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Caccioppoli's inequalities on constant mean curvature hypersurfaces in Riemannian manifolds

S. Ilias*, B. Nelli†, M.Soret

Abstract

We prove some Caccioppoli's inequalities for the traceless part of the second fundamental form of a complete, non compact, finite index, constant mean curvature hypersurface of a Riemannian manifold, satisfying some curvature conditions. This allows us to unify and clarify many results in the literature and to obtain some new results. For example, we prove that there is no stable, complete, non compact hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature $H \neq 0$, provided that, for suitable p, the L^p-norm of the traceless part of second fundamental form satisfies some growth condition.

Keywords. Caccioppoli's inequality, Simons' equation, constant mean curvature hypersurfaces, stable, finite index.
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1 Introduction

In this article, we give a general setting that unifies and clarifies the proofs of many results present in the literature on nonexistence of stable constant mean curvature hypersurfaces. We obtain some new results in the subject as well. The key result of this article is the following generalization of a result of R. Schoen, L. Simon, S.T. Yau [35]. Throughout the article, $\mathcal{N}$ is an orientable Riemannian manifold with bounded sectional curvature and such that the norm of the derivative of the curvature tensor is bounded.

Theorem (see Theorem 4.1). *Let M be a complete, non compact hypersurface with constant mean curvature H and finite index, immersed in $\mathcal{N}$. Denote by φ the norm of the traceless part of the second fundamental form of M. Then, there exists a compact subset K of M, such that for any $q > 0$ and for any $f \in C_0^\infty(M \setminus K)$ one has*

$$\int_{M\setminus K} f^2\varphi^{2q+2}(\mathcal{A}\varphi^2 + \mathcal{B}H\,\varphi + \mathcal{C}H^2 + \mathcal{E}) \leq \mathcal{D}\int_{M\setminus K}\varphi^{2q+2}|\nabla f|^2 + \mathcal{F}\int_{M\setminus K} f^2\varphi^{2q+1} + \mathcal{G}\int_{M\setminus K} f^2\varphi^{2q}$$

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where the constants are as in Theorem 4.1. Moreover, if M is stable, $K = \emptyset$.

The inequality of the previous Theorem is a Caccioppoli's inequality for finite index hypersurfaces with constant mean curvature. In order to get it, we first prove a Simons' inequality for constant mean curvature hypersurfaces of a Riemannian manifold (Theorem 3.1). Beyond their own interest, Caccioppoli's inequalities are useful to deduce nonexistence results for stable constant mean curvature hypersurfaces in space forms, under some restrictions on the dimension of the space and on the growth of the L^p norm of the traceless part of the second fundamental form, for suitable p. For example, we prove that there is no stable, complete, non compact hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature $H \neq 0$, provided that, for suitable p, the L^p-norm of the traceless second fundamental form satisfies some growth condition. This is an answer to a do Carmo's question in a particular case (see pg. 133 in [19]).
It is worth to point out that the analogous of the previous Theorem can be proved in the setting of δ-stable hypersurfaces, and for hypersurfaces with constant H_r-curvature, that is the r-th symmetric function of the principal curvatures (in [3] and [4] one can find some related results). Moreover, we restrict ourselves to hypersurfaces, but, in fact, the previous Theorem can be easily adapted to submanifolds of any codimension with parallel mean curvature.
We analyze the consequences of the previous Theorem in the case where the ambient manifold has constant sectional curvature. In a forthcoming article we will analyze the case where the ambient space is either a product or a warped product of constant sectional curvature manifolds.
As the proof of the previous Theorem relies on a Simons' inequality in this setting, it is worth to recall some literature about the classical Simons' inequality. In the pioneer paper [37], J. Simons proved an identity for the Laplacian of the norm of the second fundamental form of a minimal submanifold of the Euclidean space. Such identity is known as *Simons' formula.* Using Simons' formula and some more work on minimal cones, J. Simons was able to deduce that, for $n \leq 7$, the only entire solutions of the minimal surface equation in $\mathbb{R}^n$ are linear functions. Concerning the restriction on the dimension in the result of Simons, we recall that E. Bombieri, E. De Giorgi and E. Giusti [12] proved that, for $n > 7$, there are solutions of the minimal surface equation that are not linear. Then, Bernstein's question [11] was completely answered.
Later, there has been a lot of work about Simons' formula. In [35], R. Schoen, L. Simon, S.T. Yau generalized Simons' formula to a inequality (known as *Simons' inequality*) for minimal hypersurfaces in a Riemannian manifold. Then, they applied Simons' inequality in order to prove an estimate for the L^p norm of the second fundamental form of a stable minimal hypersurface in a Riemannian manifold, for a suitable p (see Theorem 1 in [35]). Among many interesting consequences of the L^p estimate in [35], we point out the following one: there is no non totally geodesic, area minimizing hypersurfaces of dimension $n \leq 5$, in a flat Riemannian manifold.
Later, some authors proved generalizations of Simons' inequality. We quote the article of P. Bérard, [9], where the author deduced a very general Simons' identity satisfied by the second fundamental form of a submanifold of arbitrary codimension of a Riemannian manifold.
Our article is organized as follows.
In Section 2, we recall some generalities about stability.
In Section 3, we obtain a Simons' inequality for the traceless part of the second fundamental form of a constant mean curvature hypersurface in a Riemannian manifold (see Theorem 3.2). We deduce it from the very general formula obtained by P. Bérard in [8], [9]. Our computations are strongly inspired by those of R. Schoen, L. Simon, S.T. Yau [35]. We give them, because, to our knowledge they are not present in the literature, except for hypersurfaces in space forms [2].

Then, we evaluate the terms of Simons' inequality depending on the curvature of the ambient space, in order to obtain a handier inequality (Theorem 3.1).
In Section 4, we prove the Theorem that we stated at the beginning of the Introduction (Theorem 4.1).
In Section 5, we deduce, from Theorem 4.1, three different kinds of Caccioppoli's inequalities on a constant mean curvature hypersurface with finite index. The first one is the analogous, for finite index constant mean curvature hypersurfaces, of the inequality by R. Schoen, L. Simon, S. T. Yau (see Theorem 1 in [35]). The second one is different in nature because it involves the gradient of the norm of the second fundamental form of the hypersurface. The third one is obtained by a careful study of the sign of the coefficients of the inequality of Theorem 4.1, in the case where the ambient space has constant sectional curvature. Later on, we obtain some refinements of Caccioppoli's inequality of third type, that will be useful for the applications.
In Section 6, we show how our Cacioppoli's inequalities can be used to obtain some nonexistence results for stable constant mean curvature hypersurfaces. In many cases, we recover the known results present in the literature.

2 Stability notions and finite index hypersurfaces

The following assumptions will be maintained throughout this article. Let $\mathcal{N}$ be an orientable Riemannian manifold and let M be an orientable hypersurface immersed in $\mathcal{N}$. Assume that M has constant mean curvature. When the mean curvature is non zero, we orient M by its mean curvature vector $\vec{H}$. Then $\vec{H} = H\vec{\nu}$ and H is positive. When the mean curvature is zero, we choose, once for all, an orientation $\vec{\nu}$ on M.
We introduce the *stability operator* L, defined on smooth functions with compact support in M, that is $L := \Delta + Ric(\nu,\nu) + |A|^2$, where $\Delta = tr \circ Hess$ and A is the shape operator on M.
Let Ω be a relatively compact domain of M. We denote by Index(Ω) (respectively WIndex(Ω)) the number of negative eigenvalues of the operator $-L$, for the Dirichlet problem on Ω

$$-Lf = \lambda f, \quad f_{|\partial\Omega} = 0$$

$$(\text{respectively } -Lf = \lambda f, \ f_{|\partial\Omega} = 0, \ \int_\Omega f = 0).$$

The Index(M) (respectively WIndex(M)) is defined as follows

$$\text{Index}(M) := \sup\{\text{Index}(\Omega) \mid \Omega \subset M \text{ rel. comp.}\}$$
$$(\text{respectively WIndex}(M) := \sup\{\text{WIndex}(\Omega) \mid \Omega \subset M \text{ rel. comp.}\})$$

It is easy to see that $-\int_M fL(f)$ is the second derivative of the volume in the direction of $f\nu$ (see [6]), then Index(M) (respectively WIndex(M)) measures the number of linearly independent normal deformations with compact support of M, decreasing area (respectively decreasing area, leaving fixed a volume). When $H = 0$, one can drop the condition $\int_M f = 0$, then, for a minimal hypersurface one consider only the Index(M).
The hypersurface M is said *stable* (respectively *weakly stable*) if Index(M) $= 0$ (respectively WIndex(M) $= 0$). This means that

$$Q(f,f) := -\int_M fL(f) \geq 0, \quad \forall f \in C_0^\infty(M) \ \ \text{(respectively } \forall f \in C_0^\infty(M) \ \int_M f = 0).$$

It is proved in [5] that Index(M) is finite if and only if WIndex(M) is finite. So, when we deal with finite index questions, we will refer to the finiteness of one of the two indices without distinction. Let us recall some well known relations between stability and finite index.

Proposition 2.1. *Let M be a complete, non compact constant mean curvature hypersurface in $\mathcal{N}$.*
(1) If M is weakly stable, then there exists a compact subset K in M such that $M \setminus K$ is stable.
(2) The hypersurface M has finite index if and only if there exists a compact subset K in M such that $M \setminus K$ is stable.

Proof. (1) If M is stable, we choose $K = \emptyset$ and (1) is proved. Assume M is not stable, then there exists $f \in C_0^\infty(M)$ such that $Q(f,f) < 0$. Let $K = supp(f)$, we will prove that $M \setminus K$ is stable, i.e. for any $g \in C_0^\infty(M \setminus K)$, one has $Q(g,g) \geq 0$. Denote by $\alpha = \int_M g$ and $\beta = \int_M f$ and define $h := \alpha f - \beta g$. By a straightforward computation, one has that $\int_M h = 0$. As M is weakly stable, one has $Q(h,h) \geq 0$. As $supp(f) \cap supp(g) = \emptyset$, using the bi-linearity of Q one has

$$0 \leq Q(h,h) = \alpha^2 Q(f,f) + \beta^2 Q(g,g). \tag{1}$$

As $Q(f,f) < 0$, inequality (1) implies that $\beta \neq 0$ and $Q(g,g) \geq 0$. Hence $M \setminus K$ is stable.
(2) Assume that M has finite index, then, by a proof similar to the proof of Proposition 1 in [25], one obtains that $M \setminus K$ is stable for a suitable compact subset K. The vice versa is proved by B. Devyver (see Theorem 1.2 in [18]). □

In the literature there are interesting relations between the stability of a minimal hypersurface M in $\mathbb{R}^{n+1}$ and the finiteness of $\int_M |A|^n$. Let us recall some of them. P. Berard [8] proved that a complete stable minimal hypersurface of $\mathbb{R}^{n+1}$, $n \leq 5$ such that $\int_M |A|^n < \infty$, must be a hyperplane. In [36], Y.B. Shen and X.H. Zhu stated that the previous result holds for any n but in the proof, they use an unpublished result by Anderson. P. Bérard, M. do Carmo and W. Santos [10] proved that if M is a complete hypersurface in $\mathbb{H}^{n+1}$, with constant mean curvature $H^2 < 1$, such that $\int_M |A - HI|^n < \infty$, then M has finite index. Notice that the converse is not true as it is showed by the examples by A. da Silveira [16]. In [38], J. Spruck proved that, if M is a minimal submanifold of dimension m of $\mathbb{R}^{n+1}$ such that $\int_M |A|^m$ is small, then M is stable (the definition of stability can be easily extended to submanifolds of arbitrary codimension). In the same spirit of [38], one can prove the following well known result (we give a proof because we were not able to find one in the literature).

