Generalized Ricci flow I: Local existence and uniqueness

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Abstract

In this paper we investigate a kind of generalized Ricci flow which possesses a gradient form. We study the monotonicity of the given function under the generalized Ricci flow and prove that the related system of partial differential equations are strictly and uniformly parabolic. Based on this, we show that the generalized Ricci flow defined on a n-dimensional compact Riemannian manifold admits a unique short-time smooth solution. Moreover, we also derive the evolution equations for the curvatures, which play an important role in our future study.

Key words and phrases: Generalized Ricci flow, uniformly parabolic system, short-time existence, Thurston's eight geometries.

1 Introduction

In the early eighties R. Hamilton introduced the Ricci flow to construct canonical metrics for some manifolds. Since then many mathematicians, including Hamilton, Yau, Perelman and others, developed many tools and techniques to study the Ricci flow. The latest developments confirmed that the Ricci flow

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approach is very powerful in the study of three-manifolds. In fact, a complete proof of Poincare's conjecture and Thurston's geometrization conjecture has been offered in Cao-Zhu's paper [3] and others after Perelman's breakthrough.

It is useful to observe that, in Perelman's work [10], a key step is to introduce a functional for a metric g and a function f on a manifold M

$$W(g,f) = \int_{M^3} d^3x\sqrt{g}e^{-f}(R + |\nabla f|^2).$$

The variation of this functional generates a gradient flow which is a system of partial differential equations

$$\dot{g}_{ij} = -2(R_{ij} + \nabla_i \nabla_j f),$$

$$\dot{f} = -(R + \triangle f).$$

If we fix a measure for the conformal class of metrics $e^f ds^2$ of a metric, i.e., let $dm = e^{-f}dV$ be fixed, then we get back to the original Ricci flow after we apply a transformation of diffeomorphism generated by the vector field $\nabla_i f$ to the metric. In this way, we express the Ricci flow as a gradient flow. Dynamics of a gradient flow is much easier to handle. The functional generating the flow gives a monotone functional along the orbit of the flow automatically. If the flow exists for all time, then it shall flow to a critical point which leads to the existence of a canonical metric. Even for a flow which does not exist for all time, the generating functional helps very much in the analysis of singularities.

Perelman's above idea came from physics. Ricci flow arises as the first order approximation of the renormalization flow of a sigma model. Since there are many kinds of sigma models, it would be interesting to try some other models. Indeed such a generalization was made by physicists in [11]. For a three-manifold M^3, they proposed to add a $U(1)$ gauge field with potential 1-form A and field strength F which are coupled as a *Maxwell-Chern-Simons theory*. The corresponding action given by [6] or [5] reads

$$S = \int_M d^3x\sqrt{g}e^{-f}(-\chi + R + |\nabla f|^2) - \frac{1}{2}e^{-f}H \wedge *H - e^{-f}F \wedge *F.$$

The $U(1)$ gauge field A is a one-form potential whose field strength $F = dA$. The Wess-Zumino field B is a two-form potential whose field strength $H = dB$, f is a dilaton. In their paper, they find that Thurston's eight geometries appear as critical points of the above functional. Furthermore they show that there are no other critical points. So basically critical points of the above functional are eight geometries of Thurston. They also propose to study the gradient flow of the functional S as a generalization of the Ricci flow. Unfortunately, they modify the gradient flow in a way to change sign for the variable of gauge fields. Although the modified flow shares the same set of critical points they lost the important monotone property (along an orbit).

In addition, we are also able to consider a flow for a similar functional for a four-dimension manifold

$$S_1 = \int_M d^4x\sqrt{g}e^{-f}(\chi + R + |\nabla f|^2) - \frac{1}{2}e^{-f}H \wedge *H - e^{-f}F \wedge *F + \frac{e}{2}F \wedge F$$

where e is the Euler number $e(\eta)$ of the bundle η. The corresponding flow is given by

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} + \nabla_i \nabla_j f - \dfrac{1}{4}H_{ikl}H_j^{\ kl} - F_i^{\ k}F_{jk}], \\ \dfrac{\partial B_{ij}}{\partial t} = e^f \nabla_k (e^{-f} H^k_{\ ij}), \\ \dfrac{\partial A_i}{\partial t} = -e^f \nabla_k (e^{-f} F_i^{\ k}), \\ \dfrac{\partial f}{\partial t} = \chi - 2R - 3\triangle f + |\nabla f|^2 + \dfrac{1}{3}H^2 + \dfrac{3}{2}F^2. \end{cases}$$

The generalization to four-manifolds is probably more interesting. It may offer a systematic way to study four-manifolds.

The success of studying three-manifolds relies on a program proposed by Thurston, i.e., his geometrization conjecture. He conjectures and proves for several large classes of three-manifolds, that every three-manifold can be decomposed into pieces of three-manifolds of canonical metrics, i.e., those manifolds carrying one of the eight geometries of Thurston.

For four-dimension manifolds the critical points of S_1 might play a similar role as building blocks of smooth four-dimension manifolds. It would be interesting to study those critical points and to study what other four manifolds one can get by performing surgeries and gluing on those manifolds. We shall address this problem in the future.

As a first step, we shall show that the flow does exist. We shall also prove that the modified system of partial differential equations are strictly and uniformly parabolic.

The paper is organized as follows. Section 2 is devoted to the proof of local existences and uniqueness. In section 3 we study the monotonicity of S under the modified flow. In Section 4 we investigate the equations for the critical points of S and point out that fields F and H do not provide any help for the case of compact manifold but maybe play an important role for the noncompact case. In Section 5, we derive the evolution equations for the curvatures, which play an important role in our future study.

2 Local Existences and Uniqueness

In this section, we mainly establish the short-time existence and uniqueness result for the gradient flow (6), (7) and (8) on a compact 3-dimensional manifold M. It is known that the gradient flow (6), (7) and (8) is a system of second order nonlinear weakly parabolic partial differential equations. By the proof of the local existence and uniqueness of the Ricci flow (for example see [3] [4],), we can obtain a modified evolution equations by the diffeomorphism φ of M, which is a strictly parabolic system. Then, by the standard theory of parabolic equations, the modified evolution equations has a uniqueness solution.

Let us choose a normal coordinate $\{x^i\}$ around a fixed point $x \in M$ such that $\dfrac{\partial g_{ij}}{\partial x^k} = 0$ and $g_{ij}(p) = \delta_{ij}$.

