A note on exceptional unimodal singularities and K3 surfaces

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Abstract

This is a short note on the relation between the graded stable derived categories of 14 exceptional unimodal singularities and the derived category of K3 surfaces obtained as compactifications of the Milnor fibers. As a corollary, we obtain a basis of the numerical Grothendieck group similar to the one given by Ebeling and Ploog [EP10].

1 Introduction

Let $f \in \mathbb{C}[x, y, z]$ be a weighted homogeneous polynomial defining one of *Arnold's 14 exceptional unimodal singularities* [Arn75]. The list of corresponding weight systems $(a, b, c; h) = \deg(x, y, z; f)$ is given in Table 1.1. The quotient ring $R = \mathbb{C}[x, y, z]/(f)$ is the homogeneous coordinate ring of a weighted projective line $\mathbb{X}$ in the sense of Geigle and Lenzing [GL87, Len94], and the *Dolgachev number* $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)$ of the singularity is defined as the orders of the isotropy groups of $\mathbb{X}$.

On the other hand, one can choose a distinguished basis $(\alpha_i)_{i=1}^{\gamma_1+\gamma_2+\gamma_3}$ of vanishing cycles of f so that the Coxeter-Dynkin diagram is given by the diagram $\widehat{T}(\gamma_1, \gamma_2, \gamma_3)$ shown in Figure 1.1. The triple $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ defined in this way is called the *Gabrielov number* of the singularity. The *strange duality* discovered by Arnold [Arn75] states that 14 exceptional unimodal singularities come in pairs $(f, \check{f})$ such that the Dolgachev number of f is equal to the Gabrielov number of $\check{f}$ and vice versa. Pinkham [Pin77] and Dolgachev and Nikulin gave an interpretation of the strange duality in terms of algebraic cycles and transcendental cycles of K3 surfaces.

Let $g(x, y, z, w) \in \mathbb{C}[x, y, z, w]$ be a general weighted homogeneous polynomial with $\deg(x, y, z, w; g) = (a, b, c, 1; h)$ and $S = \mathbb{C}[x, y, z, w]/(g)$ be the quotient ring. The Deligne-Mumford stack

$$\mathcal{Y} = \operatorname{Proj} S = [(g^{-1}(0) \setminus \mathbf{0})/\mathbb{C}^\times]$$

is a compactification of the Milnor fiber of f. The *stable derived category* of S is defined as the quotient category

$$D^b_{\text{sing}}(\operatorname{gr} S) = D^b(\operatorname{gr} S)/D^{\text{perf}}(\operatorname{gr} S)$$

of the bounded derived category $D^b(\operatorname{gr} S)$ of finitely-generated $\mathbb{Z}$-graded S-modules by the full triangulated subcategory $D^{\text{perf}}(\operatorname{gr} S)$ consisting of bounded complexes of projective

name	$(a,b,c;h)$	$\boldsymbol{\delta}$	$\boldsymbol{\gamma}$	dual
E_{12}	$(6,14,21;42)$	$(2,3,7)$	$(2,3,7)$	E_{12}
E_{13}	$(4,10,15;30)$	$(2,4,5)$	$(2,3,8)$	Z_{11}
Z_{11}	$(6,8,15;30)$	$(2,3,8)$	$(2,4,5)$	E_{13}
E_{14}	$(3,8,12;24)$	$(3,3,4)$	$(2,3,9)$	Q_{10}
Q_{10}	$(6,8,9;24)$	$(2,3,9)$	$(3,3,4)$	E_{14}
Z_{12}	$(4,6,11;22)$	$(2,4,6)$	$(2,4,6)$	Z_{12}
W_{12}	$(4,5,10;20)$	$(2,5,5)$	$(2,5,5)$	W_{12}
Z_{13}	$(3,5,9,18)$	$(3,3,5)$	$(2,4,7)$	Q_{11}
Q_{11}	$(4,6,7;18)$	$(2,4,7)$	$(3,3,5)$	Z_{13}
W_{13}	$(3,4,8,16)$	$(3,3,4)$	$(2,5,6)$	S_{11}
S_{11}	$(4,5,6,16)$	$(2,5,6)$	$(3,3,4)$	W_{13}
Q_{12}	$(3,5,6;15)$	$(3,3,6)$	$(3,3,6)$	Q_{12}
S_{12}	$(3,4,5;13)$	$(3,4,5)$	$(3,4,5)$	S_{12}
U_{12}	$(3,4,4;12)$	$(4,4,4)$	$(4,4,4)$	U_{12}

Table 1.1: 14 exceptional unimodal singularities

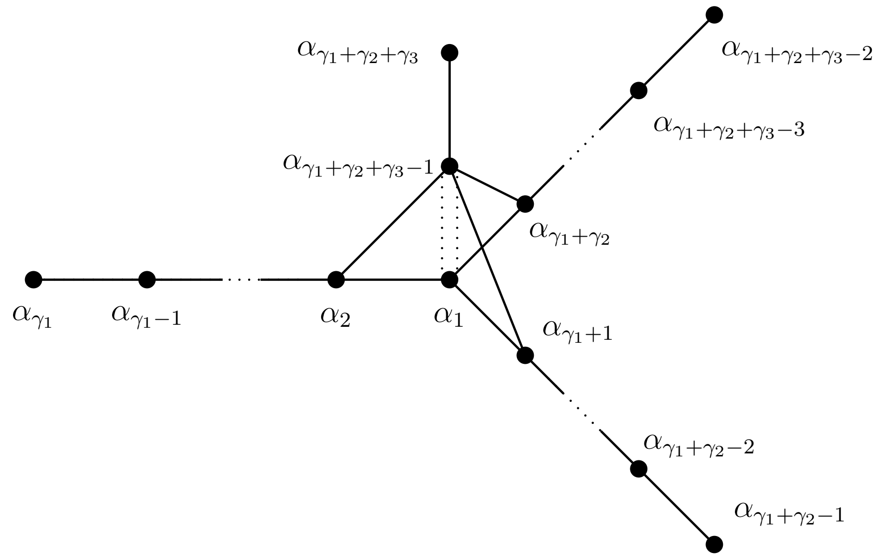

Figure 1.1: The diagram $\widehat{T}(\gamma_1,\gamma_2,\gamma_3)$

modules [Buc87, Hap91, Kra05, Orl09]. Since S is Gorenstein with parameter zero, one has an equivalence

$$\Psi_S : D^b_{\mathrm{sing}}(\operatorname{gr} S) \xrightarrow{\sim} D^b \operatorname{coh} \mathcal{Y}$$

by Orlov [Orl09, Theorem 2.5]. The stable derived category of $R = S/(w)$ is defined similarly as $D^b_{\mathrm{sing}}(\operatorname{gr} R) = D^b(\operatorname{gr} R)/D^{\mathrm{perf}}(\operatorname{gr} R)$, and studied by Kajiura, Saito and Takahashi [KST09] and Lenzing and de la Peña [LdlP06]. Since R is Gorenstein with parameter -1, one has a functor

$$\Psi_R : D^b \operatorname{coh} \mathcal{Y}_\infty \to D^b_{\mathrm{sing}}(\operatorname{gr} R)$$

and a semiorthogonal decomposition

$$D^b_{\mathrm{sing}}(\operatorname{gr} R) = \left\langle R/\mathfrak{m}, \Psi_R(D^b \operatorname{coh} \mathcal{Y}_\infty)\right\rangle$$

by Orlov [Orl09, Theorem 2.5], where $R/\mathfrak{m}_R$ is the residue field by the maximal ideal $\mathfrak{m}_R = (x, y, z)$ of the origin and $\mathcal{Y}_\infty := \operatorname{Proj} R$ is the substack at infinity.

