EMBEDDING SMOOTH AND FORMAL DIFFEOMORPHISMS THROUGH THE JORDAN-CHEVALLEY DECOMPOSITION

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ABSTRACT

In [Xiang Zhang, The embedding flows of C^∞ hyperbolic diffeomorphisms, J. Differential Equations 250 (2011), no. 5, 2283-2298] Zhang proved that any local smooth hyperbolic diffeomorphism whose eigenvalues are weakly nonresonant is embedded in the flow of a smooth vector field. We present a new, simpler and more conceptual proof of such result using the Jordan-Chevalley decomposition in algebraic groups and the properties of the exponential operator.

We characterize the hyperbolic smooth (resp. formal) diffeomorphisms that are embedded in a smooth (resp. formal) flow. We introduce a criterium showing that the presence of weak resonances for a diffeomorphism plus two natural conditions imply that it is not embeddable. This solves a conjecture of Zhang. The criterium is optimal, we provide a method to construct embeddable diffeomorphisms with weak resonances if we remove any of the conditions.

1. INTRODUCTION

We are interested on studying embedding flows for real analytic, complex analytic and C^∞ local diffeomorphisms.

We denote by $\mathcal{X}_\infty(\mathbb{R}^n, 0)$ and $\mathrm{Diff}_\infty(\mathbb{R}^n, 0)$ the C^∞ local singular vector fields and diffeomorphisms respectively defined in a neighborhood of $0 \in \mathbb{R}^n$.

We denote by $\mathcal{X}(\mathbb{R}^n, 0)$ (resp. $\mathcal{X}(\mathbb{C}^n, 0)$) the set of germs of real analytic (resp. complex analytic) vector fields which are singular at 0. The formal completion of these spaces are denoted by $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ and $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ respectively. Indeed a formal vector field $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ is an

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expression of the form

$$\sum_{j=1}^{n} a_j(z_1, \ldots, z_n) \frac{\partial}{\partial z_j} \text{ where } a_1, \ldots, a_n \in \mathfrak{m}$$

and $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[z_1, \ldots, z_n]]$. Moreover X belongs to $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ if and only if all the coefficients of the power series $a_1, \ldots, a_n$ are real.

We define $\text{Diff}\,(\mathbb{R}^n, 0)$ (resp. $\text{Diff}(\mathbb{C}^n, 0)$) the group of local real analytic (resp. complex analytic) diffeomorphisms defined in a neighborhood of $0 \in \mathbb{C}^n$. We denote by $\widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ and $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ respectively their formal completions. A formal diffeomorphism $\varphi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is an expression of the form

$$(a_1(z_1, \ldots, z_n), \ldots, a_n(z_1, \ldots, z_n)) \text{ where } a_1, \ldots, a_n \in \mathfrak{m}$$

such that its first jet is an invertible linear operator. The set $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is a group for the composition. The composition in $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is defined in the natural way by taking the composition in $\text{Diff}\,(\mathbb{C}^n, 0)$ and passing to the limit in the Krull topology (see [6], page 204).

We say that $\varphi \in \text{Diff}_{\infty}(\mathbb{R}^n, 0)$ (resp. $\text{Diff}\,(\mathbb{R}^n, 0)$, $\text{Diff}\,(\mathbb{C}^n, 0)$, $\widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$, $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$) has an *embedding flow* if there exists $X \in \mathcal{X}_{\infty}(\mathbb{R}^n, 0)$ (resp. $\mathcal{X}(\mathbb{R}^n, 0)$, $\mathcal{X}(\mathbb{C}^n, 0)$, $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$, $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$) such that $\varphi = \exp(X)$, i.e. φ is the 1 time flow of X. This concept is defined even if X is formal, in fact $\exp(X)$ is a formal diffeomorphism such that $j^k\exp(X) = j^k\exp(X_k)$ for any $k \in \mathbb{N}$ where X_k is an analytic vector field such that $j^k X = j^k X_k$.

The embedding flow problem is classical. For instance the embedding flow problem has been deeply studied for 1-dimensional real diffeomorphisms (see [1] [2] [11] [12]). Palis proved for arbitrary dimension that the C^1 diffeomorphisms in a compact manifold that are embedded in a C^1 flow form a meagre set [15].

Let $\varphi \in \text{Diff}\,(\mathbb{C}, 0)$ be a one variable complex analytic diffeomorphism such that $j^1\varphi \neq Id$. The embedding flow problem is equivalent to the linearization problem. Indeed φ is embedded in a local complex analytic flow if and only φ is analytically linearizable. Moreover φ has a formal embedding flow in $\hat{\mathcal{X}}(\mathbb{C}, 0)$ if and only if φ is formally linearizable. In the case $j^1\varphi = Id$ and $\varphi \neq Id$ the diffeomorphism φ is always embedded in a formal flow whereas it has an analytic embedding flow if and only if the Ecalle-Voronin invariants of φ are trivial [18].

Even in the one dimensional case there are consequences regarding integrability of complex analytic foliations. Given a complex analytic codimension 1 foliation $\mathcal{F}$ and a leaf $\mathcal{L}$ we can associate to $\mathcal{L}$ its holonomy group $\mathcal{H}$. It can be interpreted as a subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$. The integrability of the foliation is related to the solvable nature of these holonomy groups [16]. The existence of embedding flows, in the solvable case, for the elements of the group is related to the existence of analytic first integrals, integrating factors... In a different but analogous context the existence of embedding flows has been applied to find analytic inverse integrating factors in the neighborhood of limit cycles and elementary singular points of real analytic planar vector fields [7].

Our point of view in the embedding flow problem is based on taking profit of the Jordan-Chevalley decomposition in algebraic groups. More precisely any $\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ can be written uniquely in the Jordan multiplicative form $\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$ where $\varphi_s, \varphi_u \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$, φ_s is formally conjugated to a diagonal linear transformation (semisimple part) and $j^1\varphi_u$ is a unipotent linear operator (unipotent part). Analogously any $X \in \hat{\mathcal{X}}(\mathbb{C}^n,0)$ can be written in a unique way in the Jordan additive form $X = X_s + X_N$ where $X_s, X_N \in \hat{\mathcal{X}}(\mathbb{C}^n,0)$, X_s is formally conjugated to a diagonal linear vector field (semisimple part), j^1X_N is a nilpotent linear operator (nilpotent part) and $[X_s, X_N] = 0$. A positive outcome of the decomposition is that we obtain a natural normal form for φ by linearizing φ_s. Moreover, since affine algebraic groups contain the semisimple and unipotent parts of all their elements (see 15.3, page 99 [10]) we can use the Jordan-Chevalley decomposition to study invariant structures by the action of a diffeomorphism. For instance given $f \in \mathbb{C}[[z_1,\ldots,z_n]]$ the set $G = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) : f \circ \varphi = f\}$ is a group defined by algebraic equations on the coefficients of φ. Hence $\varphi \in G$ implies $\varphi_s \in G$ and $\varphi_u \in G$. This is a simplification since φ_s is formally linearizable and the properties of φ_u can be interpreted on terms of the properties of a formal vector field (the so called infinitesimal generator). This perspective was used to study invariant and periodic (invariant for an iterate) analytic and formal curves by elements of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ [17].

Let us focus on the embedding problem for elements φ of $\widehat{\mathrm{Diff}}\,(\mathbb{R}^n,0)$. Consider a real linear vector field $X_1 \in \mathcal{X}(\mathbb{R}^n,0)$ such that $j^1\varphi = \exp(X_1)$. We can suppose that X_1 is in Jordan normal form. The linear operators X_1 and $j^1\varphi$ have eigenvalues μ_1, …, μ_n and $\lambda_1 = e^{\mu_1}$, …, $\lambda_n = e^{\mu_n}$ respectively. Consider the Jordan additive (resp. multiplicative) decomposition $X_{1,s} + X_{1,N}$ (resp. $(j^1\varphi)_s \circ (j^1\varphi)_u$) of X_1 (resp. of $j^1\varphi$). We say that a monomial $w_1^{a_1}\ldots w_n^{a_n}e_j$, where e_j is the jth element of the canonical base of $\mathbb{C}^n$, is

- *resonant* if $\lambda_1^{a_1}\ldots\lambda_n^{a_n}\lambda_j^{-1} = 1$.

- *strongly resonant* if $a_1\mu_1 + \ldots + a_n\mu_n - \mu_j = 0$.
- *weakly resonant* if $a_1\mu_1 + \ldots + a_n\mu_n - \mu_j \in 2\pi i\mathbb{Z}^*$.

Resonant implies either strongly or weakly resonant. A monomial $w_1^{a_1} \ldots w_n^{a_n} e_j$ is resonant if and only if it commutes with $(j^1\varphi)_s$. Moreover $w_1^{a_1} \ldots w_n^{a_n} e_j$ is strongly resonant if and only if the Lie bracket $[w_1^{a_1} \ldots w_n^{a_n} \partial/\partial w_j, X_{1,s}]$ is equal to 0. Resonances and strong resonances are resonances of the semisimple parts of $j^1\varphi$ and X_1 respectively. We say that the eigenvalues of X_1 are not weakly resonant if there is no weakly resonant monomial of degree greater or equal than 2. In such a case both concepts of resonance coincide. Zhang proves in this setting that there is existence and uniqueness of the embedding flow.

Theorem 1.1. [19] *Let $\varphi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ be a formal diffeomorphism. Let X_1 be a linear vector field such that $j^1\varphi = \exp(X_1)$ and whose eigenvalues are not weakly resonant. Then there exists a unique $\hat{X}$ in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ such that $j^1\hat{X} = X_1$ and $\varphi = \exp(\hat{X})$. Moreover $\hat{X}$ belongs to $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ if $\varphi \in \widehat{\text{Diff}}(\mathbb{R}^n, 0)$ and $X_1 \in \mathcal{X}(\mathbb{R}^n, 0)$.*

The solution of the embedding flow problem in the formal setting has implications in the C^∞ setting. The existence of an embedding flow for a hyperbolic element φ of $\text{Diff}_\infty(\mathbb{R}^n, 0)$ is equivalent to the existence of an embedding flow for its asymptotic development $\hat{\varphi} \in \widehat{\text{Diff}}(\mathbb{R}^n, 0)$ at the origin by a theorem of Chen [3].

