A Central Limit Theorem for a sequence of Brownian motions in the unit sphere in $\mathbb{R}^n$

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Abstract

We use a Stochastic Differential Equation satisfied by Brownian motion taking values in the unit sphere $S_{n-1} \subset \mathbb{R}^n$ and we obtain a Central Limit Theorem for a sequence of such Brownian motions. We also generalize the results to the case of the n-dimensional Ornstein-Uhlenbeck processes.

Key words: Central Limit Theorem, Brownian motion in the unit sphere in $\mathbb{R}^n$, Ornstein-Uhlenbeck processes.

1 Introduction

This paper may be regarded as an extension to higher dimensions of the 2-dimensional study made in [VYH11].

We now consider a sequence of Brownian motions $\left(\Theta_t^{(k)}, t \geq 0\right)$, $k \in \mathbb{N}$ taking values in the unit sphere $S_{n-1}(\subset \mathbb{R}^n)$, all starting from the same point on the sphere. In Section 2, we introduce a general representation of $\Theta^{(k)}$ in terms of a Stochastic Differential Equation. Using this representation, we describe in detail in Section 3 the limit in law, as $K \to \infty$, for the renormalized sum:

$$Z_t^K \equiv \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \left(\Theta_t^{(k)} - E\left[\Theta_t^{(k)}\right] \right)$$

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of these processes, indexed by $t \geq 0$, and taking values in $\mathbb{R}^n$. Of course, one could invoke the classical Central Limit Theorem (CLT), at least for the finite dimensional marginals of $\left(Z_t^K, t \geq 0\right)$, as $K \to \infty$. However, with the help of stochastic calculus, there is much more to say about the description of the asymptotics. Finally, in this Section, we remark that the CLT can be generalized to Ornstein-Uhlenbeck processes taking values in the unit sphere $S_{n-1}(\subset \mathbb{R}^n)$. Three technical points are gathered in an Appendix.

Further extensions may also be obtained, by following e.g. [Itô83] or [Ochi85] and studying for which class of functions $f(\Theta)$ we can obtain a functional CLT such as (12) (see below) for $f(\Theta_t)$, instead of the unique function $f_*(\Theta) = \Theta$ which we study here.

2 A presentation of Brownian motion in the sphere S_{n-1}

As remarked in [Str71] and [Yor84] (eq. (4.j), p.34), Brownian motion $(\Theta_t, t \geq 0)$ in the unit sphere $S_{n-1} \subset \mathbb{R}^n$ may be viewed as the solution of a Stochastic Differential Equation:

$$\Theta_t = \Theta_0 + \int_0^t \sigma^{0,1}(\Theta_s) \cdot dB_s - \frac{n-1}{2} \int_0^t ds \; \Theta_s \; . \tag{1}$$

In (1), $B_t \equiv \left(B_t^{(i)}, i \leq n\right)$, $t \geq 0$, denotes a n-dimensional Brownian motion starting from $a \neq 0$, while $(\sigma^{0,1}(x), x \in S_{n-1})$ denotes the family of $n \times n$ matrices [Kry80], defined by:

$$\sigma^{0,1}(x) = (\delta_{i,j} - x_i x_j)_{i,j \leq n}, \quad (x \in S_{n-1}) \tag{2}$$

and/or characterized by:

$$\sigma^{0,1}(x) \cdot x = 0, \quad \text{and} \quad \sigma^{0,1}(x) \cdot y = y, \quad \text{if } y \cdot x = 0. \tag{3}$$

Note that $\sigma^{0,1}(x)$ is symmetric and satisfies: $\sigma^{0,1}(x)\sigma^{0,1}(x) = \sigma^{0,1}(x)$.
Thus, from (3), we deduce that:

$$\sigma^{0,1}(x)m = m - (m \cdot x)\,x, \; m \in \mathbb{R}^n, \tag{4}$$

$$\left(\sigma^{0,1}(x)m\right) \cdot \left(\sigma^{0,1}(x)m'\right) = (m \cdot m') - (x \cdot x')(m \cdot m'), m, m' \in \mathbb{R}^n. \tag{5}$$