Theorem 2.1 (Local existences and uniqueness) *Let $(M, g_{ij}(x))$ be a three-dimensional compact Riemannian manifold. Then there exists a constant $T > 0$*

such that the evolution equations

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} - \dfrac{1}{4} H_{ikl} H_j{}^{kl} - F_i{}^k F_{jk}], \\ \dfrac{\partial A_i}{\partial t} = -\nabla_k F_i{}^k, \\ \dfrac{\partial B_{ij}}{\partial t} = \nabla_k H^k{}_{ij}. \end{cases} \tag{1}$$

has a unique smooth solution on $M \times [0, T)$ for every initial fields.

Lemma 2.1 *For each gauge equivalent class of a gauge field A, there exists an $A^{'}$ such that $d(*A^{'}) = 0$.*

The lemma can be proved by the Hodge decomposition.

Proof. For each one-form A, by the Hodge decomposition, there exists an one-form A_0, a function α and a two-form β such that

$$A = A_0 + d\alpha + d^*\beta,$$

$$dA_0 = 0, \quad d^*A_0 = 0.$$

Let $A' = A - d\alpha$. A' is in the same gauge equivalent class of A. Since $d(*A_0) = 0$, $d(*d^*\beta) = 0$, then we have $d(*A') = 0$. □

Lemma 2.2 *The differential operator of the right hand of (7) with respect to the gauge equivalent class of a gauge field A is uniformly elliptic.*

Proof. Let $A = A_i dx^i$ be a gauge field. By Lemma 4.1 we can choose an $A^{'}$ in the gauge equivalent class of A such that $d(*A^{'}) = 0$. We still denote A' as A . Since $d(*A) = 0$, we have $dd^*A = 0$, then $\sum_{k=1}^{3} \dfrac{\partial^2 A_k}{\partial x^k \partial x^i} = 0, \forall\ i = 1, \cdots, 3$. Noting that $F = dA$ and $F_{ij} = \dfrac{\partial A_j}{\partial x^i} - \dfrac{\partial A_i}{\partial x^j}$, We have

$$\frac{\partial A_i}{\partial t} = -\nabla_k F_i{}^k = -\nabla_k (g^{kl} F_{il}) = -g^{kl} (\frac{\partial^2 A_l}{\partial x^k \partial x^i} - \frac{\partial^2 A_i}{\partial x^k \partial x^l}) = g^{kl} \frac{\partial^2 A_i}{\partial x^k \partial x^l}.$$

The right hand side of above equation is clearly elliptic at point x. If we apply a diffeomorphism to the metric it won't change the positivity property of the second order operator of the right hand side. □

Now let us consider the equation for B_{ij}.

Lemma 2.3 *For each gauge equivalent class of a B-field B, i.e., a two-form B on M, there exists a $B^{'}$ such that $d(*B^{'}) = 0$.*

Proof. Again we use the Hodge decomposition. For a two-form B, there exist a one-form α, a two-form B_0 and a three-form β such that

$$B = B_0 + d\alpha + d^*\beta,$$

$$dB_0 = 0, d^*B_0 = 0.$$

Let $B^{'} = B - d\alpha$. Since $B^{'}$ is in the same gauge equivalent class of B, we have $d(*B^{'}) = 0$. $\square$

Lemma 2.4 *The differential operator of the right hand side of (8) with respect to the gauge equivalent class of a B-field B is uniformly elliptic.*

Proof. Let us consider the equation for B-field. Without loss of generality, we assume $d(*B) = 0$. Thus $dd^*B = 0$. Then $\sum_{k=1}^{3}(\frac{\partial^2 B_{ki}}{\partial x^k \partial x^j} + \frac{\partial^2 B_{jk}}{\partial x^k \partial x^i}) = 0, \forall\ i, j = 1, \cdots, 3$. We have

$$\frac{\partial B_{ij}}{\partial t} = \nabla_k H^k{}_{ij} = g^{kl}(\frac{\partial^2 B_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 B_{jl}}{\partial x^k \partial x^i} + \frac{\partial^2 B_{li}}{\partial x^k \partial x^j}) = g^{kl}\frac{\partial^2 B_{ij}}{\partial x^k \partial x^l}.$$

The right hand side is clearly elliptic at the point x. If we apply a diffeomorphism to the metric it does not change the positivity property of the second order operator of the right hand side. $\square$

Suppose $\hat{g}_{ij}(x,t)$ is a solution of the equations (1), and $\varphi_t : M \to M$ is a family of diffeomorphisms of M. Let

$$g_{ij}(x,t) = \varphi_t^* \hat{g}_{ij}(x,t),$$

where φ_t^* is the pull-back operator of φ_t. We now want to find the evolution equations for the metric $g_{ij}(x,t)$.

Denote

$$y(x,t) = \varphi_t(x) = \{y^1(x,t), y^2(x,t), \cdots, y^n(x,t)\}$$

in local coordinates. Then

$$g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t)$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}g_{ij}(x,t) &= \frac{\partial}{\partial t}\left[\hat{g}_{\alpha\beta}(y,t)\cdot\frac{\partial y^\alpha}{\partial x^i}\cdot\frac{\partial y^\beta}{\partial x^j}\right] \\
&= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t) + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\hat{g}_{\alpha\beta}(y,t) \\
&\quad +\hat{g}_{\alpha\beta}(y,t)\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j} + \hat{g}_{\alpha\beta}(y,t)\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}) \ .
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial}{\partial y^\gamma}\hat{g}_{\alpha\beta} &= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}g_{kl}\frac{\partial}{\partial y^\gamma}(\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}) \\
&= \frac{\partial y^\beta}{\partial t}\frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\alpha}{\partial x^i}g_{jk} + \frac{\partial y^\alpha}{\partial t}\frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\beta}{\partial x^j}g_{ik} \ , \\
\Gamma^k_{jl} &= \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^l}\frac{\partial x^k}{\partial y^\gamma}\hat{\Gamma}^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^l} \ ,
\end{aligned}
$$

then

$$
\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{H}_{\alpha\rho\delta}\hat{H}_\beta{}^{\rho\delta} = H_{ikl}H_j{}^{kl}, \qquad \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{F}_\alpha{}^\rho\hat{F}_{\beta\rho} = F_i{}^kF_{jk} \ .
$$