Let $\Phi_{\mathrm{gr}} : \operatorname{gr} R \to \operatorname{gr} S$ be the functor sending an R-module to the same module considered as an S-module by the natural projection $\varphi : S \to R$. Since R is perfect as an S-module, the functor Φ_{gr} sends a perfect complex of R-modules to a perfect complex of S-modules and induces the *push-forward functor*

$$\Phi_{\mathrm{sing}} : D^b_{\mathrm{sing}}(\operatorname{gr} R) \to D^b_{\mathrm{sing}}(\operatorname{gr} S)$$

studied in [DM, AP]. Let further $\iota : \mathcal{Y}_\infty \hookrightarrow \mathcal{Y}$ be the inclusion.

Theorem 1.1. *The composite functor*

$$\Psi_S \circ \Phi_{\mathrm{sing}} \circ \Psi_R : D^b \operatorname{coh} \mathcal{Y}_\infty \to D^b \operatorname{coh} \mathcal{Y}$$

is isomorphic to the push-forward functor

$$\iota_* : D^b \operatorname{coh} \mathcal{Y}_\infty \to D^b \operatorname{coh} \mathcal{Y},$$

and the image of the residue field $R/\mathfrak{m}_R$ in $D^b_{\mathrm{sing}}(\operatorname{gr} R)$ by $\Psi_S \circ \Phi_{\mathrm{sing}}$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathcal{Y}}[2]$ shifted by two.

Let Σ be a fan in $\boldsymbol{N} \cong \mathbb{Z}^3$ such that the associated toric 3-fold X_Σ is a weak Fano manifold and the strict transform $Y \subset X_\Sigma$ of $\mathcal{Y} \subset \mathbb{P}(a, b, c, 1)$ is the minimal resolution of the coarse moduli space of $\mathcal{Y}$ [Kob08]. The McKay correspondence as a derived equivalence [KV00, BKR01] gives

$$\Upsilon : D^b \operatorname{coh} \mathcal{Y} \xrightarrow{\sim} D^b \operatorname{coh} Y. \tag{1.1}$$

Recall that the *numerical Grothendieck group* $\mathcal{N}(Y)$ is the quotient of the Grothendieck group $K(Y)$ of Y by the radical of the Euler form

$$\chi([\mathcal{E}], [\mathcal{F}]) = \sum_i \dim \operatorname{Ext}^i(\mathcal{E}, \mathcal{F}).$$

The integral cohomology ring

$$H^\bullet(Y, \mathbb{Z}) = H^0(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \oplus H^4(Y, \mathbb{Z})$$

equipped with the *Mukai pairing*

$$((a_0, a_2, a_4),\ (b_0, b_2, b_4)) = (a_2, b_2) - (a_0, b_4) - (a_4, b_0)$$

is called the *Mukai lattice*. For a coherent sheaf $\mathcal{E}$, its *Mukai vector* is defined by

$$v(\mathcal{E}) = \operatorname{ch}(\mathcal{E})\sqrt{\operatorname{td}(Y)}.$$

Riemann-Roch theorem states that

$$\chi(\mathcal{E}, \mathcal{F}) = -(v(\mathcal{E}), v(\mathcal{F})),$$

so that $\mathcal{N}(Y)$ can be identified with the image of $K(Y)$ in the Mukai lattice $H^\bullet(Y, \mathbb{Z})$ under the map $v : K(Y) \to H^\bullet(Y, \mathbb{Z})$.

Corollary 1.2. *There is a full exceptional collection* $(\mathcal{S}_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}$ *in* $D^b_{\mathrm{sing}}(\operatorname{gr} R)$ *whose images* $\mathcal{E}_\alpha = \Upsilon \circ \Psi_S \circ \Phi_{\mathrm{sing}}(\mathcal{S}_\alpha)$ *satisfy the following:*

- *The endomorphism dg algebra of* $\bigoplus_{\alpha=1}^{\delta_1+\delta_2+\delta_3-1} \mathcal{E}_\alpha$ *is the trivial extension of the endomorphism dg algebra of* $\bigoplus_{\alpha=1}^{\delta_1+\delta_2+\delta_3-1} \mathcal{S}_\alpha$.
- *The sequence* $(\mathcal{E}_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}$ *is a spherical collection.*
- *The Coxeter-Dynkin diagram of the spherical collection is* $\widehat{T}(\delta_1, \delta_2, \delta_3)$.
- *The spherical collection is a basis of the numerical Grothendieck group* $\mathcal{N}(Y)$.
- *The spherical collection split-generates* $D^b \operatorname{coh} Y$.

Recall that an object $\mathcal{E}$ is said to be *spherical* if $\operatorname{Hom}^i(\mathcal{E}, \mathcal{E})$ is isomorphic to $\mathbb{C}$ for $i = 0, 2$ and zero otherwise [ST01, Definition 1.1]. A sequence of objects is called a *spherical collection* if each object is spherical. The definition of the *trivial extension* can be found in [Sei10], which is called the *cyclic completion* in [Seg08]. The endomorphism dg algebra of $\bigoplus_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1} \mathcal{E}_\alpha$ is not the trivial extension of the endomorphism dg algebra of $\bigoplus_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1} \mathcal{S}_\alpha$; otherwise, the derived category $D^b \operatorname{coh} Y$ will not depend on the defining equation of Y. The spherical collection $(\mathcal{E}_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}$ has the same properties as the collection given by Ebeling and Ploog [EP10].

Let $(Y, \check{Y})$ be a pair of K3 surfaces obtained as compactifications of Milnor fibers of a dual pair of exceptional unimodal singularities. The *A-model VHS* $(H_{A,\mathbb{Z}}, \nabla^A, \mathscr{F}_A^\bullet, Q_A)$ associated with Y is an integral variation of pure and polarized Hodge structures of weight 2 in a neighborhood

$$U = \left\{ \beta + \sqrt{-1}\omega \in \operatorname{NS}(Y) \otimes \mathbb{C} \;\middle|\; \langle \omega, d \rangle \gg 0 \text{ for any } d \in \operatorname{Eff}(Y) \right\}$$

of the large radius limit in the complexified Kähler moduli space of Y. Here $\operatorname{NS}(Y) \subset \mathcal{N}(Y)$ is the Néron-Severi group of Y, $\operatorname{Eff}(Y) \subset \mathcal{N}(Y)$ is the semigroup of effective curves on Y, $H_{A,\mathbb{Z}}$ is the trivial local system on U with fiber $\mathcal{N}(Y)$ and $\nabla^A = d$ is the associated trivial flat connection on $\mathscr{H}_A = H_{A,\mathbb{Z}} \otimes \mathcal{O}_U$. The polarization Q_A is given by the Mukai pairing, and the Hodge filtration is such that

$$\mho = \exp(\beta + \sqrt{-1}\omega) = \left(1,\ (\beta + \sqrt{-1}\omega),\ \frac{1}{2}(\beta + \sqrt{-1}\omega)^2\right)$$

spans the $(2,0)$-part of $H_{A,\mathbb{C}} = H_{A,\mathbb{Z}} \otimes \mathbb{C}$.

