An efficient algorithm for the computation of the trace of the symmetrized product of an arbitrary number of Dirac matrices with two indices

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Abstract

Let Γ_a be Dirac matrices in d-dimensional Minkowski spacetime, and let $\beta_i = B_i^{ab}\Gamma_{ab}$, where $\Gamma_{ab} = \Gamma_{[a}\Gamma_{b]}$ and B_i^{ab} are arbitrary antisymmetric tensors. The trace of the symmetrized product of an odd number of β-matrices vanishes identically. The trace of the symmetrized product of $2n$ β-matrices can be written as a sum of certain B-contractions over the integer partitions of n, with every term being multiplied by a numerical factor α. We provide a general algorithm to compute these α-coefficients for any d and up to any desired value of n. The algorithm uses random matrices to generate a linear system of equations whose solution is the set of coefficients for a given n. A recurrence relation among these coefficients is shown to hold in all analyzed cases and is used to greatly simplify the computation for large values of n. Numerical values for the α-coefficients are given for $n = 1, \ldots, 7$.

Keywords: Clifford algebra and Dirac matrices, Field theories in higher dimensions, Computer algebra

1. Introduction

Dirac matrices are ubiquitous in field theory. The calculation of Feynman diagrams in the Standard Model of Particle Physics, for instance, requires computation of the trace of the product of a number of Dirac matrices, a calculation that is somewhat cumbersome but a part of the daily lives of QFT-theorists.

Field theories in spacetime dimensions higher than four continue to make use of Dirac matrices. Dirac matrices with one and two indices [cf. eq. (2)], for instance, provide a representation of the anti-de Sitter algebra, and are thus relevant for theories of gravity.[1] The trace of the symmetrized product of Dirac matrices with two indices produces a symmetric invariant polynomial for the Lorentz algebra [1].

The calculation of these traces is essential for the computation of explicit expressions for (i) characteristic classes for symmetry groups with an $\mathrm{SO}(n)$ subgroup [2, 3], (ii) topological invariants and (iii) generalized Racah–Casimir operators [4]. In the context of high-energy physics, being able to efficiently perform this kind of calculation is crucial for Chern–Simons (CS) gravity and supergravity theories (see, e.g., Refs. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). In order to express a CS (super)gravity Lagrangian in the Lorentz basis (i.e., in terms of the spacetime curvature and torsion) one needs to (i) compute the required symmetric invariant polynomial and (ii) use the mathematical techniques developed in Ref. [16].

The trace problem is deceptively simple in appearance. When there are just a few matrices involved (which corresponds in physics to theories with low spacetime dimensionality, e.g., $d = 3$ or $d = 5$), it indeed

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[1] The de Sitter algebra is obtained by replacing Γ_a with $i\Gamma_a$. The Poincaré algebra can be recovered from any of the de Sitter algebras by means of an İnönü–Wigner contraction.

presents no difficulty whatsoever. However, when more Dirac matrices are involved (i.e., in calculations for higher-dimensional theories, as $d = 11$), the calculations become extremely cumbersome and difficult in practice. This is true not only for analytical but also for numerical computations; calculation time of explicit expressions for the invariant polynomials grows exponentially with the size of the input when the straightforward algorithms in symbolic algebra programs are used.

This problem and similar ones have been analyzed analytically [1], but no general formula or efficient algorithm to solve the issue for an arbitrary number of matrices has been put forward.

In this paper we present a novel approach to the computation of the symmetrized product of an arbitrary number of Dirac matrices with two indices. Our method makes use of random matrices and the partitions of the integers to generate a linear system of equations whose solution provides a set of coefficients α_s that completely characterize the trace.

We give a self-contained description of the problem in section 2, where we also summarize our results. In section 3 we explain our method and give two algorithms that can be used to determine the α-coefficients. We close in section 4 with a brief discussion and conclusions.

2. Formulation of the Problem and Results

Let us consider Dirac matrices Γ_a, $a = 0, \ldots, d-1$, in d-dimensional Minkowski spacetime. By definition, they satisfy the Clifford algebra [17]

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\eta_{ab} \mathbb{1}, \tag{1}$$

where $\eta_{ab} = (- + \cdots +)$ is the usual Minkowski metric and $\mathbb{1}$ stands for the $m \times m$ unit matrix, with $m = 2^{\lfloor d/2 \rfloor}$.