Therefore, in the normal coordinate, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}g_{ij}(x,t) &= \frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}g_{kl}\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})g_{kl}\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta} + \frac{\partial y^\alpha}{\partial t}\frac{\partial}{\partial x^i}(\frac{\partial x^k}{\partial y^\alpha})g_{jk} \\
&\quad +\frac{\partial y^\beta}{\partial t}\frac{\partial}{\partial x^j}(\frac{\partial x^k}{\partial y^\beta})g_{ik} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\left[-2(\hat{R}_{\alpha\beta} - \frac{1}{4}\hat{H}_{\alpha\rho\delta}\hat{H}_\beta{}^{\rho\delta} - \hat{F}_\alpha{}^\rho\hat{F}_{\beta\rho})\right] \\
&= \frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha})g_{jk} + \frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}\frac{\partial x^k}{\partial y^\beta})g_{ik} - 2R_{ij} + \frac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk} \\
&= -2R_{ij} + \nabla_i(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}) + \nabla_j(\frac{\partial y^\beta}{\partial t}\frac{\partial x^k}{\partial y^\beta}g_{ik}) + \frac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk} \ .
\end{aligned}
$$

If we define $y(x,t) = \varphi_t(x)$ by the equations

$$
\begin{cases}
\dfrac{\partial y^\alpha}{\partial t} = \dfrac{\partial y^\alpha}{\partial x^k}(g^{jl}(\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl})), \\
y^\alpha(x,0) = x^\alpha
\end{cases}
\tag{2}
$$

and $V_i = g_{ik}g^{jl}(\Gamma^k_{jl} - \tilde{\Gamma}^k_{jl})$, we get the following evolution equations for the pull-back metric

$$
\begin{cases}
\dfrac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij} + \nabla_iV_j + \nabla_jV_i + \dfrac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk}, \\
g_{ij}(x,0) = \tilde{g}_{ij}(x),
\end{cases}
\tag{3}
$$

where $\tilde{g}_{ij}(x)$ is the initial metric and $\tilde{\Gamma}^k_{jl}$ is the connection of the initial metric. The initial value problem (2) can be rewritten as

$$\begin{cases} \frac{\partial y^\alpha}{\partial t} = g^{jl}(\frac{\partial^2 y^\alpha}{\partial x^j \partial x^l} + \frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^l}\hat{\Gamma}^\alpha_{\beta\gamma} - \frac{\partial y^\alpha}{\partial x^k}\tilde{\Gamma}^k_{jl}) \ , \\ y^\alpha(x,0) = x^\alpha. \end{cases} \tag{4}$$

Equation (4) is clearly a strictly parabolic system. Then, we have

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij}(x,t) &= \frac{\partial}{\partial x^i}\left\{g^{kl}\frac{\partial g_{kl}}{\partial x^j}\right\} - \frac{\partial}{\partial x^k}\left\{g^{kl}(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})\right\} \\ &\quad + \frac{\partial}{\partial x^i}\left\{g_{jk}g^{pq}\tfrac{1}{2}g^{km}(\frac{\partial g_{mq}}{\partial x^p} + \frac{\partial g_{mp}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^m})\right\} \\ &\quad + \frac{\partial}{\partial x^j}\left\{(g_{ik}g^{pq}\tfrac{1}{2}g^{km}(\frac{\partial g_{mq}}{\partial x^p} + \frac{\partial g_{mp}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^m})\right\} + \frac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk} \\ &= g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk} \ . \end{aligned}$$

As a result, from the original equations, we can obtain

$$\begin{cases} \frac{\partial g_{ij}(x,t)}{\partial t} = g^{kl}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{1}{2}H_{ikl}H_j{}^{kl} + 2F_i{}^kF_{jk}, \\ \frac{\partial A_i}{\partial t} = g^{kl}\frac{\partial^2 A_i}{\partial x^k \partial x^l}, \\ \frac{\partial B_{ij}}{\partial t} = g^{kl}\frac{\partial^2 B_{ij}}{\partial x^k \partial x^l}. \end{cases} \tag{5}$$

Let

$$u_1 = g_{11}, u_2 = g_{12}, u_3 = g_{13}, u_4 = g_{22}, u_5 = g_{23}, u_6 = g_{33},$$

$$u_7 = A_1, u_8 = A_2, u_9 = A_3, u_{10} = B_{12}, u_{11} = B_{13}, u_{12} = B_{23} \ .$$

The above equations can be rewritten as the following form

$$\frac{\partial u_i}{\partial t} = \sum_{j\,k\,l} a_{ikjl}\frac{\partial^2 u_j}{\partial x^k \partial x^l} + (\text{lower order terms}) \quad (k,l = 1,2,3;\ i,j = 1,2,\cdots,12),$$

in which

$$a_{ikjl} = g^{kl}\ (j = i), \quad a_{ikjl} = 0\ (j \neq i) \quad (i = 1,\cdots,12),$$

For arbitrary $\xi \in \mathbb{R}^{4\times 11} \setminus \{0\}$, we have

$$\sum_{ijkl} a_{ikjl}\xi^i_k\xi^j_l = \sum_{kl}\sum_i g^{kl}\xi^i_k\xi^i_l > 0.$$

Summarize the above discussions, we have the following lemma.

Lemma 2.5 *The differential operator of the right hand side of (5) with respect to the metric g is uniformly elliptic.*

Proof of Theorem 4.1. Noting Lemmas 4.2, 4.4, 4.5 and the compactness property of M, and using the standard theorem of partial differential equations (see [1], [2], [7]), we can immediately obtain the local existence of smooth solution of the modified system (5) with the initial value

$$g_{ij}(x,0)=\tilde{g}_{ij}(x), \quad A_i(x,0)=\tilde{A}_i(x), \quad B_{ij}(x,0)=\tilde{B}_{ij}(x).$$

In turn the solution of the gradient flow (1) can be obtained from (4) (or (2)). The proof of the existence of smooth solution is completed.

Now we argue the uniqueness of the solution of the gradient flow (1).