3 A Central Limit Theorem for a sequence of Brownian motions in the sphere $S_{n-1}(\subset \mathbb{R}^n)$

Let $\Theta^{(1)}, \ldots, \Theta^{(k)}, \ldots$ be a sequence of such identically distributed Brownian motions in the sphere S_{n-1}. We aim for a Central Limit Theorem concerning:

$$Z_t^K \equiv \frac{1}{\sqrt{K}} \sum_{k=1}^K \left(\Theta_t^{(k)} - E\left[\Theta_t^{(k)}\right]\right). \tag{6}$$

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Adding K equations of the kind of (1) term by term, for $\left(\Theta_t^{(k)}, k \leq K\right)$, it is immediate that:

$$Z_t^K = M_t^K - \frac{n-1}{2} \int_0^t ds \, Z_s^K, \tag{7}$$

with

$$M_t^K = \frac{1}{\sqrt{K}} \sum_{k=1}^K \int_0^t \sigma^{0,1}(\Theta_s^{(k)}) \cdot dB_s^{(k)}. \tag{8}$$

Thus, from (7), we obtain:

$$Z_t^K = \exp\left(\frac{-(n-1)t}{2}\right) \int_0^t \exp\left(\frac{(n-1)s}{2}\right) dM_s^K. \tag{9}$$

Now, clearly, the Central Limit Theorem for $\left(Z_t^K\right)$, $K \to \infty$, which we are seeking, will follow from the limit in law of the martingales $\left(M_t^K, t \geq 0\right)$, as $K \to \infty$. We now state both limit results in the following:

Theorem 3.1 a) *The sequence of martingales $\left(M_t^K, t \geq 0\right)$ converges in law, as $K \to \infty$, towards:*

$$M_t^{(\infty)} = \sqrt{1 - \frac{1}{n}} \left\{ \Theta(0) \int_0^t \sqrt{1 - e^{-ns}} d\beta_s + \int_0^t \sqrt{1 + \frac{e^{-ns}}{n-1}} dB_s' \right\}, \tag{10}$$

where $(\beta_s, s \geq 0)$ is a 1-dimensional BM and $(B_s', s \geq 0)$ is a $(n-1)-$dimensional BM taking values in the hyperplane which is orthogonal to $\Theta(0)$, and B' is independent of β.
b) *Consequently, $\left(Z_t^K, t \geq 0\right)$ converges in law, as $K \to \infty$, towards:*

$$Z_t^{(\infty)} = \exp\left(-\frac{n-1}{2}t\right) \int_0^t \exp\left(\frac{n-1}{2}s\right) dM_s^{(\infty)}. \tag{11}$$

Proof of Theorem 3.1: Using the Law of Large Numbers, it is not difficult to show that[††]:

$$\left(M_t^K, t \geq 0\right) \xrightarrow[K \to \infty]{(law)} \int_0^t \sqrt{E\left[\sigma^{0,1}(\Theta_s^{(1)})\right]} \cdot dB_s^{(1)} \equiv \left(M_t^\infty, t \geq 0\right), \tag{12}$$

where $Q(s) \equiv E\left[\sigma^{0,1}(\Theta_s^{(1)})\right]$ is a deterministic matrix, depending on s. The RHS of (12) is a centered Gaussian martingale in $\mathbb{R}^n$. Before computing the square root involved in (12), we shall first calculate (see (2) for the definition of $\sigma^{0,1}$):

$$E\left[\sigma^{0,1}(\Theta_s^{(1)})\right] = \left(\delta_{i,j} - E\left[\Theta_s^{(i)}\Theta_s^{(j)}\right]\right)_{i,j \leq n}. \tag{13}$$

[††]In the Appendix A.1, a more general result, concerning $\frac{1}{\sqrt{K}} \int_0^t \sum_{k=1}^K H_s^{(k)} \cdot dB_s^{(k)}$ is presented, where $(B^{(k)}, k = 1, \ldots, K)$ are K independent BMs and $(H^{(k)}, B^{(k)})$ are $k \leq K$ iid random vectors.