Dirac matrices with two indices, which are the subject of this work, are defined as

$$\Gamma_{ab} = \Gamma_{[a}\Gamma_{b]} = \frac{1}{2}\left(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a\right). \tag{2}$$

Experience shows that the trace of a product of Dirac matrices is most efficiently written with all matrices multiplied by arbitrary antisymmetric tensors. Take, for instance, the trace of the symmetrized product of two Dirac matrices with two indices, and compare the following equations:

$$\mathrm{Tr}\left\{\Gamma_{ab}\Gamma_{cd}\right\} = m\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \tag{3}$$

$$A^{ab} B^{cd} \mathrm{Tr}\left\{\Gamma_{ab}\Gamma_{cd}\right\} = 2m A^{a}{}_{b} B^{b}{}_{a}. \tag{4}$$

The two terms on the right-hand side of eq. (3) have collapsed into one in eq. (4). Greater simplifications are achieved for more complicated cases. If desired, eq. (3) can be recovered from eq. (4) by means of the formal replacement $A^{ab} \to \delta^{ab}_{cd}$, $B^{ab} \to \delta^{ab}_{cd}$, where δ^{ab}_{cd} is the generalized Kronecker delta.

For completeness, let us define the symmetrized product of n matrices M_i, $i = 1, \ldots, n$ as

$$\{M_1 \cdots M_n\} = \frac{1}{n!} \sum_{\pi \in S_n} M_{\pi(1)} \cdots M_{\pi(n)}, \tag{5}$$

where the sum extends over all permutations π in the symmetric group S_n.

Let B_i^{ab}, $i = 1, 2, 3, \ldots$, be arbitrary antisymmetric tensors, and let us define

$$\beta_i = B_i^{ab} \Gamma_{ab}. \tag{6}$$

The symmetrized product of n β-matrices can be written as a linear combination of $\Gamma_{[p]}$-matrices,[2] with $p = 0, 4, 8, \ldots, 2n$ (for n even) or $p = 2, 6, 10, \ldots, 2n$ (for n odd). The only term that contributes to the trace is that proportional to the identity matrix ($p = 0$). For odd d, however, the $\Gamma_{[d]}$ matrix is also proportional

[2] Here $\Gamma_{[p]}$ is a shorthand notation for $\Gamma_{a_1 \cdots a_p} = \Gamma_{[a_1} \cdots \Gamma_{a_p]}$, where we understand $\Gamma_{[0]} = \mathbb{1}$.

to the identity and must be generically taken into account when computing the trace. The expansion of the symmetrized product of the β-matrices includes only Γ-matrices with an even number of indices, so that the $\Gamma_{[d]}$-term never actually shows up in our case. In particular, this means that the trace of the symmetrized product of an odd number of β-matrices vanishes identically.

The trace of the symmetrized product of $2n$ β-matrices, on the other hand, can be written as

$$\mathrm{Tr}\,\{\beta_1 \cdots \beta_{2n}\} = m \sum_{s \vdash n} \alpha_s \mathcal{B}^{(s)}, \tag{7}$$

where the notation $s \vdash n$ [18] indicates that the sum must be performed over all integer partitions s of n, and $\mathcal{B}^{(s)}$ stands for the following sum of contractions of B-tensors:

$$\mathcal{B}^{(s)} = \sum_{\langle i_1 \cdots i_{2n} \rangle} \Big\langle B_{i_1} \cdots B_{i_{2s_1}} \Big\rangle \Big\langle B_{i_{2s_1+1}} \cdots B_{i_{2(s_1+s_2)}} \Big\rangle \cdots \Big\langle B_{i_{2(s_1+\cdots+s_{r-1})+1}} \cdots B_{i_{2(s_1+\cdots+s_r)}} \Big\rangle. \tag{8}$$

In eq. (8), the notation $\langle i_1 \cdots i_{2n} \rangle$ is used to indicate that the sum must be performed over all $i_1, \ldots, i_{2n} \in \{1, \ldots, 2n\}$, with the restriction that they be all different. This implements the permutation of all β-matrices. Every term in the sum contains the product of r factors of the form $\langle B_1 \cdots B_q \rangle$, where r is the length of the partition $s = (s_1, \ldots, s_r)$, $n = s_1 + \cdots + s_r$. The j-th factor in the product represents the trace of the product of $2s_j$ B-tensors, i.e.,

$$\langle B_1 \cdots B_q \rangle = (B_1)^{c_1}{}_{c_2} \, (B_2)^{c_2}{}_{c_3} \cdots (B_q)^{c_q}{}_{c_1} \,, \tag{9}$$

with $q = 2s_j$.