By Lemma 4.2, 4.4 and the standard theorem of partial differential equations, we can obtain the uniqueness of A and B . For any two solutions $\hat{g}_{ij}^{(1)}$ and $\hat{g}_{ij}^{(2)}$ of the gradient flow (1) with the same initial data, we can solve the initial value problem (4) (or (2)) to get two families $\varphi^{(1)}$ and $\varphi^{(2)}$ of diffeomorphisms of M. Thus we get two solutions

$$g_{ij}^{(1)}(\cdot,t)=(\varphi_t^{(1)})^*\hat{g}_{ij}^{(1)}(\cdot,t), \qquad g_{ij}^{(2)}(\cdot,t)=(\varphi_t^{(2)})^*\hat{g}_{ij}^{(2)}(\cdot,t),$$

to the modified evolution (5) equations with the same initial value $g_{ij}(x,0)=\tilde{g}_{ij}(x)$. The uniqueness result for the strictly parabolic equation implies that $g_{ij}^{(1)}=g_{ij}^{(2)}$. Since the initial value problem (4) is clearly a strictly parabolic system, the corresponding solutions $\varphi^{(1)}$ and $\varphi^{(2)}$ of (4) must agree. Consequently, the metrics $\hat{g}_{ij}^{(1)}$ and $\hat{g}_{ij}^{(2)}$ must agree also. Thus, we have proved Theorem. □

Remark 2.1 *we are also able to consider a flow for a similar functional for a four-dimension manifold*

$$S_1=\int_M d^4x\sqrt{g}e^{-f}(\chi+R+4|\nabla\phi|^2)-\frac{\epsilon_H}{2}e^{-f}H\wedge *H-\epsilon_F e^{-f}F\wedge *F+\frac{e}{2}F\wedge F$$

where e is the Euler number $e(\eta)$ of the bundle η. The corresponding flow is *given by*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} + \nabla_i\nabla_j f - \dfrac{1}{4}H_{ikl}H_j{}^{kl} - F_i{}^k F_{jk}], \\ \dfrac{\partial B_{ij}}{\partial t} = e^f \nabla_k (e^{-f} H^k{}_{ij}), \\ \dfrac{\partial A_i}{\partial t} = -e^f \nabla_k (e^{-f} F_i{}^k), \\ \dfrac{\partial f}{\partial t} = \chi - 2R - 3\triangle f + |\nabla f|^2 + \dfrac{1}{3}H^2 + \dfrac{3}{2}F^2. \end{cases}$$

By the same argument, we can obtain the same results in section 3-4.

3 The Monotonicity Formula

Let M be a n-dimensional compact Riemannian manifold with metric g_{ij} , the Levi-Civita connection is given by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

where (g^{ij}) is the inverse of (g_{ij}). The Riemannian curvature tensors read

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp}R^p_{ijl}.$$

The Ricci tensor is the contraction

$$R_{ik} = g^{jl} R_{ijkl}$$

and the scalar curvature is

$$R = g^{ij} R_{ij}.$$

For each field we shall consider the gauge equivalent classes of fields. Two metrics g_1, g_2 are in the same equivalent class if and only if they are differ by a diffeomorphism, i.e., there exists a diffeomorphism $f : M \to M$ such that $g_2 = f^* g_1$. Two gauge fields A_1 and A_2 are equivalent if and only if there exists a function α on M such that $A_2 = A_1 + d\alpha$. Two B-fields B_1 and B_2 are equivalent if and only if there exists an one-form β on M such that $B_2 = B_1 + d\beta$.

From the first variation of S, we can obtain the flow equations

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} + \nabla_i \nabla_j f - \dfrac{1}{4} H_{ikl} H_j{}^{kl} - F_i{}^k F_{jk}], \\ \dfrac{\partial B_{ij}}{\partial t} = e^f \nabla_k (e^{-f} H^k{}_{ij}), \\ \dfrac{\partial A_i}{\partial t} = -e^f \nabla_k (e^{-f} F_i{}^k), \\ \dfrac{\partial \phi}{\partial t} = \chi - 2R - 3\triangle f + |\nabla f|^2 + \dfrac{1}{3} H^2 + \dfrac{3}{2} F^2. \end{cases}$$

If φ_t is a one-parameter group of diffeomorphisms generated by a vector field ∇f, we have

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} - \frac{1}{4} H_{ikl} H_j{}^{kl} - F_{ik} F_j{}^k),$$

$$\frac{\partial A_i}{\partial t} = -\nabla_k F_i{}^k + \frac{\partial}{\partial x^i} (\nabla^k f A_k),$$

$$\frac{\partial B_{ij}}{\partial t} = \nabla_k H^k{}_{ij} + \frac{\partial}{\partial x^i} (\nabla^k f B_{kj}) + \frac{\partial}{\partial x^j} (\nabla^k f B_{ik}).$$

Let $\tilde{A} = A - d\beta$ where $\dfrac{\partial \beta}{\partial t} = \nabla^k f A_k$, then $\tilde{F} = F$ and

$$\frac{\partial \tilde{A}_i}{\partial t} = -\nabla_k \tilde{F}_i{}^k.$$

Similarly, let $\tilde{B} = B + d\omega$ where $\dfrac{\partial \omega_i}{\partial t} = \nabla^k f B_{ik}$, then

$$\frac{\partial \tilde{B}_{ij}}{\partial t} = \nabla_k (\tilde{H}^k{}_{ij}).$$

Because A and $\tilde{A}$ (B and $\tilde{B}$) are in the same gauge equivalent class, we still denote $\tilde{A}$ ($\tilde{B}$) as A (B). Now we consider the flow equation

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} - \frac{1}{4} H_{ikl} H_j{}^{kl} - F_{ik} F_j{}^k), \tag{6}$$

$$\frac{\partial A_i}{\partial t} = -\nabla_k F_i{}^k, \tag{7}$$

$$\frac{\partial B_{ij}}{\partial t} = \nabla_k (H^k{}_{ij}). \tag{8}$$

Theorem 3.1 *Let g_{ij}, A_i, B_{ij} and f evolve according to the coupled flow*

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2[R_{ij} - \dfrac{1}{4}H_{ikl}H_j{}^{kl} - F_i{}^k F_{jk}], \\ \dfrac{\partial B_{ij}}{\partial t} = \nabla_k H^k{}_{ij}, \\ \dfrac{\partial A_i}{\partial t} = -\nabla_k F_i{}^k, \\ \dfrac{\partial f}{\partial t} = \chi - 2R - 3\triangle f + 2|\nabla f|^2 + \dfrac{1}{3}H^2 + \dfrac{3}{2}F^2. \end{cases}$$

Then

$$\frac{dS}{dt} = \int \Big[(-\chi + R - |\nabla f|^2 + 2\triangle f - \frac{1}{12}H^2 - \frac{1}{2}F^2)^2 + 2(R_{ij} + \nabla_i\nabla_j f - \frac{1}{4}H_{ikl}H_j{}^{kl} - F_i{}^k F_{jk})^2$$
$$+ 2(\nabla_k F_i{}^k - F_i{}^k\nabla_k f)^2 + \frac{1}{2}(\nabla_k H_k{}^{ij} - H_k{}^{ij}\nabla_k f)^2\Big] e^{-f} dV.$$

In particular S is nondecreasing in time and the monotonicity is strict unless we are on the critical points.

