THE α-CALCULUS-CUM-α-ANALYSIS OF $\frac{\partial^r}{\partial s^r}\zeta(s,\alpha)$

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Abstract : For Hurwitz zeta function $\zeta(s,\alpha)$, as a function of α , we discuss the location and the nature of singularities of $\zeta^{(r)}(s,\alpha)=\frac{\partial^r}{\partial s^r}\zeta(s,\alpha),$ the formulae for the derivatives and the primitives of $\zeta^{(r)}(s,\alpha)$, the Riemann-integrability of $\zeta^{(r)}(s,\alpha)$ on the intervals [0,1] and [1,∞) ; and the evaluation of integrals $\int_0^1 \zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)............\zeta(-m_k,\alpha)\cdot\zeta^{(r)}(s,\alpha)d\alpha$, where $m_1,m_2,............,m_k \geq 0$ are integers and Re s<1 .

Keywords : Hurwitz zeta function , Bernoulli polynomials /numbers .

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Let $r \geq 0$ be an integer and for the complex variables s,α , let $\zeta(s,\alpha)$ be the Hurwitz zeta function and let $\zeta^{(r)}(s,\alpha)=\frac{\partial^r}{\partial s^r}\zeta(s,\alpha)$. The object of this paper is to bring out the behaviour of $\zeta^{(r)}(s,\alpha)$ as an analytic function of the complex variable α ; to determine the location and the nature of the singularities of $\zeta^{(r)}(s,\alpha)$; to determine the Riemann integrability of $\zeta^{(r)}(s,\alpha)$ on intervals [0,1] and [1, ∞) ; to obtain the formulae for its derivatives and primitives with respect to α ; to evaluate the integrals $\int_0^1 \zeta(-m_1,\alpha)\zeta(-m_2,\alpha)..........\zeta(-m_r,\alpha)\zeta(s,\alpha)$ and $\int_0^1 \zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)..........\zeta(-m,\alpha)\cdot\zeta'(s,\alpha)d\alpha$, where $m_1,m_2,..........,m_r \geq 0$ are integers and Re s<1 .

Next , we formally introduce our notation and terminology .

For complex $\alpha \neq 0,-1,-2,..............,$ and for the complex variable s , let $\zeta(s,\alpha)$ be the Hurwitz zeta function defined by $\zeta(s,\alpha)=\sum_{n\geq 0}(n+\alpha)^{-s}$ for Re s>1 and its analytic continuation . Let $\zeta(s,1)=\zeta(s),$ the Riemann zeta function . In what follows, $\Gamma(s)$ stands for gamma function . $B_n(\alpha)=\sum_{i=0}^{n}\binom{n}{i}B_{n-i}\alpha^i$ stands for Bernoulli polynomial of degree n . Here $B_n = B_n(0)$ are Bernoulli numbers , which are known to be rational numbers . We note that if $n \geq 0$ is an integer , then $\zeta(-n,\alpha)=-\frac{B_{n+1}(\alpha)}{n+1}$ and $B_n(1-\alpha)=(-1)^n B_n(\alpha)$. We write $\psi(\alpha)=\frac{\Gamma'}{\Gamma}(\alpha)$. Note that $\frac{d}{d\alpha}\zeta'(0,\alpha)=\frac{d}{d\alpha}\log\frac{\Gamma(\alpha)}{\sqrt{2\pi}}=\frac{\Gamma'}{\Gamma}(\alpha)=\psi(\alpha)$. Thus $\frac{d}{d\alpha}\zeta'(0,\alpha)=\psi(\alpha)$. We shall see that $\frac{d}{d\alpha}\psi(\alpha)=\zeta(2,\alpha)$. In view of the fact $\frac{d}{d\alpha}\zeta(s,\alpha)=-s\zeta(s+1,\alpha)$, we have $\frac{d^r}{d\alpha^r}\psi(\alpha)=(-1)^{r-1}r!\zeta(r+1,\alpha)$. We also note that if s_1, s_2 are complex numbers , then

$$\int_0^1 \zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)d\alpha = 2(2\pi)^{s_1+s_2-2}\cdot\Gamma(1-s_1)\Gamma(1-s_2)\cdot\cos\frac{\pi}{2}(s_1-s_2)\cdot\zeta(2-s_1-s_2),$$

barring the singularities of either side. In author [3] , we study evaluation of Tornheim double zeta function $T(s_1,s_2,s_3)$ for various values of the complex variables s_1,s_2,s_3 in terms of integrals of the type $\int_0^1 \zeta(s_1,\alpha)\zeta(s_2,\alpha)\zeta^{(r)}(s_3,\alpha)d\alpha$ for $r=0$ or 1 . In view of this , we shall find some integrals of the type $\int_0^1 \zeta(s_1,\alpha)\zeta(s_2,\alpha)\zeta^{(r)}(s_3,\alpha)d\alpha$, which are explicitly computable in terms of values of Riemann zeta function and its derivatives . Apart from several other facts , in particular , we prove the following .

