THE TWISTED ALEXANDER POLYNOMIAL FOR FINITE ABELIAN COVERS OVER THREE MANIFOLDS WITH BOUNDARY

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Abstract. We consider a sign–determined Reidemeister torsion with multi-variables for three manifolds with boundary. This invariant is often called the twisted Alexander polynomial when we consider link exteriors. Our purpose is to provide the twisted Alexander invariant of finite abelian covers over three manifolds with boundary, which is given by the product of those of the base space manifold. This is a generalization of the Alexander polynomial of knots in finite cyclic branched covers over the three sphere. We show examples for fibered manifolds in details and also show an example in a non–fibered case.

1. Introduction

The classical Alexander polynomial is defined for null–homologous knots in rational homology spheres, where null–homologous means that the homology class of a knot is trivial in the first homology group of the ambient space.

If the pair $(\hat{M}, \hat{K})$ of a rational homology sphere $\hat{M}$ and a null-homologous knot $\hat{K}$ in $\hat{M}$ is given by a finite cyclic branched cover over S^3 branched along a knot K, where $\hat{K}$ is the lift of K, then we can compute the Alexander polynomial of $\hat{K}$ by using the well–known formula:

$$\Delta_{\hat{K}}(t) = \prod_{\xi \in \{x \in \mathbb{C} \mid x^k = 1\}} \Delta_K(\xi t) \quad \text{up to a factor } \pm t^a \ (a \in \mathbb{Z})$$

where k is the order of the covering transformation group, ξ runs all over the k-th roots of unity and $\Delta_K(t)$ is the Alexander polynomial of K. Such formulas have been investigated from the viewpoint of Reidemeister torsion for a long time. In particular, V. Turaev gave a formula for the Alexander polynomial of $\hat{K}$ in a finite cyclic branched cover over S^3, and a generalization in the case of links in general three–dimensional manifolds (we refer to [Tur86, Theorem 1.9.2 and 1.9.3]).

The purpose of this paper is to provide the generalization of the above formula giving the Alexander polynomial of a knot in a finite cyclic branched cover over S^3 to a formula for the twisted Alexander polynomial of finite abelian covers, which is a special kind of Reidemeister torsion. Especially, we also consider the twisted Alexander polynomial for a link in a three–dimensional manifold from the viewpoint of Reidemeister torsion in the same way as V. Turaev. But to deal with finite abelian covers beyond finite cyclic covers, we adopt the approach of J. Porti in his work [Por04]. Porti gave a new proof of Mayberry–Murasugi's formula, which gives the order of the first homology group of finite abelian branched covers over S^3 branched along links, by using Reidemeister torsion theory. We call the twisted Alexander polynomial *the polynomial torsion* regarded as a kind of Reidemeister torsion.

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In this paper, we are interested in the Reidemeister torsion for a finite sheeted abelian covering. We are mainly intested in link exteriors in homology three–spheres and its abelian covers. Our main theorem (see Theorem 1) is stated for an abelian cover $\hat{M} \to M$ between two three–dimensional manifolds with boundary as follows:

$$\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\boldsymbol{t}) = \epsilon \cdot \prod_{\xi\in\hat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\boldsymbol{t})$$

where $\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\boldsymbol{t})$ and $\Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\boldsymbol{t})$ are the signed twisted Alexander polynomials with multivariable, $\hat{G}$ is the Pontrjagin dual of the covering transformation group G and ϵ is a sign determined by the homology orientations of $\hat{M}$ and M.

To be more precise, we need two homomorphisms of the fundamental group to define the twisted Alexander polynomial of a manifold. The symbol φ denotes a surjective homomorphism from $\pi_1(M)$ to a multiplicative group $\mathbb{Z}^n$ and ρ denotes a homomorphism to a linear automorphism group of some vector space (see Section 3 for the definition of the polynomial torsion). To define the twisted Alexander polynomial of $\hat{M}$, we use the pull–backs $\widehat{\varphi}$ and $\widehat{\rho}$ of φ and ρ to $\pi_1(\hat{M})$ (see Section 4).

When we choose $\hat{M} \to M$ as a finite cyclic cover of a knot exterior E_K of K in S^3, φ is the abelianization homomorphism $\pi_1(E_K) \to \pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)] \simeq \mathbb{Z}$ and ρ is the one–dimensional trivial representation, Theorem 1 reduces to the classical formula for the Alexander polynomial of $\hat{K}$, where $\hat{K}$ is the lift of the knot in the finite cyclic branched cover over S^3:

$$\Delta_{\hat{K}}(t) = \prod_{\xi\in\{x\in\mathbb{C}\,|\,x^k=1\}} \Delta_K(\xi t),$$

up to a factor $\pm t^a$ $(a \in \mathbb{Z})$, where k is the order of $\pi_1(M)/\pi_1(\hat{M})$. Our formula also provides the Alexander polynomial of a link in finite abelian branched covers over S^3 branched along the link.

We show some explicit computation examples in the last section 5. In particular, for three–dimensional manifolds fiber over the circle, the polynomial torsion for such manifold is expressed as the characteristic polynomial of the twisted monodromy induced by the fibered structure (see Theorem 2). When we construct an m–fold cyclic cover over a fibered three–dimensional manifold by the m-times composition of the twisted monodromy ϕ_* of the base space, we can realize the polynomial torsion as the characteristic polynomial $\det(t^m\mathbf{1} - \phi_*^m)$, which gives easily comprehensible examples of Theorem 1. We give the characteristic polynomial for the figure eight knot exterior as an example of computation for fibered manifolds. We also write down an example of non–fibered knot exterior in details.

Organization

The outline of the paper is as follows. Section 2 deals with some reviews on the sign–determined Reidemeister torsion for a manifold and on the multiplicative property of Reidemeister torsions (the Multiplicativity Lemma) which is the main tool for computing Reidemeister torsions by using a cut and past argument. In Section 3, we give the definition of the polynomial torsion (the twisted Alexander polynomial) for a manifold with boundary. In Section 4, we consider the polynomial torsion of finite abelian covering spaces (see Theorem 1) and the problem of the sign are stated and proved. In the last section 5, we will see several examples of our main theorem and computation methods for the polynomial torsion of knot exteriors.

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2. Preliminaries

2.1. **The Reidemeister torsion.** We review the basic notions and results about the sign–determined Reidemeister torsion introduced by V. Turaev which are needed in this paper. Details can be found in Milnor's survey [Mil66] and in Turaev's monograph [Tur02].

Torsion of a chain complex. Let $C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over a field $\mathbb{F}$. Choose a basis $\mathbf{c}^{(i)}$ of C_i and a basis $\mathbf{h}^i$ of the i-th homology group $H_i(C_*)$. The torsion of C_* with respect to these choices of bases is defined as follows.

For each i, let $\mathbf{b}^i$ be a set of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \mathrm{im}(d_i \colon C_i \to C_{i-1})$ and let $\tilde{\mathbf{h}}^i$ denote a lift of $\mathbf{h}^i$ in $Z_i = \ker(d_i \colon C_i \to C_{i-1})$. The set of vectors $d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i$ is a basis of C_i. Let $[d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i] \in \mathbb{F}^*$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i$ with respect to $\mathbf{c}^i$). The *sign-determined Reidemeister torsion* of C_* (with respect to the bases $\mathbf{c}^*$ and $\mathbf{h}^*$) is the following alternating product (see [Tur01, Definition 3.1]):

$$\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^{n} [d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{F}^*. \tag{1}$$

Here

$$|C_*| = \sum_{k \geqslant 0} \alpha_k(C_*)\beta_k(C_*),$$

where $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$ and $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k(C_*)$.

The torsion $\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choices of $\mathbf{b}^i$ nor on the lifts $\tilde{\mathbf{h}}^i$. Note that if C_* is acyclic (*i.e.* if $H_i = 0$ for all i), then $|C_*| = 0$.

Torsion of a CW-complex. Let W be a finite CW-complex and (V, ρ) be a pair of a vector space with an inner product over $\mathbb{F}$ and a homomorphism of $\pi_1(W)$ into $Aut(V)$. The vector space V turns into a right $\mathbb{Z}[\pi_1(W)]$-module denoted V_ρ by using the right action of $\pi_1(W)$ on V given by $v \cdot \gamma = \rho(\gamma)^{-1}(v)$, for $v \in V$ and $\gamma \in \pi_1(W)$. The complex of the universal cover with integer coefficients $C_*(\widetilde{W}; \mathbb{Z})$ also inherits a left $\mathbb{Z}[\pi_1(W)]$-module via the action of $\pi_1(W)$ on $\widetilde{W}$ as the covering group. We define the V_ρ-twisted chain complex of W to be

$$C_*(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z}).$$

The complex $C_*(W; V_\rho)$ computes the V_ρ-*twisted homology* of W which is denoted $H_*(W; V_\rho)$.

Let $\{e^i_1, \ldots, e^i_{n_i}\}$ be the set of i-dimensional cells of W. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\{\tilde{e}^i_1, \ldots, \tilde{e}^i_{n_i}\}$. If we choose an orthonormal basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ of V, then we consider the corresponding basis

$$\mathbf{c}^i = \left\{\boldsymbol{v}_1 \otimes \tilde{e}^i_1, \ldots, \boldsymbol{v}_m \otimes \tilde{e}^i_1, \cdots, \boldsymbol{v}_1 \otimes \tilde{e}^i_{n_i}, \ldots, \boldsymbol{v}_m \otimes \tilde{e}^i_{n_i}\right\}$$

of $C_i(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z})$. We call the basis $\mathbf{c}^* = \oplus_i \mathbf{c}^i$ *a geometric basis* of $C_*(W; V_\rho)$. Now choosing for each i a basis $\mathbf{h}^i$ of the V_ρ-twisted homology $H_i(W; V_\rho)$, we can compute the torsion

$$\mathrm{Tor}(C_*(W; V_\rho), \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{C}^*.$$

The cells $\left\{ \tilde{e}^i_j \,\middle|\, 0 \leqslant i \leqslant \dim W, 1 \leqslant j \leqslant n_i \right\}$ are in one–to–one correspondence with the cells of W, their order and orientation induce an order and an orientation for the cells $\left\{ \tilde{e}^i_j \,\middle|\, 0 \leqslant i \leqslant \dim W, 1 \leqslant j \leqslant n_i \right\}$. Again, corresponding to these choices, we get a basis $\mathbf{c}^i_{\mathbb{R}}$ over $\mathbb{R}$ of $C_i(W; \mathbb{R})$.

Choose an *homology orientation* of W, which is an orientation of the real vector space $H_*(W; \mathbb{R}) = \bigoplus_{i \geqslant 0} H_i(W; \mathbb{R})$. Let $\mathfrak{o}$ denote this chosen orientation. Provide each vector space $H_i(W; \mathbb{R})$ with a reference basis $\mathbf{h}^i_{\mathbb{R}}$ such that the basis $\left\{\mathbf{h}^0_{\mathbb{R}}, \ldots, \mathbf{h}^{\dim W}_{\mathbb{R}}\right\}$ of $H_*(W; \mathbb{R})$ is positively oriented with respect to $\mathfrak{o}$. Compute the sign–determined Reidemeister torsion $\mathrm{Tor}(C_*(W; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}}) \in \mathbb{R}^*$ of the resulting based and homology based chain complex and consider its sign

$$\tau_0 = \mathrm{sgn}\left(\mathrm{Tor}(C_*(W; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})\right) \in \{\pm 1\}.$$

We define the *sign–refined twisted Reidemeister torsion* of W (with respect to $\mathbf{h}^*$ and $\mathfrak{o}$) to be

$$\tau_0 \cdot \mathrm{Tor}(C_*(W; V_\rho), \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{F}^*. \tag{2}$$

This definition only depends on the combinatorial class of W, the conjugacy class of ρ, the choices of $\mathbf{c}^*$, $\mathbf{h}^*$ and the homology orientation $\mathfrak{o}$. It is independent of the orthonormal basis of V and of the choice of the positively oriented basis of $H_*(W; \mathbb{R})$. Moreover, it is independent of the order and orientation of the cells (because they appear twice).