Proposition 2.2. *Let M be a complete minimal hypersurface in $\mathbb{R}^{n+1}$, such that $\int_M |A|^n < \infty$. Then M has finite index.*

Proof. As M has infinite volume, the hypothesis yields that there exists a compact subset K of M depending on the Sobolev constant $c(n)$ such that inequality

$$\int_{M \setminus K} |A|^n \leq c(n)^{-\frac{n}{2}} \tag{2}$$

is satisfied. Then, one apply Theorem 2 in [38], to obtain that $M \setminus K$ is stable. Then, by (2) of Proposition 2.1, one obtains that M has finite index. □

The idea of Spruck's proof of Theorem 2 in [38] allows us to prove that if $\int_M H^n$ small, without any further assumption on M, one has a lower bound on the volume of balls in M.

Proposition 2.3. *Let M be a complete submanifold of dimension n in a simply connected manifold with sectional curvature bounded from above by a negative constant. Denote by H the mean curvature of M and assume that $\int_M H^n \leq \frac{1}{2c}$, where c is a constant depending only on n and on the bound on the curvature of the ambient space. Then, for any R*

$$|B_R| \geq \frac{1}{(2cn)^n} R^n$$

where $|B_R|$ is the volume of the geodesic ball in M of radius R.

Proof. Applying the isoperimetric inequality of D. Hoffman and J. Spruck [26] we obtain

$$|B_R|^{\frac{n-1}{n}} \leq c \left[|\partial B_R| + \int_{B_R} |H| \right]. \tag{3}$$

where c is a constant depending only on n and on the bound on the curvature of the ambient space. By Hölder inequality one has

$$\int_{B_R} |H| \leq |B_R|^{\frac{n-1}{n}} \left(\int_{B_R} |H|^n \right)^{\frac{1}{n}}. \tag{4}$$

Replacing (4) in (3) yields

$$|B_R|^{\frac{n-1}{n}} (1 - c||H||_n) \leq c |\partial B_R|. \tag{5}$$

By hypothesis, (5) gives

$$|B_R|^{\frac{n-1}{n}} \leq 2c |\partial B_R| = 2c \frac{d|B_R|}{dR}. \tag{6}$$

By integrating inequality (6) one has the result.

□

Remark 2.1. *It is clear that any minimal complete submanifold satisfies the assumptions of the Proposition 2.3.*

3 Simons' inequality for constant mean curvature hypersurfaces

The following notations (that closely follow those of J. Simons [37] and P. Bérard [9]) will be maintained througout the article. Assume that $\mathcal{N}$, M H and ν are as at the beginning of Section 2. We denote by $\overline{\nabla}$ the Levi-Civita connection in $\mathcal{N}$ and by R (respectively Ric) the curvature tensor (respectively Ricci curvature) of $\mathcal{N}$. We denote by ∇ the connection on M for the induced metric g. Let A (respectively B) be the shape operator (respectively the second fundamental form) of M, i.e. $\langle A(x), y \rangle = \langle \nu, B(x, y) \rangle$ for any x, y in TM. Furthermore, if w is any vector field normal to M, denote by A^w the tensor field defined by $\langle A^w(x), y \rangle = \langle w, B(x, y) \rangle$ (note that with this notation, $A = A^\nu$). Denote by Φ the traceless part of the second fundamental form, defined by $\Phi = B - H\nu g$ and by ϕ the endomorphim of TM associated Φ, i.e. $\langle \phi(x), y\rangle = \langle \nu, \Phi(x, y)\rangle$, for any x, y in TM. Finally, let ∇^2 the rough Laplacian of the normal bundle, associated to the connection ∇, defined by $\nabla^2 = \sum_1^n \nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}$, where $\{e_i\}$ is a local orthonormal frame in a neighborhood of a point of M.

From now on, the ambient manifold $\mathcal{N}$ will satisfy the following assumptions. We denote by $sec(X, Y)$ the sectional curvature of $\mathcal{N}$ for the two plane generated by $X, Y \in T\mathcal{N}$. We assume that K_1 and K_2 are an upper bound and a lower bound of the sectional curvatures, i.e. for any X, $Y \in T\mathcal{N}$, $K_2 \leq sec(X, Y) \leq K_1$. Furthermore we assume that the derivative of the curvature tensor is bounded, that is, there exists a constant K' such that, for any elements e_i, e_j, e_k, e_s, e_t of a local orthonormal frame, one has $\sqrt{\sum_{ijkst} \langle (\nabla_{e_t} R)(e_i, e_j)e_k, e_s\rangle^2} \leq K'$.

Theorem 3.1 (Simons' inequality). *Let M be a hypersurface of constant mean curvature H immersed in $\mathcal{N}$ and let $\varphi = |\phi|$. Then, there exists $\varepsilon > 0$ such that the following inequality holds*

$$
\begin{aligned}
&\varphi\Delta\varphi \geq \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2 - \varphi^4 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\varphi^3 \\
&+ n\left(H^2 + \frac{(K_2 - K_1)H}{2} + 2K_2 - K_1\right)\varphi^2 - 2nK'\varphi \\
&+ \frac{n^2 H(K_2 - K_1)}{2} - \frac{n(n-1)}{2\varepsilon}(K_1 - K_2)^2
\end{aligned}
\tag{7}
$$

In order to prove Theorem 3.1, we need some preliminary results. First, we have to establish a Simons' identity for constant mean curvature hypersurfaces. We prove it here for the sake of completeness but the proof is essentially contained in [9].

Proposition 3.1. *Let M be a hypersurface of constant mean curvature, immersed in $\mathcal{N}$. Then the following relation is satisfied at any point p of M*

$$
\begin{aligned}
&\langle \nabla^2\Phi, \Phi\rangle = -|\phi|^4 + nHtr(\phi^3) + nH^2|\phi|^2 - |\phi|^2 Ric(\nu, \nu) + nH\sum_{i=1}^n R(\nu, e_i, \nu, \phi(e_i)) \\
&+ \sum_{i,k=1}^n \{2R(e_k, \phi(e_i), e_k, \phi(e_i)) + 2R(e_k, e_i, \phi(e_i), \phi(e_k))\} \\
&+ \sum_{i,k=1}^n \{\langle (\overline{\nabla}_{e_i} R)(e_k, \phi(e_i))e_k, \nu\rangle + \langle (\overline{\nabla}_{e_i} R)(e_i, \phi(e_k))e_k, \nu\rangle\},
\end{aligned}
\tag{8}
$$

where $\{e_i\}$ is a local orthonormal frame at p.
Moreover, if $\mathcal{N}$ has constant sectional curvature c, one has

$$\langle \nabla^2\Phi, \Phi\rangle = -|\phi|^4 + nHtr(\phi^3) + n(H^2 + c)|\phi|^2 \tag{9}$$

Proof. By definition

$$\langle \nabla^2\Phi, \Phi\rangle = \langle \nabla^2 B, \Phi\rangle = \langle \nabla^2 B, B\rangle - H\langle \nabla^2 B, g\nu\rangle.$$

Then, at p

$$\langle \nabla^2 \Phi, \Phi \rangle = \sum_{i,j=1}^{n} \langle \nabla^2 B(e_i, e_j), B(e_i, e_j) \rangle - H \sum_{i=1}^{n} \langle \nabla^2 B(e_i, e_i), \nu \rangle. \tag{10}$$

We need to compute the two terms in the right hand side of equation (10). First we observe that, for any normal vector field w and any tangent vector fields x, y one has (see Theorem 2 in [9])

$$\begin{aligned} \langle \nabla^2 B(x,y), w \rangle = &-|A|^2 \langle A^w(x), y \rangle + \langle R(A)^w(x), y \rangle \\ &+ \langle R(nH\nu, x)y, w \rangle + \langle A^w(y), A^{nH\nu}(x) \rangle + \langle R'^w(x), y \rangle, \end{aligned} \tag{11}$$

where R'^w and $R(A)^w$ are defined as follows:

$$\langle R'^w(x), y \rangle = \sum_{k=1}^{n} \{ \langle (\nabla_x R)(e_k, y)e_k, w \rangle + \langle (\nabla_{e_k} R)(e_k, x)y, w \rangle \}$$

$$\begin{aligned} \langle R(A)^w(x), y \rangle = &\sum_{k=1}^{n} \{ 2\langle R(e_k, y)B(x, e_k), w \rangle + 2\langle R(e_k, x)B(y, e_k), w \rangle \\ &- \langle A^w(x), R(e_k, y)e_k \rangle - \langle A^w(y), R(e_k, x)e_k \rangle + \langle R(e_k, B(x,y))e_k, w \rangle - 2\langle A^w(e_k), R(e_k, x)y \rangle \} \end{aligned}$$

In order to compute the first term $\sum_{i,j=1}^{n} \langle \nabla^2 B(e_i, e_j), B(e_i, e_j) \rangle$ in (10), we take $x = e_i$, $y = e_j$, $w = B(e_i, e_j)$ in (11) and sum on i and j. The computation of all the terms is as follows.

1. The first term is

$$-|A|^2 \sum_{i,j} \langle A^{B(e_i,e_j)}(e_i), e_j \rangle = -|A|^2 \sum_{i,j} \langle A^2(e_j), e_j \rangle = -|A|^4. \tag{12}$$

2. The second term is

$$\begin{aligned} \sum_{i,j=1}^{n} \langle R(A)^{B(e_i,e_j)}(e_i), e_j \rangle = &\sum_{k,i,j=1}^{n} \{ 2\langle R(e_k, e_j)B(e_i, e_k), B(e_i, e_j) \rangle + 2\langle R(e_k, e_i)B(e_i, e_k), B(e_i, e_j) \rangle \\ &- \langle A^{B(e_i,e_j)}(e_i), R(e_k, e_j)e_k \rangle - \langle A^{B(e_i,e_j)}(e_j), R(e_k, e_i)e_k \rangle \\ &+ \langle R(e_k, B(e_i, e_j))e_k, B(e_i, e_j) \rangle - 2\langle A^{B(e_i,e_j)}(e_k), R(e_k, e_i)e_j \rangle \}. \end{aligned} \tag{13}$$

The first two terms in the right-hand side of (13) are zero, then, rearranging terms

$$\begin{aligned} \sum_{i,j=1}^{n} \langle R(A)^{B(e_i,e_j)}(e_i), e_j \rangle = &\sum_{k,i=1}^{n} \{ 2R(e_k, A(e_i), e_k, A(e_i)) + 2R(e_k, e_i, A(e_i), A(e_k)) \} \\ &- |A|^2 Ric(\nu, \nu). \end{aligned} \tag{14}$$

3. The third term is

$$\sum_{i,j=1}^{n} nH\langle R(\nu, e_i)(e_j), B(e_i, e_j)\rangle = nH\sum_{i=1}^{n} R(\nu, e_i, \nu, A(e_i)). \tag{15}$$

4. The fourth term is

$$\sum_{i,j=1}^{n}\langle A^{B(e_i,e_j)}(e_j), A^{nHN}(e_i)\rangle = nH\sum_{i=1}^{n}\langle A^3(e_i), e_i\rangle = nHtr(A^3). \tag{16}$$

5. The fifth term is

$$\sum_{i,j=1}^{n}\langle R'^{B(e_i,e_j)}(e_i), e_j\rangle = \sum_{i,k=1}^{n}\{\langle(\overline{\nabla}_{e_i}R)(e_k, A(e_i))e_k, \nu\rangle + \langle(\overline{\nabla}_{e_i}R)(e_i, A(e_k))e_k, \nu\rangle\}. \tag{17}$$

By summing up all the term in (12), (14), (15), (16) and (17) and rearranging terms, one has

$$\begin{aligned}
&\sum_{i,j=1}^{n}\langle\nabla^2 B(e_i, e_j), B(e_i, e_j)\rangle = -|A|^4 + nHtr(A^3) + nH\sum_{i=1}^{n} R(\nu, e_i, \nu, A(e_i))\\
&+\sum_{i,k=1}^{n}\{2R(e_k, A(e_i), e_k, A(e_i)) + 2R(e_k, e_i, A(e_i), A(e_k))\} - |A|^2 Ric(\nu, \nu)\\
&+\sum_{i,k=1}^{n}\{\langle(\overline{\nabla}_{e_i}R)(e_k, A(e_i))e_k, \nu\rangle + \langle(\overline{\nabla}_{e_i}R)(e_i, A(e_k))e_k, \nu\rangle\}.
\end{aligned} \tag{18}$$

In order to compute the second term $\langle\nabla^2 B(e_i, e_i), \nu\rangle$ of (10), we take $x = y = e_i$, and $w = \nu$ in (11). The computation of all the terms is as follows.