The K3 surface $\check{Y}$ comes in a family $\check{\varphi} : \check{\mathfrak{Y}} \to \check{\mathcal{M}}$ where $\check{\mathcal{M}}$ is an algebraic torus of the same dimension as U. Let $\check{U}'$ be a neighborhood of the *large complex structure limit* in $\check{\mathcal{M}}$ and $\check{U}$ be its universal cover. The local system $R^2 \check{\varphi}_! \mathbb{Z}_{\check{\mathfrak{Y}}}$ carries an integral variation of polarized mixed Hodge structures, and the *B-model VHS* $H_{B,\mathbb{Z}}$ is defined as the pull-back to $\check{U}$ of the graded subquotient $\mathrm{gr}_2^W R^2 \check{\varphi}_! \mathbb{Z}_{\check{\mathfrak{Y}}}$ of weight two. There is a biholomorphic map $\varsigma : \check{U} \to U$ called the *mirror map*. Iritani has introduced certain subsystems $H_{A,\mathbb{Z}}^{\mathrm{amb}} \subset H_{A,\mathbb{Z}}$ and $H_{B,\mathbb{Z}}^{\mathrm{vc}} \subset H_{B,\mathbb{Z}}$ and given an isomorphism

$$\mathrm{Mir}_{\mathcal{Y}} : \varsigma^*(H_{A,\mathbb{Z}}^{\mathrm{amb}}, \nabla^A, \mathscr{F}_A^\bullet, Q_A) \xrightarrow{\sim} (H_{B,\mathbb{Z}}^{\mathrm{vc}}, \nabla^B, \mathscr{F}_B^\bullet, Q_B) \tag{1.2}$$

of integral variations of pure and polarized Hodge structures [Iri, Theorem 6.9].

Corollary 1.3. *One has $H_{A,\mathbb{Z}}^{\mathrm{amb}} = H_{A,\mathbb{Z}}$ and $H_{B,\mathbb{Z}}^{\mathrm{vc}} = H_{B,\mathbb{Z}}$, so that the isomorphism* (1.2) *gives an isomorphism*

$$\mathrm{Mir}_{\mathcal{Y}} : \varsigma^*(H_{A,\mathbb{Z}}, \nabla^A, \mathscr{F}_A^\bullet, Q_A) \xrightarrow{\sim} (H_{B,\mathbb{Z}}, \nabla^B, \mathscr{F}_B^\bullet, Q_B).$$

of integral variations of pure and polarized Hodge structures.

The equalities $H_{A,\mathbb{Z}}^{\mathrm{amb}} = H_{A,\mathbb{Z}}$ and $H_{B,\mathbb{Z}}^{\mathrm{vc}} = H_{B,\mathbb{Z}}$ fail in general for a K3 hypersurface in a smooth toric weak Fano variety. A typical example is the case when Y is the quartic surface in the projective space (cf. [Iri, Section 6.6]). Hodge-theoretic mirror symmetry for the quartic surface is studied in detail by Hartmann [Har].

The organization of this paper is as follows: We prove Theorem 2.1 in Section 2, which is slightly more general than Theorem 1.1. In Section 3, we use an exceptional collection given by Lenzing and de la Penã [LdlP06] to prove Corollary 1.2. Variations of Hodge structures is discussed in Section 4.

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2 Push-forward in stable derived categories

Let k be a field and $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian graded k-algebra. We assume that A is *connected* in the sense that $A_0 = k$, and write the maximal ideal as $\mathfrak{m}_A = \bigoplus_{i \geq 1} A_i$. The graded ring A is said to be *Gorenstein* if A has a finite injective dimension n and

$$\mathbb{R}\mathrm{Hom}_A(k, A) = k(a)[-n]$$

for some integer a, which is called the Gorenstein parameter of A. If $A = k[x_1, \dots, x_n]/f$ for $\deg(x_1, \dots, x_n; f) = (a_1, \dots, a_n; h)$, then A is Gorenstein with parameter $a = a_1 + \cdots + a_n - h$.

Let $\mathrm{gr}\, A$ be the abelian category of finitely-generated $\mathbb{Z}$-graded right A-modules, and $\mathrm{tor}\, A$ be the full subcategory consisting of graded modules which are finite-dimensional over k. The quotient category $\mathrm{gr}\, A / \mathrm{tor}\, A$ will be denoted by $\mathrm{qgr}\, A$, which is equivalent to the abelian category of coherent sheaves on the quotient stack $\mathrm{Proj}\, A = [(\mathrm{Spec}\, A \setminus \mathbf{0}) / \mathbb{G}_{\mathrm{m}}]$ by Serre's theorem [Orl09, Proposition 2.16].

Let $D^b(\operatorname{gr} A)$ be the bounded derived category of $\operatorname{gr} A$. An object of $D^b(\operatorname{gr} A)$ is said to be *perfect* if it is quasi-isomorphic to a bounded complex of projective modules. The full subcategory of $D^b(\operatorname{gr} A)$ consisting of perfect complexes will be denoted by $D^{\mathrm{perf}}(\operatorname{gr} A)$. The quotient category

$$D^b_{\mathrm{sing}}(\operatorname{gr} A) = D^b(\operatorname{gr} A)/D^{\mathrm{perf}}(\operatorname{gr} A)$$

is called the bounded *stable derived category* of $\operatorname{gr} A$ [Buc87, Hap91, Kra05, Orl04].

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. The *right orthogonal* to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects M satisfying $\operatorname{Hom}(N, M) = 0$ for any $N \in \mathcal{N}$. The *left orthogonal* ${}^\perp\mathcal{N}$ is defined similarly. The subcategory $\mathcal{N}$ is said to be *right admissible* if the embedding $I : \mathcal{N} \hookrightarrow \mathcal{D}$ has a right adjoint functor $Q : \mathcal{D} \to N$. Left admissibility is defined similarly as the existence of a left adjoint functor, and $\mathcal{N}$ is said to be admissible if it is both right and left admissible. A subcategory $\mathcal{N}$ is right admissible if and only if for any $X \in \mathcal{D}$, there exists a distinguished triangle $N \to X \to M \to N[1]$ with $N \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Such a triangle is unique up to isomorphism, and one has $Q(X) = N$ in this case. If $\mathcal{N}$ is right admissible, then the quotient category $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^\perp$. Analogous statement also holds for left admissible categories. A sequence $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ of triangulated subcategories in a triangulated category $\mathcal{D}$ is called a *weak semiorthogonal decomposition* if there is a sequence $\mathcal{D}_1 = \mathcal{N}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ of left admissible subcategories such that $\mathcal{N}_i$ is left orthogonal to $\mathcal{D}_{i-1}$ in $\mathcal{D}_i$.

For an integer i, let $\operatorname{gr} A_{\geq i}$ be the full abelian subcategory of $\operatorname{gr} A$ consisting of modules M such that $M_e = 0$ for $e < i$. Let further $\mathcal{S}^A_{\geq i}$ and $\mathcal{P}^A_{\geq i}$ be the full triangulated subcategories of $D^b(\operatorname{gr} A)$ generated by the graded torsion modules $A/\mathfrak{m}_A(e)$ for $e \leq -i$ and graded free modules $A(e)$ for $e \leq -i$ respectively. By [Orl09, Lemma 2.4], the subcategories $\mathcal{S}^A_{\geq i}$ and $\mathcal{P}^A_{\geq i}$ are right and left admissible respectively in $D^b \operatorname{gr} A_{\geq i}$, and let $\mathcal{D}^A_i$ and $\mathcal{T}^A_i$ be their right and left orthogonal subcategories. It follows that one has weak semiorthogonal decompositions

$$D^b(\operatorname{gr} A_{\geq i}) = \left\langle \mathcal{D}^A_i, \mathcal{S}^A_{\geq i} \right\rangle, \tag{2.1}$$

$$D^b(\operatorname{gr} A_{\geq i}) = \left\langle \mathcal{P}^A_{\geq i}, \mathcal{T}^A_i \right\rangle, \tag{2.2}$$

where $\mathcal{D}^A_i$ is equivalent to the quotient category $D^b(\operatorname{gr} A_{\geq i})/\mathcal{S}^A_{\geq i}$ which in turn is equivalent to $D^b(\operatorname{qgr} A)$, and $\mathcal{T}^A_i$ is equivalent to the quotient category $D^b(\operatorname{gr} A_{\geq i})/\mathcal{P}^A_{\geq i}$ which in turn is equivalent to $D^b_{\mathrm{sing}}(\operatorname{gr} A)$. In addition, one has a semiorthogonal decomposition

$$\mathcal{T}^A_0 = \left\langle A/\mathfrak{m}_A, A/\mathfrak{m}_A(-1), \ldots, A/\mathfrak{m}_A(a+1), \mathcal{D}^A_{-a} \right\rangle \tag{2.3}$$

if $a \leq 0$ by Orlov [Orl09, Equation (12)].