Proposition : For any integer $r \geq 0$ and for Re s<o , $\zeta^{(r)}(s,\alpha)$ is a continuous function of α in the whole complex $\alpha-$plane and we have the following .

1) $\zeta^{(r)}(s,0)=\zeta^{(r)}(s)=\zeta^{(r)}(s,1)$ for Re s<o and $\lim_{s\to 0-}\zeta^{(r)}(s,0)=\zeta^{(r)}(0)$,

where $s \to 0$ through real values from the left of 0 .

2) $\frac{\partial}{\partial \alpha}\zeta^{(r)}(s,\alpha) = -r \cdot \zeta^{(r-1)}(s+1,\alpha) - s\zeta^{(r)}(s+1,\alpha)$ for $s \neq 0$ and for $r \geq 0$.

3) We have $\frac{\partial}{\partial \alpha}\zeta^{(r)}(0,\alpha) = -r!\gamma_{r-1}(\alpha)$ for $r \geq 0$,

where $s\zeta(s+1,\alpha) = \sum_{n\geq 0}\gamma_{n-1}(\alpha)s^n$ with $\gamma_{-1}(\alpha) = 1$.

4) We have $\frac{\partial}{\partial \alpha}\gamma_{r-1}(\alpha) = -\frac{1}{r!}\frac{\partial^2}{\partial \alpha^2}\zeta^{(r)}(0,\alpha) = -\frac{1}{r!}\cdot\frac{\partial^r}{\partial s^r}\left(s(s+1)\zeta(s+2,\alpha)\right)\big|_{s=0}$ for $r \geq 1$.

Note : 1) of Proposition follows from the fact $\zeta(s,\alpha) - \zeta(s,\alpha+1) = \alpha^{-s}$

and consequently , $\zeta^{(r)}(s,\alpha) - \zeta^{(r)}(s,\alpha+1) = \alpha^{-s}(-\log\alpha)^r$ for an integer $r \geq 0$,

after letting $\alpha \to 0$.

Corollaries of Proposition :

1) For $s \neq 1$, the primitives (or the indefinite integrals) of $\zeta(s,\alpha), \zeta'(s,\alpha)$ and $\zeta''(s,\alpha)$ are as follows :

a) $\int \zeta(s,\alpha)d\alpha = \frac{\zeta(s-1,\alpha)}{1-s}$

b) $\int \zeta'(s,\alpha)d\alpha = \frac{\zeta(s-1,\alpha)}{(1-s)^2} + \frac{\zeta'(s-1,\alpha)}{1-s}$

c) $\int \zeta''(s,\alpha)d\alpha = 2\frac{\zeta(s-1,\alpha)}{(1-s)^3} + 2\frac{\zeta'(s-1,\alpha)}{(1-s)^2} + \frac{\zeta''(s-1,\alpha)}{1-s}$

d) $\int \zeta(s,1-\alpha)d\alpha = \frac{\zeta(s-1,1-\alpha)}{s-1}$

e) $\int \zeta'(s,1-\alpha)d\alpha = -\left(\frac{\zeta(s-1,1-\alpha)}{(1-s)^2} + \frac{\zeta'(s-1,1-\alpha)}{1-s}\right)$

f) In general , for an integer $r \geq 0$,

$$\int \zeta^{(r)}(s,\alpha)d\alpha = \sum_{\ell=0}^{r} c_\ell \frac{\zeta^{(\ell)}(s-1,\alpha)}{(1-s)^{r+1-\ell}}$$

for some absolute constants c_ℓ's .