Remark 1. The torsion (2) depends on the choice of the lifts $\tilde{e}^i_j$ under the action of $\pi_1(W)$ by ρ. The effect of different lift of a cell is expressed as the determinant of $\rho(\gamma)$ for some γ in $\pi_1(W)$. To avoid this problem, we often use representations into $\mathrm{SL}(V)$ in applications.

Remark 2. In particular, if the Euler characteristic $\chi(W)$ is zero, then we can use any basis of V because the effect of change of bases is the determinant of base change matrix with power of $\chi(W)$.

One can prove that the sign–refined Reidemeister torsion is invariant under cellular subdivision, homeomorphism and simple homotopy equivalences. In fact, it is precisely the sign $(-1)^{|C_*|}$ in Equation (1) which ensures all these important invariance properties to hold (see [Tur02]).

2.2. **The Multiplicativity Lemma for torsions.** In this section, we briefly review the Multiplicativity Lemma for Reidemeister torsions (with sign).

First, we review *compatible bases* of vector spaces. Let $0 \to E' \xrightarrow{i} E \xrightarrow{j} E'' \to 0$ be a short exact sequence of finite dimensional vector spaces and let s denotes a section of j. Thus, $i \oplus s \colon E' \oplus E'' \to E$ is an isomorphism. We equip the three vector spaces E', E and E'' respectively with the following three bases : $\mathbf{b}' = \left(b'_1, \ldots, b'_p\right)$, $\mathbf{b} = (b_1, \ldots, b_n)$, and $\mathbf{b}'' = \left(b''_1, \ldots, b''_q\right)$. With such notation, one has $n = p + q$, and we say that the bases $\mathbf{b}'$, $\mathbf{b}$ and $\mathbf{b}''$ are compatible if the isomorphism $i \oplus s$:

$E' \oplus E'' \to E$ has determinant 1 in the bases $\mathbf{b}' \cup \mathbf{b}'' = \left(b'_1, \ldots, b'_p, b''_1, \ldots, b''_q\right)$ of $E' \oplus E''$ and $\mathbf{b}$ of E. If it is the case, we write $\mathbf{b} \sim \mathbf{b}' \cup \mathbf{b}''$.

We review the multiplicativity property of the Reidemeister torsion (with sign).

Multiplicativity Lemma. *Let*

$$0 \to C'_* \to C_* \to C''_* \to 0 \tag{3}$$

be an exact sequence of chain complexes. Assume that C'_*, C_* and C''_* are based and homology based. For all i, let ${\mathbf{c}'}^i$, $\mathbf{c}^i$ and ${\mathbf{c}''}^i$ denote the reference bases of C'_i, C_i and C''_i respectively. Associated to (3) is the long sequence in homology

$$\cdots \to H_i(C'_*) \to H_i(C_*) \to H_i(C''_*) \to H_{i-1}(C'_*) \to \cdots$$

Let $\mathcal{H}_*$ denote this acyclic chain complex and base $\mathcal{H}_{3i+2} = H_i(C'_*)$, $\mathcal{H}_{3i+1} = H_i(C_*)$ and $\mathcal{H}_{3i} = H_i(C''_*)$ with the reference bases of $H_i(C'_*)$, $H_i(C_*)$ and $H_i(C''_*)$ respectively. If for all i, the bases ${\mathbf{c}'}^i$, $\mathbf{c}^i$ and ${\mathbf{c}''}^i$ are compatible, i.e. $\mathbf{c}^i \sim {\mathbf{c}'}^i \cup {\mathbf{c}''}^i$, *then the torsion $\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ is expressed as*

$$\begin{aligned}\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{\alpha(C'_*, C''_*) + \varepsilon(C'_*, C_*, C''_*)} &\cdot \mathrm{Tor}(C'_*, {\mathbf{c}'}^*, {\mathbf{h}'}^*) \\ &\cdot \mathrm{Tor}(C''_*, {\mathbf{c}''}^*, {\mathbf{h}''}^*) \cdot \mathrm{Tor}(\mathcal{H}_*, \{\mathbf{h}^*, {\mathbf{h}'}^*, {\mathbf{h}''}^*\}, \emptyset)\end{aligned}$$

where

$$\alpha(C'_*, C''_*) = \sum_{i \geqslant 0} \alpha_{i-1}(C'_*)\alpha_i(C''_*) \in \mathbb{Z}/2\mathbb{Z}$$

and

$$\varepsilon(C'_*, C_*, C''_*) = \sum_{i \geqslant 0} \big((\beta_i(C_*) + 1)(\beta_i(C'_*) + \beta_i(C''_*)) + \beta_{i-1}(C'_*)\beta_i(C''_*)\big) \in \mathbb{Z}/2\mathbb{Z}.$$

The proof is a careful computation based on linear algebra, we refer to [Tur86, Lemma 3.4.2] and [Mil66, Theorem 3.2]. This lemma appears to be a very powerful tool for computing Reidemeister torsions. It will be used all over this paper.

3. Definition of the polynomial torsion

In this section we define the polynomial torsion. This gives a point of view from the Reidemeister torsion to polynomial invariants of topological space.

Hereafter M denotes a compact and connected three–dimensional manifold such that its boundary ∂M is empty or a disjoint union of b two–dimensional tori:

$$\partial M = T_1^2 \cup \ldots \cup T_b^2.$$

In the sequel, we denote by V a vector space over $\mathbb{C}$ and by ρ a representation of $\pi_1(M)$ into $Aut(V)$, and such that $\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(M)$.

Next we introduce a twisted chain complex with some variables. It will be done by using a $\mathbb{Z}[\pi_1(M)]$–module with variables to define a new twisted chain complex. We regard $\mathbb{Z}^n$ as the multiplicative group generated by n variables $t_1, \ldots, t_n$, *i.e.*,

$$\mathbb{Z}^n = \left\langle t_1, \ldots, t_n \,|\, t_i t_j = t_j t_i \; (\forall i, j)\right\rangle$$

and consider a surjective homomorphism $\varphi\colon \pi_1(W) \to \mathbb{Z}^n$. We often abbreviate the n variables $(t_1, \ldots, t_n)$ to $\boldsymbol{t}$ and the rational functions $\mathbb{C}(t_1, \ldots, t_n)$ to $\mathbb{C}(\boldsymbol{t})$.

When we consider the right action of $\pi_1(M)$ on $V(\boldsymbol{t}) = \mathbb{C}(\boldsymbol{t}) \otimes V$ by the tensor representation

$$\varphi \otimes \rho^{-1}\colon \pi_1(M) \to Aut(V(\boldsymbol{t})), \quad \gamma \mapsto \varphi(\gamma) \otimes \rho^{-1}(\gamma),$$

we have the associated twisted chain $C_*(M; V_\rho(\boldsymbol{t}))$ given by:

$$C_*(M; V_\rho(\boldsymbol{t})) = V_\rho(\boldsymbol{t}) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M}; \mathbb{Z})$$

where $f \otimes v \otimes \gamma \cdot \sigma$ is identified with $f\varphi(\gamma) \otimes \rho(\gamma)^{-1}(v) \otimes \sigma$ for any $\gamma \in \pi_1(M)$, $\sigma \in C_*(\widetilde{M};\mathbb{Z})$, $v \in V$ and $f \in \mathbb{C}(\boldsymbol{t})$. We call this complex *the $V_\rho(\boldsymbol{t})$-twisted chain complex* of M.

Definition 1. Fix a homology orientation on M. If $C_*(M;V_\rho(\boldsymbol{t}))$ is acyclic, then the sign–refined Reidemeister torsion of $C_*(M;V_\rho(\boldsymbol{t}))$:

$$\Delta_M^{\varphi\otimes\rho}(t_1,\ldots,t_n) = \tau_0 \cdot \mathrm{Tor}(C_*(M;V_\rho(\boldsymbol{t})),\mathbf{c}^*,\emptyset) \in \mathbb{C}(t_1,\ldots,t_n) \setminus \{0\}.$$

is called the *polynomial torsion* of M.

Observe that the sign–refined Reidemeister torsion $\Delta_M^{\varphi\otimes\rho}(t_1,\ldots,t_n)$ is determined up to a factor $t_1^{m_1}\cdots t_n^{m_n}$ such as the classical Alexander polynomial.

Example 1 (J. Milnor [Mil68a], P. Kirk & C. Livingston [KL99])**.** Suppose that M is the knot exterior $E_K = S^3 \setminus N(K)$ of a knot K in S^3.

If the representation $\rho \in \mathrm{Hom}(\pi_1(E_K);\mathbb{Q})$ is the trivial homomorphism and φ is the abelianization of $\pi_1(E_K)$, *i.e.*, $\varphi : \pi_1(E_K) \to H_1(E_K;\mathbb{Z}) \simeq \langle t\rangle$, then the twisted chain complex $C_*(E_K;\mathbb{Q}(t)_\rho)$ is acyclic and the Reidemeister torsion $\Delta_{E_K}^{\varphi\otimes\rho}(t)$ is expressed as a rational function which is the Alexander polynomial $\Delta_K(t)$ divided by $(t-1)$ (see [Tur02]).

In the case of a one–dimensional representation $\rho\colon \pi_1(E_K) \to \mathrm{GL}(1;\mathbb{C}) = \mathbb{C}\setminus\{0\}$, such that $\rho(\mu) = \xi$, the twisted chain complex $C_*(E_K;\mathbb{C}(t)_\rho)$ is also acyclic, and the Reidemeister torsion $\Delta_{E_K}^{\varphi\otimes\rho}(t)$ is given by (up to $\pm\xi t^k$, $k \in \mathbb{Z}$):

$$\Delta_{E_K}^{\varphi\otimes\rho}(t) = \frac{\Delta_K(\xi t)}{\xi t - 1},$$

where $\Delta_K(t)$ is the Alexander polynomial.

Example 2. Suppose now that M is the link exterior $E_L = S^3 \setminus N(L)$ of a link L in S^3. We suppose that L has 2 or more components and n denotes the number of that components. We denote by μ_i the meridian of the i-th component. Consider the abelianization $\varphi\colon \pi_1(E_L) \to \mathbb{Z}^n$ defined by $\varphi(\mu_i) = t_i$. Let $\rho\colon \pi_1(E_L) \to \mathrm{GL}(1;\mathbb{C}) = \mathbb{C} \setminus \{0\}$ be the one–dimensional representation such that $\rho(\mu_i) = \xi_i$. Then the twisted chain complex $C_*(E_L;\mathbb{C}(\boldsymbol{t})_\rho)$ is acyclic and the Reidemeister torsion $\Delta_{E_L}^{\varphi\otimes\rho}(t_1,\ldots,t_n)$ is given by (up to $\pm(\xi_1 t_1)^{k_1}\cdots(\xi_n t_n)^{k_n}$, $k_i \in \mathbb{Z}$):

$$\Delta_{E_L}^{\varphi\otimes\rho}(t_1,\ldots,t_n) = \Delta_L(\xi_1 t_1,\ldots,\xi_n t_n)$$

where $\Delta_L(t_1,\ldots,t_n)$ is the Alexander polynomial of L.

4. Torsion for finite sheeted abelian coverings

4.1. **Statement of the result.** Let $\hat{M}$ be a finite sheeted abelian covering of M. We denote by p the induced homomorphism from $\pi_1(\hat{M})$ to $\pi_1(M)$ by the covering map $\hat{M} \to M$. The associated deck transformation group is a finite abelian group G of order $|G|$. We endow the manifolds M and $\hat{M}$ with some arbitrary homology orientations.