Let $\varphi : S \to R$ be a morphism of graded connected Gorenstein rings such that the Gorenstein parameters of S and R are $a_S = 0$ and $a_R = -1$ respectively. Let $\Phi_{\mathrm{gr}} : \operatorname{gr} R \to \operatorname{gr} S$ be the exact functor which sends an R-module to the same module considered as an S-module via φ. The functor Φ_{gr} sends finite-dimensional R-modules to finite-dimensional S-modules, and induces an exact functor $\Phi_{\mathrm{qgr}} : \operatorname{qgr} R \to \operatorname{qgr} S$.

Assume that R has finite projective dimension as an S-module. Then φ sends perfect complexes of R-modules to perfect complexes of S-modules, and induces a functor $\Phi_{\mathrm{sing}} : D^b_{\mathrm{sing}}(\operatorname{gr} R) \to D^b_{\mathrm{sing}}(\operatorname{gr} S)$ of stable derived categories.

Theorem 2.1. *Let $\varphi : S \to R$ be a morphism of graded connected Gorenstein rings with Gorenstein parameters $a_S = 0$ and $a_R = -1$ such that R is perfect as an S-module. Then the composite functor*

$$D^b(\operatorname{qgr} R) \xrightarrow{\sim} \mathcal{D}_1^R \hookrightarrow \mathcal{T}_0^R \xrightarrow{\sim} D^b_{\mathrm{sing}}(\operatorname{gr} R) \xrightarrow{\Phi_{\mathrm{sing}}} D^b_{\mathrm{sing}}(\operatorname{gr} S) \xrightarrow{\sim} \mathcal{T}_0^S = \mathcal{D}_0^S \xrightarrow{\sim} D^b(\operatorname{qgr} S)$$

is isomorphic to the functor $\Phi_{\mathrm{qgr}} : D^b \operatorname{qgr} R \to D^b \operatorname{qgr} S$, and the image of $R/\mathfrak{m}_R \in \mathcal{T}_0^R$ by

$$\mathcal{T}_0^R \xrightarrow{\sim} D^b_{\mathrm{sing}}(\operatorname{gr} R) \xrightarrow{\Phi_{\mathrm{sing}}} D^b_{\mathrm{sing}}(\operatorname{gr} S) \xrightarrow{\sim} \mathcal{T}_0^S = \mathcal{D}_0^S \xrightarrow{\sim} D^b(\operatorname{qgr} S)$$

is isomorphic to $\mathcal{O}[\dim S - 1]$, where $\mathcal{O}$ is the image of the free module $S \in \operatorname{gr} S$ by the projection $\operatorname{gr} S \to \operatorname{qgr} S$ and $[\dim S - 1]$ is the shift in the derived category.

Proof. The first statement is clear from the definitions of the functors Φ_{qgr} and Φ_{sing}. For the second statement, note that the image of $R/\mathfrak{m}_R \in \mathcal{T}_0^R$ by the composition

$$\mathcal{T}_0^R \xrightarrow{\sim} D^b_{\mathrm{sing}}(\operatorname{gr} R) \xrightarrow{\Phi_{\mathrm{sing}}} D^b_{\mathrm{sing}}(\operatorname{gr} S)$$

is $S/\mathfrak{m}_S$. Its image by the equivalence

$$D^b_{\mathrm{sing}}(\operatorname{gr} S) \cong D^b(\operatorname{gr} S_{\geq 0})/\mathcal{P}_{\geq 0}^S \xrightarrow{\sim} \mathcal{T}_0^S,$$

is characterized as the object $N \in \mathcal{T}_0^S$ such that there is a distinguished triangle

$$N \to S/\mathfrak{m}_S \to M \to N[1]$$

with $M \in \mathcal{P}_{\geq 0}^S$. Since S is Gorenstein with parameter zero, one has

$$\operatorname{Hom}(S/\mathfrak{m}_S, S(i)) = \begin{cases} k[-\dim S] & i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

which shows that the cone $N = \operatorname{Cone}(S/\mathfrak{m}_S[-1] \to S[\dim S - 1])$ belongs to $\mathcal{T}_0^S$ and satisfies the desired property with $M = S[\dim S]$. It is clear that $S/\mathfrak{m}_S[-1] \in D^b(\operatorname{gr} A_{\geq 0})$ goes to $0 \in D^b(\operatorname{qgr} A)$ and $S[\dim S - 1] \in D^b(\operatorname{gr} A_{\geq 0})$ goes to $\mathcal{O}[\dim S - 1] \in D^b(\operatorname{qgr} A)$, so that $N \in \mathcal{T}_0^S = \mathcal{D}_0^S \subset D^b(\operatorname{gr} A_{\geq 0})$ goes to $\mathcal{O}[\dim S - 1] \in D^b(\operatorname{qgr} A)$, and Theorem 2.1 it proved. $\square$

3 Spherical collections on K3 surfaces

Let $\mathbb{X}$ be the weighted projective line with weight $\boldsymbol{p} = (p_1, p_2, p_3)$ in the sense of Geigle and Lenzing [GL87]. The abelian category $\operatorname{coh} \mathbb{X}$ of coherent sheaves on $\mathbb{X}$ is equivalent by Serre's theorem [GL87, Section 1.8] to the quotient category $\operatorname{gr} T / \operatorname{tor} T$ of the abelian category $\operatorname{gr} T$ of finitely-generated L-graded T-modules by the full subcategory $\operatorname{tor} T$ consisting of torsion modules. Here L is the abelian group of rank one generated by four elements $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, and $\vec{c}$ with relations $p_1 \vec{x}_1 = p_2 \vec{x}_2 = p_3 \vec{x}_3 = \vec{c}$, and $T = \mathbb{C}[x_1, x_2, x_3]/(x_1^{p_1} + x_2^{p_2} + x_3^{p_3})$ is an L-graded ring of Krull dimension two. Let

$$(\mathcal{P}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1} = (\mathcal{O}, U_1^{(1)}, \ldots, U_1^{(p_1-1)}, U_2^1, \ldots, U_2^{(p_2-1)}, U_3^1, \ldots, U_3^{(p_3-1)}, \mathcal{O}(-\vec{\omega} - \vec{c})[1])$$

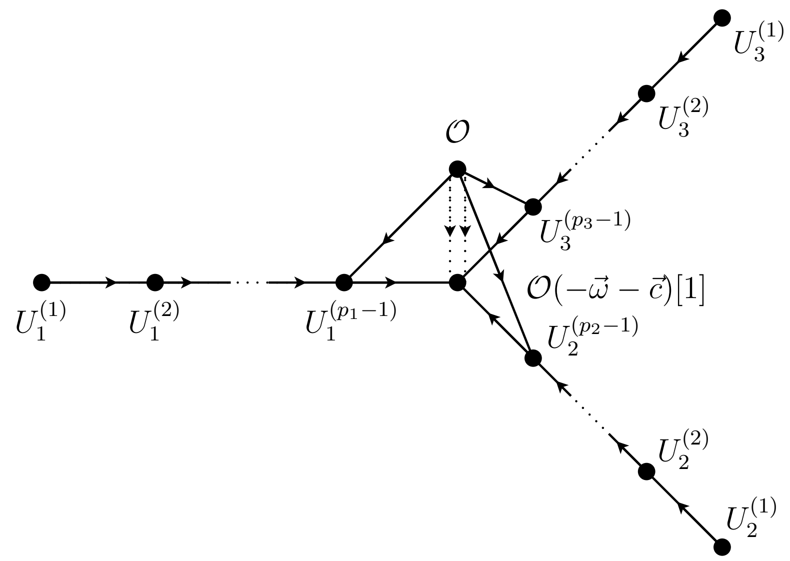

Figure 3.1: The full strong exceptional collection on $\mathbb{X}_{\boldsymbol{p}}$

be the full strong exceptional collection given by Lenzing and de la Penã [LdlP06, Proposition 3.9], where $U_i^{(j)}$ are defined by

$$U_i^{(j)} = \operatorname{coker}(\mathcal{O}(-(p_i-1)\vec{x}_i) \hookrightarrow \mathcal{O}((-p_i+1+j)\vec{x}_i).$$