2) Under the action of the operator $\frac{\partial}{\partial \alpha}$, we have the following diagram

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \to \zeta(-1,\alpha) \to \zeta(0,\alpha). \to -1 \to 0 \to \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots \to \zeta'(-1,\alpha) + \zeta(-1,\alpha) \to \zeta'(0,\alpha) \to -\gamma_0(\alpha) \to \zeta(2,\alpha) \to \ldots\ldots\ldots\ldots$$

$$\ldots \to \zeta''(-1,\alpha) + 2\zeta'(-1,\alpha) + 2\zeta(-1,\alpha) \to \zeta''(0,\alpha) \to -2\gamma_1(\alpha) \to 2(\zeta'(2,\alpha) + \zeta(2,\alpha)) \to \ldots\ldots\ldots$$

3) We have $\int_1^\infty \zeta^{(r)}(s,\alpha)d\alpha = -\sum_{\ell=o}^{r} c_\ell \cdot \frac{\zeta^{(\ell)}(s-1)}{(1-s)^{r+1-\ell}}$ for Re s>2 ,

where c_ℓ's are absolute constants as in f) of Corollary 1) above .

Note :This is so, because for Re s>1 ,as $\alpha \to \infty$,

$$\zeta^{(r)}(s,\alpha) = (-1)^r \sum_{n \geq 0} (n+\alpha)^{-s} \log^r (n+\alpha) = 0$$

4) We have $\int_0^1 \zeta^{(r)}(s,\alpha)d\alpha = 0$ for Re s<1.

Note : This follows from f) of corollary 1) and 1) of Proposition above .

5) For Re s_1, Re s_2 <0 and for real s, we have

$$\lim_{s \to 1-} \int_0^1 \zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\cdot(s-1)\zeta(s,\alpha)d\alpha = \int_0^1 \zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)d\alpha - \zeta(s_1)\cdot\zeta(s_2)$$

$$= 2(2\pi)^{s_1+s_2-2}\cdot\Gamma(1-s_1)\Gamma(1-s_2)\cos\frac{\pi}{2}(s_1-s_2)\cdot\zeta(2-s_1-s_2) - \zeta(s_1)\cdot\zeta(s_2) \ ,$$

where $s \to 1$ from left through real values .

6) Let $m_1, m_2, \ldots\ldots\ldots\ldots m_r \geq 1$ be integers and let $N = \sum_{i=1}^{r} m_i$. Then

$$\int_0^1 \zeta(1-m_1,\alpha)\cdot\zeta(1-m_2,\alpha)\ldots\ldots\ldots\ldots\cdot\zeta(1-m_r,\alpha)d\alpha$$

$= \dfrac{(-1)^r}{m_1 m_2 \ldots\ldots\ldots\ldots m_r} \int_0^1 B_{m_1}(\alpha)\cdot B_{m_2}(\alpha)\ldots\ldots\ldots\ldots B_{m_r}(\alpha)d\alpha$ is explicitly computable as

a rational number , which equals zero when N is odd .

Remark : Note that $\prod_{\ell=1}^{r} B_{m_2}(\alpha)$ is a polynomial with rational coefficients .

Hence $\int_0^1 \prod_{\ell=1}^{r} B_{m_2}(\alpha)d\alpha$ is a rational number **.**

Note that $\int_0^1 B_{m_1}(\alpha)B_{m_2}(\alpha)\ldots\ldots\ldots\ldots B_{m_r}(\alpha)d\alpha$

$= \int_0^1 B_{m_1}(1-\alpha)\cdot B_{m_2}(1-\alpha)\ldots\ldots\ldots\ldots B_{m_r}(1-\alpha)d\alpha = (-1)^N \int_0^1 B_{m_1}(\alpha)\cdot B_{m_2}(\alpha)\ldots\ldots\ldots\ldots B_{m_r}(\alpha)d\alpha$

=0 , when N is odd .

7) If $m_1, m_2, \ldots\ldots\ldots\ldots, m_r \geq 0$ are integers and Re s<1 , then

$$\int_0^1 \zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)\ldots\ldots\ldots\ldots\cdot\zeta(-m_r,\alpha)\zeta(s,\alpha)d\alpha$$

is explicitly computable as a linear combination of

$\zeta(s-1), \zeta(s-2), \ldots\ldots\ldots\ldots, \zeta(s-N)$ with coefficients dependent on s,

where $N = \sum_{i=1}^{r}(m_i+1)$ is the degree of the product polynomial

$\zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)\ldots\ldots\ldots\ldots\cdot\zeta(-m_r,\alpha)$.