In order to calculate $E\left[\Theta_s^{(i)}\Theta_s^{(j)}\right]$ as "naturally" as possible, we consider two generic vectors m and m' in $\mathbb{R}^n$, and we compute:

$$\varphi_{m,m'}(t) \equiv E\left[(m \cdot \Theta_t)(m' \cdot \Theta_t)\right]. \tag{14}$$

Using (1) and the (special) properties of the matrices $\{\sigma^{0,1}(x)\}$, we easily deduce from Itô's formula, that:

$$
\begin{aligned}
E\left[(m \cdot \Theta_t)(m' \cdot \Theta_t)\right] &= (m \cdot \Theta_0)(m' \cdot \Theta_0) - (n-1)\int_0^t ds\, E\left[(m \cdot \Theta_s)(m' \cdot \Theta_s)\right] \\
&\quad + \int_0^t ds\, E\left[\left(\sigma^{0,1}(\Theta_s)m\right) \cdot \left(\sigma^{0,1}(\Theta_s)m'\right)\right].
\end{aligned} \tag{15}
$$

Using (5), (15) simplifies as:

$$
\begin{aligned}
E\left[(m \cdot \Theta_t)(m' \cdot \Theta_t)\right] &= (m \cdot \Theta_0)(m' \cdot \Theta_0) - (n-1)\int_0^t ds\, E\left[(m \cdot \Theta_s)(m' \cdot \Theta_s)\right] \\
&\quad + \int_0^t ds\, \left(m \cdot m' - E\left[(m \cdot \Theta_s)(m' \cdot \Theta_s)\right]\right) \\
&= (m \cdot \Theta_0)(m' \cdot \Theta_0) + (m \cdot m')\,t - n\int_0^t ds\, E\left[(m \cdot \Theta_s)(m' \cdot \Theta_s)\right].
\end{aligned} \tag{16}
$$

Consequently, the function $\varphi_{m,m'}(t) = E\left[(m \cdot \Theta_t)(m' \cdot \Theta_t)\right]$ is the solution of a first order linear differential equation, hence:

$$E\left[(m \cdot \Theta_t)(m' \cdot \Theta_t)\right] = e^{-nt}\left\{(m \cdot \Theta_0)(m' \cdot \Theta_0) + (m \cdot m')\int_0^t e^{ns}ds\right\}. \tag{17}$$

Now, taking $m = e_i$ and $m' = e_j$, where $(e_k; k \leq n)$ is the canonical basis of $\mathbb{R}^n$, the matrix $Q(s)$ has elements:

$$\text{for } i \neq j, \quad (Q(s))_{i,j} = -E\left[\Theta_i(s)\Theta_j(s)\right] = -\Theta_i(0)\Theta_j(0)e^{-ns}, \tag{18}$$

$$
\begin{aligned}
\text{for } i = j, \quad (Q(s))_{i,i} &= 1 - E\left[\Theta_i(s)\Theta_i(s)\right] \\
&= 1 - \left\{(\Theta_i(0))^2 e^{-ns} + e^{-ns}\left(\frac{e^{ns}-1}{n}\right)\right\} \\
&= \left(1 - \frac{1}{n}\right) + e^{-ns}\left(\frac{1}{n} - (\Theta_i(0))^2\right).
\end{aligned} \tag{19}
$$

Finally:

$$
\begin{aligned}
Q(s) &= \left(1 - \frac{1}{n}\right) Id + e^{-ns}\left(\frac{1}{n}\delta_{ij} - \Theta_i(0)\Theta_i(s)\right)_{i,j \leq n} \\
&\equiv \left(1 - \frac{1}{n}\left(1 - e^{-ns}\right)\right) Id + e^{-ns}\left(\Theta_i(0)\Theta_j(0)\right)_{i,j \leq n}.
\end{aligned} \tag{20}
$$

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Using (20) in the RHS of (12), we obtain:

$$M_t^{(\infty)} \equiv \int_0^t \sqrt{Q(s)} \, dB_s. \tag{21}$$

Now $\sqrt{Q(s)} \equiv \Lambda(s)$, where (for the explicit calculation, see Appendix A.2):

$$
\begin{aligned}
\Lambda(s) &\equiv \sqrt{1-\frac{1}{n}}\sqrt{1-e^{-ns}}\,Id + \sqrt{1-\frac{1}{n}}\left(\sqrt{1+\frac{e^{-ns}}{n-1}}-\sqrt{1-e^{-ns}}\right)\sigma^{0,1}\left(\Theta(0)\right)\\
&= \sqrt{1-\frac{1}{n}}\left\{\sqrt{1-e^{-ns}}\,Id + \left(\sqrt{1+\frac{e^{-ns}}{n-1}}-\sqrt{1-e^{-ns}}\right)\sigma^{0,1}\left(\Theta(0)\right)\right\}.
\end{aligned}
\tag{22}
$$