Let $(\mathcal{S}_\beta)_{\beta=1}^{p_1+p_2+p_3-1}$ be the right dual collection to $(\mathcal{P}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$, which is characterized by the property

$$\dim \operatorname{Hom}(\mathcal{P}_\alpha, \mathcal{S}_\beta) = \delta_{\alpha,\, p_1+p_2+p_3-\beta},$$

and given explicitly as

$$(\mathcal{S}_\beta)_{\beta=1}^{p_1+p_2+p_3-1} = (\mathcal{O}(-\vec{c})[2], \mathcal{O}(-\vec{x}_1)[1], S_1^{(p_1-2)}, \dots, S_1^{(1)}, \dots, \mathcal{O}(-\vec{x}_3)[1], S_3^{(p_3-2)}, \dots, S_3^{(1)}, \mathcal{O}),$$

where

$$S_i^{(j)} = \operatorname{coker}(\mathcal{O}(-(p_i-j-2)\vec{x}_i) \hookrightarrow \mathcal{O}(-(p_i-j-1)\vec{x}_i)).$$

The total morphism algebra of the collection $(\mathcal{P}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$ is isomorphic to the path algebra of the quiver shown in Figure 3.1, where two dotted arrows represent two relations. In terms of quiver representations, $\mathcal{P}_\alpha$ are projective modules and $\mathcal{S}_\alpha$ are simple modules, and one has

$$\dim \operatorname{Hom}^i(\mathcal{S}_\alpha, \mathcal{S}_\beta) = \begin{cases} \delta_{\alpha\beta} & i=0, \\ \#(\text{solid arrows from } \beta \text{ to } \alpha) & i=1, \\ \#(\text{dotted arrows from } \beta \text{ to } \alpha) & i=2. \end{cases}$$

Let $\mathcal{K}$ be the total space of the canonical bundle of $\mathbb{X}$. Since the collection $(\mathcal{S}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$ is full, the push-forward $(\iota_* \mathcal{S}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$ generates the derived category $D^b \operatorname{coh}_{\mathbb{X}} \mathcal{K}$ of coherent sheaves on $\mathcal{K}$ supported on the image of the zero section $\iota : \mathbb{X} \to \mathcal{K}$.

Theorem 3.1 (Segal [Seg08, Theorem 4.2], Ballard [Bal, Proposition 4.14])**.** *Let $\mathcal{S}$ be an object of $D^b \operatorname{coh} \mathbb{X}$ and $\iota_* \mathcal{S}$ be the push-forward of $\mathcal{S}$ along the zero-section. Then the endomorphism dg algebra of $\iota_* \mathcal{S}$ is the trivial extension of the endomorphism dg algebra of $\mathcal{S}$.*

It follows that

$$\operatorname{Hom}^i(\iota_* \mathcal{S}_\alpha, \iota_* \mathcal{S}_\beta) = \operatorname{Hom}^i(\mathcal{S}_\alpha, \mathcal{S}_\beta) \oplus \operatorname{Hom}^{2-i}(\mathcal{S}_\beta, \mathcal{S}_\alpha)^\vee,$$

so that

$$\chi(\iota_* \mathcal{S}_\alpha, \iota_* \mathcal{S}_\beta) = \begin{cases} 2 & \text{if } \alpha = \beta, \\ -1 & \text{if } \alpha \text{ and } \beta \text{ are connected by a solid arrow,} \\ 2 & \text{if } \alpha \text{ and } \beta \text{ are connected by two dotted arrows,} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for $1 \leq \alpha, \beta \leq p_1 + p_2 + p_3 - 1$.

Let $\mathcal{Y}$ be a very general hypersurface of degree h in $\mathbb{P}(a, b, c, 1)$, where $(a, b, c; h)$ is a weight system in Table 1.1. The divisor $\mathcal{Y}_\infty = \{w = 0\} \subset \mathcal{Y}$ at infinity is a weighted projective line whose weight is given by the Dolgachev number of the singularity; $(p_1, p_2, p_3) = (\delta_1, \delta_2, \delta_3)$. Since the formal neighborhood of $\mathcal{Y}_\infty$ in $\mathcal{Y}$ is isomorphic to the formal neighborhood of $\mathbb{X}$ in $\mathcal{K}$, one has an equivalence

$$D^b \operatorname{coh}_{\mathbb{X}} \mathcal{K} \cong D^b \operatorname{coh}_{\mathcal{Y}_\infty} \mathcal{Y} \tag{3.2}$$

of triangulated categories. We fix an isomorphism of the formal neighborhoods and identify $D^b \operatorname{coh}_{\mathcal{Y}_\infty} \mathcal{Y}$ with $D^b \operatorname{coh}_{\mathbb{X}} \mathcal{K}$. Since

$$\operatorname{Hom}^*(\mathcal{O}_\mathcal{Y}, \iota_* \mathcal{S}_\alpha) \cong H^*(\iota_* \mathcal{S}_\alpha) \cong H^*(\mathcal{S}_\alpha) \cong \operatorname{Hom}^*(\mathcal{S}_{p_1+p_2+p_3-1}, \mathcal{S}_\alpha),$$

one has

$$\dim \operatorname{Hom}^i(\mathcal{O}_\mathcal{Y}[1], \iota_* \mathcal{S}_\alpha) = \delta_{i1} \delta_{\alpha, p_1+p_2+p_3-1}, \tag{3.3}$$

so that the Euler form on the spherical collection $(\mathcal{O}_\mathcal{Y}[1], \iota_* \mathcal{S}_1, \ldots, \iota_* \mathcal{S}_n)$ is identical to the spherical collection in Figure 3.3 given by Ebeling and Ploog [EP10].

Lemma 3.2. *The spherical collection*

$$(\mathcal{O}_\mathcal{Y}, \iota_* \mathcal{S}_1, \ldots, \iota_* \mathcal{S}_n)$$

split-generates $D^b \operatorname{coh} \mathcal{Y}$.