1. The first term is

$$-\sum_{i=1}^{n}|A|^2\langle A(e_i), e_i\rangle = -nH|A|^2. \tag{19}$$

2. The second term is

$$\begin{aligned}
&\sum_{i=1}^{n}\langle R(A)^{\nu}(e_i), e_i\rangle = \sum_{k,i=1}^{n}\{2\langle R(e_k, e_i)B(e_i, e_k), \nu\rangle + 2\langle R(e_k, e_i)B(e_i, e_k), \nu)\rangle\\
&-\langle A(e_i), R(e_k, e_i)e_k\rangle - \langle A(e_i), R(e_k, e_i)e_k\rangle + \langle R(e_k, B(e_i, e_i))e_k, \nu\rangle - 2\langle A(e_k), R(e_k, e_i)e_i\rangle.\}
\end{aligned} \tag{20}$$

The first two terms in the right-hand side of (20) are zero, then

$$\sum_{i=1}^{n}\langle R(A)^{\nu}(e_i), e_i\rangle = \sum_{k,i=1}^{n}\{-2\langle A(e_i), R(e_k, e_i)e_k\rangle - 2\langle A(e_k), R(e_k, e_i)e_i\rangle\} - nHRic(\nu, \nu). \tag{21}$$

The first two terms in the right-hand side of (21) are opposite, then

$$\sum_{i=1}^{n} \langle R(A)^{\nu}(e_i), e_i\rangle = -nHRic(\nu,\nu). \tag{22}$$

3. The third term is

$$\sum_{i=1}^{n} nH\langle R(\nu,e_i)e_i,\nu\rangle = nHRic(\nu,\nu). \tag{23}$$

4. The fourth term is

$$\sum_{i=1}^{n} \langle A(e_i), A^{nH\nu}(e_i)\rangle = nH|A|^2. \tag{24}$$

5. The fifth term is

$$\sum_{i=1}^{n} \langle R'^{\nu}(e_i), e_i\rangle = \sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i}R)(e_k,e_i)e_k,\nu\rangle + \langle(\overline{\nabla}_{e_k}R)(e_k,e_i)e_i,\nu\rangle\}. \tag{25}$$

The sum of the terms in (19), (22), (23) and (24) is zero, hence one has

$$\sum_{i=1}^{n} \langle \nabla^2 B(e_i,e_i),\nu\rangle = \sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i}R)(e_k,e_i)e_k,\nu\rangle + \langle(\overline{\nabla}_{e_k}R)(e_k,e_i)e_i,\nu\rangle\}. \tag{26}$$

Replacing (18) and (26) in (10), one obtains

$$\begin{aligned}
\langle \nabla^2\Phi,\Phi\rangle &= -|A|^4 + nHtr(A^3) + nH\sum_{i=1}^{n} R(\nu,e_i,\nu,A(e_i)) \\
&+ \sum_{i,k=1}^{n} \{2R(e_k,A(e_i),e_k,A(e_i)) + 2R(e_k,e_i,A(e_i),A(e_k))\} - |A|^2 Ric(\nu,\nu) \\
&+ \sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i}R)(e_k,A(e_i))e_k,\nu\rangle + \langle(\overline{\nabla}_{e_i}R)(e_i,A(e_k))e_k,\nu\rangle\} \\
&- H\sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i}R)(e_k,e_i)e_k,\nu\rangle + \langle(\overline{\nabla}_{e_k}R)(e_k,e_i)e_i,\nu\rangle\}.
\end{aligned} \tag{27}$$

In order to obtain (8), one needs to write the right-hand side of (27) in terms of ϕ. This is straightforward by replacing in (27) the following identities

$$|A|^2 = |\phi|^2 + H^2 n,\ |A|^4 = |\phi|^4 + n^2H^4 + 2nH^2|\phi|^2,\ tr(A^3) = tr(\phi^3) + nH^3 + 3H|\phi|^2.$$

Equality (9) is a straightforward consequence of (8). □

A key step towards Theorem 3.1 is the following Proposition.

Proposition 3.2. *Let M be a hypersurface of constant mean curvature, immersed in $\mathcal{N}$. Then the following relation is satisfied at any point p of M*

$$
\begin{aligned}
&|\phi|\Delta|\phi| + |\phi|^4 + \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi|^3 - nH^2|\phi|^2 + |\phi|^2 Ric(\nu,\nu) - nH\sum_{i=1}^{n} R(\nu, e_i, \nu, \phi(e_i)) \\
&- \sum_{i,k=1}^{n} \{2R(e_k, \phi(e_i), e_k, \phi(e_i)) + 2R(e_k, e_i, \phi(e_i), \phi(e_k))\} \qquad (28) \\
&- \sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i} R)(e_k, \phi(e_i))e_k, \nu\rangle + \langle(\overline{\nabla}_{e_i} R)(e_i, \phi(e_k))e_k, \nu\rangle\} \\
&\geq \frac{2}{n(1+\varepsilon)}|\nabla|\phi||^2 - \frac{2}{\varepsilon}\sum_{ij} R(\nu, e_i, e_i, e_j)^2,
\end{aligned}
$$

where $\{e_i\}$ is a local orthonormal frame at p.

Proof. As in the Weitzenböck formulas, by a straightforward computation one has

$$\langle\nabla^2\Phi, \Phi\rangle = -|\nabla\Phi|^2 + \frac{1}{2}\Delta|\Phi|^2 = -|\nabla\Phi|^2 + |\Phi|\Delta|\Phi| + |\nabla|\Phi||^2$$

That is

$$|\Phi|\Delta|\Phi| = \langle\nabla^2\Phi, \Phi\rangle + |\nabla\Phi|^2 - |\nabla|\Phi||^2 \qquad (29)$$

In order to obtain inequality (28), we will use a Kato's inequality to estimate the difference $|\nabla\Phi|^2 - |\nabla|\Phi||^2$. R. Schoen, L. Simon and S. T. Yau [35] did such estimate in the case of minimal hypersurfaces. One can easily adapt their computation to the case $H \neq 0$, in order to obtain the following result.

Lemma 3.1 (Kato's inequality). *Assume the hypothesis of Proposition 3.2 are satisfied. Then, for any positive ε*

$$|\nabla\Phi|^2 - |\nabla|\Phi||^2 \geq \frac{2}{n(1+\varepsilon)}|\nabla|\Phi||^2 - \frac{2}{\varepsilon}\sum_{ij} R(\nu, e_i, e_i, e_j)^2. \qquad (30)$$

Furthermore, if $\mathcal{N}$ has constant curvature, then $|\Phi|$ satisfies the simpler inequality

$$|\nabla\Phi|^2 - |\nabla|\Phi||^2 \geq \frac{2}{n}|\nabla|\Phi||^2. \qquad (31)$$

Let us finish the proof of Proposition 3.2.
Replacing (30) in (29) one has, for any positive ε

$$|\Phi|\Delta|\Phi| \geq \langle\nabla^2\Phi, \Phi\rangle + \frac{2}{n(1+\varepsilon)}|\nabla|\Phi||^2 - \frac{2}{\varepsilon}\sum_{ij} R(\nu, e_i, e_i, e_j)^2. \qquad (32)$$

Replacing (8) in (32) and using $|\Phi| = |\phi|$, we get

$$
\begin{aligned}
&|\phi|\Delta|\phi| + |\phi|^4 - nHtr(\phi^3) - nH^2|\phi|^2 + |\phi|^2 Ric(\nu,\nu) - nH\sum_{i=1}^{n} R(\nu, e_i, \nu, \phi(e_i)) \\
&- \sum_{i,k=1}^{n} \{2R(e_k, \phi(e_i), e_k, \phi(e_i)) + 2R(e_k, e_i, \phi(e_i), \phi(e_k))\} \\
&- \sum_{i,k=1}^{n} \{\langle(\overline{\nabla}_{e_i} R)(e_k, \phi(e_i))e_k, \nu\rangle + \langle(\overline{\nabla}_{e_i} R)(e_i, \phi(e_k))e_k, \nu\rangle\} \\
&\geq \frac{2}{n(1+\varepsilon)}|\nabla|\phi||^2 - \frac{2}{\varepsilon}\sum_{ij} R(\nu, e_i, e_i, e_j)^2.
\end{aligned}
\tag{33}
$$

Now, in order to estimate $\mathrm{tr}(\phi^3)$ we need the following Lemma by H. Okumura [32], [1].

Lemma 3.2. *The following algebraic inequality holds for any traceless operator ϕ :*

$$
-\frac{n-2}{\sqrt{n(n-1)}}|\phi|^3 \leq tr(\phi^3) \leq \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3.
$$

Then, we replace the inequality of the last lemma in (33) and obtain (28).

□

When the ambient space has a particular geometry, one can simplify inequality (28). The first part of next Corollary 3.1 is proved in [2] (inequality (10) there).

Corollary 3.1. *(1) Assume that the ambient manifold has constant curvature c. Then*

$$
|\phi|\Delta|\phi| + |\phi|^4 + \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi|^3 - n(H^2+c)|\phi|^2 \geq \frac{2}{n}|\nabla|\phi||^2. \tag{34}
$$

(2) Assume that the ambient manifold is locally symmetric, that is $\overline{\nabla} R \equiv 0$. Then at any point $p \in M$

$$
\begin{aligned}
&|\phi|\Delta|\phi| + |\phi|^4 + \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi|^3 - n(H^2 + Ric(\nu,\nu))|\phi|^2 - nH\sum_{i=1}^{n} R(\nu, e_i, \nu, \phi(e_i)) \\
&- \sum_{i,k=1}^{n} \{2R(e_k, \phi(e_i), e_k, \phi(e_i)) + 2R(e_k, e_i, \phi(e_i), \phi(e_k))\} \\
&\geq \frac{2}{n(1+\varepsilon)}|\nabla|\phi||^2 - \frac{2}{\varepsilon}\sum_{ij} R(\nu, e_i, e_i, e_j)^2.
\end{aligned}
\tag{35}
$$

where $\{e_i\}$ is a local orthonormal frame at p.

Now we are ready to prove Theorem 3.1 (Simons' inequality).

