller S., *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., 20 (2007), pp. 829-930.

[13] Crapper G. D., *An exact solution for progressive capillary waves of arbitrary amplitude*, J. Fluid Mech., 2 (1957), pp. 532–540.

[14] Ehrnström M. , *On the streamlines and particle paths of gravitational water waves*, Nonlinearity, 21 (2008), pp. 1141–1154.

[15] Ehrnström M. and Villari G., *Linear water waves with vorticity: Rotational features and particle paths*, J. Differential Equations, 244 (2008), pp. 1888-1909.

[16] Gerstner F., *Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile*, Ann. Phys. 2 (1809), pp. 412–445.

[17] Henry D., *The trajectories of particles in deep-water Stokes waves*, Int. Math. Res. Not. (2006), Art. ID 23405, 13 pp.

[18] Henry D., *Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves*, J. Nonlinear Math. Phys., 14 (2007), pp. 1-7.

[19] Henry D., *Particle trajectories in linear periodic capillary and capillary-gravity water waves*, Phil. Trans. R. Soc. A, 365 (2007), pp. 2241–2251.

[20] Henry D., *On Gerstner's water wave*, J. Nonlinear Math. Phys., 15 (2008), pp. 87–95.

[21] Hur V. M., *Symmetry of steady periodic water waves with vorticity*, Phil. Trans. R. Soc. A, 365 (2007), pp 2203-2214.

[22] Ionescu-Kruse D., *Particle trajectories in linearized irrotational shallow water flows*, J. Nonlinear Math. Phys., 15 (2008), pp. 13-27.

[23] Ionescu-Kruse D., *Particle trajectories beneath small amplitude shallow water waves in constant vorticity flows*, Nonlinear Anal-Theor, 2009, in press.

[24] Johnson R. S., *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge Univeristy Press, 1997.

[25] Kamke E., *Differentialgleichungen, Lösungsmethoden und Lösungen*, vol. I, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1967.

[26] Kinnersley W., *Exact large amplitude capillary waves on sheets of fluids*, J. Fluid Mech., 77 (1976), pp. 229–241.

[27] Lighthill J., *Waves in Fluids*, Cambridge University Press, 2001.

[28] Okamoto H. and Shoji M., *The Mathematical Theory of Permanent Progressive Water Waves*, World Scientific, River Edge, NJ, 2001.

[29] Polyanin A.D and Zaitsev V.F., *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003.

[30] Rankine W. J. M., *On the exact form of waves near the surface of deep water*, Phil. Trans. R. Soc. A, 153 (1863), 127–138.

[31] Smirnov V., *Cours de Mathématiques supérieures*, Tome III, deuxième partie, Mir, Moscou, 1972.

[32] Wahlen E., *On rotational water waves with surface tension*, Phil. Trans. R. Soc. A, 365 (2007), pp. 2215–2225.

[33] Wahlen E., *Steady water waves with a critical layer*, J. Differential Eq., 246 (2009), 2468–2483.