Proof. The line bundle $\mathcal{O}_\mathcal{Y}(-k\mathcal{Y}_\infty)$ is contained in the full triangulated subcategory of $D^b \operatorname{coh} \mathcal{Y}$ generated by the above spherical collection for any $k \in \mathbb{N}$, since the cokernel of the inclusion $\mathcal{O}_\mathcal{Y}(-k\mathcal{Y}_\infty) \hookrightarrow \mathcal{O}_\mathcal{Y}$ is supported on $\mathcal{Y}_\infty$ and hence contained in $\operatorname{coh}_{\mathcal{Y}_\infty} \mathcal{Y}$. For any coherent sheaf $\mathcal{E}$, there is a surjection

$$\varphi_0 : \mathcal{O}_\mathcal{Y}(-n_0 \mathcal{Y}_\infty)^{\oplus k_0} \to \mathcal{E}$$

for sufficiently large n_0 and k_0 (i.e. the hyperplane section $\mathcal{Y}_\infty$ is ample). Let $\mathcal{E}_1 = \ker \varphi_0$ be the kernel of this morphism. Then there is a surjection

$$\varphi_1 : \mathcal{O}_{\mathcal{Y}}(-n_1 \mathcal{Y}_\infty)^{\oplus k_1} \to \mathcal{E}_1$$

for sufficiently large n_1 and k_1, and one can set $\mathcal{E}_2 = \ker \varphi_1$. By repeating this process, one obtains a distinguished triangle

$$\mathcal{E}_{k+1}[k] \to \mathcal{F} \to \mathcal{E} \xrightarrow{[+1]} \mathcal{E}_{k+1}[k+1],$$

where $\mathcal{E}_{k+1}$ is a coherent sheaf and

$$\mathcal{F} = \left\{ \mathcal{O}_{\mathcal{Y}}(-n_k \mathcal{Y}_\infty)^{\oplus m_k} \xrightarrow{\varphi_k} \mathcal{O}_{\mathcal{Y}}(-n_{k-1} \mathcal{Y}_\infty)^{\oplus m_{k-1}} \xrightarrow{\varphi_{k-1}} \cdots \xrightarrow{\varphi_0} \mathcal{O}_{\mathcal{Y}}(-n_0 \mathcal{Y}_\infty)^{\oplus k_0} \right\}$$

for any $k \geq 0$. Since $\mathcal{Y}$ is smooth, the homological dimension of $\operatorname{coh} \mathcal{Y}$ is equal to the dimension of $\mathcal{Y}$, and this triangle splits for $k > \dim \mathcal{Y}$. It follows that any coherent sheaf is a direct summand of a complex of locally-free sheaves contained in the full triangulated subcategory of $D^b \operatorname{coh} \mathcal{Y}$ generated by $(\mathcal{S}_\beta)_{\beta=0}^{p_1+p_2+p_3-1}$, and Lemma 3.2 is proved. $\square$

Let Y be the minimal resolution of the coarse moduli space of $\mathcal{Y}$. It can be realized as an anticanonical K3 hypersurface in a toric weak Fano manifold X [Kob08]. It contains the Milnor fiber as an open subset, and the complement is a chain of a chain of (-2)-curves intersecting as in Figure 3.2. It follows that the transcendental lattice of Y is isomorphic to the Milnor lattice of Y. By the McKay correspondence as a derived equivalence [KV00, BKR01], one has an equivalence

$$\Upsilon : D^b \operatorname{coh} \mathcal{Y} \xrightarrow{\sim} D^b \operatorname{coh} Y \tag{3.4}$$

of triangulated categories. Set $\mathcal{E}_0 = \mathcal{O}_Y[1]$ and $\mathcal{E}_\alpha = \Upsilon \circ \iota_*(\mathcal{S}_\alpha)$ for $\alpha = 1, \dots, p_1+p_2+p_3-1$.

Proposition 3.3. *The numerical Grothendieck group $\mathcal{N}(Y)$ is spanned by $([\mathcal{E}_\alpha])_{\alpha=0}^{p_1+p_2+p_3-1}$ and isomorphic to the lattice $\widehat{T}(p_1, p_2, p_3)$.*

Proof. The numerical Grothendieck group $\mathcal{N}(Y)$ is generated by the class $[\mathcal{O}_Y]$ of the structure sheaf, the Néron-Severi group $\operatorname{NS}(Y)$, and the class $[\mathcal{O}_p]$ of a skyscraper sheaf. The structure of $\operatorname{NS}(Y)$ for very general Y is well studied (see e.g. [Bel02]), and generated by the irreducible components of the divisor $E = E_\infty \cup \bigcup_{i=1}^3 \bigcup_{j=1}^{p_i-1} E_j^i$ at infinity. Both the structure sheaves of irreducible components of E and a skyscraper sheaf $\mathcal{O}_p$ on E belong to $D^b \operatorname{coh}_E Y$, which is equivalent to $D^b \operatorname{coh}_{\mathcal{Y}_\infty} \mathcal{Y}$ by the functor Υ. Since $D^b \operatorname{coh}_{\mathcal{Y}_\infty} \mathcal{Y}$ is generated by $([\iota_* \mathcal{S}_\alpha])_{\alpha=1}^{p_1+p_2+p_3-1}$, the collection $([\mathcal{E}_\alpha])_{\alpha=1}^{p_1+p_2+p_3-1}$ generates $\operatorname{NS}(Y)$ and $[\mathcal{O}_p]$, so that the collection $([\mathcal{E}_\alpha])_{\alpha=0}^{p_1+p_2+p_3-1}$ generates $\mathcal{N}(Y)$. Since $\operatorname{rank} \mathcal{N}(Y) = p_1 + p_2 + p_3$, the collection $([\mathcal{E}_\alpha])_{\alpha=0}^{p_1+p_2+p_3-1}$ is a basis of $\mathcal{N}(Y)$. It is clear from (3.1) and (3.3) that $\mathcal{N}(Y)$ is isomorphic to $\widehat{T}(p_1, p_2, p_3)$ as a lattice, and Proposition 3.3 is proved. $\square$

It is an interesting problem to see if the collection $(\mathcal{E}_\alpha)_{\alpha=0}^{p_1+p_2+p_3-1}$ can be related to the collection of Ebeling and Ploog [EP10] shown in Figure 3.3 by an autoequivalence of $D^b \operatorname{coh} Y$.