We have the following exact sequence:

$$1 \to \pi_1(\hat{M}) \xrightarrow{p} \pi_1(M) \xrightarrow{\pi} G \to 1. \tag{4}$$

When we consider the polynomial torsion for $\hat{M}$, we use the pull–back of homomorphisms of $\pi_1(M)$ as homomorphisms of $\pi_1(\hat{M})$. We denote by φ the surjective homomorphism from $\pi_1(M)$ to $\mathbb{Z}^n$, which mainly means that the quotient homomorphism $\pi_1(M) \to H_1(M;\mathbb{Z})/\mathrm{Tors}$, and by $\widehat{\varphi}$ the pull-back by p:

$$\begin{tikzcd}
\pi_1(\hat{M}) \arrow[r, "p_*"] \arrow[dr, "\widehat{\varphi}"'] & \pi_1(M) \arrow[d, "\varphi"] \\
 & \mathbb{Z}^n.
\end{tikzcd}$$

The above sequence (4) also induces homomorphism from $H_1(\hat{M};\mathbb{Z})$ into $H_1(M;\mathbb{Z})$, whose image is of maximal rank in $H_1(M;\mathbb{Z})$. Since the homomorphism φ factors through $H_1(M;\mathbb{Z})$, the image of $\widehat{\varphi}$ is also of maximal rank in $\mathbb{Z}^n$. Thus we can regard $\widehat{\varphi}$ as a surjective homomorphism from $\pi_1(\hat{M})$ to $\mathrm{im}\,\widehat{\varphi} \simeq \mathbb{Z}^n$. Similarly we use the symbol $\widehat{\rho}$ for the pull–back of $\rho\colon \pi_1(M) \to Aut(V)$ by p, where V is a vector space. For homomorphisms of the quotient group $G \simeq \pi_1(M)/\pi_1(\hat{M})$, we use the Pontrjagin dual of G which is the set of all representations $\xi\colon G \to \mathbb{C}^*$ from G to non–zero complex numbers. Let $\hat{G}$ denote this space.

We give the statement of the polynomial torsion for abelian coverings via that of the base manifold.

Theorem 1. *With the above notation, we suppose that the twisted chain complex $C_*(M;V_\rho)$ is acyclic. Then the twisted chain complex $C_*(\hat{M};V_{\widehat{\rho}})$ is also acyclic and the polynomial torsion is expressed as*

$$\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(\boldsymbol{t}) = \epsilon \cdot \prod_{\xi\in\hat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}(\boldsymbol{t}) \tag{5}$$

where ϵ is a sign equal to $\tau_0(\hat{M}) \cdot \tau_0(M)^{|G|}$.

Remark 3. When the order $|G|$ is odd, the sign term ϵ in Equation (5) is described by the homology orientations of M and $\hat{M}$ and by the torsion of the covering map. We will discuss how to determine ϵ in the subsequent section of the proof of Theorem 1.

Remark 4 (Explanation of Formula (5) with variables). Formula (5) can be written concretely as follows:

$$\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\widehat{\rho}}(t_1,\ldots,t_n) = \epsilon \cdot \prod_{\xi\in\hat{G}} \Delta_M^{\varphi\otimes\rho}(t_1\xi(t_1),\ldots,t_n\xi(t_n))$$

In the special case where $n=1$, $G=\mathbb{Z}/m\mathbb{Z}$ and $\hat{M}$ is the m–fold cyclic covering M_m of M, then we have that $\xi(t) = e^{2\pi k\sqrt{-1}/m}$. Hence we have the following covering formula for the polynomial torsion.

Corollary 1. *Suppose that $\varphi(\pi_1(M)) = \langle t\rangle$ and $\widehat{\varphi}(\pi_1(M_m)) = \langle s\rangle \subset \langle t\rangle$, where we suppose that $s = t^m$. We have*

$$\Delta_{M_m}^{\widehat{\varphi}\otimes\widehat{\rho}}(s) = \Delta_{M_m}^{\widehat{\varphi}\otimes\widehat{\rho}}(t^m) = \epsilon \cdot \prod_{k=0}^{m-1} \Delta_M^{\varphi\otimes\rho}(e^{2\pi k\sqrt{-1}/m}t).$$

The torsion $\Delta_{M_m}^{\widehat{\varphi}\otimes\rho}(t^m)$ in Corollary 1 can be regarded as a kind of the total twisted Alexander polynomial introduced in [SW]. Hirasawa and Murasugi [HM07] worked on the total twisted Alexander polynomial for abelian representations as in Example 1 and they observed the similar formula as in Corollary 1 in terms of the total Alexander polynomial and the Alexander polynomial of a knot in the cyclic branched coverings over S^3.

4.2. **Proof of Theorem 1.** We use the same notation as in Remark 4.

First observe the following key facts:

(1) the universal cover $\widetilde{M}$ of M is also the one of $\hat{M}$,

(2) the torsion $\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\rho}$ is computed using the twisted complex

$$V_\rho(t_1,\ldots,t_n)\otimes_{\mathbb{Z}[\pi_1(\hat{M})]} C_*(\widetilde{M};\mathbb{Z}),$$

(3) whereas the torsion $\Delta_M^{\varphi\otimes\rho}$ is computed using

$$V_\rho(t_1,\ldots,t_n)\otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}),$$

(4) finally there is a natural action of G on $V_\rho(t_1,\ldots,t_n)\otimes_{\mathbb{Z}[\pi_1(\hat{M})]} C_*(\widetilde{M};\mathbb{Z})$. This action is defined as follows: for $x\in V_\rho(t_1,\ldots,t_m)$, $c\in C_*(\widetilde{M};\mathbb{Z})$ and $g\in G$, choose $\gamma\in\pi_1(M)$ such that $\pi(\gamma)=g$ the action is given by:

$$g\star(x\otimes_{\pi_1(\hat{M})} c) = x\gamma^{-1}\otimes_{\pi_1(\hat{M})}\gamma c. \tag{6}$$

Further observe that, since for any lift γ of g, γ is not contained in $p_*(\pi_1(\hat{M}))$, we can not reduce the right hand side in Equation (6).

Lemma 2. *The G-action in Equation* (6) *is independent of the choice of lifts of g.*

Proof. Take another lift γ' of $g\in G$ and define $\star'$ as follows:

$$g\star'(x\otimes_{\pi_1(\hat{M})} c) = x\gamma'^{-1}\otimes_{\pi_1(\hat{M})}\gamma' c.$$

Observe that $\gamma^{-1}\gamma'$ is contained in $p_*(\pi_1(\hat{M}))$. Hence the action $g^{-1}\star(g\star'(x\otimes c))$ turns into $x\otimes c$ from the following calculation:

$$\begin{aligned} g^{-1}\star(g\star'(x\otimes_{\pi_1(\hat{M})} c)) &= g^{-1}\star(x\gamma'^{-1}\otimes_{\pi_1(\hat{M})}\gamma' c)\\ &= x(\gamma^{-1}\gamma')^{-1}\otimes_{\pi_1(\hat{M})}(\gamma^{-1}\gamma')c\\ &= x\otimes_{\pi_1(\hat{M})} c \end{aligned}$$

This gives the equality $g\star(x\otimes_{\pi_1(\hat{M})} c) = g\star'(x\otimes_{\pi_1(\hat{M})} c)$. □

The proof of Theorem 1 is based on the following technical lemma.

Lemma 3. *The map*

$$\Phi\colon V_\rho(\boldsymbol{t})\otimes_{\mathbb{Z}[\pi_1(\hat{M})]} C_*(\widetilde{M};\mathbb{Z})\to (V_\rho(\boldsymbol{t})\otimes_{\mathbb{C}}\mathbb{C}[G])\otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}) \tag{7}$$

given by

$$\Phi(x\otimes_{\pi_1(\hat{M})} c) = (x\otimes 1)\otimes c$$

is an isomorphism of complexes of $\mathbb{C}[G]$-modules.

The action of G on $(V_\rho(\boldsymbol{t})\otimes_{\mathbb{C}}\mathbb{C}[G])\otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$ is given by

$$g\cdot(x\otimes g'\otimes c) = x\otimes gg'\otimes c.$$

Proof of Lemma 3. We first observe that Φ is a well–defined chain map of $\mathbb{C}$–vector spaces since $\pi_1(\hat{M})$ is a normal subgroup of $\pi_1(M)$. Let us check that Φ is G-equivalent. Let $x\in V_\rho(\boldsymbol{t})$, $c\in C_*(\widetilde{M};\mathbb{Z})$, $g\in G$ and choose $\gamma\in\pi_1(M)$ such that $\pi(\gamma)=g$, then we have the following G-equivalence:

$$\begin{aligned} \Phi(g\star(x\otimes_{\pi_1(\hat{M})} c)) &= \Phi(x\gamma^{-1}\otimes_{\pi_1(\hat{M})}\gamma c)\\ &= (x\gamma^{-1}\otimes 1)\otimes\gamma c\\ &= (x\gamma^{-1}\gamma\otimes g)\otimes c\\ &= (x\otimes g)\otimes c\\ &= g\cdot\Phi(x\otimes_{\pi_1(\hat{M})} c). \end{aligned}$$

Since $\partial(x \otimes_{\pi_1(\hat{M})} c) = x \otimes_{\pi_1(\hat{M})} \partial c = \sum_j n_j (x \otimes_{\pi_1(\hat{M})} \gamma_j e_j^{i-1})$, $n_j \in \mathbb{Z}$ and $\gamma_i \in \pi_1(M)$, we must check the compatibility of Φ with the tensor products in both chain complexes. The G-equivalence guarantees that Φ is well-defined as a chain map under the actions of $\pi_1(\hat{M})$ and $\pi_1(M)$. For every $g \in G$, let $\gamma_g \in \pi_1(M)$ be a lift of g, *i.e.*, $\pi(\gamma_g) = g$. Set $\gamma \in \pi_1(M)$ as $\pi(\gamma) = g$. This means that $\gamma_g^{-1}\gamma \in \pi_1(\hat{M})$. Then the tensor product $x \otimes_{\pi_1(\hat{M})} \gamma e_j^{i-1}$ turns into

$$\begin{aligned} x \otimes_{\pi_1(\hat{M})} \gamma e_j^{i-1} &= g g^{-1} \star (x \otimes_{\pi_1(\hat{M})} \gamma e_j^{i-1}) \\ &= g \star (x \gamma_g \otimes_{\pi_1(\hat{M})} \gamma_g^{-1} \gamma e_j^{i-1}) \\ &= g \star (x \gamma \otimes_{\pi_1(\hat{M})} e_j^{i-1}). \end{aligned}$$

The map Φ sends $x \otimes_{\pi_1(\hat{M})} \gamma e_j^{i-1}$ to $(x \otimes 1) \otimes \gamma e_j^{i-1}$. On the other hand, these chains are also expressed as $x \otimes_{\pi_1(\hat{M})} \gamma e_j^{i-1} = g \star (x\gamma \otimes_{\pi_1(\hat{M})} e_j^{i-1})$ and $(x \otimes 1) \otimes \gamma e_j^{i-1} = (x\gamma \otimes g) \otimes e_j^{i-1}$. By the G-equivalence, we can see that the map Φ is well-defined.

To prove that Φ is an isomorphism we look at the geometric basis for the complexes and prove that Φ maps geometric bases to geometric bases. Let $\{e_j^i\}_j$ be the i-dimensional cells of M and choose $\{x_1, \ldots, x_n\}$ a $\mathbb{C}$-basis of the vector space V. The corresponding i-dimensional cells of $\hat{M}$ are given by $\{g\hat{e}_j^i \mid 1 \leqslant j \leqslant n_i, g \in G\}$. We denote by $\{\widetilde{e}_j^i \mid 1 \leqslant j \leqslant n_i\}$ the i–dimensional cells of the universal cover $\widetilde{M}$.