8) If $m_1, m_2, \ldots\ldots\ldots\ldots, m_r \geq 0$ are integers and Re s<1 ,

then $\int_0^1 \zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)...........\zeta(-m_r,\alpha)\cdot\zeta'(s,\alpha)d\alpha$

is explicitly computable as a linear combination of

$\zeta'(s-1),\zeta'(s-2),...........,\zeta'(s-N)$; $\zeta(s-1),\zeta(s-2),...........,\zeta(s-N)$

with coefficients dependent upon s , where $N=\sum_{i=1}^{r}(m_i+1)$.

9) We have for Re s>1 ,

$$\int_0^1 \zeta(0,\alpha)\zeta(1-s,\alpha)\cdot\zeta(2-s,\alpha)d\alpha=\frac{1}{2(s-1)}\left(2(2\pi)^{-2s}\cdot\Gamma^2(s)\zeta(2s)-\zeta^2(1-s)\right).$$

Assuming Proposition , we shall prove the corollaries .

Proof of Corollary 2) : We shall sketch the proof of

$$\frac{\partial}{\partial\alpha}(-\gamma_0(\alpha))=\frac{\partial^2}{\partial\alpha^2}\zeta'(0,\alpha)=\zeta(2,\alpha).$$

We have $\frac{\partial^2}{\partial\alpha^2}\zeta'(0,\alpha)=\frac{\partial^2}{\partial\alpha^2}\frac{\partial}{\partial s}\zeta(s,\alpha)|_{s=0}$

$$=\frac{\partial}{\partial s}\frac{\partial^2}{\partial\alpha^2}\zeta(s,\alpha)|_{s=0}=\frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}\left(-s\zeta(s+1,\alpha)\right)|_{s=0}=\frac{\partial}{\partial s}\left(s(s+1)\zeta(s+2,\alpha)\right)|_{s=0}$$

$$=\left((s+1)\zeta(s+2,\alpha)+s\zeta(s+2,\alpha)+s(s+1)\zeta'(s+2,\alpha)\right)_{s=0}=\zeta(2,\alpha).$$

Next , we show $\frac{\partial}{\partial\alpha}\zeta''(0,\alpha)=-2\gamma_1(\alpha)$

and $\frac{\partial}{\partial\alpha}\left(-2\gamma_1(\alpha)\right)=\frac{\partial^2}{\partial\alpha^2}\zeta''(0,\alpha)=2\left(\zeta'(2,\alpha)+\zeta(2,\alpha)\right)$.

We have $\frac{\partial}{\partial\alpha}\zeta''(0,\alpha)=\frac{\partial}{\partial\alpha}\frac{\partial^2}{\partial s^2}\zeta(s,\alpha)|_{s=0}$

$$=\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial\alpha}\zeta(s,\alpha)|_{s=0}=\frac{\partial^2}{\partial s^2}\left(-s\zeta(s+1,\alpha)\right)|_{s=0}=-\left(s\zeta(s+1,\alpha)\right)''|_{s=0}=-2\gamma_1(\alpha)\ .$$

Next , $\frac{\partial}{\partial\alpha}\left(-2\gamma_1(\alpha)\right)=\frac{\partial^2}{\partial\alpha^2}\zeta''(0,\alpha)=\frac{\partial^2}{\partial\alpha^2}\frac{\partial^2}{\partial s^2}\zeta(s,\alpha)\,|_{s=0}$

$$=\frac{\partial^2}{\partial s^2}\frac{\partial^2}{\partial\alpha^2}\zeta(s,\alpha)\,|_{s=0}=\frac{\partial^2}{\partial s^2}\left(s(s+1)\zeta(s+2,\alpha)\right)|_{s=0}$$

$$=\frac{\partial}{\partial s}\left\{s\cdot\zeta(s+2,\alpha)+(s+1)\zeta(s+2,\alpha)+s(s+1)\zeta'(s+2,\alpha)\right\}_{s=0}$$

$$=\left\{\begin{matrix}\zeta(s+2,\alpha)+s\zeta'(s+2,\alpha)+\zeta(s+2,\alpha)+(s+1)\zeta'(s+2,\alpha)\\+(s+1)\zeta'(s+2,\alpha)+s\zeta'(s+2,\alpha)+s(s+1)\zeta''(s+2,\alpha)\end{matrix}\right\}_{s=0}=2\left(\zeta(2,\alpha)+\zeta'(2,\alpha)\right).$$