Proof.

$$\begin{aligned} \frac{dS}{dt} &= \int d^3x\sqrt{g}e^{-f}(\frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial t} - \frac{\partial f}{\partial t})(-\chi + R + 2\triangle f - |\nabla f|^2 - \frac{1}{12}H^2 - \frac{1}{2}F^2) \\ &\quad + \int d^3x\sqrt{g}e^{-f}\frac{\partial g_{ij}}{\partial t}(-R_{ij} - \nabla_i\nabla_j f + \frac{1}{4}H_{ikl}H_j{}^{kl} + F_i{}^k F_{jk}) \\ &\quad + \int d^3x\sqrt{g}e^{-f}\frac{\partial A_i}{\partial t}(-2\nabla_k(F_i{}^k e^{-f})e^f) + \frac{\partial B_{ij}}{\partial t}(\frac{1}{2}\nabla_k(H^k{}_{ij}e^{-f})e^f) \\ &= \int (\triangle f - |\nabla f|^2)(-\chi + R - |\nabla f|^2 + 2\triangle f - \frac{1}{12}H^2 - \frac{1}{2}F^2)e^{-f}dV \\ &\quad + \int [-\chi + R - |\nabla f|^2 + 2\triangle f - \frac{1}{12}H^2 - \frac{1}{2}F^2]^2 e^{-f}dV \\ &\quad + \int 2(R_{ij} + \nabla_i\nabla_j f - \frac{1}{4}H_{ikl}H_j{}^{kl} - F_{ik}F_j{}^k)^2 e^{-f}dV \\ &\quad + \int -2\nabla_i\nabla_j f(R_{ij} + \nabla_i\nabla_j f - \frac{1}{4}H_{ikl}H_j{}^{kl} - F_{ik}F_j{}^k)e^{-f}dV \\ &\quad + \int 2(\nabla_k F_i{}^k - F_i{}^k\nabla_k f)^2 e^{-f}dV + \int \frac{1}{2}(\nabla_k H^k{}_{ij} - H^k{}_{ij}\nabla_k f)^2 e^{-f}dV \\ &\quad + \int 2F_i{}^k\nabla_k f(\nabla_k F_i{}^k - F_i{}^k\nabla_k f)e^{-f}dV + \int \frac{1}{2}H^k{}_{ij}\nabla_k f(\nabla_k H^k{}_{ij} - H^k{}_{ij}\nabla_k f)e^{-f}dV. \end{aligned}$$

By the similar argument of Ricci flow, we have

$$\int (\triangle f - |\nabla f|^2)(R - |\nabla f|^2 + 2\triangle f)e^{-f}dV = 2\int \nabla_i\nabla_j f(\nabla_i\nabla_j f + R_{ij})e^{-f}dV.$$

And noting the following properties

$$\nabla_m F_{ij} + \nabla_j F_{mi} + \nabla_i F_{jm} = 0,$$

$$\nabla_m H_{ijk} = \nabla_i H_{mjk} + \nabla_j H_{imk} + \nabla_k H_{ijm},$$

we have

$$\begin{aligned}
&\int (\triangle f - |\nabla f|^2)(-\chi - \frac{1}{12}H^2 - \frac{1}{2}F^2)e^{-f}dV \\
= &\int g^{ij}(\nabla_i\nabla_j f - \nabla_i f\nabla_j f)(-\chi - \frac{1}{12}H^2 - \frac{1}{2}F^2)e^{-f}dV \\
= &\int g^{ij}\nabla_i f\nabla_j(\chi + \frac{1}{12}H^2 + \frac{1}{2}F^2)e^{-f}dV \\
= &\int g^{ij}\nabla_i f(\frac{1}{6}\nabla_j H_{pkl}H^{pkl} + \nabla_j F_{kl}F^{kl})e^{-f}dV \\
= &\int g^{ij}\nabla_i f(\frac{1}{6}(\nabla_p H_{jkl} + \nabla_k H_{pjl} + \nabla_l H_{pkj})H^{pkl} + (-\nabla_k F_{lj} - \nabla_l F_{jk})F^{kl})e^{-f}dV \\
= &\int g^{ij}\nabla_i f(\frac{1}{2}\nabla_p H_{jkl}H^{pkl} + 2\nabla_k F_{jl}F^{kl})e^{-f}dV \\
= &\int (-\frac{1}{2}g^{ij}\nabla_p\nabla_i f H_{jkl}H^{pkl} - 2g^{ij}\nabla_k\nabla_i f F_{jl}F^{kl})e^{-f}dV \\
&+ \int \frac{1}{2}g^{ij}H_{jkl}\nabla_i f(-\nabla_p H^{pkl} + \nabla_p f H^{pkl})e^{-f}dV + \int 2g^{ij}\nabla_i f F_{jl}(\nabla_k f F^{kl} - \nabla_k F^{kl})e^{-f}dV \\
= &\int 2\nabla_i\nabla_j f(-\frac{1}{4}H_{ikl}H_j{}^{kl} - F_{ik}F_j{}^k)e^{-f}dV + \int \frac{1}{2}\nabla_k f H^k{}_{ij}(H^p{}_{ij}\nabla_p f - \nabla_p H^p{}_{ij})e^{-f}dV \\
&+ \int 2\nabla_k f F_i{}^k(F_i{}^k\nabla_k f - \nabla_k F_i{}^k)e^{-f}dV.
\end{aligned}$$

Combining with the above argument, we finish the proof.

Let $u = e^{-f}$ be the lowest eigenfunction of the Schrodinger operator, i.e.

$$(R - \frac{1}{12}H^2 - \frac{1}{2}F^2 - 4\Delta)u = \lambda u,$$

or,

$$R - \frac{1}{12}H^2 - \frac{1}{2}F^2 + 2\Delta f - |\nabla f|^2 = \lambda.$$

It minimizes the functional

$$S(g, A, B, f) = \int_M dV e^{-f/2}(R - \frac{1}{12}H^2 - \frac{1}{2}F^2 - 4\Delta)e^{-f/2} / \int_M e^{-f}dV.$$

We have a new functional

$$\lambda(g,A,B)=inf_{\{f|\int_M e^{-f}dV=1\}}S(g,A,B,f).$$

Let $\lambda(t)=\lambda(g(t),A(t),B(t))$, we have

$$\frac{d\lambda}{dt}=\int_M(|R_{ij}+\nabla_i\nabla_j f-\frac{1}{4}H_{ikj}H_j^{kl}-F_{ik}F_j^k|^2+\frac{1}{4}|\nabla^k H_{kij}-H_{kij}\nabla^k f|^2+|\nabla_k F_i^k-F_i^k\nabla_k f|^2)e^{-f}dV.$$

We have then (see also [9]):

1) $\lambda(t)$ is monotone, i.e. $\frac{d\lambda(t)}{dt}\geq 0$.