To every term in eq. (7), i.e., to every partition s of n, there corresponds an α_s coefficient. Numerical values for the α-coefficients corresponding to the partitions of $n = 1, \ldots, 7$ are given in Table 1.

The following examples for $n = 1, \ldots, 4$ should help clarify the meaning of eqs. (7) and (8):

$$\mathrm{Tr}\,\{\beta_1\beta_2\} = m \sum_{\langle ij \rangle} \alpha_1 \langle B_i B_j \rangle \,, \tag{10}$$

$$\mathrm{Tr}\,\{\beta_1 \cdots \beta_4\} = m \sum_{\langle ijkl \rangle} \left[\alpha_2 \langle B_i B_j B_k B_l \rangle + \alpha_{11} \langle B_i B_j \rangle \langle B_k B_l \rangle \right], \tag{11}$$

$$\begin{aligned} \mathrm{Tr}\,\{\beta_1 \cdots \beta_6\} = m \sum_{\langle i_1 \cdots i_6 \rangle} & \left[\alpha_3 \langle B_{i_1} \cdots B_{i_6} \rangle + \alpha_{21} \langle B_{i_1} \cdots B_{i_4} \rangle \langle B_{i_5} B_{i_6} \rangle + \right. \\ & \left. + \alpha_{111} \langle B_{i_1} B_{i_2} \rangle \langle B_{i_3} B_{i_4} \rangle \langle B_{i_5} B_{i_6} \rangle \right], \end{aligned} \tag{12}$$

$$\begin{aligned} \mathrm{Tr}\,\{\beta_1 \cdots \beta_8\} = m \sum_{\langle i_1 \cdots i_8 \rangle} & \left[\alpha_4 \langle B_{i_1} \cdots B_{i_8} \rangle + \alpha_{31} \langle B_{i_1} \cdots B_{i_6} \rangle \langle B_{i_7} B_{i_8} \rangle + \right. \\ & + \alpha_{22} \langle B_{i_1} \cdots B_{i_4} \rangle \langle B_{i_5} \cdots B_{i_8} \rangle + \alpha_{211} \langle B_{i_1} \cdots B_{i_4} \rangle \langle B_{i_5} B_{i_6} \rangle \langle B_{i_7} B_{i_8} \rangle + \\ & \left. + \alpha_{1111} \langle B_{i_1} B_{i_2} \rangle \langle B_{i_3} B_{i_4} \rangle \langle B_{i_5} B_{i_6} \rangle \langle B_{i_7} B_{i_8} \rangle \right]. \end{aligned} \tag{13}$$

The proof of eq. (7) is by exhaustion; the right-hand side includes all possible terms that may contribute to the trace of the symmetrized product of $2n$ β-matrices.[3]

Our approach to the computation of the α-coefficients is the subject of section 3.

3. Method

3.1. General Algorithm

The central observation behind the algorithm used in the computation of the α-coefficients shown in Table 1 is the fact that eq. (7) is valid for *arbitrary* tensors B_i^{ab}.

[3]The formula for $\mathrm{Tr}\,(\Gamma_* \{\beta_1 \cdots \beta_n\})$ includes pseudoscalar terms that appear in certain dimensions d (e.g., $\epsilon_{abcd} B_i^{ab} B_j^{cd}$ for $d = 4$) but are absent from $\mathrm{Tr}\,\{\beta_1 \cdots \beta_{2n}\}$, where only Lorentz scalars are allowed. Here $\Gamma_* = \Gamma_0 \cdots \Gamma_{d-1}$ is the d-dimensional generalization of γ_5 in $d = 4$.