Proof of Corollary 5) : For Re s_1 , Re s_2 <0 and for real s with s<1 ,

$$\int_0^1\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\cdot(s-1)\zeta(s,\alpha)d\alpha=-\int_0^1\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\frac{\partial}{\partial\alpha}\zeta(s-1,\alpha)d\alpha$$

$$=-\left\{\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\cdot\zeta(s-1,\alpha)\ \Big|_{\alpha=0}^{1}-\int_0^1\zeta(s-1,\alpha)\cdot\frac{\partial}{\partial\alpha}(\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha))d\alpha\right\}$$

$$=0+\int_0^1\zeta(s-1,\alpha)\cdot\frac{\partial}{\partial\alpha}\left(\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\right)d\alpha\ .$$

Thus $\lim_{s\to1-}\int_0^1\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)(s-1)\zeta(s,\alpha)d\alpha$

$$=\lim_{s\to1-}\int_0^1\zeta(s-1,\alpha)\cdot\frac{\partial}{\partial\alpha}\left(\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\right)d\alpha=\int_0^1\zeta(0,\alpha)\cdot\frac{\partial}{\partial\alpha}\left(\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\right)d\alpha$$

$$=\int_0^1(\frac{1}{2}-\alpha)\frac{\partial}{\partial\alpha}(\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha))d\alpha=(\frac{1}{2}-\alpha)\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)\ \Big|_{\alpha=0}^{1}+\int_0^1\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)d\alpha\,,$$

on integration by parts .

This , in turn , $=\int_0^1\zeta(s_1,\alpha)\cdot\zeta(s_2,\alpha)d\alpha-\zeta(s_1)\cdot\zeta(s_2)$

$$=2(2\pi)^{s_1+s_2-2}\cdot\Gamma(1-s_1)\Gamma(1-s_2)\cos\frac{\pi}{2}(s_1-s_2)\cdot\zeta(2-s_1-s_2)-\zeta(s_1)\cdot\zeta(s_2).$$

Proof of Corollary 7) : We have

$$\zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)............\zeta(-m_r,\alpha)=\left(-\frac{B_{m+1}(\alpha)}{m_1+1}\right)\left(-\frac{B_{m_2+1}(\alpha)}{m_2+1}\right)............\left(-\frac{B_{m_r+1}(\alpha)}{m_r+1}\right)$$

= $P_N(\alpha)$, say , where $P_N(\alpha)$ is a polynomial of degree N$=\sum_{i=1}^{r}(m_i+1)$.

Let $P_N(\alpha)=\sum_{\ell=0}^{N}a_i\alpha^i$, where $a_i's$ are rational numbers .

Thus $\int_0^1 P_N(\alpha)\zeta(s,\alpha)d\alpha=\sum_{i=0}^{N}a_i\int_0^1\alpha^i\zeta(s,\alpha)d\alpha$.

Note that for each i , $\int_0^1\alpha^i\zeta(s,\alpha)d\alpha=\int_0^1\alpha^i d\left(\frac{\zeta(s-1,\alpha)}{1-s}\right)$

$$=\alpha^i\,\frac{\zeta(s-1,\alpha)}{1-s}\,\Big|_{\alpha=0}^{1}-\int_0^1\frac{\zeta(s-1,\alpha)}{1-s}\cdot i\alpha^{i-1}d\alpha=\frac{\zeta(s-1)}{1-s}+\frac{i}{s-1}\int_0^1\alpha^{i-1}\zeta(s-1,\alpha)d\alpha\ .$$

We write $\int_0^1\alpha^{i-1}\zeta(s-1,\alpha)d\alpha=\int_0^1\alpha^{i-1}d\left(\frac{\zeta(s-2,\alpha}{2-s}\right)$ and continue the process of integration by parts till we reach the stage , when we come across the integral $\int_0^1\zeta(s-i,\alpha)d\alpha$, which is equal to zero .

Thus $\int_0^1\alpha^i\zeta(s,\alpha)d\alpha$ is a linear combination of $\zeta(s-1),\zeta(s-2),............,\zeta(s-i)$

for each i=0,1,2,............, N.

Proof of corollary 8) : We have $\int_0^1\zeta(-m_1,\alpha)\cdot\zeta(-m_2,\alpha)............\zeta(-m_r,\alpha)\cdot\zeta'(s,\alpha)d\alpha$

$$=\int_0^1 P_N(\alpha)\cdot\zeta'(s,\alpha)d\alpha=\sum_{j=0}^{N}a_j\int_0^1\alpha^j\cdot\zeta'(s,\alpha)d\alpha\ ,$$

where $P_N(\alpha)$ is the polynomial as in corollary 7) .