2) Critical points of (*) are the same as critical points of λ.

4 Critical points

Consider the functional

$$\begin{aligned} S&=\int_M d^3x\sqrt{g}e^{-f}(-\chi+R+|\nabla f|^2)-\frac{1}{2}e^{-f}H\wedge *H-e^{-f}F\wedge *F \\ &=\int d^3x\sqrt{g}e^{-f}(-\chi+R+|\nabla f|^2-\frac{1}{12}H^2-\frac{1}{2}F^2). \end{aligned} \tag{9}$$

Its first variation can be expressed as follows

$$\begin{aligned} \delta S=&\int d^3x\sqrt{g}e^{-f}(\frac{1}{2}g^{ij}\delta g_{ij}-\delta f)(-\chi+R+2\triangle f-|\nabla f|^2-\frac{1}{12}H^2-\frac{1}{2}F^2) \\ &+\int d^3x\sqrt{g}e^{-f}\delta g_{ij}(-R_{ij}-\nabla_i\nabla_j f+\frac{1}{4}H_{ikl}H_j{}^{kl}+F_i{}^kF_{jk}) \\ &+\int d^3x\sqrt{g}e^{-f}\delta A_i(-2\nabla_k(F_i{}^ke^{-f})e^f)+\delta B_{ij}(\frac{1}{2}\nabla_k(H^k{}_{ij}e^{-f})e^f). \end{aligned} \tag{10}$$

The $U(1)$ gauge field A is a one-form potential whose field strength $F=dA$. The Wess-Zumino field B is a two-form potential whose field strength $H=dB,\eta$ is the volume form, f is a dilaton. And in 3-dimension manifold, the field strength is proportional to the Levi-Civita tensor $H_{\mu\nu\rho}=H(x)\eta_{\mu\nu\rho}$, where $H(x)$ is a scalar field and $\eta^{\mu\nu\rho}=\epsilon^{\mu\nu\rho}/\sqrt{g}$ is the completely skewsymmetric Levi-Civita tensor. Therefore, the critical points satisfy the following equations

$$R_{ij}+\nabla_i\nabla_j f-\frac{1}{4}H_{ikl}H_j{}^{kl}-F_i{}^kF_{jk}=0, \tag{11}$$

$$\nabla_k(F_i{}^k e^{-f}) = 0, \tag{12}$$

$$\nabla_k(H^k_{ij} e^{-f}) = 0, \tag{13}$$

$$-\chi + R + 2\triangle f - |\nabla f|^2 - \frac{1}{12}H^2 - \frac{1}{2}F^2 = 0. \tag{14}$$

Suppose M is a compact Riemannian manifold. From (12) and (13), we can obtain $F = H = 0$ at the critical points of the general Ricci flow on M. In fact,

$$\begin{aligned}\int_M F^2 e^{-f} dV &= \int_M F^{ij} F_{ij} e^{-f} dV = \int_M F^{ij}(\nabla_i A_j - \nabla_j A_i) e^{-f} dV \\ &= 2\int_M F^{ij} \nabla_i A_j e^{-f} dV = -2\int_M \nabla_i(F^{ij} e^{-f}) A_j dV = 0,\end{aligned}$$

$$\begin{aligned}\int_M H^2 e^{-f} dV &= \int_M H^{ijk} H_{ijk} e^{-f} dV = \int_M H^{ijk}(\nabla_k B_{ij} + \nabla_i B_{jk} + \nabla_j B_{ki}) e^{-f} dV \\ &= 3\int_M H^{ijk} \nabla_i B_{jk} e^{-f} dV = -3\int_M \nabla_i(H^{ijk} e^{-f}) B_{jk} dV = 0.\end{aligned}$$

Remark: Although the fields F and H do not provide any help in the study of critical points of general Ricci flow for compact Riemannian manifold, they maybe play an important role for the noncompact case.

5 Evolution of Curvatures

By virtue of the curvature tensor evolution equations of the Ricci flow, we can obtain the curvature tensor evolution equations under the gradient flow (1). Let us choose a normal coordinate system $\{x^i\}$ around a fixed point $x \in M$ such that $\frac{\partial g_{ij}}{\partial x^k} = 0$ and $g_{ij}(p) = \delta_{ij}$.

Theorem 5.1 *Under the gradient flow (1), the curvature tensor satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t} R_{ijkl} &= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\ &+ \frac{1}{4}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\ &+ \frac{1}{4}g^{mn}(H_{kpq}H_m^{pq}R_{ijnl} + H_{mpq}H_l{}^{pq}R_{ijkn}) \\ &+ \nabla_i\nabla_l(F_k{}^p F_{jp}) - \nabla_i\nabla_k(F_j{}^p F_{lp}) - \nabla_j\nabla_l(F_k{}^p F_{ip}) + \nabla_j\nabla_k(F_i{}^p F_{lp}) \\ &+ g^{mn}(F_k{}^p F_{mp}R_{ijnl} + F_m^{\ p} F_{lp}R_{ijkn}),\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\triangle$ is the Laplacian with respect to the evolving metric.

Proof. At the point $x \in M$, which we has chosen a normal coordinate system such that $\frac{\partial g_{ij}}{\partial x^k} = 0$, we compute

$$\frac{\partial}{\partial t}\Gamma^h_{jl} = \frac{1}{2}\frac{\partial}{\partial t}g^{hm}\left(\frac{\partial g_{ml}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^m}\right) + \frac{1}{2}g^{hm}\left[\frac{\partial}{\partial x^j}(\frac{\partial g_{ml}}{\partial t}) + \frac{\partial}{\partial x^l}(\frac{\partial g_{mj}}{\partial t}) - \frac{\partial}{\partial x^m}(\frac{\partial g_{jl}}{\partial t})\right],$$