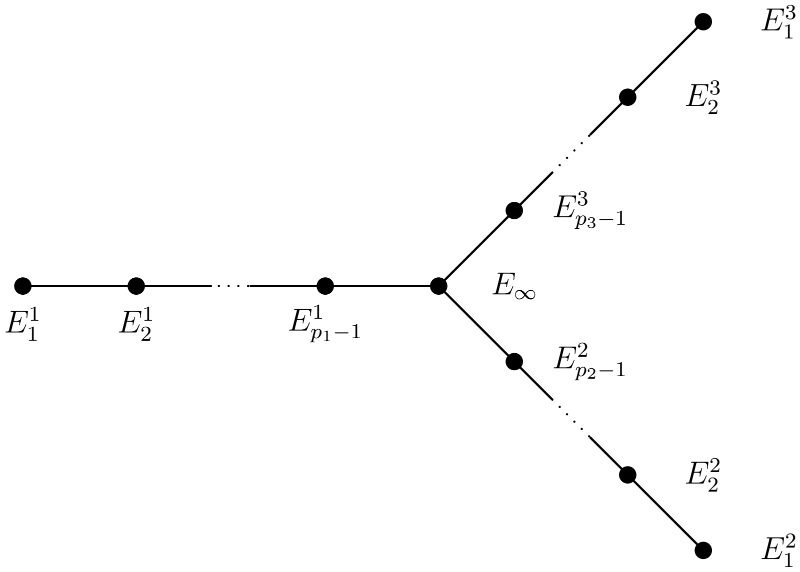

Figure 3.2: The configuration of (-2)-curves at infinity

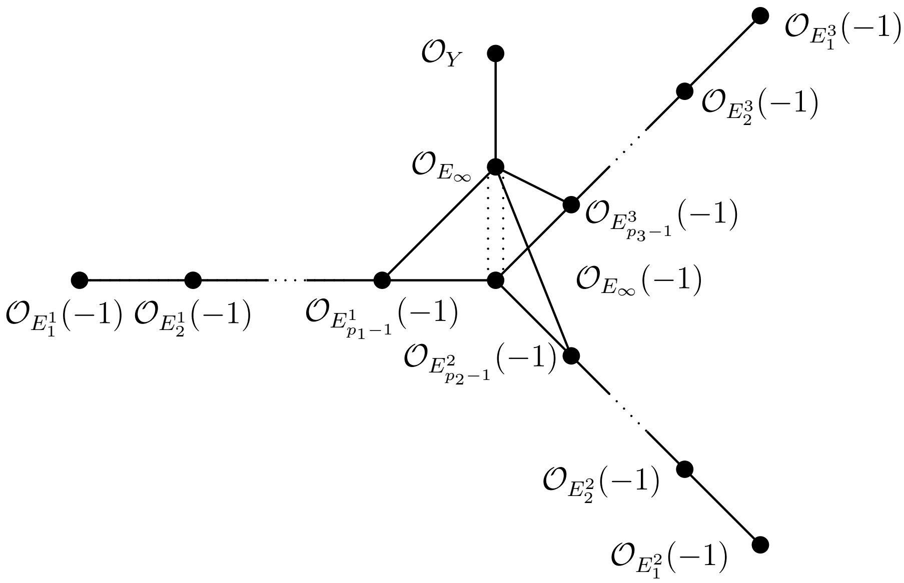

Figure 3.3: A spherical collection of Ebeling and Ploog

4 Variations of Hodge structures

We discuss Hodge-theoretic aspect of mirror symmetry for K3 surfaces in this section [AM97, Dol96, Mor97, KKP08, Iri]. Let $\boldsymbol{N} \cong \mathbb{Z}^3$ be a free abelian group of rank three and $\boldsymbol{M} = \operatorname{Hom}(\boldsymbol{N}, \mathbb{Z})$ be the dual group. Let further (Δ, Δ^*) be a pair of a reflexive polytope $\Delta \subset \boldsymbol{N}_{\mathbb{R}} = \boldsymbol{N} \otimes \mathbb{R}$ and its polar dual polytope $\Delta^* \subset \boldsymbol{M}_{\mathbb{R}}$. Recall that the *fan polytope* of a fan is the convex hull of primitive generators of one-dimensional cones. Choose a pair $(\Sigma, \check{\Sigma})$ of fans in $\boldsymbol{N}$ and $\boldsymbol{M}$, such that the fan polytopes of Σ and $\check{\Sigma}$ are Δ and Δ^* respectively, and the corresponding toric varieties X and $\check{X}$ are smooth. Let $\{b_1, \dots, b_m\} \subset \boldsymbol{N}$ be the set of generators of one-dimensional cones of the fan Σ. One has the *fan sequence*

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} \boldsymbol{N} \to 0$$

and the *divisor sequence*

$$0 \to \boldsymbol{M} \xrightarrow{\beta^*} (\mathbb{Z}^m)^* \to \mathbb{L}^* \to 0$$

where β sends the i-th coordinate vector to b_i and

$$\operatorname{Pic}(X) \cong H^2(X; \mathbb{Z}) \cong \mathbb{L}^*.$$

Set $\mathcal{M} = \mathbb{L}^* \otimes \mathbb{C}^\times$ and $\check{\mathbb{T}} = \boldsymbol{M} \otimes \mathbb{C}^\times$ so that one has the exact sequence

$$1 \to \check{\mathbb{T}} \to (\mathbb{C}^\times)^m \to \mathcal{M} \to 1.$$

The uncompactified mirror $\check{Y}_\alpha$ of a general anticanonical hypersurface $Y \subset X$ is defined by

$$\check{Y}_\alpha = \{y \in \check{\mathbb{T}} \mid W_\alpha(y) = \sum_{i=1}^m \alpha_i y^{b_i} = 1\}$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{C}^\times)^m$. Let $\widetilde{\check{\varphi}} : \widetilde{\check{\mathfrak{Y}}} \to (\mathbb{C}^\times)^m$ be the second projection from

$$\widetilde{\check{\mathfrak{Y}}} = \{(y, \alpha) \in \check{\mathbb{T}} \times (\mathbb{C}^\times)^m \mid W_\alpha(y) = 1\}.$$

The quotient of the family $\widetilde{\check{\varphi}} : \widetilde{\check{\mathfrak{Y}}} \to (\mathbb{C}^\times)^m$ by the free $\check{\mathbb{T}}$-action

$$t \cdot (y, (\alpha_1, \dots, \alpha_m)) = (t^{-1} y, (t^{b_1} \alpha_1, \dots, t^{b_m} \alpha_m))$$

will be denoted by $\check{\varphi} : \check{\mathfrak{Y}} \to \mathcal{M}$ where $\mathcal{M} = (\mathbb{C}^\times)^m / \check{\mathbb{T}}$. Choose an integral basis $p_1, \dots, p_r$ of $\mathbb{L}^* \cong \operatorname{Pic} X$ such that each p_i is nef. This gives the corresponding coordinate $q_1, \dots, q_r$ on $\mathcal{M} = \mathbb{L}^* \otimes \mathbb{C}^\times$. Let $\check{U}' \subset \mathcal{M}$ be a neighborhood of $q_1 = \cdots = q_r = 0$ and $\check{U}$ be its universal cover. The *B-model VHS* $(H_{B,\mathbb{Z}}, \nabla^B, \mathscr{F}_B^\bullet, Q_B)$ on $\check{U}$ consists of the pull-back $H_{B,\mathbb{Z}}$ of the local system $\operatorname{gr}_2^W R^2 \check{\varphi}_! \mathbb{Z}_{\check{\mathfrak{Y}}}$, the Gauss-Manin connection ∇^B on $\mathscr{H}_B = H_{B,\mathbb{Z}} \otimes \mathcal{O}_{\check{U}}$, the Hodge filtration $\mathscr{F}_B^\bullet$, and the polarization Q_B given by

$$Q_B(\omega_1, \omega_2) = \int_{\check{Y}_\alpha} \omega_1 \cup \omega_2.$$

The subsystem of $H_{B,\mathbb{Z}}$ consists of vanishing cycles of W_α will be denoted by $H_{B,\mathbb{Z}}^{\mathrm{vc}}$.

On the A-model side, let

$$H^\bullet_{\mathrm{amb}}(Y;\mathbb{C}) = \mathrm{Im}(\iota^* : H^\bullet(X;\mathbb{C}) \to H^\bullet(Y;\mathbb{C}))$$

be the subspace of $H^\bullet(Y;\mathbb{C})$ coming from the cohomology classes of the ambient toric variety, and set

$$U = \{\sigma = \beta + \sqrt{-1}\omega \in H^2_{\mathrm{amb}}(Y;\mathbb{C}) \mid \langle \omega, d\rangle \gg 0 \text{ for any } d \in \mathrm{Eff}(Y)\}.$$

where $\mathrm{Eff}(Y)$ is the semigroup of effective curves. This open subset U is considered as a neighborhood of the large radius limit point. Since Y is chosen to be general, the restriction map $\iota^* : \mathrm{NS}(X) \to \mathrm{NS}(Y)$ is surjective, so that U here coincides with U given in Section 1. Let $(\sigma^i)_{i=1}^r$ be the coordinate on $H^2_{\mathrm{amb}}(Y;\mathbb{C})$ dual to the basis $(p_i)_{i=1}^r$; $\sigma = \sum_{i=1}^r \sigma^i p_i$.