Note that $\int_0^1 \alpha^j \cdot \zeta'(s,\alpha) d\alpha = \int_0^1 \alpha^j d\left(\frac{\zeta'(s-1,\alpha)}{1-s} + \frac{\zeta(s-1,\alpha)}{(1-s)^2}\right) d\alpha$

$$= \left[\alpha^j \left(\frac{\zeta'(s-1,\alpha)}{1-s} + \frac{\zeta(s-1,\alpha)}{(1-s)^2}\right)\right]_{\alpha=0}^1 - \int_0^1 \left(\frac{\zeta'(s-1,\alpha)}{(1-s)} + \frac{\zeta(s-1,\alpha)}{(1-s)^2}\right) \cdot j\alpha^{j-1} d\alpha$$

$$= \frac{\zeta'(s-1)}{1-s} + \frac{\zeta(s-1)}{(1-s)^2} - j\int_0^1 \left(\frac{\zeta'(s-1,\alpha)}{(1-s)} + \frac{\zeta(s-1,\alpha)}{(1-s)^2}\right) \cdot \alpha^{j-1} d\alpha \ .$$

We continue to integrate by parts j times .

Thus we find that $\int_0^1 \alpha^j \zeta'(s,\alpha) d\alpha$ is a linear combination of

$\zeta'(s-1), \zeta'(s-2), \ldots\ldots\ldots\ldots, \zeta'(s-j)$; $\zeta(s-1), \zeta(s-2), \ldots\ldots\ldots\ldots, \zeta(s-j)$

for each $j = 1,2,\ldots\ldots\ldots\ldots, N$.

Proof of Corollary 9) : We have for Re s>1 ,

$$\int_0^1 \zeta(0,\alpha)\zeta(1-s,\alpha)\cdot\zeta(2-s,\alpha) d\alpha = \frac{1}{2(s-1)} \int_0^1 (\frac{1}{2}-\alpha) \frac{\partial}{\partial\alpha} \zeta^2(1-s,\alpha) d\alpha$$

$$= \frac{1}{2(s-1)} \left\{ (\frac{1}{2}-\alpha)\zeta^2(1-s,\alpha) |_{\alpha=0}^1 + \int_0^1 \zeta^2(1-s,\alpha) d\alpha \right\}$$

$$= \frac{1}{2(s-1)} \left(-\zeta^2(1-s) + \int_0^1 \zeta^2(1-s,\alpha) d\alpha \right) = \frac{1}{2(s-1)} \left(2(2\pi)^{-2s} \Gamma^2(s) \zeta(2s) - \zeta^2(1-s) \right).$$

It is known that $\zeta(s,\alpha)$ is an analytic function of the complex variable s with a simple pole at s=1 . In author [1] , [2] it has been shown that $\zeta(s,\alpha)$ is an analytic function of the complex variable α except for possible singularities at non-positive integer values of α . For an integer $r \geq 0$, we write

$\zeta^{(r)}(s,\alpha)=\dfrac{\partial^r}{\partial s^r}\zeta(s,\alpha)$. In author [1] , [2] it has been shown that $\zeta^{(r)}(s,\alpha)$ is an analytic function of the complex variable α except for possible singularities at $\alpha=0,-1,-2,\ldots\ldots\ldots$.

We have $\zeta(s,\alpha)=\sum_{n\geq 0}(n+\alpha)^{-s}$ for Re s>1 ,

so that $(-1)^r\zeta^{(r)}(s,\alpha)=\sum_{n\geq 0}(n+\alpha)^{-s}\log^r(n+\alpha)$ for $r\geq 0$.

Thus $(-1)^r\dfrac{\partial}{\partial\alpha}\zeta^{(r)}(s,\alpha)=r\sum_{n\geq 0}(n+\alpha)^{-s-1}\log^{r-1}(n+\alpha)-s\sum_{n\geq 0}(n+\alpha)^{-s-1}\log^r(n+\alpha)$.

This gives $\dfrac{\partial}{\partial\alpha}\zeta^{(r)}(s,\alpha)=-r\zeta^{(r-1)}(s+1,\alpha)-s\zeta^{(r)}(s+1,\alpha)$ for Re s>1 and $r\geq 1$.

By analyticity of $\zeta^{(r)}(s,\alpha)$ as a function of complex variables s and α , the above result holds everywhere . This proves 2) of Proposition .