Thus, (21) now writes:

$$
\begin{aligned}
M_t^{(\infty)} &\equiv \int_0^t \Lambda(s)\, dB_s\\
&= \sqrt{1-\frac{1}{n}}\left\{\int_0^t \sqrt{1-e^{-ns}}\,dB_s + \int_0^t\left[\sqrt{1+\frac{e^{-ns}}{n-1}}-\sqrt{1-e^{-ns}}\right]\sigma^{0,1}\left(\Theta(0)\right)dB_s\right\}.
\end{aligned}
\tag{23}
$$

We remark here that, with $\beta_s \equiv \Theta(0) \cdot B_s$,

$$B'_s = B_s - \Theta(0)\beta_s \equiv \sigma^{0,1}\left(\Theta(0)\right)B_s \tag{24}$$

is a $(n-1)$-dimensional BM taking values in the hyperplane which is orthogonal to $\Theta(0)$. Thus, from (23) we deduce (10).
From (9), letting $K \to \infty$, we obtain (11). ∎
Moreover, changing the variables $s = t - u$ and using the dominated convergence Theorem, we have:

$$
\begin{aligned}
Z_t^{(\infty)} &\overset{(law)}{=} \sqrt{1-\frac{1}{n}}\exp\left(-\frac{n-1}{2}t\right)\int_0^t \exp\left(\frac{n-1}{2}s\right)\times\\
&\qquad\qquad\qquad \times\left\{\Theta(0)\sqrt{1-e^{-ns}}\,d\beta_s + \sqrt{1+\frac{e^{-ns}}{n-1}}\,dB'_s\right\}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\overset{s=t-u}{\underset{(law)}{=}} \sqrt{1-\frac{1}{n}}\int_0^t \exp\left(-\frac{n-1}{2}u\right)\left\{\sqrt{1-e^{-n(t-u)}}\,\Theta(0)d\beta_u + \sqrt{1+\frac{e^{-n(t-u)}}{n-1}}\,dB'_u\right\}\\
&\overset{t\to\infty}{\longrightarrow} \sqrt{1-\frac{1}{n}}\int_0^\infty \exp\left(-\frac{n-1}{2}u\right)\Theta(0)d\beta_u + \sqrt{1-\frac{1}{n}}\int_0^\infty \exp\left(-\frac{n-1}{2}u\right)dB'_u.
\end{aligned}
\tag{26}
$$

Proposition 3.2 *The following asymptotic results hold:*

a)

$$Z_t^{(\infty)} \xrightarrow[t\to\infty]{(law)} Z_\infty^{(\infty)}, \qquad (27)$$

where:

$$Z_\infty^{(\infty)} \equiv \sqrt{1 - \frac{1}{n}} \int_0^\infty \exp\left(-\frac{n-1}{2}u\right) dB_u . \qquad (28)$$

b)

$$Z_t^{(\infty)} - \exp\left(-\frac{n-1}{2}t\right) \int_0^t \sqrt{1 - \frac{1}{n}} \exp\left(\frac{n-1}{2}s\right) dB_s \xrightarrow[t\to\infty]{L^2} 0. \qquad (29)$$

Part *a)* of Proposition 3.2 follows from the previous calculations, using (24). In order to prove part *b)*, it suffices to use the expression (25) and the following Proposition, which reinforces the convergence in L^2 result in (29).