Proof of Theorem 3.1. We choose the local orthonormal frame such that it diagonalizes the endomorphism ϕ and we denote by λ_i its eigenvalue associated to e_i (i.e. $\phi(e_i) = \lambda_i\, e_i$). The proof is an estimation of the terms of (28) depending on R and $\overline{\nabla} R$.
The first term of (28) to estimate is

$$\begin{aligned}
&\sum_{i=1}^{n} R(\nu, e_i, \nu, \phi(e_i)) \\
&= \sum_{i=1}^{n} R\left(\nu, \frac{e_i + \phi(e_i)}{\sqrt{2}}, \nu, \frac{e_i + \phi(e_i)}{\sqrt{2}}\right) - \frac{1}{2}\sum_{i=1}^{n} \left(R(\nu, e_i, \nu, e_i) + R(\nu, \phi(e_i), \nu, \phi(e_i))\right) \\
&= \sum_{i=1}^{n} sec\left(\nu, \frac{e_i + \phi(e_i)}{\sqrt{2}}\right) \left\| \frac{e_i + \phi(e_i)}{\sqrt{2}} \right\|^2 - \frac{1}{2} Ric(\nu, \nu)) - \frac{1}{2}\sum_{i=1}^{n} sec(\nu, \phi(e_i)) \|\phi(e_i)\|^2 \\
&= \sum_{i=1}^{n} sec\left(\nu, \frac{e_i + \phi(e_i)}{\sqrt{2}}\right) \frac{(1 + \lambda_i)^2}{2} - \frac{1}{2} Ric(\nu, \nu) - \frac{1}{2}\sum_{i=1}^{n} sec(\nu, \phi(e_i)) \lambda_i^2 \\
&\geq \frac{nK_2}{2} - \frac{(K_1 - K_2)}{2} |\phi|^2 - \frac{1}{2} Ric(\nu, \nu).
\end{aligned} \tag{36}$$

The second term of (28) to estimate is

$$\begin{aligned}
&\sum_{i,k=1}^{n} \{2R(e_k, \phi(e_i), e_k, \phi(e_i)) + 2R(e_k, e_i, \phi(e_i), \phi(e_k))\} \\
&= \sum_{i,k=1}^{n} \{2\lambda_i^2 R(e_k, e_i, e_k, e_i) + 2\lambda_i \lambda_k R(e_k, e_i, e_i, e_k)\} \\
&= \sum_{i,k=1}^{n} (\lambda_k - \lambda_i)^2 sec(e_i, e_k) \geq K_2 \sum_{i,k=1}^{n} (\lambda_k - \lambda_i)^2 = 2nK_2 |\phi|^2.
\end{aligned} \tag{37}$$

The third term of (28) to estimate is

$$\begin{aligned}
&- \sum_{i,k=1}^{n} \{\langle (\overline{\nabla}_{e_i} R)(e_k, \phi(e_i)) e_k, \nu\rangle + \langle (\overline{\nabla}_{e_i} R)(e_i, \phi(e_k)) e_k, \nu\rangle\} \\
&= - \sum_{i,k=1}^{n} \{\lambda_i \langle (\overline{\nabla}_{e_i} R)(e_k, e_i) e_k, \nu\rangle + \lambda_k \langle (\overline{\nabla}_{e_i} R)(e_i, e_k) e_k, \nu\rangle\} \\
&\leq 2 \sum_k \sqrt{\sum_i \lambda_i^2} \sqrt{\sum_i \langle (\overline{\nabla}_{e_i} R)(e_k, e_i) e_k, \nu\rangle^2} \leq 2nK' |\phi|.
\end{aligned} \tag{38}$$

where in the first inequality we have used Cauchy-Schwarz inequality.
The fourth and last term of (28) to estimate is $\sum_{ij} R(\nu, e_i, e_i, e_j)^2$. One has

$$R(\nu, e_i, e_j, e_i) = \frac{1}{2}\left(R\left(\frac{\nu + e_j}{\sqrt{2}}, e_i, \frac{\nu + e_j}{\sqrt{2}}, e_i\right) - R\left(\frac{\nu - e_j}{\sqrt{2}}, e_i, \frac{\nu - e_j}{\sqrt{2}}, e_i\right)\right).$$

Hence

$$\frac{K_2 - K_1}{2} \leq R(\nu, e_i, e_j, e_i) \leq \frac{K_1 - K_2}{2}.$$

That is

$$\sum_{i,j} R(\nu, e_i, e_i, e_j)^2 \leq \frac{n(n-1)}{4}(K_1 - K_2)^2. \tag{39}$$

Replacing (36), (37), (38), (39) in inequality (28), one has (recall that $\varphi = |\phi|$)

$$\begin{aligned} \varphi\Delta\varphi \geq & -\varphi^4 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\varphi^3 + nH^2\varphi^2 + nH\left(\frac{nK_2}{2} - \frac{(K_1 - K_2)}{2}\varphi^2 - \frac{1}{2}K_1 n\right) \\ & + n(2K_2 - K_1)\varphi^2 - 2nK'\varphi - \frac{n(n-1)}{2\varepsilon}(K_1 - K_2)^2 + \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2. \end{aligned} \tag{40}$$

Rearranging terms in (40), one obtains (7).

□

Now, we state the result of Theorem 3.1 in the particular case of $\mathcal{N}$ being a product of manifolds with constant sectional curvature. Notice that in this case $K' = 0$.

Corollary 3.2. *Let $M_i(c_i)$ be a Riemannian manifold with constant sectional curvature equal to $c_i = -1, 0, 1,\ i = 1, 2$. Let M be a n-dimensional hypersurface immersed in a manifold $M_1(c_1) \times M_2(c_2)$ with constant mean curvature H. Then we have*
(1) $c_1 = c_2 = -1$ *or* $c_1 = -1,\ c_2 = 0$:

$$\varphi\Delta\varphi \geq -\varphi^4 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\varphi^3 + n(H^2 - \frac{H}{2} - 2)\varphi^2 - \frac{n}{2}\left(\frac{(n-1)}{\varepsilon} + nH\right) + \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2$$

(2) $c_1 = -1,\ c_2 = 1$:

$$\varphi\Delta\varphi \geq -\varphi^4 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\varphi^3 + n(H^2 - H - 3)\varphi^2 - n\left(\frac{2(n-1)}{\varepsilon} + nH\right) + \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2.$$

(3) $c_1 = 1,\ c_2 = 1$ *or* $c_1 = 1,\ c_2 = 0$:

$$\varphi\Delta\varphi \geq -\varphi^4 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\varphi^3 + n(H^2 - \frac{H}{2} - 1)\varphi^2 - \frac{n}{2}\left(\frac{(n-1)}{\varepsilon} + nH\right) + \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2.$$

Proof. It is enough to compute K_1 and K_2 in all the cases and replace such values in (7). In case (1), $K_1 = 0$, $K_2 = -1$, in case (2), $K_1 = 1$, $K_2 = -1$, in case (3), $K_1 = 1$, $K_2 = 0$. □

Remark 3.1. • *M. Batista [7] proved a formula analogous to the result of Corollary 3.2 for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.*
• *In the case of $\mathbb{H}^3 \times \mathbb{R}$ and $\mathbb{S}^3 \times \mathbb{R}$, D. Fectu and H. Rosenberg [24] proved a formula analogous to the result of Corollary 3.2 for surfaces with parallel mean curvature.*

4 A generalization of a R. Schoen, L. Simon, S.T. Yau's inequality for finite index hypersurfaces with constant mean curvature

In this section we prove a generalization of one of the integral inequalities in [35], for finite index, constant mean curvature hypersurfaces in a Riemannian manifold (Theorem 4.1). The analogous inequality for minimal hypersurfaces is not explicitly stated in [35]. There, it is a key step towards the L^p estimate of the norm of the second fundamental form.
We recall that we maintain the notation and the conditions on $\mathcal{N}$, established at the beginning of Section 3. From now on, for any $\Omega \subset M$, we denote by $\Omega_+ := \{p \in \Omega \mid \varphi(p) \neq 0\}$.

Theorem 4.1. *Let M be a complete, non compact hypersurface with constant mean curvature H and finite index, of a manifold $\mathcal{N}$. Then, there exists a compact subset K of M such that, for any $q > -\frac{n+2}{n}$ and for any $f \in C_0^\infty((M \setminus K)_+)$ one has*

$$\begin{aligned}&\int_{(M\setminus K)_+} f^2\varphi^{2q+2}(\mathcal{A}\varphi^2 + \mathcal{B}H\,\varphi + \mathcal{C}H^2 + \mathcal{E})\\ &\leq \mathcal{D}\int_{(M\setminus K)_+}\varphi^{2q+2}|\nabla f|^2 + \mathcal{F}\int_{(M\setminus K)_+} f^2\varphi^{2q+1} + \mathcal{G}\int_{(M\setminus K)_+} f^2\varphi^{2q}\end{aligned} \tag{41}$$

where

$$\mathcal{D} = \left(\frac{q+1+\tilde{\varepsilon}}{\tilde{\varepsilon}}\right)\left(\frac{2}{n(1+\varepsilon)} + (3q+2) - \tilde{\varepsilon}\right),$$

$$\mathcal{A} = \left(\frac{2}{n(1+\epsilon)} + (2q+1) - \tilde{\varepsilon}\right) - (q+1)(q+1+\tilde{\varepsilon}),$$

$$\mathcal{B} = -a_1(q+1)(q+1+\tilde{\varepsilon}),$$

$$\mathcal{C} = n\left(\frac{2}{n(1+\varepsilon)} + (2q+1) - \tilde{\varepsilon}\right) + n(q+1)(q+1+\tilde{\varepsilon}),$$

$$\mathcal{E} = a_2(q+1)(q+1+\tilde{\varepsilon}) + nK_2\left(\frac{2}{n(1+\epsilon)} + (2q+1) - \tilde{\varepsilon}\right),$$

$$\mathcal{F} = 2nK'(q+1)(q+1+\tilde{\varepsilon}), \quad \mathcal{G} = -a_3(q+1)(q+1+\tilde{\varepsilon}),$$

with

$$a_1 = \frac{n(n-2)}{\sqrt{n(n-1)}}, \quad a_2 = n\frac{(K_2-K_1)H}{2} + n(2K_2-K_1), \quad a_3 = \frac{n^2H(K_2-K_1)}{2} - \frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2$$

for any ε, $\tilde{\varepsilon} > 0$.
Moreover if, in addition, $q \geq 0$, then we can replace $(M \setminus K)_+$ with $M \setminus K$ and, if M is stable, then $K = \emptyset$.

Proof. Using the same notations as before, Simons' inequality (7) yields

$$\varphi\Delta\varphi \geq \frac{2}{n(1+\varepsilon)}|\nabla\varphi|^2 - \varphi^4 - a_1 H\,\varphi^3 + \varphi^2(nH^2+a_2) - 2nK'\varphi + a_3. \tag{42}$$

By Proposition 2.1, there exists a compact subset K in M such that $M \setminus K$ is stable. Notice that, if M is stable, then $K = \emptyset$.
Multiplying by $\varphi^{2q} f^2$, with $f \in C_0^\infty((M\setminus K)_+)$ and integrating we obtain

$$\begin{aligned}
&-(2q+1)\int_{(M\setminus K)_+} \varphi^{2q} f^2 |\nabla\varphi|^2 - 2\int_{(M\setminus K)_+} \varphi^{2q+1} f\langle\nabla f, \nabla\varphi\rangle \\
&+\int_{(M\setminus K)_+} \varphi^{2q+4} f^2 + a_1 H\int_{(M\setminus K)_+} \varphi^{2q+3} f^2 - (n\,H^2+a_2)\int_{(M\setminus K)_+} \varphi^{2q+2} f^2 \\
&+2\,K'\int_{(M\setminus K)_+} \varphi^{2q+1} f^2 - a_3\int_{(M\setminus K)_+} \varphi^{2q} f^2 \geq \frac{2}{n(1+\varepsilon)}\int_{(M\setminus K)_+} |\nabla\varphi|^2\varphi^{2q} f^2.
\end{aligned} \tag{43}$$

We observe that, as we allow q to be negative, we restrict to the subset $(M\setminus K)_+$. If $q \geq 0$, f can be taken in $C_0^\infty(M\setminus K)$ and the set of integration in all the integrals in the following of the proof can be taken as $M \setminus K$.
Young's inequality gives for $\tilde{\varepsilon} > 0$,

$$|2\,\varphi^{2q+1} f\langle\nabla f, \nabla\varphi\rangle| \leq 2\,(\varphi^q f|\nabla\varphi|)(\varphi^{q+1}|\nabla f|) \leq \tilde{\varepsilon}\varphi^{2q} f^2|\nabla\varphi|^2 + \frac{1}{\tilde{\varepsilon}}\varphi^{2q+2}|\nabla f|^2. \tag{44}$$