$$\begin{aligned}
\frac{\partial}{\partial t}R^h_{ijl} &= \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial t}\Gamma^h_{jl}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial t}\Gamma^h_{il}\right) \\
&= -\frac{1}{2}g^{hp}g^{qm}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial^2 g_{ml}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{mj}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^m}\right) \\
&\quad + \frac{1}{2}g^{hp}g^{qm}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial^2 g_{ml}}{\partial x^j \partial x^i} + \frac{\partial^2 g_{mi}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^m}\right) \\
&\quad + \frac{1}{2}g^{hm}\left[\frac{\partial^2}{\partial x^i \partial x^l}\left(\frac{\partial g_{mj}}{\partial t}\right) - \frac{\partial^2}{\partial x^i \partial x^m}\left(\frac{\partial g_{jl}}{\partial t}\right) - \frac{\partial^2}{\partial x^j \partial x^l}\left(\frac{\partial g_{mi}}{\partial t}\right) + \frac{\partial^2}{\partial x^j \partial x^m}\left(\frac{\partial g_{il}}{\partial t}\right)\right] \\
&= \frac{1}{2}g^{hm}\left[\frac{\partial^2}{\partial x^i \partial x^l}\left(\frac{\partial g_{mj}}{\partial t}\right) - \frac{\partial^2}{\partial x^i \partial x^m}\left(\frac{\partial g_{jl}}{\partial t}\right) - \frac{\partial^2}{\partial x^j \partial x^l}\left(\frac{\partial g_{mi}}{\partial t}\right) + \frac{\partial^2}{\partial x^j \partial x^m}\left(\frac{\partial g_{il}}{\partial t}\right)\right] \\
&\quad - g^{hp}\frac{\partial g_{pq}}{\partial t}R^q_{ijl}\ ,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ijkl} &= \frac{\partial}{\partial t}R^h_{ijl}g_{kh} + R^h_{ijl}\frac{\partial}{\partial t}g_{kh} \\
&= \frac{1}{2}\left[\frac{\partial^2}{\partial x^i \partial x^l}\left(\frac{\partial g_{kj}}{\partial t}\right) - \frac{\partial^2}{\partial x^i \partial x^k}\left(\frac{\partial g_{jl}}{\partial t}\right) - \frac{\partial^2}{\partial x^j \partial x^l}\left(\frac{\partial g_{ki}}{\partial t}\right) + \frac{\partial^2}{\partial x^j \partial x^k}\left(\frac{\partial g_{il}}{\partial t}\right)\right]\ ,
\end{aligned}$$

then we have

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ijkl} &= \frac{\partial^2}{\partial x^i \partial x^k}R_{jl} - \frac{\partial^2}{\partial x^i \partial x^l}R_{kj} + \frac{\partial^2}{\partial x^j \partial x^l}R_{ki} - \frac{\partial^2}{\partial x^j \partial x^k}R_{il} \\
&\quad + \frac{1}{4}\left[\frac{\partial^2}{\partial x^i \partial x^l}(H_{kpq}H_j{}^{pq}) - \frac{\partial^2}{\partial x^i \partial x^k}(H_{jpq}H_l{}^{pq}) - \frac{\partial^2}{\partial x^j \partial x^l}(H_{kpq}H_i{}^{pq}) + \frac{\partial^2}{\partial x^j \partial x^k}(H_{ipq}H_l{}^{pq})\right] \\
&\quad + \frac{\partial^2}{\partial x^i \partial x^l}(F_k{}^pF_{jp}) - \frac{\partial^2}{\partial x^i \partial x^k}(F_j{}^pF_{lp}) - \frac{\partial^2}{\partial x^j \partial x^l}(F_k{}^pF_{ip}) + \frac{\partial^2}{\partial x^j \partial x^k}(F_i{}^pF_{lp}) \\
&\triangleq I_1 + \frac{1}{4}I_2 + I_3.
\end{aligned}$$

By the identity (see [3])

$$\begin{aligned}
&\nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} \\
&= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj})
\end{aligned}$$

and

$$\nabla_i\nabla_k R_{jl} = \frac{\partial^2 R_{jl}}{\partial x^i \partial x^k} - R_{ml}\frac{\partial}{\partial x^i}\Gamma^m_{kj} - R_{jm}\frac{\partial}{\partial x^i}\Gamma^m_{kl}\ ,$$

we have

$$
\begin{aligned}
I_1 &= \nabla_i\nabla_k R_{jl} + R_{ml}\frac{\partial}{\partial x^i}\Gamma^m_{kj} + R_{jm}\frac{\partial}{\partial x^i}\Gamma^m_{kl} - \nabla_i\nabla_l R_{kj} - R_{km}\frac{\partial}{\partial x^i}\Gamma^m_{lj} - R_{mj}\frac{\partial}{\partial x^i}\Gamma^m_{lk} \\
&\quad - \nabla_j\nabla_k R_{il} - R_{ml}\frac{\partial}{\partial x^j}\Gamma^m_{ki} - R_{im}\frac{\partial}{\partial x^j}\Gamma^m_{kl} + \nabla_j\nabla_l R_{ki} + R_{km}\frac{\partial}{\partial x^j}\Gamma^m_{li} + R_{mi}\frac{\partial}{\partial x^j}\Gamma^m_{lk} \\
&= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} - R_{km}R^m_{ijl} + R_{ml}R^m_{ijk} \\
&= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),
\end{aligned}
$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$.

Now we compute I_2.

It is easily verified that

$$
\nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) = \frac{\partial^2}{\partial x^i\partial x^k}(H_{jpq}H_l{}^{pq}) - H_{mpq}H_l{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{kj} - H_{jpq}H_m{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{kl}\ .
$$

As a result, we obtain

$$
\begin{aligned}
I_2 &= \nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) + H_{mpq}H_j{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{lk} + H_{kpq}H_m{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{lj} - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) \\
&\quad - H_{mpq}H_l{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{kj} - H_{jpq}H_m{}^{pq}\frac{\partial}{\partial x^i}\Gamma^m_{kl} - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) - H_{mpq}H_i{}^{pq}\frac{\partial}{\partial x^j}\Gamma^m_{lk} \\
&\quad - H_{kpq}H_m{}^{pq}\frac{\partial}{\partial x^j}\Gamma^m_{li} + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq}) + H_{mpq}H_l{}^{pq}\frac{\partial}{\partial x^j}\Gamma^m_{ki} + H_{ipq}H_m{}^{pq}\frac{\partial}{\partial x^j}\Gamma^m_{kl} \\
&= \nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq}) \\
&\quad + H_{kpq}H_m{}^{pq}R^m_{ijl} + H_{mpq}H_l{}^{pq}R^m_{jik} \\
&= \nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq}) \\
&\quad + g^{mn}(H_{kpq}H_m{}^{pq}R_{ijnl} + H_{mpq}H_l{}^{pq}R_{ijkn})\ .
\end{aligned}
$$