Theorem 1.2. [19] *Let $\varphi \in \text{Diff}_\infty(\mathbb{R}^n, 0)$ be a hyperbolic diffeomorphism. Let X_1 be a linear real vector field such that $j^1\varphi = \exp(X_1)$ and whose eigenvalues are not weakly resonant. Then there exists $X \in \mathcal{X}_\infty(\mathbb{R}^n, 0)$ such that $j^1X = X_1$ and $\varphi = \exp(X)$.*

The proof of theorem 1.1 is obtained by doing an inductive process of calculations in the jet level. We introduce a simpler and much more conceptual proof by using the Jordan-Chevalley decomposition of vector fields and diffeomorphisms and the properties of the exponential operator. Calculations are almost no longer required since they are encapsulated in the linearization of the semisimple parts. Zhang shows that the formal diffeomorphism can be considered in normal form up to a formal change of coordinates and then calculates the embedding flow. The first step can be achieved directly by linearizing the semisimple part of the formal diffeomorphism. Then it is easy to check out that $X_{1,s} + \log\varphi_u$ is the expression of the embedding flow. The formal vector field $\log\varphi_u \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ is the infinitesimal generator of φ_u, i.e. the unique nilpotent vector field such that $\varphi_u = \exp(\log\varphi_u)$.

We compare the concepts of embedding flow in $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ and $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ for formal diffeomorphisms in $\widehat{\mathrm{Diff}}\,(\mathbb{R}^n, 0)$. They coincide if the linear part of the embedding flow is required to be real.

Theorem 1.3. *Let $\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{R}^n, 0)$. Suppose that φ is of the form $\exp(\hat{X})$ for some $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ with $j^1\hat{X} \in \mathcal{X}(\mathbb{R}^n, 0)$. Then there exists $\hat{Y} \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ such that $\varphi = \exp(\hat{Y})$ and $j^1\hat{Y} = j^1\hat{X}$.*

We characterize the diffeomorphisms in $\widehat{\mathrm{Diff}}\,(\mathbb{R}^n, 0)$ and $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ having an embedding flow via a normal form theorem.

Theorem 1.4. *Let $\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{R}^n, 0)$ (resp. $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$). Let X_1 be a linear element of $\mathcal{X}(\mathbb{R}^n, 0)$ (resp. $\mathcal{X}(\mathbb{C}^n, 0)$) such that $X_{1,s}$ is diagonal and $j^1\varphi = \exp(X_1)$. Then φ is embedded in a formal flow X with $j^1X = X_1$ if and only if there exists a tangent to the identity $\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{R}^n, 0)$ (resp. $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$) such that $\eta^{\circ(-1)} \circ \varphi \circ \eta$ is strongly resonant (with respect to $X_{1,s}$).*

Let us remind that $\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is tangent to the identity if $j^1\eta = Id$. By definition a formal diffeomorphism is strongly resonant if all its non-vanishing monomials are strongly resonant.

The first examples of hyperbolic diffeomorphisms $\varphi = Az + O(|z|^2) \in \mathrm{Diff}_{\infty}(\mathbb{R}^n, 0)$ (resp. $\mathrm{Diff}\,(\mathbb{R}^n, 0)$), such that A has a real logarithm B, without embedding vector fields are provided by Zhang [19]. He conjectures that if φ has a non-vanishing weakly resonant monomial then it can not be embedded in a C^∞ (resp. real analytic flow). We provide a method to build counterexamples to the conjecture given by resonant diffeomorphisms. We single out two counterexamples that are particularly relevant. Each of them implies that an extra condition should be added to obtain a positive result. Then we prove that in the new setup the conjecture is true. As a consequence our examples can be considered as a classification of the type of counterexamples to the original conjecture.

Theorem 1.5. *Let $A \in GL(n, \mathbb{C})$ be a matrix. Let X_1 be a logarithm of A such that $X_{1,s}$ is diagonal and $[X_{1,N}, Y] = 0$ for any weakly resonant vector field Y. Consider $\varphi = Az + f_2 + \ldots + f_k + \ldots$ in $\mathrm{Diff}_{\infty}(\mathbb{R}^n, 0)$ (resp. $\mathrm{Diff}\,(\mathbb{R}^n, 0)$, $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$). Suppose that φ satisfies one of the following conditions:*

(a) *$f_2 = \ldots = f_{k-1} = 0$ and f_k contains non-vanishing weakly resonant monomials.*
(b) *$Az + f_2 + \ldots + f_{k-1}$ is strongly resonant, there is no weakly resonant monomial of degree $2 \leq d \leq k-1$ and f_k contains non-vanishing weakly resonant monomials.*

Then φ is non-embeddable in the flow of a vector field $X \in \mathcal{X}_{\infty}(\mathbb{R}^n, 0)$ (resp. $\mathcal{X}(\mathbb{R}^n, 0)$, $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$) such that $j^1 X = X_1$.

Let us clarify that f_j is homogeneous of degree j for $j \geq 2$. Resonances are considered with respect to $X_{1,s}$.

The condition (a) is weaker than (b) if $f_2 = \ldots = f_{k-1} = 0$. Otherwise no condition is stronger than the other.

Let us remark that for instance in conditions (a) and (b) of the previous theorem the diffeomorphism φ is not suppposed to be in normal form, or in other words to be resonant. Moreover we do not suppose in condition (a) that there are no weakly resonant monomials of degree $2 \leq d \leq k-1$, or equivalently that there is uniqueness of the embedding flow until order at most $k-1$.

2. Real vector fields

We introduce some useful concepts to study real C^∞ or analytic diffeomorphisms. They include real vector fields, the exponential operator, the Jordan-Chevalley decomposition of diffeomorphisms, analysis of resonances and linearization. The results in this section are classical and they are included for the sake of completeness.

Definition 2.1. *We say that X is a formal vector vector field and we denote $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ if X is a derivation of the $\mathbb{C}$-algebra $\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[z_1, \ldots, z_n]]$. We can express $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ in the more usual notation*

$$X = X(z_1)\frac{\partial}{\partial z_1} + \ldots + X(z_n)\frac{\partial}{\partial z_n}.$$

We say that X is a holomorphic vector field if $X(\mathfrak{m}_0) \subset \mathfrak{m}_0$ where $\mathfrak{m}_0$ is the maximal ideal of $\mathbb{C}\{z_1, \ldots, z_n\}$ We denote by $\mathcal{X}(\mathbb{C}^n, 0)$ the set of local holomorphic vector fields in a neighborhood of 0 in $\mathbb{C}^n$.

Definition 2.2. *We denote by $\mathcal{X}_{\infty}(\mathbb{R}^n, 0)$ the set of C^∞ singular vector fields defined in a neighborhood of 0 in $\mathbb{R}^n$. We denote by $\mathrm{Diff}_{\infty}(\mathbb{R}^n, 0)$ the set of C^∞ diffeomorphisms defined in a neighborhood of 0 in $\mathbb{R}^n$.*

Definition 2.3. *Let $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ be a formal vector field. We say that X is real if*

$$\sigma^* X = \sigma^* \left(\frac{1}{2} \left(Re(X) - iIm(X) \right) \right) = \frac{1}{2} \left(Re(X) + iIm(X) \right)$$

for $\sigma(z_1, \ldots, z_n) = (\overline{z_1}, \ldots, \overline{z_n})$. We define $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ the set of real formal vector fields. We define $\mathcal{X}(\mathbb{R}^n, 0) = \mathcal{X}(\mathbb{C}^n, 0) \cap \hat{\mathcal{X}}(\mathbb{R}^n, 0)$.

A good example is the vector field $\partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$. We have $Re(\partial/\partial z) = \partial/\partial x$ and $Im(\partial/\partial z) = \partial/\partial y$. The complex conjugation $\sigma = \overline{z}$ is of the form $\sigma(x, y) = (x, -y)$ in real coordinates ($z = x + iy$). Then σ preserves $Re(\partial/\partial z)$ whereas it conjugates $Im(\partial/\partial z)$ and $-Im(\partial/\partial z)$. The vector field $\partial/\partial z$ is real. On the contrary $i\partial/\partial z = (1/2)(\partial/\partial y + i\partial/\partial x)$ is not real since $Re(i\partial/\partial z) = \partial/\partial y$ is not preserved by σ. The real vector field $Re(\partial/\partial z)$ preserves the real line $\mathbb{R}$ whereas $Re(i\partial/\partial z)$ does not. The proof of the next lemma is straightforward and it is omitted.

Lemma 2.1. *Let $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$. The following conditions are equivalent:*

- *X is real.*
- *$\sigma^* Re(X) = Re(X)$.*
- *$\sigma^* Im(X) = -Im(X)$.*
- *X is of the form*

$$\left(\sum a^1_{j_1 \dots j_n} z_1^{j_1} \dots z_n^{j_n} \frac{\partial}{\partial z_1}\right) + \dots + \left(\sum a^n_{j_1 \dots j_n} z_1^{j_1} \dots z_n^{j_n} \frac{\partial}{\partial z_n}\right)$$

with $a^k_{j_1 \dots j_n} \in \mathbb{R}$ $\forall 1 \leq k \leq n$ $\forall 0 \leq j_1, \dots, j_n$.

Definition 2.4. *We say that φ is a formal endomorphism and we denote $\varphi \in \widehat{\text{End}}(\mathbb{C}^n, 0)$ if φ is a $\mathbb{C}$-algebra homomorphism of the maximal ideal of $\mathbb{C}[[z_1, \dots, z_n]]$. We can express $\varphi \in \widehat{\text{End}}(\mathbb{C}^n, 0)$ in the more usual notation*

$$\varphi = (z_1 \circ \varphi, \dots, z_n \circ \varphi), \quad z_j \circ \varphi \stackrel{def}{=} \varphi(z_j) \text{ for } 1 \leq j \leq n.$$

We say that φ is a formal diffeomorphism if φ is an isomorphism. We say that φ is holomorphic if $\varphi(\mathfrak{m}_0) \subset \mathfrak{m}_0$ where $\mathfrak{m}_0$ is the maximal ideal of $\mathbb{C}\{z_1, \dots, z_n\}$.