Proposition 3.3 *As $t \to \infty$, the Gaussian martingales:*

$$\left(G_t^{(0)}\Theta(0), t \geq 0\right) \equiv \Theta(0)\left(\int_0^t \sqrt{1 - e^{-ns}}e^{\frac{n-1}{2}s}d\beta_s - \int_0^t e^{\frac{n-1}{2}s}d\beta_s, t \geq 0\right), \qquad (30)$$

and

$$(G_t', t \geq 0) \equiv \left(\int_0^t \sqrt{1 + \frac{e^{-ns}}{n-1}}e^{\frac{n-1}{2}s}dB_s' - \int_0^t e^{\frac{n-1}{2}s} dB_s', t \geq 0\right) \qquad (31)$$

converge a.s. and in L^2, and the limit variables are Gaussian, with variances, respectively:
$\left(\frac{\sqrt{\pi}\Gamma(-1+\frac{1}{n})}{n\Gamma(\frac{1}{2}+\frac{1}{n})} - \frac{n+1}{n-1}\right)$, *and* $\frac{2\,_2F_1\left(-\frac{1}{2}, -1+\frac{1}{n}, \frac{1}{n}, \frac{1}{1-n}\right)-1}{n-1}$.

Proof of Proposition 3.3:
a) The increasing process of the real-valued Gaussian martingale $G_t^{(0)}$ is:

$$\int_0^t e^{(n-1)s}\left(\sqrt{1-e^{-ns}}-1\right)^2 ds,$$

which converges, as $t \to \infty$; thus:

$$G_t^{(0)} \xrightarrow[t\to\infty]{} \int_0^\infty \left(\sqrt{1-e^{-ns}}e^{\frac{n-1}{2}s} - e^{\frac{n-1}{2}s}\right) d\beta_s,$$

where the convergence holds both a.s. and in every L^p. Of course, the limit variable is Gaussian and its variance is given by (we change the variables $u = e^{-ns}$ and $B(a,b)$ denotes

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the Beta function with arguments a and b**):

$$\int_0^\infty ds\, e^{(n-1)s} \left(\sqrt{1 - e^{-ns}} - 1\right)^2 = \frac{1}{n} \int_0^1 du\, u^{-2+\frac{1}{n}} \left(\sqrt{1-u} - 1\right)^2$$

$$= \frac{1}{n} \left[\int_0^1 du\, u^{-2+\frac{1}{n}} \left((1-u) - 2\sqrt{1-u} + 1\right) \right]$$

$$= \frac{1}{n} \left\{ B\left(-1 + \frac{1}{n}, 2\right) - 2B\left(-1 + \frac{1}{n}, \frac{3}{2}\right) - \frac{n}{n-1} \right\}$$

$$= \frac{\sqrt{\pi}\,\Gamma(-1 + \frac{1}{n})}{n\Gamma(\frac{1}{2} + \frac{1}{n})} - \frac{n+1}{n-1} \ .$$

To be rigorous, the integral $\int_0^1 du\, u^{-\alpha} \left(\sqrt{1-u} - 1\right)^2$, which is well defined for $0 < \alpha < 1$, can be extended analytically for any complex α with $\mathrm{Re}(\alpha) < 3$.

b) Likewise, the "increasing process" of the vector-valued Gaussian martingale G_t' is:

$$\int_0^t e^{(n-1)s} \left(\sqrt{1 + \frac{e^{-ns}}{n-1}} - 1\right)^2 ds$$

which also converges as $t \to \infty$. The limit variable:

$$\int_0^\infty \left(\sqrt{1 + \frac{e^{-ns}}{n-1}}\, e^{\frac{n-1}{2}s} - e^{\frac{n-1}{2}s}\right) dB_s',$$

is also Gaussian and, by repeating the previous calculation, we easily compute its variance.

∎

Proof of Proposition 3.2:

From Proposition 3.3, by multiplying both processes by $e^{\left(-\frac{(n-1)t}{2}\right)}\sqrt{1 - \frac{1}{n}}$, we obtain (29).

∎

Remark 3.4 (The Ornstein-Uhlenbeck case)
In fact, for every process satisfying:

$$dZ_s = d\mathbb{B}_s + h(|Z_s|)Z_s ds, \tag{32}$$

where $(\mathbb{B}_t, t \geq 0)$ is a n-dimensional Brownian motion (BM) and $h : \mathbb{R}_+ \to \mathbb{R}$ is a bounded function, there is a CLT of the kind of Theorem 3.1. See Appendix A.3 for the proof.