Using the estimate (44) in (43), we obtain

$$\begin{aligned}
&\left(\frac{2}{n(1+\varepsilon)} + (2q+1) - \tilde{\varepsilon}\right)\int_{(M\setminus K)_+} \varphi^{2q} f^2|\nabla\varphi|^2 \leq \frac{1}{\tilde{\varepsilon}}\int_{(M\setminus K)_+} |\nabla f|^2\varphi^{2q+2} \qquad (45)\\
&+\int_{(M\setminus K)_+} (\varphi^2 + a_1\,H\varphi - (n\,H^2+a_2))\varphi^{2q+2} f^2 + 2nK'\int_{(M\setminus K)_+} \varphi^{2q+1} f^2 - a_3\int_{(M\setminus K)_+} \varphi^{2q} f^2.
\end{aligned}$$

The stability inequality restricted to $(M\setminus K)_+$ yields, for any $\psi \in \mathcal{C}_0^\infty((M\setminus K)_+)$

$$\int_{(M\setminus K)_+} |\nabla\psi|^2 \geq \int_{(M\setminus K)_+} (|A|^2 + \mathrm{Ric}(\nu,\nu))\psi^2 \geq \int_{(M\setminus K)_+} (\varphi^2 + n\,H^2 + n\,K_2)\psi^2. \tag{46}$$

Taking $\psi = f\,\varphi^{q+1}$ in (46), we get

$$\begin{aligned}
\int_{(M\setminus K)_+} \varphi^{2q+2}|\nabla f|^2 + (q+1)^2\int_{(M\setminus K)_+} \varphi^{2q} f^2|\nabla\varphi|^2 + 2(q+1)\int_{(M\setminus K)_+} \varphi^{2q+1} f\langle\nabla f, \nabla\varphi\rangle \\
\geq \int_{(M\setminus K)_+} \varphi^{2q+4} f^2 + n\,(H^2+K_2)\int_{(M\setminus K)_+} \varphi^{2q+2} f^2.
\end{aligned} \tag{47}$$

Young's inequality gives

$$
\begin{aligned}
&2(q+1)\Big|\int_{(M\setminus K)_+}\varphi^{2q+1} f\langle\nabla f,\nabla\varphi\rangle\Big| \le 2(q+1)\int_{(M\setminus K)_+}\varphi^{2q+1} f|\nabla f||\nabla\varphi| \\
&\le (q+1)\left(\tilde{\varepsilon}\int_{(M\setminus K)_+}\varphi^{2q} f^2|\nabla\varphi|^2+\frac{1}{\tilde{\varepsilon}}\int_{(M\setminus K)_+}\varphi^{2q+2}|\nabla f|^2\right) \qquad (48)
\end{aligned}
$$

and using (48) in (47), we obtain

$$
\begin{aligned}
&-(q+1)(q+1+\tilde{\varepsilon})\int_{(M\setminus K)_+}\varphi^{2q} f^2|\nabla\varphi|^2 \\
&\le (1+\frac{(q+1)}{\tilde{\varepsilon}})\int_{(M\setminus K)_+}|\nabla f|^2\varphi^{2q+2}-\int_{(M\setminus K)_+} f^2\varphi^{2q+2}(\varphi^2+n\,(H^2+K_2)). \qquad (49)
\end{aligned}
$$

Now we make a linear combination of the equations (45) and (49) in order to eliminate the term $\int\varphi^{2q} f^2|\nabla\varphi|^2$. One needs $(q+1)(q+1+\tilde{\varepsilon})>0$ and $(\frac{2}{n(1+\epsilon)}+(2q+1)-\tilde{\varepsilon})>0$ for $\tilde{\varepsilon}$ small enough. These conditions are satisfied if

$$
q>-\frac{n+2}{2n}. \qquad (50)
$$

Therefore, $(q+1)(q+1+\tilde{\varepsilon})(45)+(\frac{2}{n(1+\epsilon)}+(2q+1)-\tilde{\varepsilon})(49)$ gives

$$
\begin{aligned}
&0\le [\frac{(q+1)(q+1+\tilde{\varepsilon})}{\tilde{\varepsilon}}+(\frac{2}{n(1+\epsilon)})(1+\frac{q+1}{\tilde{\varepsilon}})]\int_{(M\setminus K)_+}|\nabla f|^2\varphi^{2q+2} \\
&+\int_{(M\setminus K)_+}[(q+1)(q+1+\tilde{\varepsilon})(\varphi^2+a_1H\,\varphi-(n\,H^2+a_2) \\
&-(\frac{2}{n(1+\epsilon)}+(2q+1)-\tilde{\varepsilon})(\varphi^2+n\,H^2+n\,K_2)]\varphi^{2q+2}f^2 \\
&+2nK'(q+1)(q+1+\tilde{\varepsilon})\int_{(M\setminus K)_+}\varphi^{2q+1}f^2-a_3(q+1)(q+1+\tilde{\varepsilon})\int_{(M\setminus K)_+}\varphi^{2q}f^2 \qquad (51)
\end{aligned}
$$

which gives (41), where the constants are as in the statement of Theorem 4.1. □

Remark 4.1. *As we observed in the Introduction:*

- *any of our results of this section can be easily adapted to the case of δ-stable minimal hypersurfaces. More generally one can give a definition of δ-stable constant mean curvature hypersurface and study the corresponding inequalities;*
- *an inequality analogous to (41) can be obtained for a hypersurface with constant H_r-curvature, that is the r-th symmetric function of the principal curvatures (in [3] and [4] one can find some related results).*

Remark 4.2. *In the following, any interesting application of inequality* (41) *is obtained for* $\mathcal{A} > 0$. *So, we determine the condition on* q *in order to have* $\mathcal{A} > 0$. *As* $\mathcal{A}$ *is continuous with respect to* ε *and* $\tilde{\varepsilon}$ *the sign of* $\mathcal{A}$ *for* $\varepsilon = \tilde{\varepsilon} = 0$ *is preserved for* ε *and* $\tilde{\varepsilon}$ *small, so we study the sign of* $\left(\frac{2}{n} + 2q + 1\right) - (q+1)^2$. *By a straightforward computation one obtains that* $\mathcal{A} > 0$ *if and only if*

$$-\sqrt{\frac{2}{n}} < q < \sqrt{\frac{2}{n}} \tag{52}$$

With a technique analogous to that of the proof of Theorem 4.1 we are able to prove a kind of reversed Hölder inequality. L. F. Cheung and D. Zhou [15] proved such inequality in the case $q = 0$.

Theorem 4.2. *Let* M *be a complete hypersurface immersed with constant mean curvature* H *in a manifold with constant sectional curvature* c. *Assume* M *has finite index. Then there exists a ball* B_{R_0} *in* M *and a positive constant* $\mathcal{S}$ *such that, for any* $q \in \left[0, \sqrt{\frac{2}{n}}\right)$

$$\int_{M\setminus B_{R_0}} \varphi^{2q+4} \leq \mathcal{S} \int_{M\setminus B_{R_0}} \varphi^{2q+2} \tag{53}$$

provided the integral in the right hand side is finite.
Moreover, if M *is stable, then we can replace* $B_{R_0} = \emptyset$.

Proof. Let K be the subset of M such that $M \setminus K$ is stable. Inequality (45) in the present case yields ($\varepsilon = 0$, $K' = a_3 = 0$, $a_2 = nc$, $q \geq 0$)

$$\begin{aligned}(\frac{2}{n} + (2q+1) - \tilde{\varepsilon}) \int_{M\setminus K} \varphi^{2q} f^2 |\nabla\varphi|^2 &\leq \frac{1}{\tilde{\varepsilon}} \int_{M\setminus K} |\nabla f|^2 \varphi^{2q+2} \\ &+ \int_{M\setminus K} \varphi^{2q+4} f^2 + a_1 H \int_{M\setminus K} \varphi^{2q+3} f^2 - n(H^2+c) \int_{M\setminus K} \varphi^{2q+2} f^2.\end{aligned} \tag{54}$$

Young's inequality implies, for any positive δ,

$$\varphi^{2q+3} f^2 \leq \frac{\delta}{2} \varphi^{2q+4} f^2 + \frac{1}{2\delta} \varphi^{2q+2} f^2. \tag{55}$$

Replacing (55) in (54) one has

$$\begin{aligned}(\frac{2}{n} + (2q+1) - \tilde{\varepsilon}) \int_{M\setminus K} \varphi^{2q} f^2 |\nabla\varphi|^2 &\leq \frac{1}{\tilde{\varepsilon}} \int_{M\setminus K} |\nabla f|^2 \varphi^{2q+2} \\ &+ \left(1 + \frac{a_1 H \delta}{2}\right) \int_{M\setminus K} \varphi^{2q+4} f^2 + \left(\frac{a_1 H}{2\delta} - n(H^2+c)\right) \int_{M\setminus K} \varphi^{2q+2} f^2.\end{aligned} \tag{56}$$

Inequality (49) in the present case yields ($K_2 = c$)

$$\begin{aligned}&-(q+1)(q+1+\tilde{\varepsilon}) \int_{M\setminus K} \varphi^{2q} f^2 |\nabla\varphi|^2 \\ &\leq (1 + \frac{(q+1)}{\tilde{\varepsilon}}) \int_{M\setminus K} |\nabla f|^2 \varphi^{2q+2} - \int_{M\setminus K} f^2 \varphi^{2q+2} (\varphi^2 + n(H^2+c))\end{aligned} \tag{57}$$

By doing $((q+1)(q+1+\tilde{\varepsilon})(56)+(\frac{2}{n}+2q+1-\tilde{\varepsilon})(57)$ and rearranging terms, one has

$$\mathcal{P}\int_{M\setminus K}\varphi^{2q+4}f^2 \leq \mathcal{L}\int_{M\setminus K}\varphi^{2q+2}|\nabla f|^2+\mathcal{Q}\int_{M\setminus K}\varphi^{2q+2}f^2 \tag{58}$$

where

$$\mathcal{P}=\left(\frac{2}{n}+2q+1-\tilde{\varepsilon}\right)-(q+1)(q+1-\tilde{\varepsilon})\left(1+\frac{a_1H\delta}{2}\right)$$

$$\mathcal{L}=\frac{(q+1)(q+1-\tilde{\varepsilon})}{\tilde{\varepsilon}}+\left(\frac{2}{n}+2q+1-\tilde{\varepsilon}\right)\left(1+\frac{q+1}{\tilde{\varepsilon}}\right)$$

$$\mathcal{Q}=(q+1)(q+1-\tilde{\varepsilon})\left(\frac{a_1H}{2\delta}-n(H^2+c)\right)-(\frac{2}{n}+2q+1-\tilde{\varepsilon})n(H^2+c).$$

For $\delta >> 1$ and $\tilde{\varepsilon} << 1$, the positiveness of $\mathcal{P}$ is equivalent to the positiveness of $\frac{2}{n}+2q+1-(q+1)(q+1)$, that is $q\in\left[0,\sqrt{\frac{2}{n}}\right)$.
Fix R_0 such that $K\subset B_{R_0}$, so $M\setminus B_{R_0}$ is stable. Define $f\in C_0^\infty(M\setminus B_{R_0})$ to be a radial function such that $f\equiv 0$ on B_{R_0} and on $M\setminus B_{R_0+2R+1}$, $f\equiv 1$ on $B_{R_0+R+1}\setminus B_{R_0+1}$ and $|\nabla f|\leq C$, with C a positive constant.
Replacing f in (58) yields

$$\int_{(B_{R_0+2R+1}\setminus B_{R_0})}\varphi^{2q+4}\leq \mathcal{S}\int_{M\setminus B_{R_0}}\varphi^{2q+2} \tag{59}$$

where $\mathcal{S}=\frac{\mathcal{L}+C^2\mathcal{Q}}{\mathcal{P}}$. Then, by letting R go to infinity in (59) one has

$$\int_{M\setminus B_{R_0}}\varphi^{2q+4}\leq \mathcal{S}\int_{M\setminus B_{R_0}}\varphi^{2q+2} \tag{60}$$

□

Remark 4.3. *Theorem 4.2 has interesting consequences about the relations between the volume entropy of a hypersurface M and $\int_M\varphi^p$ for suitable p (see [27]).*

5 Caccioppoli's Inequalities

In this section we give three consequences of inequality (41). Such consequences are Caccioppoli's type inequalities. The first one is a generalization of Theorem 1 in [35] and involves φ and the curvature of the ambient space. The second one involves, in addition, $|\nabla\varphi|$. In order to obtain the third one, we restrict ourselves to the case of constant sectional curvature ambient spaces and we improve inequality (41), by estimating carefully the involved constants.