For every $g \in G$, let $\gamma_g \in \pi_1(M)$ be such that $\pi(\gamma_g) = g$. With such notation, one can observe that

$$\hat{\mathbf{c}}^* = \cup_{i \geqslant 0} \{x_k \otimes_{\pi_1(\hat{M})} \gamma_g \widetilde{e}_j^i \mid 1 \leqslant j \leqslant n_i, 1 \leqslant k \leqslant n, g \in G\} \tag{8}$$

is a $\mathbb{C}[\boldsymbol{t}]$-basis for $V_rho(\boldsymbol{t}) \otimes_{\mathbb{Z}[\pi_1(\hat{M})]} C_*(\widetilde{M};\mathbb{Z})$. Since the action of every γ_g is invertible on $\mathfrak{sl}_2(\mathbb{C})$, the set

$$\{g \star (x_k \otimes_{\pi_1(\hat{M})} \widetilde{e}_j^i) = x_k \gamma_g^{-1} \otimes_{\pi_1(\hat{M})} \gamma_g \widetilde{e}_j^i \mid 1 \leqslant j \leqslant n_i, 1 \leqslant k \leqslant n, g \in G\} \tag{9}$$

also gives another basis of each $V_\rho(\boldsymbol{t}) \otimes_{\mathbb{Z}[\pi_1(\hat{M})]} C_i(\widetilde{M};\mathbb{Z})$. Observe that:

$$\Phi(g \star (x_k \otimes_{\pi_1(\hat{M})} \widetilde{e}_j^i)) = (x_k \otimes g) \otimes \widetilde{e}_j^i.$$

Further observe that $\{x_k \otimes g \mid 1 \leqslant k \leqslant n, g \in G\}$ is a $\mathbb{C}[\boldsymbol{t}]$-basis for $V_\rho[\boldsymbol{t}] \otimes_{\mathbb{C}} \mathbb{C}[G]$, thus we deduce that

$$\mathbf{c}_G^* = \cup_{i \geqslant 0} \{(x_k \otimes g) \otimes \widetilde{e}_j^i \mid 1 \leqslant j \leqslant n_i, 1 \leqslant k \leqslant 3, g \in G\} \tag{10}$$

is a basis of $(V_\rho[\boldsymbol{t}] \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$, which achieve the proof of the lemma. □

Remark 5. In particular, Lemma 3 implies that the complex

$$(V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$$

is acyclic.

The basis in Equation (8) introduced in the proof of Lemma 3 is the geometric basis used to compute the polynomial torsion $\Delta_{\hat{M}}^{\widehat{\varphi} \otimes \rho}$. One can observe that the transition matrix from the basis in Equation (8) to the basis in Equation (9) has determinant 1. Thus, $\Delta_{\hat{M}}^{\widehat{\varphi} \otimes \rho}$ can also be computed using the basis in Equation (9). Finally observe that Φ maps the basis in Equation (9) to the geometric basis in Equation (10), thus

$$\Delta_{\hat{M}}^{\widehat{\varphi} \otimes \rho} = \tau_0(\hat{M}) \cdot \mathrm{Tor}((V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \hat{\mathbf{c}}^*, \emptyset).$$

Now, we want to compute the torsion of $(V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z})$ in terms of polynomial torsions of M. To this end we use the decomposition along orthogonal idempotents of the group ring $\mathbb{C}[G]$, see [Ser78] for details. Associated to $\xi \in \hat{G}$, we define

$$f_\xi = \frac{1}{|G|}\sum_{g\in G} \xi(g^{-1})g \in \mathbb{C}[G].$$

The properties of f_ξ are the following:

$$f_\xi^2 = f_\xi, \quad f_\xi f_{\xi'} = 0 \,(\text{ if } \xi \neq \xi'), \quad \sum_{\xi\in\hat{G}} f_\xi = 1$$

and

$$g \cdot f_\xi = \xi(g) f_\xi, \text{ for all } g \in G.$$

We have the following $\mathbb{C}[G]$-modules decomposition of the group ring as a direct sum according to its representations:

$$\mathbb{C}[G] = \bigoplus_{\xi\in\hat{G}} \mathbb{C}[f_\xi]. \tag{11}$$

Here each factor is the 1-dimensional $\mathbb{C}$-vector space which is isomorphic to the $\mathbb{C}[G]$-module associated to $\xi\colon G \to \mathbb{C}^*$.

Following [Por04, Section 3], corresponding to the decomposition in Equation (11) we have a decomposition of complexes of $\mathbb{C}[G]$-modules:

$$(V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}) = \bigoplus_{\xi\in\hat{G}} (V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}).$$

The geometric basis in Equation (10) induces a basis compatible with the decomposition in Equation (11) by replacing $\{g \,|\, g \in G\}$ by $\{f_\xi \,|\, \xi \in \hat{G}\}$. The change of basis cancels when we compute the torsion because Euler characteristic is zero, see [Por04, Lemma 5.2]. And thus decomposition in Equation (11) implies that (in the natural geometric bases):

$$\begin{aligned}\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\rho} &= \tau_0(\hat{M}) \cdot \mathrm{Tor}((V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*_G, \emptyset)\\ &= \tau_0(\hat{M}) \cdot \prod_{\xi\in\hat{G}} \mathrm{Tor}((V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*, \emptyset).\end{aligned} \tag{12}$$

Each factor in the right hand side is related to the polynomial torsion of M and its relation is given by the following claim.

Lemma 4. *We have:*

$$\Delta_M^{(\varphi\otimes\rho)\otimes\xi} = \tau_0(M) \cdot \mathrm{Tor}((V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathbf{c}^*, \emptyset).$$

Proof of Lemma 4. One can observe that, as a $\mathbb{Z}[\pi_1(M)]$-module, $V_\rho(\boldsymbol{t}) \otimes_{\mathbb{C}} \mathbb{C}[f_\xi]$ is isomorphic to $V_\rho(\boldsymbol{t})$ simply by replacing the action $\varphi\otimes\rho$ by $(\varphi\otimes\rho)\otimes\xi$. This proves the equality of torsions. □

Proof of Theorem 1. Combining Equation (12) and Lemma 4, we obtain

$$\Delta_{\hat{M}}^{\widehat{\varphi}\otimes\rho} = \tau_0(\hat{M}) \cdot \tau_0(M)^{|G|} \cdot \prod_{\xi\in\hat{G}} \Delta_M^{(\varphi\otimes\rho)\otimes\xi}$$

which achieves the proof of Formula (5). □

4.3. **The sign term in covering formula.** When the order of the covering is odd, the sign term ϵ in Equation (5) is the product of two sign–torsions $\tau_0(M)$ and $\tau_0(\hat{M})$. Here, we discuss what this sign term depends on. Actually, this sign term is determined by the torsion of the covering map p and depends on the homology orientations of M and $\hat{M}$ we have chosen for the computation.

The torsion of the covering map p is by definition given by the torsion of the pair formed by the mapping cylinder M_p of p and the covering $\hat{M}$. Here the *mapping cylinder* of a continuous map $p : \hat{M} \to M$ is defined as

$$M_p = ((\hat{M} \times [0,1]) \cup M)/((x,1) \sim p(x)).$$

Since the natural projection $(\hat{M} \times [0,1]) \cup M \to M_p$ is an embedding on $\hat{M} \times 0$ and M, we regard $\hat{M} = \hat{M} \times 0$ and M as subspaces of M_p.

Remark 6. Note that M is a retract of M_p, for example, a deformation retract is given by $h : M_p \to M$, $h(x,t) = p(x)$ for $x \in \hat{M}$, $t < 1$, $h(y) = y$ for $y \in M$. In particular, the homology groups $H_*(M_p;\mathbb{R})$ can be identified with $H_*(M;\mathbb{R})$. This and the commutative diagram

$$\begin{tikzcd} \hat{M} \arrow[r, "i_{\hat{M}}"] \arrow[dr, "p"'] & M_p \arrow[d, "h"] \\ & M \end{tikzcd} \tag{13}$$

show that the inclusion $i_{\hat{M}} : \hat{M} \hookrightarrow M_p$ is a homotopy equivalence if and only if p is a homotopy equivalence.

By the abelianization of the sequence (4), we can also see that the image of $H_1(\hat{M};\mathbb{Z}) \to H_1(M;\mathbb{Z})$ has maximal rank. Thus the induced homomorphism $\bar{p} : H_1(\hat{M};\mathbb{R}) \to H_1(M;\mathbb{R})$ is onto. Actually, since Euler characteristics of $\hat{M}$ and M are zero, the homomorphisms $\bar{p} : H_i(\hat{M};\mathbb{R}) \to H_i(M;\mathbb{R})$ induced by p are onto for all i.

We can describe the difference between $H_i(\hat{M};\mathbb{R})$ and $H_i(M;\mathbb{R})$ by using the homology of the pair $(M_p, \hat{M})$.

Lemma 5. *The long exact sequence for* $(M_p, \hat{M})$ *is splitting into the following short exact sequence:*

$$0 \to H_{i+1}(M_p, \hat{M};\mathbb{R}) \to H_i(\hat{M};\mathbb{R}) \to H_i(M_p;\mathbb{R}) = H_i(M;\mathbb{R}) \to 0 \tag{14}$$

for each i.

Proof. The homology long exact sequence for the pair $(M_p, \hat{M})$ is given by

$$\cdots \to H_{i+1}(M_p, \hat{M};\mathbb{R}) \to H_i(\hat{M};\mathbb{R}) \xrightarrow{(i_{\hat{M}})_*} H_i(M_p;\mathbb{R}) \to \cdots$$

and the homomorphism $(i_{\hat{M}})_*$ induced by the usual inclusion is in fact onto.

Since the map p and h induce a surjection and an isomorphism between homology groups in the commutative diagram (13), the inclusion $i_{\hat{M}}$ induces a surjective homomorphism $(i_{\hat{M}})_*$. Hence the long exact sequence splits into the short exact sequences. □

From now on, we identify $H_i(M_p;\mathbb{R})$ with $H_i(M;\mathbb{R})$ and endow M_p with the homology orientation of the one of M.

We denote by $\mathbf{h}^i_{\mathbb{R}}$ (resp. $\hat{\mathbf{h}}^i_{\mathbb{R}}$) a basis of the homology group $H_i(M;\mathbb{R})$ (resp. $H_i(\hat{M};\mathbb{R})$) which is compatible with the given homology orientations. The short exact sequence (14) gives a basis ${\mathbf{h}'_{\mathbb{R}}}^{i+1}$ of $H_{i+1}(M_p, \hat{M};\mathbb{R})$ which is compatible with

$\mathbf{h}^i_{\mathbb{R}}$ and $\hat{\mathbf{h}}^i_{\mathbb{R}}$. More precisely, if σ is a section of $(i_{\hat{M}})_*$, then the determinant of the transition matrix of ${\mathbf{h}'}^{i+1}_{\mathbb{R}}\sigma(\mathbf{h}^i_{\mathbb{R}})$ to $\hat{\mathbf{h}}^i_{\mathbb{R}}$ is 1. The set ${\mathbf{h}'_{\mathbb{R}}}^* = \cup_{i\geqslant 0}{\mathbf{h}'_{\mathbb{R}}}^i$ gives a homology orientation of $C_*(M_p, \hat{M}; \mathbb{R})$ which is called *the induced homology orientation from those of M and $\hat{M}$*.

Applying Multiplicativity Lemma to the short exact sequence:

$$0 \to C_*(\hat{M}; \mathbb{R}) \to C_*(M_p; \mathbb{R}) \to C_*(M_p, \hat{M}; \mathbb{R}) \to 0, \tag{15}$$

we have the following result.