Definition 2.5. *We denote by $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ the set of formal diffeomorphisms. We denote by $\text{Diff}(\mathbb{C}^n, 0)$ the set of local holomorphic diffeomorphisms in a neighborhood of 0 in $\mathbb{C}^n$.*

Definition 2.6. *Let $\varphi \in \widehat{\text{End}}(\mathbb{C}^n, 0)$. If $\sigma \circ \varphi \circ \sigma = \varphi$ (see def. 2.3) we say that φ is real. We define $\widehat{\text{Diff}}(\mathbb{R}^n, 0)$ the set of real elements of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. We define $\text{Diff}(\mathbb{R}^n, 0) = \text{Diff}(\mathbb{C}^n, 0) \cap \widehat{\text{Diff}}(\mathbb{R}^n, 0)$. A formal endomorphism φ is real if and only if the formal power series $z_1 \circ \varphi$, ..., $z_n \circ \varphi$ have real coefficients.*

Remark 2.1. *A formal endomorphism $\varphi = (\varphi_1, \dots, \varphi_n) \in \widehat{\text{End}}(\mathbb{C}^n, 0)$ is real if and only if the formal vector field $\sum_{j=1}^n \varphi_j \partial/\partial z_j$ is real.*

2.1. **Exponential operator.** Consider a vector field $X \in \mathcal{X}_\infty(\mathbb{R}^n,0)$ (or $X \in \mathcal{X}(\mathbb{C}^n,0)$). We denote $\exp(tX)$ the flow of the vector field X, it is the unique solution of the differential equation

$$\frac{\partial}{\partial t}\exp(tX) = X(\exp(tX))$$

with initial condition $\exp(0X) = Id$. We define the exponential $\exp(X)$ of X as $\exp(1X)$. It is a C^∞ local diffeomorphism if $X \in \mathcal{X}_\infty(\mathbb{R}^n,0)$. Moreover $\exp(X)$ is a holomorphic diffeomorphism if $X \in \mathcal{X}(\mathbb{C}^n,0)$.

We can extend the definition of the exponential operator to formal vector fields as an operator acting on formal power series. Given $\hat{X}$ in $\hat{\mathcal{X}}(\mathbb{C}^n,0)$ we define

$$\begin{array}{rccc} \exp(\hat{X}): & \mathbb{C}[[z_1,\dots,z_n]] & \to & \mathbb{C}[[z_1,\dots,z_n]] \\ & g & \to & \sum_{j=0}^{\infty} \frac{\hat{X}^{\circ(j)}}{j!}(g) \end{array}$$

where $\hat{X}^{\circ(0)}(g) = g$ and $\hat{X}^{\circ(j+1)}(g) = \hat{X}(\hat{X}^{\circ(j)}(g))$ for $j \geq 0$. Notice again that we interpret a formal vector field as a derivation on the ring of formal power series. Both definitions of exponential coincide if $\hat{X}$ is convergent, i.e. $(\exp(\hat{X}))(g) = g \circ \exp(\hat{X})$ for any $g \in \mathbb{C}[[z_1,\dots,z_n]]$. We have

$$\exp(\hat{X})(z_1,\dots,z_n) = \left(\sum_{j=0}^{\infty} \frac{\hat{X}^{\circ(j)}}{j!}(z_1), \dots, \sum_{j=0}^{\infty} \frac{\hat{X}^{\circ(j)}}{j!}(z_n) \right)$$

in the usual notation. Since the coefficients of the exponential series are real the exponential of a real formal vector field is a real formal diffeomorphism.

Definition 2.7. *Let $\varphi \in \mathrm{Diff}_\infty(\mathbb{R}^n,0)$ (resp. $\mathrm{Diff}(\mathbb{R}^n,0)$, $\mathrm{Diff}(\mathbb{C}^n,0)$, $\widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$). We say that φ is embedded in a C^∞ flow (resp. real analytic , holomorphic, formal flow) if there exists $X \in \mathcal{X}_\infty(\mathbb{R}^n,0)$ (resp. $\mathcal{X}(\mathbb{R}^n,0)$, $\mathcal{X}(\mathbb{C}^n,0)$, $\hat{\mathcal{X}}(\mathbb{R}^n,0)$) such that $\varphi = \exp(X)$.*

Definition 2.8. *Let $X \in \hat{\mathcal{X}}(\mathbb{C}^n,0)$. We say that X is nilpotent if the first jet j^1X of X is nilpotent. We denote by $\hat{\mathcal{X}}_N(\mathbb{C}^n,0)$ the set of formal nilpotent vector fields.*

Definition 2.9. *Let $\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n,0)$. We say that φ is unipotent if $j^1\varphi$ is unipotent, i.e. 1 is the unique eigenvalue of $j^1\varphi$. We denote by $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0)$ the set of formal unipotent diffeomorphisms.*

Lemma 2.2. *(see [5], [14]) The mapping $\exp: \hat{\mathcal{X}}_N(\mathbb{C}^n,0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0)$ is a bijection.*

Definition 2.10. *Let* $\varphi \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$. *The unique nilpotent formal vector field* $\log \varphi$ *such that* $\varphi = \exp(\log \varphi)$ *is called the infinitesimal generator of* φ.

Let us consider $\varphi \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ as an operator acting on power series $Id + \Theta$. More precisely $\Theta : \mathbb{C}[[z_1, \dots, z_n]] \to \mathbb{C}[[z_1, \dots, z_n]]$ is defined by $\Theta(g) = g \circ \varphi - g$ for any $g \in \mathbb{C}[[z_1, \dots, z_n]]$. We have

$$(\log \varphi)(g) = \log(Id + \Theta)(g) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\Theta^{\circ(j)}(g)}{j} \tag{1}$$

for any $g \in \mathbb{C}[[z_1, \dots, z_n]]$. Since the coefficients of the power series $\log(1+z)$ are real then log associates real formal nilpotent vector fields to real formal unipotent diffeomorphisms.

2.2. **Jordan-Chevalley decomposition.** Let us recall here some known results [8] [13] on the jordanization of diffeomorphisms and vector fields.

Let $\mathfrak{m}$ the maximal ideal of $\mathbb{C}[[z_1, \dots, z_n]]$. Any formal diffeomorphism $\varphi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ acts on the finite dimensional complex vector space $\mathfrak{m}/\mathfrak{m}^{k+1}$ of k-jets. More precisely φ defines an element φ_k of $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$ given by

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{k+1} & \stackrel{\varphi_k}{\to} & \mathfrak{m}/\mathfrak{m}^{k+1} \\ g + \mathfrak{m}^{k+1} & \mapsto & g \circ \varphi + \mathfrak{m}^{k+1} \end{array}. \tag{2}$$

Analogously a formal vector field $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ defines an element X_k of $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$ given by

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{k+1} & \stackrel{X_k}{\to} & \mathfrak{m}/\mathfrak{m}^{k+1} \\ g + \mathfrak{m}^{k+1} & \mapsto & X(g) + \mathfrak{m}^{k+1} \end{array}. \tag{3}$$

Consider the group $D_k \subset GL(\mathfrak{m}/\mathfrak{m}^{k+1})$ defined as

$$D_k = \{\alpha \in GL(\mathfrak{m}/\mathfrak{m}^{k+1}) : \alpha(gh) = \alpha(g)\alpha(h) \ \forall g, h \in \mathfrak{m}/\mathfrak{m}^{k+1}\}.$$

We define the Lie algebra $L_k \subset End(\mathfrak{m}/\mathfrak{m}^{k+1})$ as

$$L_k = \{\gamma \in End(\mathfrak{m}/\mathfrak{m}^{k+1}) : \gamma(gh) = \gamma(g)h + g\gamma(h) \ \forall g, h \in \mathfrak{m}/\mathfrak{m}^{k+1}\}.$$

Any $\gamma \in End(\mathfrak{m}/\mathfrak{m}^{k+1})$ admits a unique additive Jordan decomposition $\gamma = \gamma_s + \gamma_N$ where γ_s is semisimple (or equivalently diagonalizable), γ_N is nilpotent and $[\gamma_s, \gamma_N] = 0$. If γ is a derivation, i.e. $\gamma \in L_k$, then both the semisimple and nilpotent parts γ_s and γ_N are derivations and belong to L_k (see Lemma B, page 18 [9]).

The equations of the form $\alpha(gh) = \alpha(g)\alpha(h)$ are algebraic in the coefficients of $\alpha \in GL(\mathfrak{m}/\mathfrak{m}^{k+1})$. Thus D_k is an algebraic group, indeed it is the subgroup $\{\varphi_k : \varphi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)\}$ of actions on $\mathfrak{m}/\mathfrak{m}^{k+1}$ given by formal diffeomorphisms. Moreover α admits a unique multiplicative Jordan decomposition $\alpha = \alpha_s \circ \alpha_u$ where α_s is semisimple, α_u is unipotent and $\alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s$. The Jordan-Chevalley decomposition in algebraic groups implies $\alpha_s, \alpha_u \in D_k$ (see section 15.3, page 99 [10]).

An element α of D_{k+1} satisfies $\alpha(\mathfrak{m}^{k+1}/\mathfrak{m}^{k+2}) = \mathfrak{m}^{k+1}/\mathfrak{m}^{k+2}$. Therefore α induces a unique element in D_k. In this way we define a morphism $\pi_k : D_{k+1} \to D_k$ of algebraic groups. It satisfies $\pi_k(\varphi_{k+1}) = \varphi_k$ for all $\varphi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ and $k \in \mathbb{N}$. The Jordan decomposition is preserved by π_k. More precisely we obtain $\pi_k(\varphi_{k+1,s}) = \varphi_{k,s}$ and $\pi_k(\varphi_{k+1,u}) = \varphi_{k,u}$ for all $\varphi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ and $k \in \mathbb{N}$. As a consequence there exist $\varphi_s, \varphi_u \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ such that $(\varphi_s)_k = \varphi_{k,s}$ and $(\varphi_u)_k = \varphi_{k,u}$ for all $\varphi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ and $k \in \mathbb{N}$.

Since $(\varphi_s)_k$ is diagonalizable for any $k \in \mathbb{N}$ it can be proved that φ_s is formally diagonalizable by induction on k (see lemma 2.9). The formal diffeomorphism φ_u satisfies that $(\varphi_u)_k$ is unipotent for any $k \in \mathbb{N}$. It is easy to see that this is equivalent to the unipotency of $j^1\varphi_u = (\varphi_u)_1$. The next proposition summarizes the previous discussion.

Proposition 2.1. *Let $\varphi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. Then there exist unique formal diffeomorphisms $\varphi_s, \varphi_u \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ such that $\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$, φ_s is formally diagonalizable and φ_u is unipotent.*

The next result is the analogue for vector fields. It is obtained by considering the additive Jordan decomposition.

Proposition 2.2. *Let $X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$. Then there exist unique formal vector fields $X_s, X_N \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ such that $X = X_s + X_N$, $[X_s, X_N] = 0$, X_s is formally diagonalizable and X_N is nilpotent.*

The following results are a direct consequence of the real nature of the Jordan decomposition.