n	s	α_s
1	1	1
2	$1+1$	$1/2$
	2	$-2/3$
3	$1+1+1$	$1/6$
	$2+1$	$-2/3$
	3	$32/45$
4	$1+1+1+1$	$1/24$
	$2+1+1$	$-1/3$
	$2+2$	$2/9$
	$3+1$	$32/45$
	4	$-272/315$
5	$1+1+1+1+1$	$1/120$
	$2+1+1+1$	$-1/9$
	$2+2+1$	$2/9$
	$3+1+1$	$16/45$
	$3+2$	$-64/135$
	$4+1$	$-272/315$
	5	$15872/14175$
6	$1+1+1+1+1+1$	$1/720$
	$2+1+1+1+1$	$-1/36$
	$2+2+1+1$	$1/9$
	$2+2+2$	$-4/81$
	$3+1+1+1$	$16/135$
	$3+2+1$	$-64/135$
	$3+3$	$512/2025$
	$4+1+1$	$-136/315$
	$4+2$	$544/945$
	$5+1$	$15872/14175$
	6	$-707584/467775$
7	$1+1+1+1+1+1+1$	$1/5040$
	$2+1+1+1+1+1$	$-1/180$
	$2+2+1+1+1$	$1/27$
	$2+2+2+1$	$-4/81$
	$3+1+1+1+1$	$4/135$
	$3+2+1+1$	$-32/135$
	$3+2+2$	$64/405$
	$3+3+1$	$512/2025$
	$4+1+1+1$	$-136/945$
	$4+2+1$	$544/945$
	$4+3$	$-8704/14175$
	$5+1+1$	$7936/14175$
	$5+2$	$-31744/42525$
	$6+1$	$-707584/467775$
	7	$89473024/42567525$

Table 1: α-coefficients corresponding to the partitions of $n = 1, \dots, 7$.

For illustration purposes, let us focus first on the $n=3$ case. Eq. (12) simplifies greatly if we choose all B-tensors to be equal, since in this case the sum over all different permutations of $i_1,\ldots,i_6\in\{1,\ldots,6\}$ is trivially performed. The result reads

$$\frac{1}{6!m}\mathrm{Tr}\left(\beta^6\right)=\alpha_3\left\langle B^6\right\rangle+\alpha_{21}\left\langle B^4\right\rangle\left\langle B^2\right\rangle+\alpha_{111}\left\langle B^2\right\rangle^3. \tag{14}$$

We wish to cast eq. (14) as a linear equation with three unknowns, namely, α_3, α_{21} and α_{111}. To do this we need to be able to assign numerical values to the left-hand side and to the various $\langle B^q\rangle$-terms that appear on the right-hand side. We accomplish this by (i) picking some antisymmetric tensor B^{ab} with random numerical entries and (ii) choosing an explicit representation for the Γ-matrices.[4] We emphasize that the possibility of choosing the B-tensors at will relies upon the fact that eq. (7) is valid for arbitrary B_i's.

To be able to solve for the α-coefficients we need two more equations. These are readily obtained by randomly selecting two further B-tensors. Denoting the three different choices for the B-tensors by B_k, with $k=1,2,3$, we obtain the following 3×3 linear system:

$$Z_1^{(111)}\alpha_{111}+Z_1^{(21)}\alpha_{21}+Z_1^{(3)}\alpha_3=T_1, \tag{15}$$
$$Z_2^{(111)}\alpha_{111}+Z_2^{(21)}\alpha_{21}+Z_2^{(3)}\alpha_3=T_2, \tag{16}$$
$$Z_3^{(111)}\alpha_{111}+Z_3^{(21)}\alpha_{21}+Z_3^{(3)}\alpha_3=T_3, \tag{17}$$

where

$$T_k=\frac{1}{6!m}\mathrm{Tr}\left(\beta_k^6\right), \tag{18}$$
$$Z_k^{(111)}=\left\langle B_k^2\right\rangle^3, \tag{19}$$
$$Z_k^{(21)}=\left\langle B_k^4\right\rangle\left\langle B_k^2\right\rangle, \tag{20}$$
$$Z_k^{(3)}=\left\langle B_k^6\right\rangle. \tag{21}$$

The method to compute the α-coefficients for any value of n is now clear and can be summarized in the following sequence:

1. Let $p=p(n)$.[5]
2. Choose an explicit representation for the Γ-matrices (see, e.g., Ref. [19]).
3. For $k=1,\ldots,p$, do:
   (a) Pick an antisymmetric tensor B_k^{ab} with random numerical entries.
   (b) Compute
   $$T_k=\frac{1}{(2n)!m}\mathrm{Tr}\left(\beta_k^{2n}\right), \tag{22}$$
   where $\beta_k=B_k^{ab}\Gamma_{ab}$.
   (c) For every partition $s\vdash n$, $n=s_1+\cdots+s_r$, compute
   $$Z_k^{(s)}=\prod_{j=1}^{r}\left\langle B_k^{2s_j}\right\rangle. \tag{23}$$
   The notation $\langle B_k^q\rangle$ stands for [see eq. (9)]
   $$\langle B_k^q\rangle=(B_k)^{c_1}{}_{c_2}(B_k)^{c_2}{}_{c_3}\cdots(B_k)^{c_q}{}_{c_1}. \tag{24}$$
4. The α-coefficients are the solution to the $p\times p$ linear system of equations
$$\sum_{s\vdash n}Z_k^{(s)}\alpha_s=T_k\qquad(k=1,\ldots,p). \tag{25}$$

[4]See section 4 for a discussion of the choice of spacetime dimension d in which to carry out the computation.
[5]The partition function $p(n)$ is the number of partitions of n [18].