Lemma 6. *Let $\mathbf{h}^i_{\mathbb{R}}$ and $\hat{\mathbf{h}}^i_{\mathbb{R}}$ respectively be basis of $H_i(M; \mathbb{R})$ and $H_i(\hat{M}; \mathbb{R})$. If ${\mathbf{h}'}^{i+1}_{\mathbb{R}}$ is the induced basis of $H_{i+1}(M_p, \hat{M}; \mathbb{R})$ obtained from $\mathbf{h}^i_{\mathbb{R}}$ and $\hat{\mathbf{h}}^i_{\mathbb{R}}$, then the torsion $\mathrm{Tor}(C_*(M_p; \mathbb{R}), \hat{\mathbf{c}}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}})$ is expressed as*

$$\begin{aligned}
&\mathrm{Tor}(C_*(M_p; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})\\
&= (-1)^{\alpha(M_p,\hat{M})+\varepsilon(M_p,\hat{M})} \cdot \mathrm{Tor}(C_*(\hat{M}; \mathbb{R}), \hat{\mathbf{c}}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}) \cdot \mathrm{Tor}(C_*(M_p, \hat{M}; \mathbb{R}), {\mathbf{c}'}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*)
\end{aligned}$$

where $\alpha(M_p, \hat{M})$ and $\varepsilon(M_p, \hat{M})$ denote

$$\alpha(C_*(\hat{M}; \mathbb{R}), C_*(M_p, \hat{M}; \mathbb{R})) \quad \textit{and} \quad \varepsilon(C_*(\hat{M}; \mathbb{R}), C_*(M_p; \mathbb{R}), C_*(M_p, \hat{M}; \mathbb{R}))$$

for the short exact sequence of complexes (15)*.*

Since M is a deformation retract of the mapping cylinder M_p, we can prove that the torsion $\mathrm{Tor}(C_*(M_p; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})$ is equal to $\mathrm{Tor}(C_*(M; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})$ up to a sign term.

Lemma 7. *The torsion of the mapping cylinder M_p is given by :*

$$\mathrm{Tor}(C_*(M_p; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}) = (-1)^{\alpha(M_p,M)} \cdot \mathrm{Tor}(C_*(M; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}}) \tag{16}$$

where $\alpha(M_p, M)$ denotes $\alpha(C_*(M; \mathbb{R}), C_*(M_p, M; \mathbb{R}))$.

Proof. If we apply Multiplicativity Lemma to

$$0 \to C_*(M; \mathbb{R}) \to C_*(M_p; \mathbb{R}) \to C_*(M_p, M; \mathbb{R}) \to 0,$$

then we obtain the following equality

$$\begin{aligned}
\mathrm{Tor}(C_*(M_p; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}) &= (-1)^{\alpha(M_p,M)+\varepsilon(M_p,M)} \cdot \mathrm{Tor}(C_*(M; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})\\
&\quad \cdot \mathrm{Tor}(C_*(M_p, M; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*) \cdot \mathrm{Tor}(\mathcal{H}, \{\mathbf{h}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*\}, \emptyset).
\end{aligned} \tag{17}$$

Observe that $H_i(M_p, M; \mathbb{R}) = 0$ for all i (because $H_i(M_p; \mathbb{R}) \simeq H_i(M; \mathbb{R})$). Hence $\beta_i(C_*(M_p, M; \mathbb{R})) = 0$, for all i. This implies that $\varepsilon(M_p, M) = 0$. Since $H_*(M; \mathbb{R})$ and $H_*(M_p; \mathbb{R})$ are endowed with the same basis, the torsion $\mathrm{Tor}(\mathcal{H}, \{\mathbf{h}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*\}, \emptyset)$ turns equal to 1. It is known that M_p is obtained by some elementary expansions from M. Therefore the torsion of the pair (M_p, M) is also 1 (for more details, we refer the reader to [Tur01, Lemma 8.4]). Substituting these results into Equation (17), we have the Equation (16). □

Lemmas 6 & 7 guarantee the equality among three torsions $\mathrm{Tor}(C_*(M; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}})$, $\mathrm{Tor}(C_*(\hat{M}; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}})$ and $\mathrm{Tor}(C_*(M_p, \hat{M}; \mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*)$ up to a sign. Now, we describe the relation among these three torsions more precisely.

Lemma 8. *We have the following equality in $\mathbb{Z}/2\mathbb{Z}$:*

$$\alpha(M_p, \hat{M}) \equiv \alpha(M_p, M) \mod 2.$$

Proof. By the definition of the mapping cylinder M_p, the cells of M_p are $\{e\}$, $\{e \times [0,1]\}$ and $\{e'\}$, where e runs over cells in $\hat{M}$ and e' runs over cells in M. Thus, the dimension of $C_k(M_p;\mathbb{R})$ is given by

$$\begin{aligned}\dim_{\mathbb{R}} C_k(M_p;\mathbb{R}) &= \dim_{\mathbb{R}} C_k(\hat{M};\mathbb{R}) + \dim_{\mathbb{R}} C_{k-1}(\hat{M};\mathbb{R}) + \dim_{\mathbb{R}} C_k(M;\mathbb{R}) \\ &= |G| \dim_{\mathbb{R}} C_k(M;\mathbb{R}) + |G| \dim_{\mathbb{R}} C_{k-1}(M;\mathbb{R}) + \dim_{\mathbb{R}} C_k(M;\mathbb{R}).\end{aligned}$$

Hence

$$\dim_{\mathbb{R}} C_k(M_p, \hat{M};\mathbb{R}) = |G| \dim_{\mathbb{R}} C_{k-1}(M;\mathbb{R}) + \dim_{\mathbb{R}} C_k(M;\mathbb{R})$$

and

$$\dim_{\mathbb{R}} C_k(M_p, M;\mathbb{R}) = |G| \dim_{\mathbb{R}} C_k(M;\mathbb{R}) + |G| \dim_{\mathbb{R}} C_{k-1}(M;\mathbb{R}).$$

We are now ready to compute $\alpha(M_p, \hat{M})$ as follows:

$$\begin{aligned}\alpha(M_p, \hat{M}) &= \alpha(C_*(\hat{M};\mathbb{R}), C_*(M_p, \hat{M};\mathbb{R})) \\ &= \sum_i \alpha_{i-1}(C_*(\hat{M};\mathbb{R})) \cdot \alpha_i(C_*(M_p, \hat{M};\mathbb{R})) \\ &= |G| \cdot \sum_i |G| \left(\alpha_{i-1}(C_*(M;\mathbb{R}))\right)^2 + \alpha_{i-1}(C_*(M;\mathbb{R})) \cdot \alpha_i(C_*(M;\mathbb{R})).\end{aligned}$$

Similarly we compute $\alpha(M_p, M)$:

$$\alpha(M_p, M) = |G| \cdot \sum_i \left(\alpha_{i-1}(C_*(M;\mathbb{R}))\right)^2 + \alpha_{i-1}(C_*(M;\mathbb{R})) \cdot \alpha_i(C_*(M;\mathbb{R})).$$

Since $|G|^2 \equiv |G| \mod 2$, then $\alpha(M_p, \hat{M})$ coincides with $\alpha(M_p, M) \mod 2$. □

Lemmas 6, 7 and 8, give us the following proposition.

Proposition 9. *If the homology groups $H_*(M;\mathbb{R})$, $H_*(\hat{M};\mathbb{R})$ and $H_*(M_p, M;\mathbb{R})$ are endowed with the same bases as in Lemma 6, then*

$$\begin{aligned}&\mathrm{Tor}(C_*(M;\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}}) \\ &= (-1)^{\varepsilon(M_p,\hat{M})} \cdot \mathrm{Tor}(C_*(\hat{M};\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}) \cdot \mathrm{Tor}(C_*(M_p, \hat{M};\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*).\end{aligned}$$

Moreover, the sign difference between torsions for M and $\hat{M}$ is expressed as

$$\begin{aligned}\epsilon &= \tau_0(M) \cdot \tau_0(\hat{M}) \\ &= \mathrm{sgn}(\mathrm{Tor}(C_*(M;\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \mathbf{h}^*_{\mathbb{R}}))/\mathrm{sgn}(\mathrm{Tor}(C_*(\hat{M};\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}})) \\ &= (-1)^{\varepsilon(M_p,\hat{M})} \cdot \mathrm{sgn}(\mathrm{Tor}(p, \mathbf{h}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}})).\end{aligned}$$

Here $\mathrm{Tor}(p, \mathbf{h}^*_{\mathbb{R}}, \hat{\mathbf{h}}^*_{\mathbb{R}}) = \mathrm{Tor}(C_*(M_p, \hat{M};\mathbb{R}), \mathbf{c}^*_{\mathbb{R}}, {\mathbf{h}'_{\mathbb{R}}}^*)$*, where ${\mathbf{h}'_{\mathbb{R}}}^*$ denotes the basis of $H_*(M_p, \hat{M};\mathbb{R})$ induced by $\mathbf{h}^*_{\mathbb{R}}$ and $\hat{\mathbf{h}}^*_{\mathbb{R}}$.*

Remark 7. If p_* gives an isomorphism between $H_i(\hat{M};\mathbb{R})$ and $H_i(M;\mathbb{R})$ for all i, then $\varepsilon(M_p, \hat{M})$ turns into zero by definition.

4.4. **The case of knots.** As an illustration of Proposition 9, we treat the case where E_K is a knot exterior in this section. Actually, we express the sign term ϵ using Alexander polynomials of E_K and $\hat{E}_K$.

Let $E_K = S^3 \setminus N(K)$ denote the exterior of the knot K in S^3 and $\hat{E}_K$ be an m–fold cyclic cover over E_K where m is supposed to be odd. We moreover assume that the covering map p induces an isomorphism between $H_i(\hat{E}_K;\mathbb{R})$ and $H_i(E_K;\mathbb{R})$, *i.e.*, the n-fold branched cover over S^3 along K is a rational homology three-sphere, *i.e.*, all m-roots of unity are not a root of $\Delta_K(t)$ by Fox formula (see [BZ03, Theorem 8.21]).

Under these conditions, the homology groups $H_*((E_K)_p, \hat{E}_K; \mathbb{R})$ are trivial and the integer $\varepsilon((E_K)_p, \hat{E}_K)$ turns into 0 (see Remark 7). We suppose that $H_*(E_K; \mathbb{R})$ is given by the following bases:

$$H_0(E_K; \mathbb{R}) = \mathbb{R}[\![pt]\!],\ H_1(E_K; \mathbb{R}) = \mathbb{R}[\![\mu]\!],\ H_i(E_K; \mathbb{R}) = 0\,(i \geqslant 2).$$

Here $[\![pt]\!]$ denotes the class of the point and $[\![\mu]\!]$ denotes the class of the meridian μ of K.

Let $[\![\hat{pt}]\!]$ and $[\![\hat{\mu}]\!]$ respectively be the bases of $H_0(\hat{E}_K; \mathbb{R})$ and $H_1(\hat{E}_K; \mathbb{R})$ such that $p_*([\![\hat{pt}]\!] = [\![\hat{pt}]\!]$ and $p_*([\![\hat{\mu}]\!]) = n[\![\mu]\!]$. We set $\mathbf{h}^*_\mathbb{R}$ and $\hat{\mathbf{h}}^*_\mathbb{R}$ as $\mathbf{h}^*_\mathbb{R} = \{[\![pt]\!], [\![\mu]\!]\}$ and $\hat{\mathbf{h}}^*_\mathbb{R} = \{[\![\hat{pt}]\!], [\![\hat{\mu}]\!]\}$. Then the sign difference $\tau_0(E_K) \cdot \tau_0(\hat{E}_K)$ given by $\mathrm{sgn}(\mathrm{Tors}(p, \mathbf{h}^*_\mathbb{R}, \hat{\mathbf{h}}^*_\mathbb{R}))$ (see Proposition 9) can be expressed using the Alexander invariant of E_K and $\hat{E}_K$ as follows.