Lemma 2.3. *Let $X \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$. Then X_s and X_N are real.*

Lemma 2.4. *Let $\varphi \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$. Then φ_s and φ_u are real.*

2.3. **Real monomials.** Let $\varphi \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$. We can suppose that $j^1\varphi_s$ is a diagonal transformation in some coordinates $(w_1, \ldots, w_n)$ of $\mathbb{C}^n$ up to a linear change of coordinates. The components of the diffeomorphism φ are not anymore real power series if there exists a complex non-real eigenvalue of $j^1\varphi$. We are interested on working on coordinates making φ_s is as simple as possible. As a consequence it is necessary to characterize the real nature of endomorphisms and vector fields in the new coordinates.

Fix a real matrix $M = M_s + M_N \in End(\mathbb{R}^n)$ such that M_s is in real Jordan normal form. For instance such a property holds true if M itself is in Jordan normal form. The matrix M_s is diagonalizable. Its real Jordan blocks are of the forms

$$J = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \text{ for } \lambda, \mu \in \mathbb{R} \text{ and } (\beta) \text{ for } \beta \in \mathbb{R}. \tag{4}$$

Consider coordinates (z_p, z_{p+1}) in the former case. The complex Jordan normal form is obtained by considering the linear change of base

$$\begin{pmatrix} z_p \\ z_{p+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} w_p \\ w_{p+1} \end{pmatrix}. \tag{5}$$

We define $\rho(p) = p+1$ and $\rho(p+1) = p$. If the block is of the form (β) in a coordinate z_q we define $w_q = z_q$ and $\rho(q) = q$. It is convenient to work in the coordinates $(w_1, \ldots, w_n)$ since the matrix on the left-hand side of expression (4) becomes

$$\begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}.$$

The matrix of the operator M_s in coordinates $(w_1, \ldots, w_n)$ is diagonal. We denote $(\gamma_1, \ldots, \gamma_n)$ the eigenvalues of the matrix M in coordinates $(w_1, \ldots, w_n)$. We obtain $\overline{\gamma_j} = \gamma_{\rho(j)}$ for any $1 \leq j \leq n$ by construction. Consider a monomial $\mathfrak{h} = \lambda w_1^{a_1} \ldots w_n^{a_n} e_j$ where $\lambda \in \mathbb{C}$ and e_j is the jth element of the canonical base of $\mathbb{C}^n$. We denote $\overline{\mathfrak{h}} = \overline{\lambda} w_{\rho(1)}^{a_1} \ldots w_{\rho(n)}^{a_n} e_{\rho(j)}$. We obtain that

$$\mathfrak{h} = \frac{\mathfrak{h} + \overline{\mathfrak{h}}}{2} + i \frac{\mathfrak{h} - \overline{\mathfrak{h}}}{2i} \tag{6}$$

is the decomposition in real and imaginary parts of $\mathfrak{h}$ (see def. 2.6).

Definition 2.11. *Let $M \in End(\mathbb{R}^n)$ be a semisimple matrix. We say that M is diagonal if it is in real Jordan normal form. In such a case we consider the coordinates $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ associated to M above.*

Remark 2.2. *Given a matrix $M \in End(\mathbb{R}^n)$ we consider that it is in normal form if M_s is diagonal. We do not require M to be in Jordan normal form. One reason is that the condition in the semisimple part is simpler. A deeper reason is that this choice of normal forms is preserved by the exponential. More precisely if M is in real Jordan normal form then* $\exp(M)$ *is not necessarily in Jordan normal form whereas if M_s is diagonal then* $\exp(M)_s$ *is diagonal.*

Definition 2.12. *We say that* $\lambda w_1^{a_1} \dots w_n^{a_n} e_j$ *(or* $\lambda w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$*) is monomial of degree* $a_1 + \ldots + a_n$*. We say that a polynomial is homogeneous of degree* k *if it is a sum of degree* k *monomials.*

Definition 2.13. *We say that* $\lambda w_1^{a_1} \dots w_n^{a_n} e_j$ *(or* $\lambda w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$*) is strongly resonant (with respect to* M*) if* $\gamma_j =< \gamma, a >= \sum_{k=1}^n \gamma_k a_k$*.*

Remark 2.3. *Let* $\mathfrak{h} = w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$*. We have*

$$\left[\sum_{k=1}^{n} \gamma_k w_k \frac{\partial}{\partial w_k}, \mathfrak{h} \right] = (< \gamma, a > - \gamma_j) \mathfrak{h}. \tag{7}$$

Then $\mathfrak{h}$ *is strongly resonant if and only if* $[\sum_{k=1}^n \gamma_k w_k \partial/\partial w_k, \mathfrak{h}] = 0$*.*

Definition 2.14. *We say that* $\lambda w_1^{a_1} \dots w_n^{a_n} e_j$ *(or* $\lambda w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$*) is weakly resonant (with respect to* M*) if* $\gamma_j - < \gamma, a > \in 2\pi i \mathbb{Z}^*$*. We say that the eigenvalues* $(\gamma_1, \ldots, \gamma_n)$ *are weakly resonant if there exists a weakly resonant monomial of degree greater than* 1*.*

Definition 2.15. *We say that* $\lambda w_1^{a_1} \dots w_n^{a_n} e_j$ *(or* $\lambda w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$*) is a resonant monomial if it is either strongly or weakly resonant. Equivalently the monomial is resonant if* $e^{\gamma_j} = (e^{\gamma_1})^{a_1} \dots (e^{\gamma_n})^{a_n}$*.*

Remark 2.4. *Let* $\mathfrak{h} = w_1^{a_1} \dots w_n^{a_n} e_j$*. We have*

$$(e^{\gamma_1} w_1, \ldots, e^{\gamma_n} w_n)^{\circ(-1)} \circ \mathfrak{h} \circ (e^{\gamma_1} w_1, \ldots, e^{\gamma_n} w_n) = e^{-\gamma_j} (e^{\gamma_1})^{a_1} \dots (e^{\gamma_n})^{a_n} \mathfrak{h}.$$

Then $\mathfrak{h}$ *is resonant if and only if it commutes with* $(e^{\gamma_1} w_1, \ldots, e^{\gamma_n} w_n)$*.*

Definition 2.16. *We say that a formal endomorphism (resp. vector field) is resonant (resp. strongly, weakly resonant, nonresonant) if all its non-vanishing monomials are resonant (resp. strongly, weakly resonant, nonresonant).*

The property $\overline{\gamma_j} = \gamma_{\rho(j)}$ for any $1 \leq j \leq n$ implies

Lemma 2.5. *We have that* $w_1^{a_1} \dots w_n^{a_n} e_j$ *is resonant (resp. strongly, weakly resonant) if and only if* $w_{\rho(1)}^{a_1} \dots w_{\rho(n)}^{a_n} e_{\rho(j)}$ *is resonant (resp. strongly, weakly resonant).*

Lemma 2.6. *Let* $\mathfrak{h} = \lambda w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$ *be a strongly resonant monomial. Let* $\mathfrak{k} = \mu w_1^{b_1} \dots w_n^{b_n} \partial/\partial w_k$ *be a monomial.*

- $[\mathfrak{h}, \mathfrak{k}]$ *is strongly resonant if* $\mathfrak{k}$ *is strongly resonant.*
- $[\mathfrak{h}, \mathfrak{k}]$ *is weakly resonant if* $\mathfrak{k}$ *is weakly resonant.*
- $[\mathfrak{h}, \mathfrak{k}]$ *is nonresonant if* $\mathfrak{k}$ *is nonresonant.*

Lemma 2.7. *Let* $\varphi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ *with* $j^1 \varphi = \exp(X_1)$ *where* X_1 *is a real linear vector field such that* $X_{1,s}$ *is diagonal. Suppose that* φ *is* *strongly resonant (resp. resonant) with respect to* $X_{1,s}$. *Then* $\log \varphi_u$ *is strongly resonant (resp. resonant).*

Proof. We have $X_{1,s} = \sum_{k=1}^{n} \gamma_k w_k \partial/\partial w_k$. Thus $\phi = (e^{\gamma_1} w_1, \ldots, e^{\gamma_n} w_n)$ commutes with φ by remark 2.4. We have that

$$j^1\varphi = \exp(X_{1,s}) \circ \exp(X_{1,N}) = \phi \circ \exp(X_{1,N})$$

is the Jordan decomposition of $j^1\varphi$. As a consequence $\varphi = \phi \circ (\phi^{\circ(-1)} \circ \varphi)$ is the Jordan-Chevalley decomposition of φ. In particular we obtain $\varphi_u = \phi^{\circ(-1)} \circ \varphi$ and the non-zero monomials of φ and φ_u coincide. The rest of the proof is a simple calculation based on equation (1). □

2.4. **Linearization of vector fields and diffeomorphisms.** Formal semisimple diffeomorphisms and vector fields are formally linearizable. If they are real we can also choose a real formal diffeomorphism as the linearizing transformation.

Lemma 2.8. *Let* $X \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ *be a formal semisimple vector field. Then there exists a formal diffeomorphism* $\eta \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ *such that* $\eta^* X = j^1 X$.

Lemma 2.9. *Let* $\varphi \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ *be a formal semisimple diffeomorphism. Then there exists a formal diffeomorphism* $\eta \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ *such that* $\eta^{\circ(-1)} \circ \varphi \circ \eta = j^1\varphi$.

proof of lemmas 2.8 and 2.9. Let us show lemma 2.8. The proof of lemma 2.9 is analogous. Up to a real linear change of coordinates we can suppose that $j^1 X_s$ is diagonal in coordinates $(z_1, \ldots, z_n)$. We obtain $j^1 X = \gamma_1 w_1 \partial/\partial w_1 + \ldots \gamma_n w_n \partial/\partial w_n$ in the coordinates $(w_1, \ldots, w_n)$ introduced in this section. Suppose that X is of the form $j^1 X + X_k + Y_{k+1}$ where X_k is homogeneous of degree k and Y_{k+1} is a sum of monomials of degree greater than k. It suffices to prove that there exists a diffeomorphism $\eta_k \in \text{Diff}\,(\mathbb{R}^n, 0)$ such that $\eta_k^* X = j^1 X + X_{k+1} + Y_{k+2}$ where X_{k+1} is homogeneous of degree $k+1$, Y_{k+2} is a sum of monomials of degree greater than $k+1$ and $\eta_k - Id$ is a sum of monomials of degree greater or equal than k. In this way we obtain $\eta = \lim_{k \to \infty} \eta_2 \circ \ldots \circ \eta_k$ in $\widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ such that $\eta^* X = j^1 X$.

