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Moments of random sums and Robbins' problem of optimal stopping

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Abstract

Robbins' problem of optimal stopping asks one to minimise the expected *rank* of observation chosen by some nonanticipating stopping rule. We settle a conjecture regarding the *value* of the stopped variable under the rule optimal in the sense of the rank, by embedding the problem in a much more general context of selection problems with the nonanticipation constraint lifted, and with the payoff growing like a power function of the rank.

1. Let $X_1, \dots, X_n$ be independent random variables sampled sequentially from the uniform $[0,1]$ distribution, and let $Y_1 < \dots < Y_n$ be their order statistics. The rank R_j of the variable X_j is defined by setting $R_j = k$ on the event $X_j = Y_k$. Robbins' problem of optimal stopping [3] asks one to minimize the expected rank $\mathbb{E}R_\tau$ over all stopping times τ that assume values in $\{1, \dots, n\}$ and are adapted to the natural filtration of the sequence $X_1, \dots, X_n$. Let τ_n be the optimal stopping time. The minimum expected rank $\mathbb{E}R_{\tau_n}$ increases as n grows, and converges to some finite limit v whose exact value is unknown. The closest known upper bound is slightly less than 7/3. Finding v or even improving the existing rough bounds remains a challenge. A major source of difficulties is that the optimal stopping time τ_n is a very complicated function of the sample. It seems that τ_n has not been computed for $n > 3$. Moreover, for large n there is no simplification, and the complexity of the optimal stopping time persists in the '$n = \infty$' limiting form of the problem [6].

In a recent paper Bruss and Swan [4] stressed that it is not even known if $\limsup_n n\mathbb{E}X_{\tau_n}$ is finite. They mentioned that the property was first conjectured in [2]. While the conjecture stems from the attempts to bound v by the comparison with much simpler problem of minimising $\mathbb{E}X_\tau$ (or minor variations of the problem), it seems that the question is of independent interest as a relation between the stopped sample value and its rank. In this note we settle the conjecture by proving a considerably more general assertion:

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Proposition 1. *Fix $p > 0$. For $n = 1, 2, \dots$ let σ_n be a random variable with range $\{1, \dots, n\}$ and arbitrary joint distribution with $X_1, \dots, X_n$. Then*

$$\limsup_n \mathbb{E}[R_{\sigma_n}]^p < \infty \quad \text{implies} \quad \limsup_n n^p \, \mathbb{E}[X_{\sigma_n}]^p < \infty. \tag{1}$$

In particular, $\lim_{n\to\infty} n^p \, \mathbb{E}[X_{\tau_n}]^p < \infty$ for τ_n the stopping time minimising $\mathbb{E}[R_\tau]^p$ over all stopping times adapted to $X_1, \dots, X_n$.

The idea is to bound X_{σ_n} by exploiting properties of a random walk with negative drift.

2. Let $\xi, \xi_1, \xi_2, \dots$ be iid nonnegative random variables with $\mu = \mathbb{E}\xi \in (0, \infty)$. Let $S_k := \xi_1 + \cdots + \xi_k$ and for $\lambda > \mu$ let $M_\lambda =: \sup_{k\geq 0}(S_k - \lambda k)$.

Proposition 2. *For $p > 0$*

$$\mathbb{E}\xi^{p+1} < \infty \quad \Longleftrightarrow \quad \mathbb{E}M_\lambda^p < \infty.$$

Proof. The moment condition on ξ is equivalent to $\mathbb{E}[(\xi - \lambda)^+]^{p+1} < \infty$, and the result follows from Lemma 3.5 in [1]. □

Corollary 3. *Suppose $\mathbb{E}\xi^{p+1} < \infty$ and let σ be a nonnegative integer random variable with $\mathbb{E}\sigma^p < \infty$. Then $\mathbb{E}S_\sigma^p < \infty$.*

Proof. This follows from $S_\sigma^p \leq (M_\lambda + \lambda\sigma)^p \leq c_p(M_\lambda^p + \lambda^p\sigma^p)$, where $c_p := 2^{p-1} \vee 1$. □

3. We can apply Corollary 3 to a Poisson-embedded, limiting form of the stopping problem with continuous time [6]. Let $\xi_1, \xi_2, \dots$ be iid rate-one exponential variables, S_k as above, and let $T_1, T_2, \dots$ be iid uniform $[0, 1]$ random times, independent of the ξ_j's. The points (T_k, S_k) are the atoms of a homogeneous planar Poisson process $\mathcal{P}$ in $[0, 1] \times [0, \infty)$. To introduce the dynamics, consider an observer whose information at time $t \in [0, 1]$ is the (infinite) configuration of points of $\mathcal{P}$ within the strip $[0, t] \times [0, \infty)$, that is $\{(T_k, S_k) : T_k \leq t\}$. The rank of point (T_k, S_k) is defined as $R_{T_k} = k$, meaning that S_k is the kth smallest value among $S_1, S_2, \dots$. The piece of information added at time T_k is the point (T_k, S_k), but not the rank R_{T_k}.

Suppose the objective of the observer is to minimize $\mathbb{E}[R_\tau]^p$ over stopping times τ that assume values in the random set $\{T_1, T_2 \dots\}$ and are adapted to the information flow of the observer. For the optimal stopping time τ_∞ it is known from the previous studies that $\mathbb{E}[R_{\tau_\infty}]^p < \infty$ (see [6] and [5]). Taking $\sigma = R_{\tau_\infty}$, we have $\mathbb{E}[S_\sigma]^p < \infty$. The case $p = 1$ corresponds to the infinite version of Robbins's problem of minimising the expected rank.

4. To apply the above to a finite sample, we shall use the familiar realisation of uniform order statistics through sums of exponential variables, as

$$(Y_k, \ 1 \leq k \leq n) \stackrel{d}{=} (S_k/S_n, \ 1 \leq k \leq n).$$

Introducing the event $A_n := \{n/S_n > 1+\epsilon\}$, we can estimate for $1 \le k \le n$

$$n^p Y_k^p = n^p Y_k^p 1_{A_n} + n^p Y_k^p 1_{A_n^c} \le n^p 1_{A_n} + (1+\epsilon)^\rho S_k^p \le n^p 1_{A_n} + c_p(1+\epsilon)^p (M_\lambda^p + \lambda^p k^p),$$

where we used $S_k \le M_\lambda + \lambda k$. Using a large deviation bound for the probability of A_n and sending $\epsilon \to 0$ we conclude that for any random variable σ_n with values in $\{1, \dots, n\}$

$$\limsup_n n^p \mathbb{E}[Y_{\sigma_n}]^p \le c_p \lambda^p \limsup_n \mathbb{E}\sigma_n^p + c_p \mathbb{E} M_\lambda^p.$$

Finally, taking $\sigma_n = R_{\tau_n}$, Proposition 2 follows from

$$\limsup_n n^p \mathbb{E}[X_{\tau_n}]^p \le c_p \lambda^p \limsup_n \mathbb{E}[R_{\tau_n}]^p + c_p \mathbb{E} M_\lambda^p < \infty,$$

since $\mathbb{E}[R_{\tau_n}]^p$ converges to a finite limit (see [6], [5]).

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