Proposition 10. *Under the assumptions given above, the sign-term in Equation (5) is expressed as*

$$\epsilon = sgn(\mathrm{Tor}(p, \mathbf{h}^*_\mathbb{R}, \hat{\mathbf{h}}^*_\mathbb{R})) = sgn(\Delta_{E_K}(1) \cdot \Delta_{\hat{E}_K}(1)).$$

Proof. This is a computation using Fox differential calculus as shown in [BZ03] for example. We choose a Wirtinger presentation $\pi_1(E_K) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1}\rangle$ of $\pi_1(E_K)$. From this presentation, we construct the usual 2-dimensional CW–complex W, which consists of a single 0–cell, k 1–cells $x_1, \ldots, x_k$ and $(k-1)$ 2–cells which correspond to relations. We know that W as the same simple homotopy type as E_K (see [Wal78]), thus their Reidemeister torsions are same. The boundary operator $\partial_2 \colon C_2(W; \mathbb{R}) \to C_1(W; \mathbb{R})$ is given by the Jacobian matrix $\left((\partial r_j / \partial x_i)^\phi\right)_{i,j}$, where $\phi \colon \mathbb{Z}[\pi_1(E_K)] \to \mathbb{Z}$ sends x_i to 1. Whereas $\partial_1 \colon C_1(W; \mathbb{R}) \to C_0(W; \mathbb{R})$ is just the zero map. From Milnor's result [Mil68a] it is easy to obtain:

$$\mathrm{Tor}(E_K, \mathbf{c}_\mathbb{R}{}^*, \mathbf{h}^*_\mathbb{R}) = \Delta_{E_K}(1).$$

Similarly we have that

$$\mathrm{Tor}(\hat{E}_K, \hat{\mathbf{c}}^*_\mathbb{R}, \hat{\mathbf{h}}^*_\mathbb{R}) = \Delta_{\hat{E}_K}(1).$$

□

5. Examples of computations

5.1. **Manifolds fiber over the circle.** Recall that a three–dimensional manifold M is *fibered over the circle* if it has the structure of a surface bundle over the circle, *i.e.* if there exists a surface F, called the fiber, and a map $\phi \colon F \to F$, called the monodromy, such that M is homeomorphic to $(F \times [0,1]) / ((x, 0) \sim (\phi(x), 1))$. The monodromy induces an endomorphism of $H_*(F; \mathbb{Z})$ denoted ϕ_*. For any representation $\rho \colon \pi_1(M) \to Aut(V)$, the monodromy ϕ also induces an endomorphism on the twisted homology group $H_1(F; V_\rho)$ which is called the *twisted monodromy* (for the precise definition the reader is invited to refer to the proof of Theorem 2).

In this Section, we prove that the polynomial torsion of M is (up to a sign) equal to the characteristic polynomial of the twisted monodromy ϕ^ρ_* (see Subsection 5.1.1). Moreover the polynomial torsion of m–fold cyclic covers over M can also be computed by using the characteristic polynomial of ϕ^m_* for any positive integer m. In Subsection 5.2 we give an illustration of the simplest examples of Theorem 1 since the eigenvalues of ϕ^m_* are given by $a, \xi_m a, \ldots, \xi_m^{m-1} a$ where a is an eigenvalue of ϕ_* and $\xi_m = e^{2\pi\sqrt{-1}/m}$. Subsection 5.2.1 deals with the sign term of the polynomial torsion for fibered manifolds. We give explicit examples of polynomial torsions for the fibered knot exterior of 4_1 in Subsection 5.3, and that of the exterior for the non–fibered knot 5_2 in the last Subsection 5.4.

5.1.1. *General formula.* For a fibered three–dimensional manifold over the circle, it is well–known that the Alexander polynomial is the characteristic polynomial of the induced linear map on the homology group of a fiber by the monodromy ϕ_* (see for example [Mil68b]). Similarly, for an irreducible and acyclic representation $\rho\colon \pi_1(M) \to Aut(V)$, the polynomial torsion is the characteristic polynomial of the twisted monodromy ϕ_*^ρ.

Theorem 2. *Let M be a fibered three–manifold over S^1 and F denote its fiber. We denote by $\varphi\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ the homomorphism induced by the projection $M \to S^1$ and assume that the fiber F is a compact connected surface with non–empty boundary and also assume that the restriction $\rho_{|\pi_1(F)}$ is irreducible and $H_*(M; V_\rho(t)) = 0$. Then $\Delta_M^{\varphi\otimes\rho}(t)$ is the characteristic polynomial of the twisted monodromy on $H_1(F; V_\rho)$ with the sign τ_0.*

Proof. The proof is based on application of Multiplicativity Lemma in Section 2.2 to the Mayer–Vietoris sequence for twisted subspace homology as in [FK06, Proposition 3.2]. First, we decompose M into a neighbourhood of a fiber F and its complement N, so that $M = N \cup (F \times [0,1])$, where N is a three–dimensional manifold homeomorphic to $F \times [0,1]$. We consider the following sequence of twisted subspace complexes as the proof of [FK06, Proposition 3.2], regarding the fiber F as the weighted surface with weight one,

$$0 \to \begin{array}{c} C_*(F \times \{1\}; V_\rho(t)) \\ \oplus \\ C_*(F \times \{0\}; V_\rho(t)) \end{array} \to \begin{array}{c} C_*(N; V_\rho(t)) \\ \oplus \\ C_*(F \times [0,1]; V_\rho(t)) \end{array} \to C_*(M; V_\rho(t)) \to 0.$$

Here the coefficients are the vector space $V_\rho(t) = \mathbb{C}(t) \otimes V$. Applying Multiplicativity Lemma, we obtain the following equation for Reidemeister torsions:

$$\begin{aligned} &\mathrm{Tor}(C_*(N; V_\rho(t))) \cdot \mathrm{Tor}(C_*(F \times [0,1]; V_\rho(t))) \\ &= \pm\mathrm{Tor}(C_*(M; V_\rho(t))) \cdot \mathrm{Tor}(C_*(F \times \{1\}; V_\rho(t))) \cdot \mathrm{Tor}(C_*(F \times \{0\}; V_\rho(t))) \cdot \mathrm{Tor}(\mathcal{H}_*) \end{aligned}$$

in which we omit symbols of bases and the sign term for simplicity and where $\mathcal{H}_*$ is the Mayer–Vietoris sequence of twisted subspace homology:

$$\cdots \to \begin{array}{c} H_i(F \times \{1\}; V_\rho(t)) \\ \oplus \\ H_i(F \times \{0\}; V_\rho(t)) \end{array} \to \begin{array}{c} H_i(N; V_\rho(t)) \\ \oplus \\ H_i(F \times [0,1]; V_\rho(t)) \end{array} \to H_i(M; V_\rho(t)) \to \cdots.$$

Using [FK06, Lemma 2.1] and our definition of φ, we can regard $C_*(F \times \{1\}; V_\rho(t))$ as $\mathbb{C}(t) \otimes C_*(F \times \{1\}; V_\rho)$. Similarly, we also regard $C_*(F \times \{0\}; V_\rho(t))$, $C_*(N; V_\rho(t))$ and $C_*(F \times [0,1]; V_\rho(t))$ as $\mathbb{C}(t) \otimes C_*(F \times \{0\}; V_\rho)$, $\mathbb{C}(t) \otimes C_*(N; V_\rho)$ and $\mathbb{C}(t) \otimes C_*(F \times [0,1]; V_\rho)$. When we choose appropriate bases of chain complexes and the homology groups, we can see that the torsions of $C_*(F \times \{1\}; V_\rho(t))$, $C_*(F \times \{0\}; V_\rho(t))$, $C_*(N; V_\rho(t))$ and $C_*(F \times [0,1]; V_\rho(t))$ are same. Therefore we have the following equation:

$$\mathrm{Tor}(C_*(M; V_\rho(t))) = \pm\mathrm{Tor}(\mathcal{H}_*)^{-1}.$$

Next we compute the torsion of the Mayer–Vietoris sequence of twisted subspace homology $\mathcal{H}_*$. From the assumption that $\rho_{|\pi_1(S)}$ is irreducible it follows that $H_0(F; V_\rho(t)) = \mathbb{C}(t) \otimes H_0(F; V_\rho) = 0$ since by the duality for twisted homology we see that $\mathrm{Hom}(H_0(F; V_\rho), \mathbb{C}) \simeq H^0(F; V_\rho)$, and it is known that $H^0(F; V_\rho)$ is generated by invariant vectors in V under the action of $\pi_1(F)$ by ρ (for the duality, we refer to [KL99, Section 2.1]). As the surface F has a non–empty boundary, it follows that $H_2(F; V_\rho(t)) = \mathbb{C}(t) \otimes H_2(F; V_\rho) = 0$. We also suppose that $H_*(M; V_\rho(t)) = 0$, thus, using [FK06, Proposition 3.2], the Mayer–Vietoris sequence $\mathcal{H}_*$ reduces to the following short exact sequence:

$$0 \to H_1(F; V_\rho) \otimes \mathbb{C}(t) \xrightarrow{\iota_- - t\iota_+} H_1(F; V_\rho) \otimes \mathbb{C}(t) \to 0.$$

Here ι_- is the homomorphism on $H_1(F; V_\rho)$ induced by the natural inclusion of $F \times \{0\} \hookrightarrow N \simeq F \times [0,1]$, and ι_+ is the homomorphism on $H_1(F; V_\rho)$ induced by the natural inclusion of $F \times \{1\} \hookrightarrow N \simeq F \times [0,1]$. Finally observe that $\iota_- = \mathbf{1}$ is the identity and that ι_+ is the twisted monodromy ϕ_*^ρ. Therefore the polynomial torsion $\Delta_M^{\varphi\otimes\rho}(t)$ is exactly $\tau_0 \cdot \det(\mathbf{1} - t\phi_*^\rho)$. □

Remark 8. We use the irreducibility of $\rho_{|\pi_1(F)}$ to show $H_0(F; V_\rho) = 0$. To obtain the same result we only have to suppose that $H_0(F; V_\rho) = 0$ on the restriction $\rho_{|\pi_1(F)}$.

Remark 9. In the case that $H_0(F; V_\rho) \neq 0$, the torsion $\Delta_M^{\varphi\otimes\rho}(t)$ is expressed as the rational function which is the characteristic polynomial of the twisted monodromy on $H_1(F; V_\rho)$ divided by that of the twisted monodromy on $H_0(F; V_\rho)$. For example, let M be a fibered link exterior E_L in S^3, if we choose ρ as the pull–back of $\pi_1(S^1) = \langle t \rangle \to \mathrm{GL}(1; \mathbb{C}) = \mathbb{C} \setminus \{0\}$, $t \mapsto \xi$, by φ, then together with the classical result of Milnor [Mil68b], we can see that, up to a factor $\pm\xi t^k$ ($k \in \mathbb{Z}$),

$$\Delta_M^{\varphi\otimes\rho}(t) = \pm\frac{\det(\mathbf{1} - t\phi_*^\rho)}{\det(\mathbf{1} - t\xi)} = \begin{cases} \frac{\Delta_K(\xi t)}{\xi t - 1} & \text{if } L \text{ is a fibered knot } K \\ \Delta_L(\xi t, \ldots, \xi t) & \text{if } L \text{ has 2 or more components,} \end{cases}$$

where Δ_L denotes the Alexander polynomial of a link L (see Examples 1 and 2).

Remark 10. Observe that, as for the usual Alexander polynomial, the torsion polynomial of a fibered manifold is monic. For representations onto finite groups, S. Friedl and S. Vidussi proved [FV08] that twisted Alexander polynomials decide fiberness for knots of genus one.