The vector field X_k is a sum of real vector fields of the form

$$X_{k,a_1,\ldots,a_n,j,\lambda} = \mathfrak{h}_\lambda \stackrel{def}{=} \lambda w_1^{a_1} \ldots w_n^{a_n} \frac{\partial}{\partial w_j} + \overline{\lambda} w_{\rho(1)}^{a_1} \ldots w_{\rho(n)}^{a_n} \frac{\partial}{\partial w_{\rho(j)}}.$$

Indeed the vector field $\mathfrak{h}_\mu$ is real for any $\mu \in \mathbb{C}$ (see equation (6)). Suppose that the monomials of $\mathfrak{h}_\lambda$ are not strongly resonant. Since we have $[\mathfrak{h}_\mu, j^1 X] = \mathfrak{h}_{\mu(\gamma_j - <\gamma, a>)}$ we define $\mu = \lambda/(\gamma_j - <\gamma, a>)$ and

$\tilde{X}_{k,a_1,\ldots,a_n,j,\lambda} = \mathfrak{h}_\mu$. We denote by X_k^0 (resp. $\tilde{X}_k^0$) the sum of the non-strongly resonant vector fields of the form $X_{k,a_1,\ldots,a_n,j,\lambda}$ (resp. of the form $\tilde{X}_{k,a_1,\ldots,a_n,j,\lambda}$). Consider the real diffeomorphism $\eta_k = \exp(-\tilde{X}_k^0)$. We obtain

$$(\eta_k^{\circ(-1)})^* j^1 X = j^1 X + [\tilde{X}_k^0, j^1 X] + \frac{1}{2!}[\tilde{X}_k^0, [\tilde{X}_k^0, j^1 X]] + \ldots =$$

$$= j^1 X + X_k^0 + h.o.t.$$

Moreover we obtain $(\eta_k^{\circ(-1)})^*(j^1 X + X_k - X_k^0) = j^1 X + X_k + h.o.t.$ Hence $\eta_k^* X = j^1 X + X_k - X_k^0 + h.o.t.$ The vector field $\eta_k^* X$ is still semisimple. We obtain

$$(\eta_k^* X)_k = j^1 X + (X_k - X_k^0)$$

as the Jordan decomposition (see equation (3)) of $(\eta_k^* X)_k$ since $j^1 X$ is semisimple, $X_k - X_k^0$ is nilpotent and $[j^1 X, X_k - X_k^0] = 0$ (see eq. (7)). Since $(\eta_k^* X)_k$ is semisimple and the Jordan-Chevalley decomposition is unique we obtain $X_k - X_k^0 \equiv 0$. □

Remark 2.5. *Let us notice that a simpler version of the previous proof shows that a formal semisimple vector field in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ is formally linearizable.*

3. Embedding flows

Given $\varphi \in \widehat{\text{Diff}}(\mathbb{R}^n, 0)$ we can consider whether it is embedded in the flow of a formal vector field in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ or $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$. A priori these properties could be different. Indeed real diffeomorphisms can be embedded in the flows of non-real vector fields, for example we have $Id = \exp(2\pi i z \partial/\partial z)$. Since Jordanization interprets elements of $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ as formal vector fields in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ theorem 1.3 justifies our approach.

proof of theorem 1.3. Let $\hat{X} = X_s + X_N$ be the Jordan-Chevalley decomposition of $\hat{X}$. Since $\exp(X_s)$ is semisimple (and then formally linearizable) and $\exp(X_N)$ is unipotent we obtain $\varphi_s = \exp(X_s)$ and $\varphi_u = \exp(X_N)$. We have $\log \varphi_u = X_N$ by lemma 2.2. Indeed φ_u and X_N are real. The difficulty of the proof is that X_s is not necessarily real.

We denote $\alpha = Re(X_s)$ and $\beta = Im(X_s)$. In fact α and β are elements of $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ such that $X_s = \alpha + i\beta$. By lemma 2.3 it suffices to prove $\varphi = \exp(\alpha_s + \log \varphi_u)$ where α_s is the semisimple part of α.

We have

$$[X_s, X_N] = 0 \implies [\alpha, X_N] = 0 \text{ and } [\beta, X_N] = 0.$$

Moreover since $j^1\hat{X}$, j^1X_s are real then $j^1\alpha = j^1X_s$ and $j^1\beta = 0$. We obtain $\exp(tX_N)^*\alpha = \alpha$ for any $t \in \mathbb{C}$ as a consequence of $[\alpha, X_N] = 0$. The uniqueness of the Jordan-Chevalley decomposition

$$\exp(tX_N)^*(\alpha_s + \alpha_N) = \exp(tX_N)^*(\alpha_s) + \exp(tX_N)^*(\alpha_N) = \alpha_s + \alpha_N$$

implies $\exp(tX_N)^*(\alpha_s) = \alpha_s$ for any $t \in \mathbb{C}$. We deduce $[\alpha_s, X_N] = 0$. We have $\varphi_s^*(\hat{X}) = \hat{X}$ since $[X_s, \hat{X}] = 0$ and $\varphi_s = \exp(X_s)$. Thus we obtain $\varphi_s^*(X_s) = X_s$ and $\varphi_s^*(X_N) = X_N$ by uniqueness of the Jordan-Chevalley decomposition. Since φ_s is real we have $\varphi_s^*\alpha = \alpha$ and then $\varphi_s^*\alpha_s = \alpha_s$. We deduce the equality $\varphi_s \circ \exp(\alpha_s) = \exp(\alpha_s) \circ \varphi_s$. Moreover, $j^1\alpha = j^1X_s$ is semisimple; thus we get $j^1\alpha_s = j^1X_s$.

Let us show that $\varphi_s = \exp(\alpha_s)$. Since we have $j^1\varphi_s = j^1\exp(\alpha_s)$ then the formal diffeomorphism $\eta = \exp(-\alpha_s) \circ \varphi_s$ is unipotent. The diffeomorphisms φ_s and $\exp(\alpha_s)$ commute, we obtain

$$\varphi_s = \varphi_s \circ Id = \exp(\alpha_s) \circ \eta$$

two Jordan-Chevalley decompositions of φ_s. We deduce $\varphi_s = \exp(\alpha_s)$. We define $\hat{Y} = \alpha_s + \log \varphi_u$. It satisfies $j^1\hat{Y} = j^1\alpha_s + j^1X_N = j^1\hat{X}$. Then $[\alpha_s, X_N] = 0$ implies $\exp(\hat{Y}) = \exp(\alpha_s) \circ \exp(\log \varphi_u) = \varphi_s \circ \varphi_u = \varphi$. $\square$

Example. It is clear that there are elements φ of $\widehat{\text{Diff}}(\mathbb{R}^n, 0)$ that can be embedded in flows in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ but not in flows in $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$. An example is provided by $\varphi = (-z_1, z_1 - z_2)$ since the linear operator φ is embedded in the flow of $\pi i z_1 \partial/\partial z_1 + (-z_1 + \pi i z_2)\partial/\partial z_2$ but not in a real one. Indeed Jordan blocks associated to negative eigenvalues of real matrices with real logarithms appear pairwise [4]. But even in the class of diffeomorphisms in $\widehat{\text{Diff}}(\mathbb{R}^n, 0)$ whose linear part has a real logarithm it is possible to find elements that are embedded in flows in $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ but not in flows in $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$. Next we introduce an example.

Consider $\mu_1 = -2 + \sqrt{2}2\pi i$, $\mu_2 = -2 - \sqrt{2}2\pi i$, $\mu_3 = 3 + (1 - 4\sqrt{2})\pi i$, $\mu_4 = 3 - (1 - 4\sqrt{2})\pi i$. A simple calculation provides that the $\mathbb{Z}$-module

$$V = \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 : m_1\mu_1 + m_2\mu_2 + m_3\mu_3 + m_4\mu_4 \in 2\pi i\mathbb{Z}\}$$

is equal to $\mathbb{Z}(3, 3, 2, 2) + \mathbb{Z}(5, 1, 3, 1)$. We define $\mu_1' = \mu_1 - 4\pi i$, $\mu_2' = \mu_2$, $\mu_3' = \mu_3 + 6\pi i$, $\mu_4' = \mu_4$. We obtain

$$m_1\mu_1' + m_2\mu_2' + m_3\mu_3' + m_4\mu_4' = 0 \ \forall (m_1, m_2, m_3, m_4) \in V.$$

On the contrary we have

$$5\mu_1'' + \mu_2'' + 3\mu_3'' + \mu_4'' \neq 0. \tag{8}$$

for any choice of logarithms μ_1'', μ_2'', μ_3'', μ_4'' of e^{μ_1}, e^{μ_2}, e^{μ_3}, e^{μ_4} respectively such that $\mu_1'' = \overline{\mu_2''}$, $\mu_3'' = \overline{\mu_4''}$. Otherwise if $\mu_1'' = \mu_1 - 2\pi i k_1$ and $\mu_3'' = \mu_3 - 2\pi i k_2$ we obtain $4k_1 + 2k_2 = 1$ for some $k_1, k_2 \in \mathbb{Z}$.

Consider the linear diffeomorphism φ_0 defined by

$$\left(-2z_1 - \pi\sqrt{8}z_2, \pi\sqrt{8}z_1 - 2z_2, 3z_3 - (1-4\sqrt{2})\pi z_4, (1-4\sqrt{2})\pi z_3 + 3z_4\right).$$

By considering a real logarithm B of φ_0 we can introduce coordinates (w_1, w_2, w_3, w_4) such that $\varphi_0 = (e^{\mu_1}w_1, e^{\mu_2}w_2, e^{\mu_3}w_3, e^{\mu_4}w_4)$ as in section 2.3. These coordinates do not depend on the choice of the matrix B since the decomposition of $\mathbb{C}^4$ as direct sum of eigenspaces of B does not depend on B. Indeed it coincides with the analogous decomposition associated to φ_0. Consider

$$\varphi = (e^{\mu_1}w_1 + w_1^6 w_2 w_3^3 w_4, e^{\mu_2}w_2 + w_1 w_2^6 w_3 w_4^3, e^{\mu_3}w_3, e^{\mu_4}w_4).$$

It is a real hyperbolic element of $\text{Diff}(\mathbb{R}^4, 0)$. We have that φ is embedded in a formal flow of linear part $\sum_{j=1}^4 \mu_j' w_j \partial/\partial w_j$ by theorem 1.1. The monomials $w_1^6 w_2 w_3^3 w_4 e_1$ and $w_1 w_2^6 w_3 w_4^3 e_2$ are weakly resonant for any choice of a real logarithm of φ_0 by equation (8). Therefore φ is not embedded in a flow in $\hat{\mathcal{X}}(\mathbb{R}^4, 0)$ by theorem 1.5 (remark that $X_{1,N}$ is a vanishing vector field). Of course $\sum_{j=1}^4 \mu_j' w_j \partial/\partial w_j$ is not real. We can enlarge the class of embeddable diffeomorphisms by considering non real logarithms of the linear part but if the linear part of the logarithm is real theorem 1.3 implies that we can not enlarge the class of embeddable diffeomorphisms by trying to consider non real formal flows.