3.2. Minimal Algorithm

Careful inspection of the α-coefficients shown in Table 1 shows that there exists a recurrence relation among different coefficients.

Let s be a partition of n. The frequency representation [18] of s is the notation $s = (1^{\mu_1}\ 2^{\mu_2}\ \cdots)$, where μ_j represents the multiplicity of j, i.e., the number of times that a given integer j appears in s.

We find that the coefficient α_s corresponding to the partition $s = (1^{\mu_1}\ 2^{\mu_2}\ \cdots)$ can be written as

$$\alpha_s = \prod_{j=1}^{n} \frac{\alpha_j^{\mu_j}}{\mu_j!}, \tag{26}$$

where α_j are the coefficients associated with the "elementary" partitions $1 = 1$, $2 = 2$, $3 = 3$, etc.

For example, all coefficients associated with the non-elementary partitions of $n = 1, 2, 3$ can be computed from α_1, α_2 and α_3 by means of the equations

$$\alpha_{11} = \frac{\alpha_1^2}{2!}\frac{\alpha_2^0}{0!} = \frac{1}{2}, \tag{27}$$

$$\alpha_{111} = \frac{\alpha_1^3}{3!}\frac{\alpha_2^0}{0!}\frac{\alpha_3^0}{0!} = \frac{1}{6}, \tag{28}$$

$$\alpha_{21} = \frac{\alpha_1^1}{1!}\frac{\alpha_2^1}{1!}\frac{\alpha_3^0}{0!} = -\frac{2}{3}. \tag{29}$$

Of course, this recurrence relation also holds for more complicated cases, such as

$$\alpha_{3211} = \frac{\alpha_1^2}{2!}\frac{\alpha_2^1}{1!}\frac{\alpha_3^1}{1!}\frac{\alpha_4^0}{0!}\frac{\alpha_5^0}{0!}\frac{\alpha_6^0}{0!}\frac{\alpha_7^0}{0!} = -\frac{32}{135}. \tag{30}$$

When applied to an elementary coefficient, eq. (26) yields an identity.

The recurrence relation in eq. (26) can be used to compute the values for the α-coefficients associated with all the non-elementary partitions of n. Its use, however, requires knowledge of the elementary coefficients, for which no closed formula is available. This situation suggests a "minimal" algorithm that (i) calculates elementary coefficients in a manner analogous to that of the "general" algorithm and (ii) computes non-elementary coefficients from eq. (26).

The following sequence describes such an algorithm:

1. Let N be the maximum integer for which we wish to calculate the α-coefficients.
2. Choose an explicit representation for the Γ-matrices.
3. Pick an antisymmetric tensor B^{ab} with random numerical entries.[6]
4. For $n = 1, \ldots, N$, do:
   (a) Compute
   $$T = \frac{1}{(2n)!m}\mathrm{Tr}\left(\beta^{2n}\right), \tag{31}$$
   where $\beta = B^{ab}\Gamma_{ab}$.
   (b) For every partition $s \vdash n$, $n = s_1 + \cdots + s_r$, compute
   $$Z^{(s)} = \prod_{j=1}^{r} \left\langle B^{2s_j} \right\rangle. \tag{32}$$
   (c) Use the recurrence relation (26) to calculate all non-elementary coefficients associated with the partitions of n (this step is empty for $n = 1$).
   (d) Solve
   $$\sum_{s\vdash n} Z^{(s)}\alpha_s = T \tag{33}$$
   for α_n (this is a linear equation with one unknown).

[6] We took $d = 2$ and $B^{01} = +1$, since a two-index antisymmetric tensor has only one degree of freedom in two spacetime dimensions, and overall numerical factors are not significant for the calculation. See section 4 for a discussion of the choice of spacetime dimension d in which to carry out the computation.