5.2. **The polynomial torsion of finite cyclic covers over a fibered manifold.** We can construct m–fold cyclic cover $\hat{M}$ of a fibered manifold $M = (F \times [0,1])/\left((x,0) \sim (\phi(x),1)\right)$ by using n–times composition of ϕ, *i.e.*,

$$\hat{M} = (F \times [0,1])/\left((x,0) \sim (\phi^m(x),1)\right).$$

Every manifold fibered over S^1 has the induced homomorphism from the fundamental group onto $\pi_1(S^1)$. We have the following diagram between these homomorphisms for a fibered manifold M and the m–fold cyclic cover $\hat{M}$

$$\begin{array}{ccc} \pi_1(\hat{M}) & \longrightarrow & \pi_1(M) \\ \downarrow & & \downarrow \\ \pi_1(S^1) = \langle s \rangle & \xrightarrow[s \mapsto t^m]{} & \pi_1(S^1) = \langle t \rangle, \end{array}$$

where we denote $\pi_1(S^1)$ by an infinite cyclic group.

When we consider that the twisted chain complex of $\hat{M}$ is given by the pull–back of a representation $\rho\colon \pi_1(M) \to Aut(V)$, the polynomial torsion of $\hat{M}$ is given by the characteristic polynomial $\det(s\mathbf{1} - \phi_*^m)$ of the twisted monodromy ϕ_*^m by Theorem 2. Moreover if we choose the variable in the polynomial torsion of $\hat{M}$ as the pull–back of $\varphi\colon \pi_1(M) \to \pi_1(S^1) = \langle t \rangle$, then the polynomial torsion $\Delta_{\hat{M}}^{p^*\varphi\otimes p^*\rho}(t)$ is given by $\det(t^m\mathbf{1} - \phi_*^m)$ where $p\colon \pi_1(\hat{M}) \to \pi_1(M)$ is the induced homomorphism by the covering map. This means that the following explicit formula of Theorem 2.

Corollary 11. *Under the conditions stated above, one can see that*

$$\begin{aligned}\Delta_{\hat{M}}^{p^*\varphi\otimes p^*\rho}(t) &= \tau_0 \det(t^m\mathbf{1} - \phi_*^m)\\ &= \tau_0 \prod_{i=0}^{m-1} \Delta_M^{\varphi\otimes\rho}(\xi_m^i t)\end{aligned}$$

where $\xi_m = e^{2\pi\sqrt{-1}/m}$.

5.2.1. *The problem of the sign for fibered knots.* In particular, we consider the exterior E_K of a fibered knot K, then we can compute the sign τ_0 as follows.

Observe first that in the case of a fibered knot, the homomorphism $\varphi\colon \pi_1(E_K) \to \mathbb{Z}$ induced by the projection $E_K \to S^1$ is the usual abelianization:

$$\pi_1(E_K) \to H_1(E_K;\mathbb{Z}) \simeq \mathbb{Z},\ [\![\mu]\!] \mapsto t.$$

We equip the knot exterior E_K with the homology orientation: $H_*(E_K;\mathbb{R})$ is based, by using the meridian μ of the knot and a point pt, as $H_1(E_K;\mathbb{R}) = \mathbb{R}[\![\mu]\!]$ and $H_0(E_K;\mathbb{R}) = \mathbb{R}[\![pt]\!]$. As in the proof of Theorem 2, the knot exterior E_K can be split along the fiber F. We continue to denote $\varphi\colon \pi_1(E_K) \to \mathbb{Z}$ the homomorphism induced by the fibration. The boundary of $F \times I$ is the disjoint union of two copies of F denoted $F \times \{0\} \simeq F$ and $F \times \{1\} \simeq F$.

Theorem 2 gives us the following expression for the polynomial torsion of E_K

$$\Delta_{E_K}^{\varphi\otimes\rho}(t) = \tau_0 \cdot \det(t\mathbf{1} - \phi_*^\rho)$$

where $\phi_*^\rho\colon H_1(F;V_\rho) \to H_1(F;V_\rho)$ is induced by the monodromy ϕ. Moreover the sign τ_0 is expressed using the isomorphism $\phi_1\colon H_1(F;\mathbb{R}) \to H_1(F;\mathbb{R})$ induced by $\phi\colon F \to F$.

Proposition 12. *We denote by* $\mathbf{h}_{\mathbb{R}}^*$ *the basis of* $H_*(E_K;\mathbb{R})$ *given by the homology orientation as above and let* ${\mathbf{h}'_{\mathbb{R}}}^*$ *be any basis of* $H_*(F;\mathbb{R})$*. The sign* τ_0 *is given by*

$$\tau_0 = -\mathrm{sgn}(\mathrm{Tor}(\mathcal{M}_{\mathbb{R}}, \{\mathbf{h}_{\mathbb{R}}^*, {\mathbf{h}'_{\mathbb{R}}}^*\}, \emptyset)) = \mathrm{sgn}(\det(\mathbf{1} - \phi_1))$$

where the isomorphism $\phi_1\colon H_1(F;\mathbb{R}) \to H_1(F;\mathbb{R})$ *is induced by the monodromy* $\phi\colon F \to F$.

The polynomial torsion of E_K *is the product:*

$$\Delta_{E_K}^{\varphi\otimes\rho}(t) = \mathrm{sgn}(\det(\mathbf{1} - \phi_1)) \cdot \det(t\mathbf{1} - \phi_*^\rho)$$

The Mayer–Vietoris sequence $\mathcal{M}_{\mathbb{R}}$ splits into the following two short exact sequences with real coefficients.

$$0 \to H_1(F;\mathbb{R}) \oplus H_1(F;\mathbb{R}) \xrightarrow{\Phi_{(1)}} H_1(F;\mathbb{R}) \oplus H_1(F;\mathbb{R}) \to 0 \tag{18}$$

and

$$\begin{aligned}0 \to H_1(E_K;\mathbb{R}) \xrightarrow{\delta} H_0(F;\mathbb{R}) &\oplus H_0(F;\mathbb{R})\\ &\xrightarrow{\Phi_{(0)}} H_0(F;\mathbb{R}) \oplus H_0(F;\mathbb{R}) \xrightarrow{\iota_*} H_0(E_K;\mathbb{R}) \to 0.\end{aligned} \tag{19}$$

Here $\Phi_{(i)} = \begin{pmatrix} \mathbf{1} & -\phi_i \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}$, where $\phi_i\colon H_i(F;\mathbb{R}) \to H_i(F;\mathbb{R})$ is induced by the monodromy, for $i = 0, 1$ and ι_* is the sum of two homomorphisms induced by the copy of inclusion $F \times I \to E_K$.

As in [Dub06], the sign τ_0 of the polynomial torsion is entirely given by the sign of Sequence (18), that is to say, Proposition 12 follows from the following:

Lemma 13. $\tau_0 = -\mathrm{sgn}(\mathrm{Tor}(\mathcal{M}_{\mathbb{R}}, \{\mathbf{h}_{\mathbb{R}}^*, {\mathbf{h}'_{\mathbb{R}}}^*\}, \emptyset)) = \mathrm{sgn}(\det(\mathbf{1} - \phi_1))$

Proof of the Lemma. The Reidemeister torsion with real coefficients is given by:

$$\begin{aligned}\mathrm{Tor}(C_*(E_K;\mathbb{R}),\mathbf{c}^*_{\mathbb{R}},\mathbf{h}^*_{\mathbb{R}}) &= -\mathrm{Tor}(\mathcal{M}_{\mathbb{R}},\{\mathbf{h}^*_{\mathbb{R}},{\mathbf{h}'_{\mathbb{R}}}^*\},\emptyset)^{-1}\frac{\mathrm{Tor}(C_*(F\times I;\mathbb{R}),{\mathbf{c}'}^*_{\mathbb{R}},{\mathbf{h}'_{\mathbb{R}}}^*)}{\mathrm{Tor}(C_*(F;\mathbb{R}),{\mathbf{c}'}^*_{\mathbb{R}},{\mathbf{h}'_{\mathbb{R}}}^*)}\\ &= -\mathrm{Tor}(\mathcal{M}_{\mathbb{R}},\{\mathbf{h}^*_{\mathbb{R}},{\mathbf{h}'_{\mathbb{R}}}^*\},\emptyset)^{-1}. \qquad (20)\end{aligned}$$

The sign in the right hand side of Equation (20) is computed using the fact that $\alpha(C'_*,C''_*)\equiv 0 \mod 2$ and $\varepsilon(C'_*,C_*,C''_*)\equiv 1 \mod 2$ in Multiplicativity Lemma since the chain complexes corresponding to C'_* and C''_* are the sum of two same chain complexes and $H_*(E_K;\mathbb{R})=\mathbb{R}[\![pt]\!]\oplus\mathbb{R}[\![\mu]\!]$. We will prove that

$$\mathrm{Tor}(\mathcal{M}_{\mathbb{R}},\{\mathbf{h}^*_{\mathbb{R}},{\mathbf{h}'_{\mathbb{R}}}^*\},\emptyset)^{-1} = -\det(\mathbf{1}-\phi_1)$$

by proving that the torsion of Sequence (19) is -1. By a diagram chasing, we have $\delta(u)=(u,u)$ for $u\in\mathbb{R}\simeq H_1(E_K;\mathbb{R})$ and by construction, $\Phi_{(0)}(a,b)=(a-b,-a+b)$ and $\iota_*(c,d)=c+d$ for vectors (a,b), (c,d) in $\mathbb{R}^2\simeq H_0(F;\mathbb{R})\oplus H_0(F;\mathbb{R})$. When we choose new bases of vector spaces in Sequence (19) as

$$\begin{aligned}H_1(E_K;\mathbb{R}) &= \langle[\![\mu]\!]\rangle,\\ H_1(F;\mathbb{R})\oplus H_1(F;\mathbb{R}) &= \langle\delta([\![\mu]\!])=[\![pt]\!]\oplus[\![pt]\!],[\![pt]\!]\oplus 0\rangle,\\ H_1(F;\mathbb{R})\oplus H_1(F;\mathbb{R}) &= \langle\Phi_{(0)}([\![pt]\!]\oplus 0)=[\![pt]\!]\oplus -[\![pt]\!],[\![pt]\!]\oplus 0\rangle,\\ H_1(E_K;\mathbb{R}) &= \langle[\![pt]\!]\rangle,\end{aligned}$$

we obtain that the torsion of Sequence (19) is -1 by a direct computation. □

5.3. **The figure eight knot exterior.** In general, it is difficult to compute the twisted monodromy for the twisted homology group of fiber in a fibered manifold. However we can find explicit examples given by computation of tangent spaces and derivation of morphisms for affine varieties in [Por97, Section 4.5]. The typical example is the figure eight knot exterior. It is a surface bundle over S^1 whose fiber F is one punctured torus. In other words, the figure eight knot is a fibered knot with genus one.

Let K denote the figure eight knot and ρ be a homomorphism from the knot group $\pi_1(E_K)$ into $\mathrm{SL}_2(\mathbb{C})$. Suppose that the there exists no common eigen space for the image $\rho(\pi_1(E_K))$. Since the Lie group $\mathrm{SL}_2(\mathbb{C})$ has the adjoint action Ad on $\mathfrak{sl}_2(\mathbb{C})=\{\boldsymbol{v}\in\mathfrak{gl}_2(\mathbb{C})\,|\,\mathrm{tr}\,\boldsymbol{v}=0\}$, given by $Ad_A(\boldsymbol{v})=A\boldsymbol{v}A^{-1}$ for any $A\in\mathrm{SL}_2(\mathbb{C})$ and $\boldsymbol{v}\in\mathfrak{sl}_2(\mathbb{C})$, we can make a representation $Ad\circ\rho$ of $\pi_1(E_K)$ on the vector space $\mathfrak{sl}_2(\mathbb{C})$ by the composition of ρ with the adjoint action Ad. We compute the polynomial torsion of $\Delta^{\varphi\otimes Ad\circ\rho}_{E_K}(t)$ for the figure knot exterior with the representation $Ad\circ\rho$ by using the twisted monodromy ϕ_* on $H_1(F;\mathfrak{sl}_2(\mathbb{C})_\rho)$ and Theorem 2. Here we denote by $\mathfrak{sl}_2(\mathbb{C})_\rho$ the vector space $\mathfrak{sl}_2(\mathbb{C})$ with the right action of $\pi_1(E_K)$ by $Ad\circ\rho^{-1}$.