A classical way of obtaining normal forms for local holomorphic vector fields and diffeomorphisms is by considering changes of coordinates in which the semisimple part is linear. We apply this ideas to characterize whether or not a diffeomorphism in $\text{Diff}(\mathbb{R}^n, 0)$ or $\text{Diff}(\mathbb{C}^n, 0)$ is embedded in a formal flow X. The embeddability of the diffeomorphism is equivalent to the existence of a strongly resonant normal form.

proof of theorem 1.4. Let us prove the result for $\varphi \in \widehat{\text{Diff}}(\mathbb{R}^n, 0)$. The proof for φ in $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ is simpler.

Suppose that $\varphi = \exp(X)$ with $X \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ and $j^1 X = X_1$. Consider the Jordan-Chevalley decomposition $X = X_s + X_N$ of X. We have $j^1 X_s = X_{1,s}$. The proof of lemma 2.8 implies the existence of a tangent to the identity $\eta \in \widehat{\text{Diff}}(\mathbb{R}^n, 0)$ such that $\eta^* X_s = j^1 X_s$. We have

$$\eta^{\circ(-1)} \circ \varphi \circ \eta = \exp(\eta^* X) = \exp(X_{1,s} + \eta^* X_N) = \exp(X_{1,s}) \circ \exp(\eta^* X_N).$$

Moreover $\eta^* X_N$ is strongly resonant since $[X_{1,s}, \eta^* X_N] = 0$ (remark 2.3). Thus $\exp(\eta^* X_N)$ is strongly resonant. Since $\exp(X_{1,s})$ is diagonal then $\eta^{\circ(-1)} \circ \varphi \circ \eta$ is strongly resonant.

Suppose that $\tilde{\varphi} = \eta^{\circ(-1)} \circ \varphi \circ \eta$ is strongly resonant. We define $\phi = \exp(-X_{1,s}) \circ \tilde{\varphi}$. Since $j^1\phi = (j^1\varphi_s)^{\circ(-1)} \circ j^1\varphi = j^1\varphi_u$ then ϕ is unipotent. Moreover $\tilde{\varphi}$ and $\exp(X_{1,s})$ commute; thus $\tilde{\varphi} = \exp(X_{1,s}) \circ \phi$ is the Jordan-Chevalley decomposition of $\tilde{\varphi}$. We apply lemma 2.7 to $\tilde{\varphi}$ to obtain that $\log \phi \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ is strongly resonant. This implies the key property $[X_{1,s}, \log \phi] = 0$ (remark 2.3). We denote $X = X_{1,s} + \log \phi$. We obtain

$$\eta^{\circ(-1)} \circ \varphi \circ \eta = \exp(X_{1,s}) \circ \exp(\log \phi) = \exp(X_{1,s} + \log \phi).$$

We have $j^1(X_{1,s} + \log \phi) = X_{1,s} + j^1 \log \varphi_u = X_1$ by lemma 2.2. Thus φ is of the form $\exp(\eta_* X)$ where $\eta_* X \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ and $j^1 \eta_* X = X_1$. □

Remark 3.1. *Consider the case* $\varphi \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ *and* $X_1 \in \mathcal{X}(\mathbb{R}^n, 0)$ *in theorem 1.4. Then a normalizing map* $\eta \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ *implies that* φ *has an embedding flow whose first jet is equal to* X_1 *by theorems 1.4 and 1.3.*

As an application of Jordanization techniques we present a new, simpler proof of theorem 1.1. The idea is that if a diffeomorphism is embedded in a flow X then the linearization of its semisimple part X_s provides a strongly resonant expression. We can always obtain a resonant expression of φ by linearizing φ_s. The hypothesis implies that these concepts are the same.

proof of th. 1.1. Let us suppose that φ and X_1 are real in order to prove the existence. The general case is simpler. Up to a real linear change of coordinates we can suppose that $X_{1,s}$ is diagonal in coordinates $(z_1, \ldots, z_n)$. Let $(w_1, \ldots, w_n)$ be the system of coordinates introduced in section 2.3. In particular $\varphi_{1,s} = \exp(X_{1,s})$ is diagonal.

Consider the Jordan-Chevalley decomposition

$$\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$$

of φ. There exists a tangent to the identity $\eta \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ such that $\eta^{\circ(-1)} \circ \varphi_s \circ \eta = j^1\varphi_s$ by lemma 2.9. We denote $\tilde{\varphi} = \eta^{\circ(-1)} \circ \varphi \circ \eta$. Since $\tilde{\varphi}$ commutes with $\tilde{\varphi}_s = j^1\exp(X_{1,s})$ then $\tilde{\varphi}$ is resonant (rem. 2.4). Moreover the properties $[X_{1,s}, X_1] = 0$ and $j^1\tilde{\varphi} = \exp(X_1)$ imply that $j^1\tilde{\varphi}$ is strongly resonant. In particular $\tilde{\varphi}$ is strongly resonant by hypothesis. There exists $\hat{X} \in \hat{\mathcal{X}}(\mathbb{R}^n, 0)$ with $j^1\hat{X} = X_1$ such that $\hat{\varphi} = \exp(\hat{X})$ (th. 1.4).

We have to prove that $\varphi = \exp(\hat{X}) = \exp(\hat{Y})$ and $j^1\hat{X} = j^1\hat{Y} = X_1$ imply $\hat{X} = \hat{Y}$. Let $\hat{X} = X_s + X_N$, $\hat{Y} = Y_s + Y_N$ be the Jordan-Chevalley decompositions of $\hat{X}$, $\hat{Y}$ respectively. We have $j^1X_s = j^1Y_s = X_{1,s}$, $\exp(X_s) = \exp(Y_s)$ and $X_N = Y_N = \log \varphi_u$. Up to a formal change of coordinates we can suppose that $X_s = j^1 X_s$ (lemma 2.8 and remark 2.5). There exists a formal tangent to the identity diffeomorphism $\eta \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ conjugating $X_s = j^1 X_s$ and Y_s (remark 2.5). We obtain

$$\eta \circ \exp(X_s) = \exp(Y_s) \circ \eta \implies \eta \circ j^1 \varphi_s = j^1 \varphi_s \circ \eta.$$

Hence η is resonant (remark 2.4) and by hypothesis strongly resonant. Thus we obtain $Y_s = \eta_* j^1 X_s = j^1 X_s = X_s$ and $\hat{X} = \hat{Y}$. □

proof of theorem 1.2. We include Zhang's proof for the sake of completeness. Let $\hat{\varphi} \in \widehat{\text{Diff}}\,(\mathbb{R}^n, 0)$ be the asymptotic development of φ. Consider the element $\hat{X}$ of $\hat{\mathcal{X}}(\mathbb{R}^n, 0)$ such that $\hat{\varphi} = \exp(\hat{X})$ and $j^1 \hat{X} = X_1$ provided by th. 1.1. Let X' be an element of $\mathcal{X}_\infty(\mathbb{R}^n, 0)$ whose asymptotic development at the origin is equal to $\hat{X}$. The hyperbolic diffeomorphisms φ and $\exp(X')$ are formally conjugated by the identity. Two formally conjugated hyperbolic C^∞ local diffeomorphisms are conjugated by a local C^∞ diffeomorphism (Chen [3]). There exists $\upsilon \in \text{Diff}_\infty(\mathbb{R}^n, 0)$ such that $\varphi = \upsilon^{\circ(-1)} \circ \exp(X') \circ \upsilon$. We obtain $\varphi = \exp(X)$ for $X = \exp(\upsilon^* X') \in \mathcal{X}_\infty(\mathbb{R}^n, 0)$. □

It is interesting to study to what extent weakly resonances are obstructions for diffeomorphisms to be embedded. In this spirit we want to address a conjecture by Zhang. Let A be a hyperbolic matrix in $GL(n, \mathbb{R})$ and let B be a real logarithm of A.

Conjecture. [19] *If $f(z) = O(|z|^2)$ is C^∞ (resp. analytic) and it has a non-vanishing weakly resonant monomial $w_1^{m_1} \dots w_n^{m_n} e_j$ (with respect to B), then the locally hyperbolic diffeomorphism $\varphi(z) = Az + f(z)$ is not embedded in a C^∞ (resp. analytic) flow.*

The conjecture as stated is false and we provide two counterexamples given by resonant diffeomorphisms.

3.1. **Building examples.** We explain a method to obtain non-vanishing weakly resonant monomials for embeddable diffeomorphisms even if the diffeomorphism is resonant. Let us consider X_s a real linear diagonal vector field. Consider a real nilpotent vector field X_N such that $[X_s, X_N] = 0$. In particular X_N is strongly resonant. We also suppose that X_N is homogeneous of degree k. It is clear that

$$\varphi = \exp(X_s + X_N) = \exp(X_s) \circ \exp(X_N) = \exp(X_N) \circ \exp(X_s)$$

is embeddable in $\text{Diff}\,(\mathbb{R}^n, 0)$. Consider $\eta = Id + \eta_l \in \text{Diff}\,(\mathbb{R}^n, 0)$ $(l \geq 2)$ where η_l is a homogeneous weakly resonant endomorphism of degree l. It is clear that $\eta^{\circ(-1)} \circ \varphi \circ \eta$ is embeddable in $\text{Diff}\,(\mathbb{R}^n, 0)$. Moreover if $j^1 \varphi$

is hyperbolic then any $\phi \in \mathrm{Diff}_{\infty}(\mathbb{R}^n, 0)$ whose asymptotic development coincides with φ is embeddable by Chen's theorem [3].