**We recall that $(\Gamma(x), x \geq 0)$ denotes the Gamma function and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

A   Appendix

A.1   Generalization for a class of symmetric matrices

For K independent Brownian motions, and a class of symmetric matrices $H^{(k)}$ such that $(H^{(k)}, B^{(k)})_{k \leq K}$ are iid, we have:

$$\tilde{M}_t^{(K)} \equiv \frac{1}{\sqrt{K}} \int_0^t \sum_{k=1}^K H_s^{(k)} \cdot dB_s^{(k)} \xrightarrow[K\to\infty]{(law)} \int_0^t h_s dB_s, \tag{33}$$

with h_s a deterministic symmetric positive definite matrix and $(B_t, t \geq 0)$ a n-dimensional BM.

Indeed, using m a generic vector in $\mathbb{R}^n$, we have:

$$
\begin{aligned}
m \cdot \tilde{M}_t^{(K)} &\equiv m \cdot \frac{1}{\sqrt{K}} \int_0^t \sum_{k=1}^K H_s^{(k)} \cdot dB_s^{(k)} = \frac{1}{\sqrt{K}} \sum_{k=1}^K \int_0^t \left(H_s^{(k)} m \right) \cdot dB_s^{(k)} \\
&\xrightarrow[K\to\infty]{(law)} \mathcal{N}\left(0; \int_0^t ds E\left[|H_s^1 \cdot m|^2 \right] \right),
\end{aligned}
\tag{34}
$$

and for the variance, we have:

$$\int_0^t ds E\left[|H_s^1 \cdot m|^2 \right] = m \cdot \int_0^t ds E\left[H_s^1 \tilde{H}_s^1 \right] m, \tag{35}$$

and

$$E\left[H_s^1 \right] \equiv h_s^2. \tag{36}$$

Remark A.1 *In our case, we have:* $H_s^k \tilde{H}_s^k = H_s^k$.

A.2   Square root of Q_s

$$
\begin{aligned}
Q(s) &= \left(1 - \frac{1}{n} \right) Id + e^{-ns} \left(\frac{1}{n} \delta_{ij} - \Theta_i(0) \Theta_j(0) \right)_{(i,j \leq n)} \\
&\equiv \left(1 - \frac{1}{n} \right) \left(1 - e^{-ns} \right) Id + e^{-ns} \cdot \sigma^{0,1} \left(\Theta(0) \right).
\end{aligned}
\tag{37}
$$

We are searching for $a(s)$ and $b(s)$ such that:

$$\left(a(s) I + b(s) \sigma^{0,1} \left(\Theta(0) \right) \right)^2 = Q(s),$$

or equivalently:

$$\left(a(s) \right)^2 I + 2 a(s) b(s) \sigma^{0,1} \left(\Theta(0) \right) + \left(b(s) \right)^2 \sigma^{0,1} \left(\Theta(0) \right) = Q(s).$$

We compare with (37) and we find:

$$(a(s))^2 = \left(1 - \frac{1}{n}\right)\left(1 - e^{-ns}\right) \quad ; \quad 2a(s)b(s) + (b(s))^2 = e^{-ns}.$$

Solving this system of equations, we easily obtain:

$$\begin{cases} a(s) = \sqrt{1 - \frac{1}{n}}\sqrt{1 - e^{-ns}} \\ b(s) = \sqrt{1 - \frac{1}{n}}\left(\sqrt{1 - \frac{e^{-ns}}{n-1}} - \sqrt{1 - e^{-ns}}\right). \end{cases} \tag{38}$$

A.3   The Ornstein-Uhlenbeck case

We consider the n-dimensional Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \mathbb{B}_t - \lambda \int_0^t Z_s ds, \tag{39}$$

where $(\mathbb{B}_t, t \geq 0)$ is a n-dimensional Brownian motion (BM), $z_0 \in \mathbb{R}^n$ and $\lambda \geq 0$.

Proposition A.2 *The Ornstein-Uhlenbeck (OU) process $\left(\tilde{\Theta}_t, t \geq 0\right)$ in the unit sphere S_{n-1} is the solution of the Stochastic Differential Equation*

$$\Theta_t^Z = \Theta_0^Z + \int_0^t \sigma^{0,1}(\Theta_s^Z) \cdot d\hat{\mathbb{B}}_s - \left(\frac{n-1}{2} + \lambda\right)\int_0^t ds\ \Theta_s^Z, \tag{40}$$

where $\left(\hat{\mathbb{B}}_t, t \geq 0\right)$ is a n-dimensional BM.