The key point is that the twisted homology group $H_1(F;\mathfrak{sl}_2(\mathbb{C})_\rho)$ is regarded as the cotangent space of an affine variety concerning $\pi_1(F)$. Note that the knot group $\pi_1(E_K)$ is expressed as

$$\pi_1(E_K)=\langle a,b,\mu\,|\,\mu a\mu^{-1}=\phi_*(a)=ab,\mu b\mu^{-1}=\phi_*(b)=bab\rangle$$

where a, b are the generators of $\pi_1(F)$ and μ is the meridian. The induced homomorphism ϕ_* of monodromy is an automorphism of $\pi_1(F)$ and ϕ_* induces a morphism from the character variety $X(F)$ to itself. Here the character variety $X(F)$ is the image of the following map:

$$\begin{aligned}R(F) &\to \mathbb{C}^3\\ \rho' &\mapsto \big(\mathrm{tr}\,\rho'(a),\mathrm{tr}\,\rho'(b),\mathrm{tr}\,\rho'(ab)\big)\end{aligned}$$

where $R(F)$ is the set of homomorphisms from $\pi_1(F)$ into $\mathrm{SL}_2(\mathbb{C})$. It is known that the set $X(F)$ has the structure of an affine variety and the pull-back of ϕ_* gives a morphism from $X(F)$ to itself. Roughly speaking, the twisted homology group $H_1(F;\mathfrak{sl}_2(\mathbb{C})_\rho)$ can be regarded as the cotangent space of $X(F)$ at the point $(\mathrm{tr}\,\rho(a), \mathrm{tr}\,\rho(b), \mathrm{tr}\,\rho(ab))$ and the differential of the induced morphism by ϕ_* gives the twisted monodromy $\phi_*^{Ad\circ\rho}$ on $H_1(F;\mathfrak{sl}_2(\mathbb{C})_\rho) \simeq \mathbb{C}^3$. For the details of character varieties, we refer to [CS83, LM85] and for the details about the above discussion to [Por97, Section 4.5]. It is also shown in [Por97, Section 4.5] that eigenvalues of the twisted monodromy $\phi_*^{Ad\circ\rho}$ consists in 1 and $\ell^{\pm 1}$. Moreover, the sum $1+\ell+\ell^{-1}$, *i.e.*, the trace of $\phi_*^{Ad\circ\rho}$, is given by $2(\mathrm{tr}\,\rho(a) + \mathrm{tr}\,\rho(b)) = 2((\mathrm{tr}\,\rho(\mu))^2 - 1)$.

Using Theorem 2 and Proposition 12, we obtain the polynomial torsion $\Delta_{E_K}^{\varphi\otimes\rho}$ as follows:

$$\begin{aligned}\Delta_{E_K}^{\varphi\otimes Ad\circ\rho} &= \mathrm{sgn}(\det(\mathbf{1}-\phi_1))\det(t\mathbf{1}-\phi_*^{Ad\circ\rho}) \\ &= -(t-1)(t^2-(\ell+\ell^{-1})t+1) \\ &= -(t-1)\left(t^2-(2(\mathrm{tr}\,\rho(\mu))^2-3)t+1\right)\end{aligned}$$

Remark 11. This result agrees with the twisted Alexander polynomial of the figure eight knot in [DY, Section 4], which is obtained without using the fibered structure of E_K.

5.4. **the 5_2 knot.** Last we give an example of the polynomial torsion for the simplest non–fibered knot, namely the 5_2 knot in Rolfsen's table [Rol90]. So, let K be the 5_2 knot as illustrated on the Figure 1.

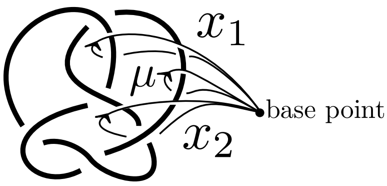

FIGURE 1. The diagram of 5_2 knot with closed loops generating the knot group

We compute the polynomial torsion of the two–fold cyclic cover $\hat{E}_K$ over the knot exterior E_K. The knot group can be expressed by the following Lin presentation:

$$\pi_1(E_K) = \langle x_1, x_2, \mu \,|\, \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2^{-2} x_1 \mu^{-1} = x_2^{-2}\rangle$$

and this presentation induced that of $\pi_1(\hat{E}_K)$ as

$$\pi_1(\hat{E}_K) = \left\langle y_1, y_2, y_3, y_4, \hat{\mu} \;\middle|\; \begin{array}{cc} y_3 = y_1 y_2^{-1}, & y_4^{-2} y_3 = y_2^{-2}, \\ \hat{\mu} y_1 \hat{\mu}^{-1} = y_3 y_4^{-1}, & \hat{\mu} y_2^{-2} y_1 \hat{\mu}^{-1} = y_4^{-2} \end{array} \right\rangle$$

where the generators satisfy that

$$p(y_1) = x_1,\ p(y_2) = x_2,\ p(y_3) = \mu x_1 \mu^{-1},\ p(y_4) = \mu x_2 \mu^{-1} \text{ and } p(\hat{\mu}) = \mu^2$$

in the image of $p\colon \pi_1(\hat{E}_K) \to \pi_1(E_K)$ (for the detail about Lin presentations, we refer to [NY]). The explicit computation of $\Delta_{E_K}^{\varphi\otimes Ad\circ\rho}(t)$ is exposed in [Yam, Section 5]. Here φ is the abelianization homomorphism of $\pi_1(E_K)$, namely, $\varphi\colon \pi_1(E_K) \to H_1(E_K;\mathbb{Z}) = \langle t\rangle$. The homomorphism ϕ sends x_i $(i=1,2)$ to 1 and μ to t. The following correspondence gives a homomorphism from $\pi_1(E_K)$ into $\mathrm{SL}_2(\mathbb{C})$:

$$x_1 \mapsto \begin{pmatrix} \zeta_7 & 0 \\ 0 & \zeta_7^{-1}\end{pmatrix},\ x_2 \mapsto \begin{pmatrix} \zeta_7^2 & 0 \\ 0 & \zeta_7^{-2}\end{pmatrix},\ \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

where $\zeta_7 = e^{2\pi\sqrt{-1}/7}$. We denote by ρ this homomorphism of $\pi_1(E_K)$. The compositions of these $\mathrm{SL}_2(\mathbb{C})$-matrices with the adjoint action is expressed as

$$Ad_{\rho(x_1)} = \begin{pmatrix} \zeta_7^2 & & \\ & 1 & \\ & & \zeta_7^{-2} \end{pmatrix},\ Ad_{\rho(x_2)} = \begin{pmatrix} \zeta_7^4 & & \\ & 1 & \\ & & \zeta_7^{-4} \end{pmatrix},\ Ad_{\rho(\mu)} = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

with respect to the basis $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ of $\mathfrak{sl}_2(\mathbb{C})$. We abbreviate the matrices $Ad_{\rho(x_1)}$, $Ad_{\rho(x_2)}$ and $Ad_{\rho(\mu)}$ to X_1, X_2 and M.

When we ignore the sign term, the polynomial torsion $\Delta_{E_K}^{\varphi\otimes Ad\circ\rho}(t)$ can be computed by using Fox differential calculus in the same way as M. Wada [Wad94] as follows:

$$\begin{aligned}
\Delta_{E_K}^{\varphi\otimes Ad\circ\rho}(t) &= \frac{\det\left(\Phi\left(\frac{\partial r_j}{\partial x_i}\right)\right)_{1\leqslant i,j\leqslant 2}}{\det \Phi(\mu - 1)} \\
&= \frac{\det \begin{pmatrix} tM^{-1} - \mathbf{1} & tX_2^2 M^{-1} \\ X_2 X_1^{-1} & -tX_2M^{-1} - X_2^2M^{-1} + X_2^2 + X_2 \end{pmatrix}}{\det \Phi(tM^{-1} - \mathbf{1})}
\end{aligned}$$

where $r_1 = \mu x_1 \mu^{-1} x_2 x_1^{-1}$, $r_2 = \mu x_2^{-2} x_1 \mu^{-1} x_2$, the symbol $\partial/\partial x_i$ is the Fox differential by x_i and Φ is the linear extension of $\varphi \otimes Ad \circ \rho^{-1}$ on the group ring $\mathbb{Z}[\pi_1(E_K)]$.

Note that the alignment of $\partial r_j/\partial x_i$ is transpose to that of the ordinary definition of twisted Alexander polynomial since the polynomial torsion is regarded as a Reidemeister torsion for the twisted chain complex $C_*(E_K; \mathfrak{sl}_2(\mathbb{C}(t))_{Ad\circ\rho})$. Here $\mathfrak{sl}_2(\mathbb{C}(t))_{Ad\circ\rho}$ is a right $\mathbb{Z}[\pi_1(E_K)]$-module $\mathbb{C}(t) \otimes \mathfrak{sl}_2(\mathbb{C})$ via $\varphi \otimes Ad \circ \rho^{-1}$.

By a direct computation of the above determinants, the twisted polynomial torsion of K turns out

$$\begin{aligned}
\Delta_{E_K}^{\varphi\otimes Ad\circ\rho}(t) &= -(t-1)(\zeta_7^3 + \zeta_7^{-3} + 2)(2t^2 - 3t + 2) \\
&= -(t-1)(\zeta_7^3 + \zeta_7^{-3} + 2)\Delta_K(-t)
\end{aligned}$$

where $\Delta_K(t) = 2t^2 - 3t + 2$ is the Alexander polynomial of K.

Similarly, when we ignore the sign terms, we can compute the polynomial torsion $\Delta_{\hat{E}_K}^{p^*\varphi\otimes Ad\circ p^*\rho}$ as follows:

$$\begin{aligned}
&\Delta_{\hat{E}_K}^{p^*\varphi\otimes Ad\circ p^*\rho}(t) \\
&= \frac{\det\left(\Phi'\left(\frac{\partial s_j}{\partial y_i}\right)\right)_{1\leqslant i,j\leqslant 2}}{\det \Phi(\mu - 1)} \\
&= \frac{\det \begin{pmatrix} -\mathbf{1} & O & t^2M^{-2} & t^2X_2^2M^{-2} \\ X_1 & X_2^2 + X_2 & O & -t^2X_2M^{-2} - t^2X_2^2M^{-2} \\ \mathbf{1} & X_2^{-2} & -\mathbf{1} & O \\ O & -X_2^{-1} - X_2^{-2} & M^2X_1^{-1}M^{-2} & X_2^{-2} + X_2^{-1} \end{pmatrix}}{\det \Phi(t^2M^{-2} - \mathbf{1})} \\
&= (t^2 - 1)(\zeta^3 + \zeta^{-3} + 2)^2(2t^2 + 3t + 2)(2t^2 - 3t + 2)
\end{aligned}$$

where Φ' is the linear extension of $p^*\varphi \otimes Ad \circ (p^*\rho)^{-1}$ on $\mathbb{Z}[\pi_1(\hat{E}_K)]$.

The above computation yields that the special case of Theorem 1 since the polynomial torsion for $\hat{E}_K$ factors into the product of the polynomial torsion for E_K and that with the variable multiplied the square root of unity.

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