The *ambient A-model VHS* $(\mathscr{H}'_A, \nabla^{A'}, \mathscr{F}'^{\bullet}_A, Q_A)$ consists ([Iri, Definition 6.2], cf. also [CK99, Section 8.5]) of the locally-free sheaf $\mathscr{H}'_A = H^\bullet_{\mathrm{amb}}(Y) \otimes \mathcal{O}_U$, the Dubrovin connection

$$\nabla^{A'} = d + \sum_{i=1}^r (p_i \circ_\sigma)\, d\sigma^i : \mathscr{H}_A \to \mathscr{H}_A \otimes \Omega^1_U,$$

the Hodge filtration

$$\mathscr{F}'^{\,p}_A = H^{4-2p}_{\mathrm{amb}}(Y) \otimes \mathcal{O}_U,$$

and the Mukai pairing

$$Q_A : \mathscr{H}_A \otimes \mathscr{H}_A \to \mathcal{O}_U,$$

which is symmetric and $\nabla^{A'}$-flat. Let $L_Y(\sigma)$ be the fundamental solution of the quantum differential equation, i.e. the $\mathrm{End}(H^\bullet_{\mathrm{amb}}(Y;\mathbb{C}))$-valued functions satisfying

$$\nabla^{A'}_i L_Y(\sigma) = 0, \qquad i = 1, \dots, r$$

and $L_Y(\sigma) = \mathrm{id} + O(\sigma)$. Since Y is a K3 surface, the quantum cup product $\circ_\sigma$ coincides with the ordinary cup product, and the fundamental solution is given by

$$L(\sigma) = \exp(-\sigma).$$

Let $H'_{A,\mathbb{C}} = \mathrm{Ker}\,\nabla^{A'}$ be the $\mathbb{C}$-local system associated with $\nabla^{A'}$ and define the integral local subsystem $H'_{A,\mathbb{Z}} \subset H'_{A,\mathbb{C}}$ as

$$H'_{A,\mathbb{Z}} = \left\{ L_Y\left(\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}(X)}\right) \,\middle|\, \mathcal{E} \in K(Y) \right\}.$$

Since L is a fundamental solution, the morphism

$$\begin{array}{rccc} L: & \mathscr{H}_A(U) & \to & \mathscr{H}'_A(U) \\ & \Downarrow & & \Downarrow \\ & s & \mapsto & L(s) \end{array}$$

of $\mathcal{O}_U$-modules is flat (i.e. $\nabla^{A'} L - L\nabla^A = 0$) and induces an isomorphism $H_A \xrightarrow{\sim} H'_A$ of $\mathbb{C}$-local systems. This isomorphism is compatible with Hodge filtrations since the generator e^σ of $\mathscr{F}^2$ goes to $1 \in \mathscr{F}'^2$. It preserves the polarizations since L is an isometry of the Mukai lattice, and it is obvious from the definition that L preserves the integral structures. The local system $H^{\mathrm{amb}}_{A,\mathbb{Z}}$ is defined as the local subsystem of $H_{A,\mathbb{Z}}$ corresponding to $\mathcal{N}(Y)^{\mathrm{amb}} = \{\iota^*\mathcal{E} \mid \mathcal{E} \in \mathcal{N}(X)\} \subset \mathcal{N}(Y)$.

Let $u_i \in H^2(X;\mathbb{Z})$ be the Poincaré dual of the toric divisor corresponding to the one-dimensional cone $\mathbb{R} \cdot b_i \in \Sigma$ and $v = u_1 + \cdots + u_m$ be the anticanonical class. Givental's *I-function* is defined as the series

$$I_{X,Y}(q,z) = e^{p \log q/z} \sum_{d \in \mathrm{Eff}(X)} q^d \frac{\prod_{k=-\infty}^{\langle d,v\rangle}(v+kz) \prod_{j=1}^m \prod_{k=-\infty}^0 (u_j+kz)}{\prod_{k=-\infty}^0 (v+kz) \prod_{j=1}^m \prod_{k=-\infty}^{\langle d,u_j\rangle}(u_j+kz)},$$

which is a multi-valued map from $\check{U}'$ (or a single-valued map from $\check{U}$) to the classical cohomology ring $H^\bullet(X;\mathbb{C}[z^{-1}])$. Givental's *$J$-function* is defined by

$$J_Y(\tau,z) = L_Y(\tau,z)^{-1}(1) = \exp(\tau/z).$$

If we write

$$I_{X,Y}(q,z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}),$$

then Givental's mirror theorem [Giv96, Giv98, CG07] states that

$$\mathrm{Euler}(\omega_X^{-1}) \cup I_{X,Y}(q,z) = F(q) \cdot \iota_* J_Y(\varsigma(q), z)$$

where $\mathrm{Euler}(\omega_X^{-1}) \in H^2(X;\mathbb{Z})$ is the Euler class of the anticanonical bundle of X, and the *mirror map* $\varsigma(q) : \check{U} \to H^2_{\mathrm{amb}}(Y;\mathbb{C})$ is defined by

$$\varsigma(q) = \iota^* \left(\frac{G(q)}{F(q)} \right).$$

The relation between $\tau = \varsigma(q)$ and $\sigma = \beta + \sqrt{-1}\omega$ is given by $\tau = \sqrt{-1}\sigma$, so that $\mathfrak{Im}(\sigma) \gg 0$ corresponds to $\exp(\tau) \sim 0$. The functions $F(q)$, $G(q)$ and $H(q)$ satisfy the Gelfand-Kapranov-Zelevinsky hypergeometric differential equations, and give periods for the B-model VHS $(\mathscr{H}_B, \nabla^B, \mathscr{F}_B^\bullet, Q_B)$. The isomorphism of integral structures is due to Iritani:

Theorem 4.1 (Iritani [Iri, Theorem 6.9]). *There is an isomorphism*

$$\mathrm{Mir}_{\mathcal{Y}} : \varsigma^*(H^{\mathrm{amb}}_{A,\mathbb{Z}}, \nabla^A, \mathscr{F}_A^\bullet, Q_A)/\iota^* H^2(\mathcal{X};\mathbb{Z}) \xrightarrow{\sim} (H^{\mathrm{vc}}_{B,\mathbb{Z}}, \nabla^B, \mathscr{F}_B^\bullet, Q_B)$$

of integral variations of pure and polarized Hodge structures.

The following lemma concludes the proof of Corollary 1.3:

Lemma 4.2. *If Δ is a reflexive fan polytope associated with any of 14 exceptional unimodal singularities, then one has equalities*

$$H^{\mathrm{amb}}_{A,\mathbb{Z}} \cong H_{A,\mathbb{Z}}$$

and

$$H^{\mathrm{vc}}_{B,\mathbb{Z}} \cong H_{B,\mathbb{Z}}$$

of integral local systems.

Proof. The equality $H^{\mathrm{amb}}_{A,\mathbb{Z}} \cong H_{A,\mathbb{Z}}$ holds since $\mathrm{NS}(Y) = \iota^* \mathrm{NS}(X)$ and the point class also belongs to $\iota^*\mathcal{N}(X)$. For the equality $H^{\mathrm{vc}}_{B,\mathbb{Z}} \cong H_{B,\mathbb{Z}}$, note that both the fibers of $H_{B,\mathbb{Z}}$ and $H^{\mathrm{vc}}_{B,\mathbb{Z}} \cong H^{\mathrm{amb}}_{A,\mathbb{Z}}$ are isomorphic to $\widehat{T}(\check{\boldsymbol{\gamma}})$, where $\check{\boldsymbol{\gamma}} = \boldsymbol{\delta}$ is the Gabrielov number of the singularity associated with $\check{Y}$ (i.e. the Dolgachev number of the singularity associated with Y). It follows that the determinants of the Gram matrices of the generators of $H_{B,\mathbb{Z}}$ and $H^{\mathrm{vc}}_{B,\mathbb{Z}}$ are the same. Since $H^{\mathrm{vc}}_{B,\mathbb{Z}}$ is a sublattice of $H_{B,\mathbb{Z}}$, this implies $H^{\mathrm{vc}}_{B,\mathbb{Z}} = H_{B,\mathbb{Z}}$ and the lemma is proved. $\square$

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