Theorem 5.1 (**Caccioppoli's inequality of type I).** *Let M be a complete hypersurface immersed with constant mean curvature H in a manifold $\mathcal{N}$. Assume M has finite index. Then, there exist a compact subset K of M and constants β_1, β_2, β_3, such that for every $f\in C_0^\infty((M\setminus K)_+)$ and $q>-\frac{n+2}{2n}$*

$$\beta_1 \int_{(M\setminus K)_+} f^{2q+4}\varphi^{2q+4} \le \beta_2 \int_{(M\setminus K)_+} |\nabla f|^{2q+4} + \beta_3 \int_{(M\setminus K)_+} f^{2q+4}. \tag{61}$$

Moreover:
(i) the constant β_1 *is positive if and only if* $|q| < \sqrt{\frac{2}{n}}$,
(ii) if, in addition, $q \ge 0$, *then we can replace* $(M\setminus K)_+$ *with* $M\setminus K$.

Proof. Let K be the compact set in M such that $M \setminus K$ is stable. Let us first write (41) taking only the term with highest exponent of φ on the left side.

$$\begin{aligned}\int_{(M\setminus K)_+} \mathcal{A} f^2\varphi^{2q+4} &\le \mathcal{D}\int_{(M\setminus K)_+} \varphi^{2q+2}|\nabla f|^2 - \mathcal{B}H\int_{(M\setminus K)_+} f^2\varphi^{2q+3}\\ &- (\mathcal{E}+\mathcal{C}H^2)\int_{(M\setminus K)_+} f^2\varphi^{2q+2} + \mathcal{F}\int_{(M\setminus K)_+} f^2\varphi^{2q+1} + \mathcal{G}\int_{(M\setminus K)_+} f^2\varphi^{2q}\end{aligned} \tag{62}$$

We will transform all the terms of the right-hand side of (62), using Young's inequality, in order to obtain terms with $f^2\varphi^{2q+4}$ that can be reabsorbed by the left-hand side. By Young's inequality one has

$$|\mathcal{B}|H\varphi^{2q+3} \le \varepsilon_1\varphi^{2q+4} + \frac{1}{\varepsilon_1}(|\mathcal{B}|H)^{2q+4} \quad \text{for any } \varepsilon_1 > 0, \tag{63}$$

$$-(\mathcal{E}+\mathcal{C}H^2)\varphi^{2q+2} \le \varepsilon_2\varphi^{2q+4} + \frac{1}{\varepsilon_2}|\mathcal{E}+\mathcal{C}H^2|^{q+2} \quad \text{for any } \varepsilon_2 > 0, \tag{64}$$

$$\mathcal{F}\varphi^{2q+1} \le \varepsilon_3\varphi^{2q+4} + \frac{1}{\varepsilon_3}\mathcal{F}^{\frac{2q+4}{3}} \quad \text{for any } \varepsilon_3 > 0, \tag{65}$$

$$\mathcal{G}\varphi^{2q} \le \varepsilon_4\varphi^{2q+4} + \frac{1}{\varepsilon_4}|\mathcal{G}|^{\frac{q+2}{2}} \quad \text{for any } \varepsilon_4 > 0. \tag{66}$$

Define $M' = \{p \in M \mid f(p) \neq 0\}$. Then, on M'

$$\begin{aligned}\varphi^{2q+2}|\nabla f|^2 &= f^2\left[\varphi^{2q+2}\frac{|\nabla f|^2}{f^2}\right]\\ &\le \varepsilon_5 f^2\varphi^{2q+4} + \frac{1}{\varepsilon_5}\frac{|\nabla f|^{2q+4}}{f^{2q+2}} \quad \text{for any } \varepsilon_5 > 0.\end{aligned} \tag{67}$$

Now, we replace (63), (64), (65), (66), (67) in (62) and we get

$$\begin{aligned}(\mathcal{A} - \sum_{i=1}^{4}\varepsilon_i - \mathcal{D}\varepsilon_5)\int_{(M\setminus K)_+} f^2\varphi^{2q+4} &\le \frac{\mathcal{D}}{\varepsilon_5}\int_{(M\setminus K)_+\cap M'} \frac{|\nabla f|^{2q+4}}{f^{2q+2}}\\ &+ \left(\frac{1}{\varepsilon_1}(|\mathcal{B}|H)^{2q+4} + \frac{1}{\varepsilon_2}|\mathcal{E}+\mathcal{C}H^2|^{q+2} + \frac{1}{\varepsilon_3}\mathcal{F}^{\frac{2q+4}{3}} + \frac{1}{\varepsilon_4}|\mathcal{G}|^{\frac{q+2}{2}}\right)\int_{(M\setminus K)_+} f^2.\end{aligned} \tag{68}$$

One obtains inequality (61), after replacing f by f^{q+2} in (68) and letting

$$\beta_1 = \mathcal{A} - \sum_{i=1}^{4} \varepsilon_i - \mathcal{D}\varepsilon_5, \quad \beta_2 = (q+2)^{2q+4}\frac{\mathcal{D}}{\varepsilon_5},$$
$$\beta_3 = \frac{1}{\varepsilon_1}(|\mathcal{B}|H)^{2q+4} + \frac{1}{\varepsilon_2}|\mathcal{E} + \mathcal{C}H^2|^{q+2} + \frac{1}{\varepsilon_3}\mathcal{F}^{\frac{2q+4}{3}} + \frac{1}{\varepsilon_4}|\mathcal{G}|^{\frac{q+2}{2}}.$$

Choosing ε_1, ε_2, ε_3, small enough and using Remark 4.2 one obtains (i). (ii) follows in the same way as in Theorem 4.1. ☐

Remark 5.1. *If M is stable and $q \geq 0$, then inequality (61) holds on M for any $f \in C_0^\infty(M)$. Therefore, fixing $t \in (0,1)$ and choosing a radial function f such that $f \equiv 1$ on B_{tR}, $f \equiv 0$ on $M \setminus B_R$ and f is linear on the annulus $B_R \setminus B_{tR}$, one has*

$$\beta_1 \int_{B_{tR}} \varphi^{2q+4} \leq |B_R| \left(\frac{\beta_2}{(1-t)^{2q+4}R^{2q+4}} + \beta_3 \right). \tag{69}$$

Inequality (69) yields interesting relations between $\int_M \varphi^{2q+4}$ and the volume entropy of M (see [27]).

Now we prove a Caccioppoli's inequality involving the gradient of the norm of φ.

Theorem 5.2 (Caccioppoli's inequality of type II). *Let M be a complete hypersurface immersed with constant mean curvature H in a manifold $\mathcal{N}$. Assume M has finite index. Then, there exist a compact subset K of M and positive constants β_4, β_5 β_6 such that, for any function $f \in C_0((M \setminus K)_+)$ and $q > -\frac{n+2}{2n}$*

$$\mathcal{A} \int_{(M\setminus K)_+} f^2 |\nabla\varphi|^{2q} \leq \beta_4 \int_{(M\setminus K)_+} |\nabla f|^2 \varphi^{2q+2} + \beta_5 \int_{(M\setminus K)_+} f^2 \varphi^{2q+3} + \beta_6 \int_{(M\setminus K)_+} f^2. \tag{70}$$

Moreover:
(i) the constant $\mathcal{A}$ is positive if and only if $|q| < \sqrt{\frac{2}{n}}$,
(ii) if, in addition, $q \geq 0$, then we can replace $(M \setminus K)_+$ with $M \setminus K$.

Proof. We sum the two inequalities (45) and (49) and we obtain

$$\begin{aligned}
&\left(\frac{2}{n(1+\varepsilon)} + (2q+1) - \tilde{\varepsilon} - (q+1)(q+1+\tilde{\varepsilon}) \right) \int_{(M\setminus K)_+} \varphi^{2q} f^2 |\nabla\varphi|^2 \\
&\leq \left(1 + \frac{q+2}{\tilde{\varepsilon}}\right) \int_{(M\setminus K)_+} |\nabla f|^2 \varphi^{2q+2} - (2nH^2 + a_2 + nK_2) \int_{(M\setminus K)_+} f^2 \varphi^{2q+2} \\
&\quad + a_1 H \int_{(M\setminus K)_+} f^2 \varphi^{2q+3} + 2K' \int_{(M\setminus K)_+} \varphi^{2q+1} f^2 - a_3 \int_{(M\setminus K)_+} \varphi^{2q} f^2.
\end{aligned} \tag{71}$$

Using Young's inequality, as in the proof of Theorem 5.1, we reduce the terms containing $f^2\varphi^{2q}$, $f^2\varphi^{2q+1}$, $f^2\varphi^{2q+2}$, to a sum of terms containing $f^2\varphi^{2q+3}$, and f^2.
Then, one has that there exist constants β_5, β_6, such that

$$\mathcal{A}\int_{(M\setminus K)_+}\varphi^{2q}f^2|\nabla\varphi|^2 \le \left(1+\frac{q+2}{\tilde{\varepsilon}}\right)\int_{(M\setminus K)_+}|\nabla f|^2\varphi^{2q+2} + \beta_5\int_{(M\setminus K)_+}f^2\varphi^{2q+3}+\beta_6\int_{(M\setminus K)_+}f^2.$$

Now we choose $\beta_4=(q+1)^{-2}\left(1+\frac{q+2}{\tilde{\varepsilon}}\right)$ and we are done.
(i) and (ii) are obtained as in Theorem 5.1.

□

Remark 5.2. *If $K_1 = K_2 = c$ and $H^2 + c \ge 0$, one obtains $\beta_6 = 0$, so the integral in* (70) *that does not contain φ disappears.*

When the ambient manifold $\mathcal{N}$ has constant sectional curvature, by studying carefully the sign of the constants involved in (41), one obtains Caccioppoli's inequalities of type III. Let us start with the minimal case.