We denote $\alpha_s = \exp(X_s)$. Since η is resonant then $\eta^{\circ(-1)} \circ \alpha_s \circ \eta = \alpha_s$ (remark 2.4). Moreover we have

$$\eta^* X_N = X_N + [\log \eta, X_N] + \frac{1}{2!}[\log \eta, [\log \eta, X_N]] + \dots \tag{9}$$

It is natural to try to find a weakly resonant $Y = w_1^{a_1} \dots w_n^{a_n} \partial/\partial w_j$ of degree l such that $[Y, X_N] \neq 0$. Then $[Re(Y) + iIm(Y), X_N] \neq 0$ (see eq. (6)) implies either $[Re(Y), X_N] \neq 0$ or $[Im(Y), X_N] \neq 0$. Anyway there exists a real weakly resonant homogeneous vector field Y' of degree l such that $[Y', X_N] \neq 0$ (lemma 2.5). We define $\eta = j^l \exp(Y')$. The formula (9) implies that $\eta^* X_N$ is of the form $X_N + [Y', X_N] + Z_{l+k}$ where $[Y', X_N]$ is weakly resonant and homogeneous of degree $l + k - 1$ by lemma 2.6 and Z_{l+k} is a sum of monomials of degree greater or equal than $l + k$. Therefore $\eta^* X_N$ is not strongly resonant. On the contrary $\eta^* X_N$ is resonant since

$$\alpha_s^*(\eta^* X_N) = (\eta \circ \alpha_s)^* X_N = (\alpha_s \circ \eta)^* X_N = \eta^*(\alpha_s^* X_N) = \eta^* X_N.$$

The second equality is as consequence of the resonant nature of η. We deduce that $\exp(\eta^* X_N)$ is resonant but not strongly resonant by lemma 2.7. Thus

$$\eta^{\circ(-1)} \circ \varphi \circ \eta = (\eta^{\circ(-1)} \circ \alpha_s \circ \eta) \circ (\eta^{\circ(-1)} \circ \exp(X_N) \circ \eta) = \alpha_s \circ \exp(\eta^* X_N)$$

is embeddable, resonant but not strongly resonant.

Next we provide a condition on the eigenvalues of X_s that guarantees that the previous method can be applied. Roughly speaking the condition is equivalent to the existence of infinitely many independent weakly linear monomials.

Lemma 3.1. *Let $X_s \in \mathcal{X}(\mathbb{R}^n, 0)$ be a linear diagonal vector field. Let $0 \neq X_N \in \mathcal{X}(\mathbb{R}^n, 0)$ be a homogeneous nilpotent vector field such that $[X_s, X_N] = 0$. Suppose that X_N is strongly resonant. Suppose that the eigenvalues $\gamma_1, \dots, \gamma_n$ of X_s satisfy $\sum_{j=1}^n m_j \gamma_j \in 2\pi i \mathbb{Z}^*$ for some $(m_1, \dots, m_n) \in (\mathbb{N} \cup \{0\})^n$. Then there exists a weakly resonant monomial vector field Y such that $[Y, X_N] \neq 0$.*

The resonances are considered with respect to X_s.

Proof. We denote $X_N = \sum_{q=1}^n b_q(w_1, \dots, w_n) \partial/\partial w_q$. Every monomial vector field $W_{k,j} = (w_1^{m_1} \dots w_n^{m_n})^k w_j \partial/\partial w_j$ is weakly resonant for all $k \geq 1$ and $1 \leq j \leq n$. It suffices to prove that we can not have $[X_N, W_{k,j}] = 0$ for all $k \geq 1$ and $1 \leq j \leq n$.

The property $[X_N, W_{k,q}](w_j) = 0$ for $q \neq j$ implies $\partial b_j/\partial w_q = 0$. In particular b_j depends only on w_j for any $1 \leq j \leq n$. Let d be the common degree of the polynomials $b_1, \ldots, b_n$. The polynomial b_j is of the form $\lambda_j w_j^d$ for some $\lambda_j \in \mathbb{C}$ and any $1 \leq j \leq n$. We have $d \geq 2$, otherwise we would get $X_N = 0$ since X_N is nilpotent. The property $[X_N, W_{k,j}](w_j) = 0$ implies

$$\lambda_j d w_j^d = \sum_{q \neq j} \lambda_q k m_q w_q^{d-1} w_j + \lambda_j (k m_j + 1) w_j^d$$

for any $k \in \mathbb{N}$. We deduce $\lambda_j = 0$ for any $1 \leq j \leq n$. We obtain $X_N = 0$ contradicting the hypothesis. $\square$

3.2. **Example.** We consider

$$X_s = -2z_1 \frac{\partial}{\partial z_1} + \left(z_2 - \frac{\pi}{2} z_3\right) \frac{\partial}{\partial z_2} + \left(\frac{\pi}{2} z_2 + z_3\right) \frac{\partial}{\partial z_3}$$

or

$$X_s = -2w_1 \frac{\partial}{\partial w_1} + \left(1 + \frac{\pi}{2} i\right) w_2 \frac{\partial}{\partial w_2} + \left(1 - \frac{\pi}{2} i\right) w_3 \frac{\partial}{\partial w_3}$$

in coordinates (w_1, w_2, w_3). We have $\rho(1) = 1$, $\rho(2) = 3$ and $\rho(3) = 2$. The monomial $X_N = w_1^2 w_2 w_3 \partial/\partial w_1$ is real, nilpotent and strongly resonant. The monomial $Y = w_1 w_3^3 \partial/\partial w_2$ is weakly resonant. We define $Y' = w_1 w_3^3 \partial/\partial w_2 + w_1 w_2^3 \partial/\partial w_3$. We obtain

$$[Y', X_N] = (w_1^3 w_3^4 + w_1^3 w_2^4) \frac{\partial}{\partial w_1} - w_1^2 w_2 w_3^4 \frac{\partial}{\partial w_2} - w_1^2 w_2^4 w_3 \frac{\partial}{\partial w_3}.$$

We define

$$\eta = j^4 \exp(Y') = (w_1, w_2 + w_1 w_3^3, w_3 + w_1 w_2^3).$$

and $\varphi = \exp(X_s + \eta^* X_N)$. We obtain $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with

$$\varphi_1 = e^{-2}(w_1 + w_1^2 w_2 w_3 + w_1^3 w_2^4 + w_1^3 w_3^4 + w_1^3 w_2^2 w_3^2) + \ldots,$$

$$\varphi_2 = e^{1+\pi i/2}(w_2 - w_1^2 w_2 w_3^4) + \ldots, \ \varphi_3 = e^{1-\pi i/2}(w_3 - w_1^2 w_2^4 w_3) + \ldots$$

The diffeomorphism φ is real, hyperbolic, resonant and embedded in an analytic flow by construction. There are 4 non-zero weakly resonant monomials of φ of degree 7. Notice that the non-linear monomial of lowest degree, i.e. $e^{-2} w_1^2 w_2 w_3 e_1$ is strongly resonant. The next example shows that we can find weakly resonant monomials even at the lowest degree.

3.3. **Example.** Let us consider a example with X_N of degree $k=1$. In particular X_s+X_N is a real non-diagonalizable linear operator. Then either all the eigenvalues of X_s+X_N are real (and the eigenvalues are not weakly resonant) or $n \geq 4$.

Let us fix $n=4$. Consider the vector field

$$X = 8x_1\frac{\partial}{\partial x_1} + (x_1+8x_2)\frac{\partial}{\partial x_2} + \left(x_3 - \frac{\pi}{4}x_4\right)\frac{\partial}{\partial x_3} + \left(\frac{\pi}{4}x_3 + x_4\right)\frac{\partial}{\partial x_4}.$$

The Jordan Chevalley decomposition $X = X_s + X_N$ of X is given by $X_N = x_1\partial/\partial x_2$ and $X_s = X - X_N$. The vector field X is of the form

$$X = 8w_1\frac{\partial}{\partial w_1} + (w_1+8w_2)\frac{\partial}{\partial w_2} + \left(1+\frac{\pi}{4}i\right)w_3\frac{\partial}{\partial w_3} + \left(1-\frac{\pi}{4}i\right)w_4\frac{\partial}{\partial w_4}$$

in the coordinates $(w_1,\ldots,w_4)$ introduced in subsection 2.3. We have $\rho(1)=1$, $\rho(2)=2$, $\rho(3)=4$ and $\rho(4)=3$. The list of weakly resonant monomials is

$$w_3^8\frac{\partial}{\partial w_1},\ w_4^8\frac{\partial}{\partial w_1},\ w_3^8\frac{\partial}{\partial w_2} \text{ and } w_4^8\frac{\partial}{\partial w_2}.$$

We define the real vector field $Y' = (w_3^8+w_4^8)\partial/\partial w_1$. We denote $\eta = \exp(Y')$. We have

$$[Y', X_N] = \left[(w_3^8+w_4^8)\frac{\partial}{\partial w_1}, w_1\frac{\partial}{\partial w_2}\right] = (w_3^8+w_4^8)\frac{\partial}{\partial w_2}.$$

We obtain $\eta(w_1,w_2,w_3,w_4) = (w_1+w_3^8+w_4^8, w_2, w_3, w_4)$. Formula (9) implies $\eta^* X_N = (w_1+w_3^8+w_4^8)\partial/\partial w_2$. We denote $\varphi = \eta^{\circ(-1)}\circ\exp(X)\circ\eta$. We obtain

$$\varphi = (e^8w_1, e^8w_2, e^{1+\frac{\pi}{4}i}w_3, e^{1-\frac{\pi}{4}i}w_4)\circ(w_1, w_1+w_2+w_3^8+w_4^8, w_3, w_4)$$

$$\implies \varphi = (e^8w_1, e^8w_1+e^8w_2+e^8w_3^8+e^8w_4^8, e^{1+\frac{\pi}{4}i}w_3, e^{1-\frac{\pi}{4}i}w_4) =$$

$$(e^8z_1, e^8z_1+e^8z_2+\frac{e^8}{2^7}\sum_{q=0}^{4}\binom{8}{2q}(-1)^q z_3^{2q}z_4^{8-2q}, e\sqrt{2}\frac{z_3-z_4}{2}, e\sqrt{2}\frac{z_3+z_4}{2}).$$

The diffeomorphism φ is real, hyperbolic, resonant and embedded in an analytic flow by construction. In spite of this all the non-linear monomials are weakly resonant. Zhang's conjecture does not hold true in this case.

3.4. **Resonances as an obstacle to embed diffeomorphisms.** In spite of the previous examples we prove theorem 1.5. It can be interpreted as a version of Zhang's conjecture.