Now it remains to compute the last term. The following identity

$$
\nabla_i\nabla_k(F_j{}^pF_{lp}) = \frac{\partial^2}{\partial x^i\partial x^k}(F_j{}^pF_{lp}) - F_m{}^pF_{lp}\frac{\partial}{\partial x^i}\Gamma^m_{kj} - F_j{}^pF_{mp}\frac{\partial}{\partial x^i}\Gamma^m_{kl}
$$

yields

$$
\begin{aligned}
I_3 &= \nabla_i\nabla_l(F_k{}^pF_{jp}) + F_m{}^pF_{jp}\frac{\partial}{\partial x^i}\Gamma^m_{lk} + F_k{}^pF_{mp}\frac{\partial}{\partial x^i}\Gamma^m_{lj} - \nabla_i\nabla_k(F_j{}^pF_{lp}) \\
&\quad - F_m{}^pF_{lp}\frac{\partial}{\partial x^i}\Gamma^m_{kj} - F_j{}^pF_{mp}\frac{\partial}{\partial x^i}\Gamma^m_{kl} - \nabla_j\nabla_l(F_k{}^pF_{ip}) - F_m{}^pF_{ip}\frac{\partial}{\partial x^j}\Gamma^m_{lk} \\
&\quad - F_k{}^pF_{mp}\frac{\partial}{\partial x^j}\Gamma^m_{li} + \nabla_j\nabla_k(F_i{}^pF_{lp}) + F_m{}^pF_{lp}\frac{\partial}{\partial x^j}\Gamma^m_{ki} + F_i{}^pF_{mp}\frac{\partial}{\partial x^j}\Gamma^m_{kl} \\
&= \nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp}) \\
&\quad + g^{mn}(F_k{}^pF_{mp}R_{ijnl} + F_m{}^pF_{lp}R_{ijkn})\ .
\end{aligned}
$$

Combining the above discussions, we complete the proof of the theorem. $\square$

Theorem 5.2 *The Ricci curvature satisfies the following evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ik} &= \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk} \\ &\quad + \frac{1}{4}g^{jl}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\ &\quad + \frac{1}{4}g^{mn}(H_{kpq}H_m^{pq}R_{in} - g^{jl}H_{mpq}H_l{}^{pq}R_{ijkn}) \\ &\quad + g^{jl}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\ &\quad + g^{mn}(F_k{}^pF_{mp}R_{in} - g^{jl}F_m^pF_{lp}R_{ijkn}).\end{aligned}$$

Proof. By Theorem 5.1, we can compute

$$\begin{aligned}\frac{\partial}{\partial t}R_{ik} &= \frac{\partial}{\partial t}R_{ijkl}g^{jl} + R_{ijkl}\frac{\partial}{\partial t}g^{jl} = \frac{\partial}{\partial t}R_{ijkl}g^{jl} - g^{jp}g^{lq}R_{ijkl}\frac{\partial}{\partial t}g_{pq} \\ &= g^{jl}[\triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} \\ &\quad + R_{ijkp}R_{ql})] + 2g^{jp}g^{lq}R_{ijkl}R_{pq} \\ &\quad + \frac{1}{4}g^{jl}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\ &\quad + \frac{1}{4}g^{jl}g^{mn}(H_{kpq}H_m^{pq}R_{ijnl} + H_{mpq}H_l{}^{pq}R_{ijkn}) - \frac{\epsilon_H}{2}R_{ijkl}g^{jp}g^{lq}H_{pmn}H_q{}^{mn} \\ &\quad + g^{jl}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\ &\quad + g^{mn}(F_k{}^pF_{mp}R_{in} + g^{jl}F_m^pF_{lp}R_{ijkn}) - 2\epsilon_F R_{ijkl}g^{jp}g^{lq}F_p{}^mF_{qm} \\ &= \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk} \\ &\quad + \frac{1}{4}g^{jl}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\ &\quad + \frac{1}{4}g^{mn}(H_{kpq}H_m^{pq}R_{in} - g^{jl}H_{mpq}H_l{}^{pq}R_{ijkn}) \\ &\quad + g^{jl}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\ &\quad + g^{mn}(F_k{}^pF_{mp}R_{in} - g^{jl}F_m^pF_{lp}R_{ijkn}).\end{aligned}$$

$\square$

Theorem 5.3 *The scalar curvature satisfies the following evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}R &= \triangle R + 2|Ric|^2 + \frac{1}{2}g^{jl}g^{ik}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq})] \\ &\quad + 2g^{jl}g^{ik}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp})] - g^{ip}R_{ik}\left(\frac{1}{2}H_{pmn}H^{kmn} + 2F_{pm}F^{km}\right).\end{aligned}$$

Proof. By a direct calculation, we have

$$\begin{aligned}
\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t}R_{ik}g^{ik} + R_{ik}\frac{\partial}{\partial t}g^{ik} = \frac{\partial}{\partial t}R_{ik}g^{ik} - R_{ik}g^{ip}g^{kq}\frac{\partial}{\partial t}g_{pq} \\
&= g^{ik}(\triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}) + 2g^{ip}g^{kq}R_{ik}R_{pq} \\
&\quad + \frac{1}{4}g^{jl}g^{ik}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq}) - \nabla_j\nabla_l(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_l{}^{pq})] \\
&\quad + \frac{1}{4}g^{ik}g^{mn}(H_{kpq}H_m^{\ pq}R_{in} - g^{jl}H_{mpq}H_l{}^{pq}R_{ijkn}) - \frac{1}{2}g^{ip}g^{kq}R_{ik}H_{pmn}H_q{}^{mn} \\
&\quad + g^{ik}g^{jl}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp}) - \nabla_j\nabla_l(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{lp})] \\
&\quad + g^{ik}g^{mn}(F_k{}^pF_{mp}R_{in} - g^{jl}F_m^{\ p}F_{lp}R_{ijkn}) - 2g^{ip}g^{kq}R_{ik}F_p{}^mF_{qm} \\
&= \triangle R + 2|Ric|^2 + \frac{1}{2}g^{jl}g^{ik}[\nabla_i\nabla_l(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_l{}^{pq})] \\
&\quad + 2g^{jl}g^{ik}[\nabla_i\nabla_l(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{lp})] - g^{ip}R_{ik}\left(\frac{1}{2}H_{pmn}H^{kmn} + 2F_{pm}F^{km}\right). \ \square
\end{aligned}$$

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