Theorem 5.3 (**Caccioppoli's inequality of type III -** $H = 0$)**.** *Let M be a complete minimal hypersurface immersed in a manifold with zero sectional curvature. Assume M has finite index. Then, there exists a compact subset K of M such that, for any $f \in C_0^\infty(M\setminus K)$, one has*

$$\mathcal{A}\int_{M\setminus K}f^2|A|^{2x+2}\le\mathcal{D}\int_{M\setminus K}|A|^{2x}|\nabla f|^2 \tag{72}$$

provided $x\in\left[1,1+\sqrt{\frac{2}{n}}\right)$.
Moreover if $x\in\left(1-\sqrt{\frac{2}{n}},1+\sqrt{\frac{2}{n}}\right)$ an inequality analogous to (72) *holds with $(M\setminus K)_+$ instead of $M\setminus K$.*

Remark 5.3. *In the case of M stable, the analogous of inequality* (72) *in $\mathbb{R}^{n+1}$ was proved by M. do Carmo and C. K. Peng [21].*

Before stating next Theorem, we give two definitions:

- For $\gamma=\frac{n-2}{n}$, $\mu=\frac{n^2}{4(n-1)}$, let g be the following function

$$g_n(x)=\frac{(2x-\gamma)^2-x^4}{(2x-\gamma)^2-\mu x^4} \tag{73}$$

- Let x_1 and x_2 be the following real numbers

$$x_1=\frac{2\sqrt{n-1}}{n}\left(1-\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right),\ x_2=\frac{2\sqrt{n-1}}{n}\left(1+\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right) \tag{74}$$

Theorem 5.4 (**Caccioppoli's inequality of type III -** $H \neq 0$**).** *Let M be a complete hypersurface immersed with constant mean curvature $H \neq 0$, in a manifold with constant sectional curvature c. Assume M has finite index and $n \leq 5$. Then there exist a compact subset K in M and a constant γ such that, for any $f \in C_0^\infty(M \setminus K)$*

$$\gamma \int_{M\setminus K} f^2 \varphi^{2x} \leq \mathcal{D} \int_{M\setminus K} \varphi^{2x} |\nabla f|^2 \tag{75}$$

provided either
(1) $c = 0$, $x \in [1, x_2)$.
or
(2) $c = -1$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 \geq g_n(x)$.
Moreover, if $n \leq 6$ and $x \in (x_1, x_2)$ in (1) (respectively $x \in [x_1 + \varepsilon, x_2 - \varepsilon]$ in (2)), an inequality analogous to (75) *holds with $(M \setminus K)_+$ instead of $M \setminus K$.*

The Caccioppoli's inequality of type III for $H \neq 0$ is strongly different from the corresponding inequality for minimal hypersurfaces. Indeed, the power of φ in the left hand term is $2x$ while in the minimal case it is $2x + 2$.

Proof of Theorems 5.3, 5.4. It is worth to write here inequality (41) and the value of the constant involved. One has $K_1 = K_2 = c$, $K' = 0$, $a_1 = \frac{n(n-2)}{\sqrt{n(n-1)}}$, $a_2 = nc$, $a_3 = 0$. Furthermore, we can take $\varepsilon = 0$ in the Kato's inequality. Therefore, we get

$$\mathcal{F} = 0, \;\; \mathcal{G} = 0$$

$$\mathcal{A} = \big(\frac{2}{n} + (2q+1) - \tilde{\varepsilon}\big) - (q+1)(q+1+\tilde{\varepsilon}).$$

$$\mathcal{B} = -a_1\,(q+1)(q+1+\tilde{\varepsilon}).$$

$$\mathcal{E} = c\,\mathcal{C} = cn\left[\left(\frac{2}{n} + (2q+1) - \tilde{\varepsilon}\right) + \;(q+1)(q+1+\tilde{\varepsilon})\right].$$

Then, inequality (41) yields (notice that we are assuming to be in the case $q \geq 0$)

$$\int_{M\setminus K} f^2 \varphi^{2q+2}(\mathcal{A}\varphi^2 + \mathcal{B}H\varphi + \mathcal{C}(H^2 + c)) \leq \mathcal{D} \int_{M\setminus K} \varphi^{2q+2} |\nabla f|^2 \tag{76}$$

We must find conditions on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, c, H such that the coefficient $\beta_7 = \mathcal{A}\varphi^2 + \mathcal{B}H\varphi + \mathcal{C}(H^2 + c)$ in (76) is positive. It is enough to do the computation for $\tilde{\varepsilon} = 0$, because all the quantities are continuous with respect to $\tilde{\varepsilon}$. We notice that in some cases the coefficient β_7 is positive without any condition on H, while in some other cases, we have to look for positiveness of β_7 provided H satisfies some conditions.
In order to simplify notation, we let $x = q + 1$. The condition (52) for the positiveness of $\mathcal{A}$ in terms of x is

$$\alpha_1 := 1 - \sqrt{\frac{2}{n}} < x < \alpha_2 := 1 + \sqrt{\frac{2}{n}} \tag{77}$$

Let us study the different cases ($c = 0, -1, 1$, $H = 0$, $H \neq 0$).
(1) $c = 0$.

- $H = 0$: in this case $\beta_7 = \mathcal{A}\varphi^2$. We only need $\mathcal{A} = -x^2 + 2x - (\dfrac{n-2}{n}) > 0$, so x must satisfy condition (77).
- $H \neq 0$: in this case $\beta_7 = \mathcal{A}\varphi^2 + \mathcal{B}H\varphi + \mathcal{C}H^2$. The quantity β_7 is positive for any value of φ if and only if $\mathcal{A} > 0$, and $\Delta_0 = H^2(\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}) < 0$. As

$$\Delta_0 = nH^2\left(x^4\frac{n^2}{n-1} - 4\left(2x - \frac{n-2}{n}\right)^2\right),$$

then, $\Delta_0 < 0$ if and only if

$$x^4\frac{n^2}{n-1} < 4\left(2x - \frac{n-2}{n}\right)^2.$$

Condition (77) guarantees that $x > \frac{n-2}{2n}$, then the previous inequality is equivalent to

$$x^2\frac{n}{\sqrt{n-1}} - 4x + \frac{2(n-2)}{n} < 0. \tag{78}$$

The discriminant of the polynomial in (78) is positive if and only if $n \leq 6$.
Then inequality (78) is satisfied for $n \leq 6$ and $x \in (x_1, x_2)$ where x_1 and x_2 are defined in (74) and are the roots of the polynomial in (78).
We observe that the value 1 is contained in (x_1, x_2), if and only if $n \leq 5$. Moreover, $\alpha_1 \leq x_1 < x_2 \leq \alpha_2$ with equality when $n = 2$. Therefore, for $n = 2$, the range of x is the same for any $H \geq 0$.
The conditions on x are summed up in Table 1.

(2) $c = -1$.

- $H \geq 0$: in this case $\beta_7 = \mathcal{A}\varphi^2 + \mathcal{B}H\varphi + \mathcal{C}(H^2 - 1)$. In order to study the positiveness of β_7 we compute the discriminant $\Delta_{-1} = (\mathcal{B}^2 - 4\mathcal{A}\mathcal{C})H^2 + 4\mathcal{A}\mathcal{C}$. The only case when one does not have a condition on φ is $\mathcal{A} > 0$, $\Delta_{-1} < 0$. We observe that for $H = 0$ the last two conditions are never satisfied together.

Looking at Table 1, in order to have $\mathcal{A} > 0$, $\Delta_{-1} < 0$, one needs that $x \in (x_1, x_2)$ and $H^2 > g_n(x)$, where g_n is the function that we have defined in (73). As the supremum of g_n on (x_1, x_2) is $+\infty$, in order to have some result one needs to restrict the interval of x to $[x_1 + \varepsilon, x_2 - \varepsilon]$, for some positive ε.

x	$\frac{n-2}{2n}$		α_1		x_1		$\frac{n-2}{n}$		x_2		α_2		$+\infty$
$g_n'(x)$	0		$-$		$\Vert$	$-$	0	$+$	$\Vert$		$+$		
$g_n(x)$	μ	$\searrow$		$-\infty$	$\Vert$	$+\infty$ $\searrow$	$g(\frac{n-2}{n})$	$\nearrow$ $+\infty$	$\Vert$	$-\infty$	$\nearrow$		μ
A		$-$	0		$+$		$+$		$+$		0	$-$	
B^2-4AC		$+$			0		$-$		0		$+$		

Table 1

Now, the results of Theorems 5.3, 5.4, are obtained just putting together the previous estimates.

□

Remark 5.4. *Let us do some comments about the case $c = 1$. If $H = 0$, then $\beta_7 = \mathcal{A}\varphi^2 + \mathcal{C}$, is positive for any value of φ if and only if $x \in [\alpha_1, \alpha_2]$ (notice that $\mathcal{C} > 0$). If $H \neq 0$, then $\beta_7 = \mathcal{A}\varphi^2 - \mathcal{B}H\varphi + \mathcal{C}(H^2+1)$. The quantity β_7 is positive for any value of φ if and only if $\mathcal{A} > 0$ and $\Delta_1 = H^2(\mathcal{B}^2 - 4\mathcal{AC}) - 4\mathcal{AC} < 0$. The two conditions are verified for any value of H if $x \in (x_1, x_2)$. While, if $x \notin (x_1, x_2)$ one needs $H^2 \leq \frac{4\mathcal{AC}}{\mathcal{B}^2 - 4\mathcal{AC}} = -g_n(x)$ (see Table 1).*

Now we refine Caccioppoli's inequalities of type III (Theorem 5.3 and 5.4) in order to allow more general exponents of φ. This will be useful in Section 6.

Theorem 5.5. *Let M be a complete minimal hypersurface immersed in a manifold with zero sectional curvature. Assume M has finite index. Let $\mu \in [2, \alpha_2 + 1)$ and $\eta > 0$, such that $\eta\mu \geq 1$. Then, there exists a compact subset K of M, and a positive constant δ_1 such that for any $f \in C_0^\infty(M \setminus K)$, one has*

$$\int_{M\setminus K} f^{2\mu\eta}|A|^{2\mu} \leq \delta_1 \int_{M\setminus K} |A|^{2\mu(1-\eta)}|\nabla f|^{2\mu\eta}. \tag{79}$$

Moreover, if $\mu \in (\alpha_1 + 1, \alpha_2 + 1)$, an inequality analogous to (79) holds with $(M \setminus K)_+$ instead of $M \setminus K$.

Proof. When $\eta\mu = 1$ inequality (79) is the same as inequality (72) with $\mu = x + 1$ and $\delta_1 = \frac{\mathcal{D}}{\mathcal{A}}$. Then assume $\mu\eta > 1$. Let $\mu = x + 1$ in inequality (72) and apply Young's inequality to the second integrand of inequality (72) on M' as follows ($y \geq 0$)

$$\begin{aligned}|A|^{2(\mu-1)}|\nabla f|^2 &= f^2\left[|A|^{2(\mu-1)}\frac{|\nabla f|^2}{f^2}\right] = f^2\left[|A|^{2(\mu-1-y)}\frac{|A|^{2y}|\nabla f|^2}{f^2}\right] \\ &\leq f^2\left[\varepsilon|A|^{2(\mu-1-y)t} + \frac{1}{\varepsilon}\frac{|A|^{2ys}|\nabla f|^{2s}}{f^{2s}}\right]\end{aligned}$$

with $t = \frac{\mu}{\mu-1-y}$, $s = \mu\eta$, $\eta = \frac{1}{y+1}$.

Then replacing the previous inequality in inequality (72) one has

$$(\mathcal{A}-\varepsilon\mathcal{D})\int_{(M\setminus K)\cap M'} f^2|A|^{2\mu} \leq \frac{\mathcal{D}}{\varepsilon}\int_{(M\setminus K)\cap M'} |A|^{2\mu(\eta-1)}\frac{|\nabla f|^{2\mu\eta}}{f^{2\mu\eta-2}}$$

Replacing f by $f^{\mu\eta}$, one has inequality (79), with $\delta_1 = \frac{\mathcal{D}}{\varepsilon(\mathcal{A}-\varepsilon\mathcal{D})}$.