Let us discuss the optimality of the conditions in the theorem. The examples represent two different kind of obstructions to get a positive result. The example in subsection 3.2 satisfies $j^1X_{1,N}=0$. It is embeddable and it contains weakly resonant monomials immediately above the lowest degree of non-linear non-vanishing monomials. It does not satisfy (a) since the lowest degree non-vanishing weakly resonant monomials have degree 7 but $f_4 \neq 0$. It does not satisfy (b) either since $w_1 w_3^3 e_2$ is a weakly resonant monomial of degree $2 \leq 4 \leq 7-1$. Notice that φ *does not have* weakly resonant monomials of degree 4. Weakly resonances of lower degree provide multiple choices for the semisimple part of the embedding flow that can make the diffeomorphism φ to be embeddable. Such an example justifies the need of restricting our study to diffeomorphisms satisfying (a) or (b).

The existence of weakly resonant vector fields Y with $[X_{1,N}, Y] \neq 0$ allows to proceed as in the example 3.3 to obtain embeddabble diffeomorphisms having non-vanishing weakly resonant monomials of the lowest degree. The example in section 3.3 satisfies both (a) and (b).

The examples can be considered as a classification of the type of counterexamples to the original conjecture.

proof of theorem 1.5. Let $\hat{\varphi} \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ be the asymptotic development of φ. Suppose that $\varphi = \exp(X)$ for some $X \in \mathcal{X}_\infty(\mathbb{R}^n, 0)$ (or $\mathcal{X}(\mathbb{R}^n, 0)\,, \hat{\mathcal{X}}(\mathbb{C}^n, 0)\,)$ with $j^1 X = X_1$. Let $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ be the asymptotic development of X. We have $\hat{\varphi} = \exp(\hat{X})$. It suffices to prove that $\hat{\varphi}$ is not embedded in a formal flow $\hat{X}$ with $j^1 \hat{X} = X_1$.

Consider the Jordan-Chevalley decomposition $\hat{\varphi} = \varphi_s \circ \varphi_u$ of $\hat{\varphi}$. We have $\varphi_{1,s} = j^1 \varphi_s = \exp(X_{1,s})$ and $j^1 \varphi_u = \exp(X_{1,N})$. As a consequence $\varphi_{1,s} = j^1 \varphi_s$ is diagonal. We have

$$\varphi_s = \varphi_{1,s} + A_2 + A_3 + \ldots, \ \varphi_u = \varphi_{1,u} + B_2 + B_3 + \ldots$$

where A_j and B_j are homogeneous of degree j for $j \geq 2$. The diffeomorphism φ_{k-1} (see eq. (2)) commutes with $\varphi_{1,s}$ by hypothesis (remark 2.4). Since the Jordan-Chevalley decomposition is compatible with the filtration in the space of jets we obtain $j^{k-1}\varphi_s = \varphi_{1,s}$ or the equivalent property $A_2 = \ldots = A_{k-1} = 0$. Let us remark that all the non-vanishing monomials of A_k are nonresonant; otherwise φ_s is not linearizable. The vector field $X_{1,N}$ is strongly resonant since $[X_{1,s}, X_{1,N}] = 0$ (remark 2.3). Therefore $\varphi_{1,u}$ is strongly resonant, it preserves resonant and nonresonant polynomials. Then $A_k \circ \varphi_{1,u}$ is a sum of nonresonant monomials. Since $f_k = A_k \circ \varphi_{1,u} + \varphi_{1,s} \circ B_k$ the expression $\varphi_{1,s} \circ B_k$ has non-vanishing weakly resonant monomials. The same property holds true for B_k.

Let $X_{1,N} + C_2 + C_3 + \ldots$ be the homogeneous decomposition of $\log \varphi_u$. Since $\varphi_{k-1} = \varphi_{1,s} \circ (\varphi_{1,s}^{\circ(-1)} \circ \varphi_{k-1})$ is the Jordan-Chevalley decomposition of φ_{k-1} then $\varphi_{1,u} + B_2 + \ldots + B_{k-1}$ is strongly resonant. We obtain that $X_{1,N}+C_2+\ldots+C_{k-1}$ is strongly resonant by using equation (1). Let B_k^w and C_k^w be the sum of the weakly resonant monomials of B_k and C_k respectively. It is easy to check out that

$$\exp(X_{1,N}+C_2+\ldots+C_{k-1}+(C_k-C_k^w))$$

is of the form $\varphi_{1,u}+B_2+\ldots+B_{k-1}+\sum_{j=k}^{\infty}\tilde{B}_j$ where $\tilde{B}_k$ is a sum of nonresonant and strongly resonant monomials. As a consequence $B_k^w \neq 0$ implies $C_k \neq C_k - C_k^w$ and $C_k^w \neq 0$.

Consider the decomposition $\hat{X} = X_s + X_N$ of $\hat{X}$. Since $\exp(X_s)$ is semisimple and $\exp(X_N)$ is unipotent we obtain $\varphi_s = \exp(X_s)$ and $\varphi_u = \exp(X_N)$. We have $\log \varphi_u = X_N$ by lemma 2.2.

Suppose that (a) holds true. We obtain $B_2 = \ldots = B_{k-1} = 0$ and then $C_2 = \ldots = C_{k-1} = 0$. We denote $X_s = X_{1,s} + X_s^2 + X_s^3 + \ldots$ where X_s^j is homogeneous of degree j for any $j \geq 2$. Let us calculate the degree k component D_k of $[X_s, \log\varphi_u] = 0$. We obtain

$$0 = D_k = [X_{1,s}, C_k] + [X_s^k, X_{1,N}].$$

Suppose that (b) holds true. Since $j^{k-1}\varphi_s = \varphi_{1,s}$ and $\varphi_s^* X_s = X_s$ we deduce that $X_s^2, \ldots, X_s^{k-1}$ are resonant. Condition (b) implies that they are also strongly resonant. We claim that $X_s^2 = \ldots = X_s^{k-1} = 0$. Otherwise $(X_s)_j = (X_s)_j + 0 = X_{1,s} + X_s^j$ are two different Jordan-Chevalley decompositions for $j = \min\{l \in \{2,\ldots,k-1\} : X_s^l \neq 0\}$ (see equation (3)). Again we obtain

$$0 = D_k = [X_{1,s}, C_k] + [X_s^k, X_{1,N}].$$

The hypothesis on $X_{1,N}$ and lemma 2.6 imply that $[X_s^k, X_{1,N}]$ does not contain non-vanishing weakly resonant monomials. But clearly $[X_{1,s}, C_k]$ does since $C_k^w \neq 0$ (rem. 2.3). We obtain a contradiction. □

Remark 3.2. *Let us remark that the condition on $X_{1,N}$ can be weakened. It is obvious from the proof that it suffices to require $[X_{1,N}, Y] = 0$ for any homogeneous weakly resonant vector field Y of degree k.*

Corollary 3.1. *Let $A \in GL(3,\mathbb{R})$ and let X_1 be a real logarithm of A such that $X_{1,s}$ is diagonal. Then any diffeomorphism $\varphi = Az + f_k + \ldots$ in $\mathrm{Diff}_\infty(\mathbb{R}^n,0)$ (resp. $\widehat{\mathrm{Diff}}(\mathbb{R}^n,0)$) such that f_k contains non-vanishing weakly resonant monomials is non-embeddable in the flow of a vector field $X \in \mathcal{X}_\infty(\mathbb{R}^n,0)$ (resp. $\hat{\mathcal{X}}(\mathbb{R}^n,0)$) such that $j^1X = X_1$.*

This is a consequence that for $n=3$ either all the eigenvalues of X_1 are real (and there are no weakly resonant monomials) or $X_{1,N}=0$. There exists a version of the result using property (b) instead of (a). Notice that for $n=2$ if A is hyperbolic and has a real logarithm then φ in $\mathrm{Diff}_\infty(\mathbb{R}^n,0)$ is always embeddable in a C^∞ flow [19].

Example. Consider the diffeomorphism $\varphi \in \text{Diff}\,(\mathbb{R}^3, 0)$ defined by

$$\varphi = \left(e^{-2} z_1, -e z_3 - \frac{3}{4} z_1 z_2 z_3^2 + \frac{z_1 z_2^3}{4}, e z_2 + \frac{z_1 z_3^3}{4} - \frac{3}{4} z_1 z_2^2 z_3 \right).$$

The eigenvalues of $j^1\varphi$ are e^{-2}, $e^{1+\pi i/2}$ and $e^{1-\pi i/2}$. We consider the change of coordinates $z_1 = w_1$, $z_2 = w_2 + w_3$, $z_3 = i(-w_2 + w_3)$ (see eq. (5)). We obtain

$$(j^1\varphi)(w_1, w_2, w_3) = \left(e^{-2} w_1, e^{1+\frac{\pi i}{2}} w_2, e^{1-\frac{\pi i}{2}} w_3 \right)$$

All eigenspaces of $j^1\varphi$ are one dimensional, hence any logarithm X_1 of $j^1\varphi$ is diagonal and $X_{1,N} = 0$. The eigenvalues of X_1 are of the form $\mu_1 = -2 + 2\pi i k_1$, $\mu_2 = 1 + \pi i/2 + 2\pi i k_2$ and $\mu_3 = 1 - \pi i/2 + 2\pi i k_3$ for some $k_1, k_2, k_3 \in \mathbb{Z}$. We have

$$\varphi(w_1, w_2, w_3) = \left(e^{-2} w_1, e^{1+\frac{\pi i}{2}} w_2 + w_1 w_3^3, e^{1-\frac{\pi i}{2}} w_3 + w_1 w_2^3 \right).$$

The diffeomorphism φ is resonant. Since

$$(\mu_1 + 3\mu_2 - \mu_3) - (\mu_1 - \mu_2 + 3\mu_3) = 4\pi i(1 + 2(k_2 - k_3))$$

either $w_1 w_3^3 e_2$ or $w_1 w_2^3 e_3$ is weakly resonant for any choice of X_1 (or equivalently for any choice of $k_1, k_2, k_3 \in \mathbb{Z}$). Condition (a) implies that φ is a real hyperbolic diffeomorphism that is not embedded in a flow of $\mathcal{X}_\infty(\mathbb{R}^n, 0)$, $\mathcal{X}(\mathbb{R}^n, 0)$, $\mathcal{X}(\mathbb{C}^n, 0)$ or $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ (th. 1.5).

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