Proof of Proposition A.2:
We shall study $\tilde{\varphi}_t \equiv \frac{Z_t}{|Z_t|}$. We remark that the Jacobi matrix and the Hessian matrix associated respectively to the functions $\Phi(x) \equiv \frac{x}{|x|}$, $(x \neq 0)$ and $g(x) \equiv |x|$ are given by:

$$\left(\frac{\partial}{\partial x_j}\Phi_i(x)\right) = \frac{1}{|x|}\sigma^{0,1}(x) \quad ; \quad \left(\frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right) = \frac{1}{|x|}\sigma^{0,1}(x).$$

Hence, using (39), $\tilde{\varphi}$ satisfies the following Stochastic Differential Equation

$$\begin{aligned} \tilde{\varphi}_t &= \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s) \cdot dZ_s - \frac{n-1}{2}\int_0^t \frac{ds}{|Z_s|^2}\tilde{\varphi}_s \tag{41} \\ &= \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s) \cdot d\mathbb{B}_s - \int_0^t \left(\frac{n-1}{2}\frac{\tilde{\varphi}_s}{|Z_s|^2} + \frac{\lambda Z_s}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s)\right) ds \\ &= \tilde{\varphi}_0 + \int_0^t \frac{1}{|Z_s|}\sigma^{0,1}(\tilde{\varphi}_s) \cdot d\mathbb{B}_s - \int_0^t \tilde{\varphi}_s\left(\frac{n-1}{2}\frac{1}{|Z_s|^2} + \lambda\ \sigma^{0,1}(\tilde{\varphi}_s)\right) ds. \tag{42} \end{aligned}$$

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We can replace the BM $\mathbb{B}$ by another BM:

$$\mathbb{B}_t^* \equiv \int_0^t \left(\sigma^{0,1}(\tilde{\varphi}_s) \cdot d\mathbb{B}_s + \sigma^{1,0}(\tilde{\varphi}_s) \cdot dW_s \right),$$

where $(W_t, t \geq 0)$ is a BM independent from $\mathbb{B}$.

Thus, $\left(\gamma_t \equiv \int_0^t \frac{Z_s}{|Z_s|} \cdot d\mathbb{B}_s, t \geq 0 \right)$ and $\mathbb{B}^*$ are two independent BMs, and from Knight's theorem [ReY99], γ is independent from the BM $\left(\hat{\mathbb{B}}_t, t \geq 0 \right)$ obtained by changing the time scale of $\left(\int_0^t \frac{1}{|Z_s|} d\mathbb{B}_s^* \right)$ with the inverse of $\int_0^t \frac{ds}{|Z_s|^2}$. Finally, $\left(\Theta_t^Z, t \geq 0 \right)$ may be obtained from $(\tilde{\varphi}_t, t \geq 0)$ by making the same change of time scale. ∎

Corollary A.3 *The angular part of a n-dimensional Ornstein-Uhlenbeck process is equal to the angular part of a Brownian motion, considered under the time scale $\alpha_t = \frac{e^{2\lambda t}-1}{2\lambda}$.*

Remark A.4 *For the case of a complex-valued OU process, see [Vak11].*

Proof of Corollary A.3:
It follows easily from equation (40) by taking the angular part. For the new time scale, it suffices to remark that, with $< \cdot >$ denoting the quadratic variation of a martingale:

$$\alpha_t \equiv < \int_0^{\cdot} \sigma^{0,1}(\Theta_s^Z) \cdot d\hat{\mathbb{B}}_s >_t = \int_0^t \left(\sigma^{0,1}(\Theta_s^Z) \right)^2 ds = \int_0^t \sigma^{0,1}(\Theta_s^Z) \, ds \ .$$

∎

Remark A.5 *Proposition A.2 is easily generalized for every process of the kind:*

$$dZ_s = d\mathbb{B}_s + h(|Z_s|)Z_s ds, \tag{43}$$

for every bounded function $h : \mathbb{R}^n \to \mathbb{R}$.

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