□

Theorem 5.6. *Let M be a complete hypersurface immersed with constant mean curvature H in a manifold with constant sectional curvature c. Assume M has finite index and $n \leq 5$. Then, there exists a compact $K \subset M$ and a positive constant δ_2 such that for any $s \geq 1$ and any $f \in C_0^\infty(M\setminus K)$, one has*

$$\int_{M\setminus K} f^{2s}\varphi^{2x} \leq \delta_2 \int_{M\setminus K} \varphi^{2x}|\nabla f|^{2s} \tag{80}$$

provided either
(1) $c = 0$, $x \in [1, x_2)$,
or
(2) $c = -1$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 \geq g(x)$.
Moreover if $n \leq 6$ and $x \in (x_1, x_2)$ in (1) (respectively $x \in [x_1 + \varepsilon, x_2 - \varepsilon]$ in (2)) an inequality analogous to (75) *holds with $(M \setminus K)_+$ instead of $M \setminus K$.*

Proof. When $s = 1$, inequality (80) is the same as (75) with $\delta_2 = \frac{\mathcal{D}}{\gamma}$. Then, assume $s > 1$ and apply Young's inequality to the second integrand of inequality (75) on M' as follows ($y \geq 0$)

$$\begin{aligned}\varphi^{2x}|\nabla f|^2 = f^2\left[\varphi^{2x}\frac{|\nabla f|^2}{f^2}\right] &= f^2\left[\varphi^{2(x-y)}\frac{\varphi^{2y}|\nabla f|^2}{f^2}\right]\\ &\leq f^2\left[\varepsilon\varphi^{2(x-y)t} + \frac{1}{\varepsilon}\frac{\varphi^{2ys}|\nabla f|^{2s}}{f^{2s}}\right]\end{aligned}$$

with $t = \frac{x}{x-y}$, $s = \frac{x}{y}$.
Then replacing the previous inequality in (75) one has

$$(\gamma-\varepsilon\mathcal{D})\int_{(M\setminus K)\cap M'} f^2\varphi^{2x} \leq \frac{\mathcal{D}}{\varepsilon}\int_{(M\setminus K)\cap M'} \varphi^{2x}\frac{|\nabla f|^{2s}}{f^{2s-2}}$$

Replacing f by f^s, one has the result with $\delta_2 = \frac{\mathcal{D}}{\varepsilon(\gamma-\varepsilon\mathcal{D})}$. □

6 Applications of the Caccioppoli's inequalities in the stable case

In this section we assume that M is stable and we discuss some consequences of Caccioppoli's inequality of type III. As the literature on the subject is wide and broken up, we will compare our results with the old ones that we are aware of.

Notice that, when M is stable, all the results of the previuos Sections hold taking the compact subset $K = \emptyset$. We split the discussion about the consequences of Caccioppoli's inequality into two parts. First we deal with minimal hypersurfaces in an ambient manifold with zero sectional curvature. We give conditions on the total curvature, that guarantee that the hypersurface is totally geodesic. Then we deal with hypersurfaces with constant mean curvature $H \neq 0$, in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$. We give non existence results, provided some conditions on the total curvature are satisfied. It will be clear in the following that all our results hold when $\int_M \varphi^p$ is finite, for suitable p (see Remark 6.4).
We recall that the classification of stable constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$, is completely known. Stable, complete, orientable, minimal surfaces in $\mathbb{R}^3$ are planes, as it was proved by M. do Carmo and C.K. Peng [20]. Later, A. Ros [34] proved that there are no non orientable stable minimal surfaces in $\mathbb{R}^3$. Finally, F. Lopez and A. Ros [29] proved that stable, complete, non compact, constant mean curvature surfaces in $\mathbb{R}^3$ are planes. A. Silveira [16] proved that, in $\mathbb{H}^3$, there are no stable complete, non compact surfaces with constant mean curvature $H \geq 1$ except horospheres, while there are many examples of stable, complete surfaces with constant mean curvature $H \in (0, 1)$. Furthermore, G. De Oliveira and the third author [17] founded many examples of minimal stable surfaces in $\mathbb{H}^3$.
It is proved in [23], [13] and [33] that, for $n = 3, 4$, in $\mathbb{R}^{n+1}$ (respectively in $\mathbb{H}^{n+1}$) there is no finite index, complete, non compact hypersurface with constant mean curvature $H \neq 0$ (respectively H great enough). The analogous problem in higher dimension is still open. We give a partial answer to it, assuming $n \leq 5$ and some growth condition on $\int_M \varphi^{2x}$, for suitable x.
We start by studying some consequences of Caccioppoli's inequality for $H = 0$. The first result is a consequence of Theorem 5.5.

Corollary 6.1. *Let M be a complete minimal stable hypersurface immersed in a manifold with zero sectional curvature. Assume that, for $\mu \in [2, \alpha_2 + 1)$, $\eta > 0$, $\eta\mu \geq 1$*

$$\lim_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} |A|^{2\mu(1-\eta)}}{R^{2\eta\mu}} = 0. \tag{81}$$

Then M is totally geodesic.

Remark 6.1. *Before proving Corollary 6.1 we observe that, in the denominator of* (81)*, one can take any power of R smaller than $\eta\mu$. In fact, for any $s \leq \eta\mu$, one has*

$$\frac{\int_{B_{2R} \setminus B_R} |A|^{2\mu(1-\eta)}}{R^{2\mu\eta}} \leq \frac{\int_{B_{2R} \setminus B_R} |A|^{2\mu(1-\eta)}}{R^{2s}}$$

Proof of Corollary 6.1. Let $f \in C_0(M)$ such that $f \equiv 1$ on B_R, $f \equiv 0$ on $B_{2R} \setminus B_R$ and $|\nabla f| \leq \frac{1}{R}$. Replacing such f in inequality (79) yields, for any $\mu \in [2, \alpha_2 + 1)$, $\eta > 0$, $\eta\mu \geq 1$

$$\int_{B_R} |A|^{2\mu} \leq \frac{\delta_1}{R^{2\eta\mu}} \int_{B_{2R} \setminus B_R} |A|^{2\mu(1-\eta)}$$

By hypothesis the second term in the previous inequality tends to zero as R tends to infinity. Hence $|A| \equiv 0$ on M and M is totally geodesic.

□

Remark 6.2. • *Notice that taking $\eta\mu = 1$ in Corollary 6.1 yields that if, for $x \in [1, \alpha_2)$, one has*

$$\limsup_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} |A|^{2x}}{R^2} = 0,$$

then M is totally geodesic.

• *Corollary 6.1 is a generalization of the result by M. do Carmo and C. K. Peng, stated in Theorem 1.3 of [21], that is: if there exists $t \in (0, 2\alpha_2)$ such that*

$$\limsup_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} |A|^{2}}{R^t} = 0$$

then M is totally geodesic. In fact, this follows by taking $\mu(1-\eta) = 1$, $t = 2\eta\mu = 2(\mu-1) \in (2, 2\alpha_2)$ in Corollary 6.1. Then we can extend the range of the power of R in the denominator to $(0, 2\alpha_2)$, as in Remark 6.1.

Now we state a particular case of Corollary 6.1, which is a generalization to higher dimension of Theorem 2 in [28] by H. Li and G. Wei. Our generalization is different from the one conjectured by H. Li and G. Wei for dimesion $n > 3$.

Corollary 6.2. *Let M be a complete stable minimal hypersurface immersed in a manifold with zero sectional curvature and let $n \leq 7$. If there exists $t \in (2\alpha_2 - 1)$, such that*

$$\lim_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} |A|^{3}}{R^t} = 0,$$

then M is totally geodesic.

Proof. In Corollary 6.1, we take $2\mu(1-\eta) = 3$ and define $t = 2\mu\eta = 2\mu - 3$. Then, in order to apply Corollary 6.1 one has to assume $2 \leq t < 2\alpha_2 - 1$. Notice that $2 < 2\alpha_2 - 1$ if and only if $n \leq 7$. Now, as in Remark 6.1, we extend the result to any value $t \in (0, 2\alpha_2 - 1)$.

□

Now we deal with the case of constant mean curvature $H \neq 0$.

Corollary 6.3. *There is no complete non compact stable hypersurface M with constant mean curvature H in $\mathcal{N} = \mathbb{R}^{n+1}$ or $\mathcal{N} = \mathbb{H}^{n+1}$, $n \leq 5$, provided there exists $s \geq 1$ such that*

$$\limsup_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} \varphi^{2x}}{R^{2s}} = 0 \tag{82}$$

and either
(1) $\mathcal{N} = \mathbb{R}^{n+1}$, $H \neq 0$, $x \in [1, x_2)$,
or
(2) $\mathcal{N} = \mathbb{H}^{n+1}$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 > g_n(x)$.

Before proving Corollary 6.3, it is worth to notice the following. Reasoning as in Remark 6.1 we can take any power of R between zero and ∞, in the denominator of (82). This means that, in the hypothesis of Corollary 6.3, $\int_{B_{2R} \setminus B_R} \varphi^{2x}$ can not be polynomial in R.

Proof of Corollary 6.3. We use the same method as in the proof of Corollary 6.1, starting with (80) instead of (79). Then we obtain that $\varphi \equiv 0$ on M, which means that M is totally umbilic. In case (1), it follows that M is contained either in a sphere or in a plane. As M is complete non compact, M is a plane, then $H = 0$. Contradiction. In case (2), it follows that M is contained either in a sphere, or in a horosphere, or in a equidistant sphere. The inequality $H^2 > g_n(x) \geq 1$ yields that M can be only contained in a sphere. As M is complete and non compact, this is a contradiction.

□

Remark 6.3. • *The proof of Corollary 6.3 yields a result in more general ambient manifolds. In fact, under the same conditions, if $\mathcal{N}$ has constant sectional curvature $c \leq 0$ but it is not simply connected, then M is totally umbilical.*

• *Taking $s = 1$ in Corollary 6.3, one has that there is no complete non compact stable hypersurface M with constant mean curvature H in $\mathcal{N} = \mathbb{R}^{n+1}$ or $\mathcal{N} = \mathbb{H}^{n+1}$, $n \leq 5$, provided*

$$\lim_{R \longrightarrow \infty} \frac{\int_{B_{2R} \setminus B_R} \varphi^{2x}}{R^2} = 0 \tag{83}$$

and provided either
(1) $\mathcal{N} = \mathbb{R}^{n+1}$, $H \neq 0$, $x \in [1, x_2)$,
or
(2) $\mathcal{N} = \mathbb{H}^{n+1}$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 > g_n(x)$.
Also in this case, if $\mathcal{N}$ has constant sectional curvature $c \leq 0$ but it is not simply connected, then M is totally umbilical.

• *Taking $x = 1$ in (1) of Corollary 6.3, we improve the result by H. Alencar and M. do Carmo, stated in Theorem 4 of [2]. Furthermore, M. do Carmo and D. Zhou [22] stated a result weaker than (1) of Corollary 6.3 and, in their proof, they use wrongly Young's inequality (see equation (3.7) there).*

Remark 6.4. *As we said before, many of the result of this article, holds for $\int_M \varphi^p < \infty$ for suitable p. As an example we use Theorem 4.2 in order to prove a result of L. F. Cheung, D. Zhou [15] in a more direct form than the one contained in [15]. The result is that for $n = 3, 4, 5$, any complete, stable hypersurface M in $\mathbb{H}^{n+1}$ with constant mean curvature $H > 1$, such that $\int_M \varphi^2 < \infty$, is compact. Theorem 4.2 yields $\int_M \varphi^4 < \infty$, ($q = 0$) and $\int_M \varphi^5 < \infty$ ($q = \frac{1}{2}$). Then, by Cauchy-Schwarz, one has $\int_M \varphi^3 < \infty$. Then one applies Corollary 2.1 in [14] and obtains the result.*

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Said Ilias
Université F. Rabelais, Dép. de Mathématiques, Tours
ilias@univ-tours.fr
Barbara Nelli
Dipartimento di Matematica Pura e Applicata,
Universitá dell'Aquila
nelli@univaq.it
Marc Soret
Université F. Rabelais, Dép. de Mathématiques, Tours
marc.soret@lmpt.univ-tours.fr