Let $k\geq 1$ be an integer . Let $\zeta_k(s,\alpha)=\sum_{n\geq k}(n+\alpha)^{-s}$ for Re s>1; and its analytic continuation . Then $\zeta_k(s,\alpha)=\zeta(s,\alpha)-\sum_{0\leq n\leq k-1}(n+\alpha)^{-s}=\zeta(s,k+\alpha)$.

Let $\zeta_k(s)=\zeta(s)-\sum_{1\leq n\leq k-1}n^{-s}$. Then in author [2] , it has been shown that

$\zeta_k(s,\alpha)=\sum_{n\geq 0}\dfrac{(-\alpha)^n}{n!}s(s+1)\ldots\ldots\ldots(s+n-1)\cdot\zeta_k(s+n)$ in the disc $|\alpha|<k,$

where empty product stands for 1 .

This gives $\zeta(s,\alpha)=\sum_{0\leq n\leq k-1}(n+\alpha)^{-s}+\sum_{n\geq 0}s(s+1)\ldots\ldots\ldots\ldots(s+n-1)\zeta_k(s+n)\cdot\dfrac{(-\alpha)^n}{n!}$

$=\sum_{0\leq n\leq k-1}(n+\alpha)^{-s}+\phi_k(s,\alpha)$, say , where $\phi_k(s,\alpha)$ is an analytic function of α in the disc $|\alpha|<k$ with $k\geq 1$ arbitrary and $\phi_k^{(r)}(s,\alpha)=\dfrac{\partial^r}{\partial s^r}\phi_k(s,\alpha)$ can be obtained by

term-by-term differentiation of $\phi_k(s,\alpha)$ with respect to s , r times .

Thus $\zeta^{(r)}(s,\alpha) = \sum_{0\le n\le k-1}(n+\alpha)^{-s}(-\log(n+\alpha))^r + \phi_k^{(r)}(s,\alpha)$,

where $\phi_k^{(r)}(s,\alpha)$ is an analytic function of α in the disc $|\alpha| < k$ and $k \ge 1$ is arbitrary .

From the expression $\zeta(s,\alpha) = \sum_{0\le n\le k-1}(n+\alpha)^{-s} + \phi_k(s,\alpha)$, where $\phi_k(s,\alpha)$ is analytic in the disc $|\alpha| < k$ with integer $k \ge 1$ arbitrary , we get the following :

1) If $s = -m$, where $m \ge 0$ is an integer , then $\zeta(-m,\alpha) = \sum_{0\le n\le k-1}(n+\alpha)^m + \phi_k(-m,\alpha)$.

   Thus $\zeta(-m,\alpha)$ is an entire function of α .

2) If $s = m \ge 1$ is an integer , then $\zeta(m,\alpha) = \sum_{0\le n\le k-1}(n+\alpha)^{-m} + \phi_k(m,\alpha)$.

   Thus $\zeta(m,\alpha)$ has a pole of order m at every non-positive integer value of α .

3) If Re s<0 , then $\zeta^{(r)}(s,\alpha) = (-1)^r \sum_{0\le n\le k-1}(n+\alpha)^{-s}\log^r(n+\alpha) + \phi_k^{(r)}(s,\alpha),$

   where $\phi_k^{(r)}(s,\alpha)$ is analytic function of α in the disc $|\alpha| < k$. This shows that $\zeta^{(r)}(s,\alpha)$ is a continuous function of α , because if $\alpha = -n_0$, where n_0 is a fixed integer from amongst 0,1,2,…………., we have $\lim_{\alpha\to -n_0}(n_0+\alpha)^{-s}\log^r(n_0+\alpha) = 0,$

   for Re s<0 . This also gives that if s is any complex number , the only possible singularities of $\zeta^{(r)}(s,\alpha)$ are non-positive integer values of α .

4) For Re s>1 , we have $\zeta(s,\alpha) = \sum_{n\ge 0}(n+\alpha)^{-s}$.

   As $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial s}(n+\alpha)^{-s} = \frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}(n+\alpha)^{-s}$ for $\alpha \ne -n$,

   we have $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial s}\left(\sum_{n\ge 0}(n+\alpha)^{-s}\right) = \frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}\left(\sum_{n\ge 0}(n+\alpha)^{-s}\right)$ for Re s>1 and for $\alpha \neq -n$, where $n = 0,1,2,..............$.

Thus $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial s}\zeta(s,\alpha) = \frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}\zeta(s,\alpha)$ for Re s>1 and for 0<α<1 .