4. Discussion and Conclusions

The algorithms described in section 3 turn around the problem of finding formulas for the trace of a product of Dirac matrices. The usual textbook approach starts with eq. (1) and *deduces* the required formulas from there. Our approach here works the other way around. We start by identifying the general form of the equation for the trace of the symmetrized product of $2n$ β-matrices. Eq. (7) amounts to such an identification, since it contains all possible sums of B-contractions that may contribute to the trace. The α-coefficients appear as undetermined parameters, which are computed by demanding validity of eq. (7) in several nontrivial cases.

As stressed in section 3, our method works because eq. (7) holds for *arbitrary* antisymmetric tensors B_i^{ab}. We have used B-tensors with random numerical entries to generate the linear system of equations whose solution provides the α-coefficients. In this sense our approach bears some resemblance to Monte Carlo methods, where random numbers play a crucial role. The use of random matrices,[7] however, is not essential to our calculation. All that is required for the general algorithm to succeed is a set of B-tensors such that every iteration produces an equation for the α-coefficients that is linearly independent from the rest, yielding a full-rank Z matrix [cf. eq. (25)].

The solution we find is, of course, independent of the choice of B-tensors; this is conceptually clear, but can also be verified by running the algorithm several times with different sets of (randomly generated) B-tensors. The fact that the same solution is obtained every time confirms the correctness of eq. (7), i.e., that no other terms can be added to the trace.

The α-coefficients are also independent of the spacetime dimension d, which means that the algorithm should in principle work for any d we choose. There is, however, an important caveat. To produce a solvable system one needs the B-tensors to have a sufficient number of independent components, so that the successive iterations of the algorithm yield linearly independent equations. We find that there is a minimum spacetime dimension $d = 2n$ that allows the Z matrix to achieve full rank. This means that the general algorithm must be run with $d \geq 2n$ in order for a solution to be produced.

The minimal algorithm, with only one linear equation to be solved, works even with a minimum spacetime dimension of $d = 2$.

Is our approach any better than the textbook method? One way to probe into this question is to compare the runtime of both. The textbook method can be implemented in, e.g., Kasper Peeters' excellent computer algebra system "Cadabra" [20, 21]. We were able to deduce, starting only from the definition of Dirac matrices, the α-coefficients for $n = 1, 2, 3$. The $n = 3$ case took some 30 min to be solved on a typical desktop computer,[8] while the $n = 4$ case caused the program to crash. This approach, of course, requires hardly any input and produces the full sought-after formula. Starting from eq. (7), we programmed our general algorithm in the computer algebra system "Maxima" [22] and were able to run it successfully for $n = 1, \ldots, 7$. The $n = 8$ case caused Maxima to run out of memory, a problem that can apparently be solved by adjusting Maxima's internal parameters. Runtime for $n = 1, \ldots, 4$ was negligible, while the $n = 7$ case took under half an hour. The minimal algorithm, which we also programmed in Maxima, had negligible runtime even for $N = 9$. Table 2 summarizes runtime for these different scenarios.

Complexity for the general algorithm grows exponentially with n. Complexity for the minimal algorithm, on the other hand, grows linearly with p.[9] All foreseeable applications of the formula for the trace of a product of $2n$ Dirac matrices with two indices are well covered by the minimal algorithm with negligible runtime.

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[7] To be precise, what we use are actually two-index antisymmetric tensors with random numerical entries.

[8] In 2011 this means a 3.20-GHz CPU, with 3.7 GB of memory.

[9] An asymptotic approximation for the partition function $p(n)$ is given by the Hardy–Ramanujan equation, $p(n) = \left(1/4n\sqrt{3}\right) \exp\left(\pi\sqrt{2n/3}\right)$ [18].

n	p	Textbook Method	General Algorithm	Minimal Algorithm
1	1	negligible	negligible	negligible
2	2	negligible	negligible	negligible
3	3	~ 30 min	negligible	negligible
4	5	crashed	negligible	negligible
5	7		negligible	negligible
6	11		~ 1 min	negligible
7	15		~ 24 min	negligible
8	22		crashed	negligible
9	30			negligible
$\vdots$	$\vdots$			$\vdots$
26	2436			~ 1 min
28	3718			~ 2 min
30	5604			~ 5 min

Table 2: Approximate runtime for the textbook method (as implemented in Cadabra) and the general and minimal algorithms (as implemented in Maxima) on a typical desktop computer. For the minimal algorithm, the first column is understood to mean N, the maximum integer for which the α-coefficients are computed. The second column lists the partition function of n, which corresponds to the number of α-coefficients to be determined.

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