Thus in view of analyticity of $\zeta\ (s,\alpha)$ as a function of each complex variable s and α , we have $\frac{\partial}{\partial\alpha}\frac{\partial}{\partial s}\zeta(s,\alpha) = \frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}\zeta(s,\alpha)$ for $s \neq 1$ and $\alpha \neq 0,-1,-2,............$.

More generally , if $r_1, r_2 \geq 0$ are integers , we have

$\frac{\partial^{r_1}}{\partial\alpha^{r_1}}\frac{\partial^{r_2}}{\partial s^{r_2}}\zeta(s,\alpha) = \frac{\partial^{r_2}}{\partial s^{r_2}}\frac{\partial^{r_1}}{\partial\alpha^{r_1}}\zeta(s,\alpha)$ for $s \neq 1$ and $\alpha \neq 0,-1,-2,............$.

In particular , $\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial s}\zeta(s,\alpha)\right)_{s=0} = \frac{\partial}{\partial s}\frac{\partial}{\partial\alpha}\zeta(s,\alpha)_{s=0}$ for $\alpha \neq 0,-1,-2,............$.

That is , $\frac{\partial}{\partial\alpha}\zeta'(0,\alpha) = \frac{\partial}{\partial s}(-s\zeta(s+1,\alpha))|_{s=0}$.

Let $\zeta(s+1,\alpha) = \frac{1}{s} + \sum_{n\geq 0}\gamma_n(\alpha)s^n$ so that $s\zeta(s+1,\alpha) = 1 + \sum_{n\geq 0}\gamma_n(\alpha)s^{n+1} = \sum_{n\geq 0}\gamma_{n-1}(\alpha)s^n$,

where we have written $\gamma_{-1}(\alpha) = 1$

Thus $(s\zeta(s+1,\alpha))^{(r)}|_{s=0} = r!\gamma_{r-1}(\alpha)$ for $r = 0,1,2,............$.

Here the superscript (r) denotes r-th order derivative with respect to s .

Thus we have $\frac{\partial}{\partial\alpha}\zeta'(0,\alpha) = -\gamma_0(\alpha)$.

However , we know $\zeta'(0,\alpha) = \log\frac{\Gamma(\alpha)}{\sqrt{2\pi}}$ for $\alpha \neq -n$, where $n = 0,1,2,..........$

Thus we have $\psi(\alpha) = -\gamma_0(\alpha)$, where $\psi(\alpha) = \frac{\Gamma'}{\Gamma}(\alpha)$.

Note that $\psi(1) = -\gamma_0(1) = -\gamma$,where γ is Euler's constant .

The fact $\left(s\zeta(s+1,\alpha)\right)^{(r)}|_{s=0}=r!\gamma_{r-1}(\alpha)$ gives $\left(\frac{\partial}{\partial\alpha}\zeta(s,\alpha)\right)^{(r)}|_{s=0}=-r!\gamma_{r-1}(\alpha)$.

Equivalently , we have $\frac{\partial}{\partial\alpha}\zeta^{(r)}(0,\alpha)=-r!\gamma_{r-1}(\alpha)$ for $r\geq 0$.

5) We have $\zeta'(0,\alpha)=-\sum_{0\leq n\leq k-1}\log\ (n+\alpha)+\phi_k^{'}(0,\alpha)$

so that $\frac{\partial}{\partial\alpha}\zeta'(0,\alpha)=-\sum_{0\leq n\leq k-1}\frac{1}{n+\alpha}+\frac{\partial}{\partial\alpha}\phi'^{'}_{k}\ (0,\alpha)$.

That is $\psi(\alpha)=-\sum_{0\leq n\leq k-1}\frac{1}{(n+\alpha)}+\frac{\partial}{\partial\alpha}\phi_k^{'}\ (0,\alpha)$.

Thus $\psi(\alpha)$ has a simple pole at each non-positive integer value of α and

consequently for $r\geq 1$, $\frac{\partial^r}{\partial\alpha^r}\psi(\alpha)=(-1)^{r-1}r!\zeta(r+1,\alpha)$. has a pole of order $(r+1)$ at each non-positive integer value of α and is analytic elsewhere .

This completes our proof .

References

1] V.V. Rane , Instant Evaluation and demystification of $\zeta(n),L(n,\chi)$ that Euler, Ramanujan missed-I (ar Xiv org. website) .

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3] V.V.Rane , Functional Equation of Tornheim Zeta function and the α-Calculus of $\frac{\partial^r}{\partial s^r}\zeta(s,\alpha)$ (ar Xiv org. website) .