

THE KÄHLER-RICCI FLOW ON PROJECTIVE BUNDLES¹

Jian Song*, Gábor Székelyhidi** and Ben Weinkove***

Abstract

We study the behaviour of the Kähler-Ricci flow on projective bundles. We show that if the initial metric is in a suitable Kähler class, then the fibers collapse in finite time and the metrics converge subsequentially in the Gromov-Hausdorff sense to a metric on the base.

1 Introduction

Let X be a projective bundle over a smooth projective variety B . This means that $X = \mathbb{P}(E)$, where $\pi : E \rightarrow B$ is a holomorphic vector bundle of rank r , say. We study the behavior of the solution $\omega = \omega(t)$ of the Kähler-Ricci flow on X starting at a Kähler metric ω_0 :

$$\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0. \quad (1.1)$$

The solution $\omega(t)$ develops a singularity after a finite time. Indeed from [TZ] a maximal smooth solution to (1.1) exists on $[0, T)$ where $T > 0$ is given by

$$T = \sup\{t > 0 \mid [\omega_0] + tc_1(K_X) > 0\}. \quad (1.2)$$

T is finite since $F \cdot c_1(-K_X)^{r-1} > 0$ for every fiber F . For $t \in [0, T)$, $\omega(t)$ lies in the class $[\omega_0] + tc_1(K_X)$. We will assume that the limiting Kähler class $[\omega_0] + Tc_1(K_X)$ satisfies

$$[\omega_0] + Tc_1(K_X) = [\pi^*\omega_B], \quad (1.3)$$

for some Kähler metric ω_B on B .

Our main result shows that a sequence of metrics along the flow converges subsequentially to a metric on B in the Gromov-Hausdorff sense as $t \rightarrow T$.

Theorem 1.1 *There exists a sequence of times $t_i \rightarrow T$ and a distance function d_B on B (which is uniformly equivalent to the distance induced by ω_B), such that $(X, \omega(t_i))$ converges to (B, d_B) in the Gromov-Hausdorff sense.*

This implies that the fibers of X collapse. In order to prove this theorem we establish (Lemmas 2.2 and 2.4) the following estimates for a uniform constant C and for all $t \in [0, T)$:

- (i) $\omega(t) \leq C\omega_0$, and

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(ii) $\text{diam}_{\omega(t)} F \leq C(T-t)^{1/3}$, for every fiber F .

That is, the metrics $\omega(t)$ are uniformly bounded from above along the flow and the diameters of the fibers tend to zero as $t \rightarrow T$.

To illustrate how Theorem 1.1 ties in with the existing literature on the Kähler-Ricci flow we discuss the example of $X = \mathbb{P}^2$ blown up at one point, which is a \mathbb{P}^1 bundle over \mathbb{P}^1 . As $t \rightarrow T$ the behavior of $\omega(t)$ in the sense of Gromov-Hausdorff depends on the point at which the Kähler classes $[\omega(t)]$ hit the boundary of the Kähler cone. Write $\alpha = [\omega_0] + Tc_1(K_X)$. Then one of the following holds:

- (1) $\alpha = 0$.
- (2) α is the pull-back of a Kähler class from the base \mathbb{P}^1 (the setting of this paper).
- (3) α is the pull-back of a Kähler class from \mathbb{P}^2 via the blow-down map $p : X \rightarrow \mathbb{P}^2$.

In each case we have a map $f : X \rightarrow M$ to a manifold M (of dimension 0, 1 and 2 respectively) and $\alpha = f^*\beta$ for β a Kähler class on M . Feldman-Ilmanen-Knopf [FIK] made a number of conjectures about the behavior of (1.1) which they established for self-similar solutions. In particular they conjectured that in each of these three cases the flow should converge in the Gromov-Hausdorff sense to a metric on M . We now briefly describe some progress on these conjectures.

In case (1) it is an immediate consequence of a result of Perelman [P] (see [SeT]) that $(X, \omega(t))$ converges in the Gromov-Hausdorff sense to a point. When the initial metric is invariant under a $U(2)$ symmetry it was shown in [SW1] that: in case (2), $(X, \omega(t))$ converges in the Gromov-Hausdorff sense to the base \mathbb{P}^1 with the Fubini-Study metric; and in case (3), $(X, \omega(t))$ converges in the Gromov-Hausdorff sense to (\mathbb{P}^2, d) where d is a metric on \mathbb{P}^2 inducing the usual topology. In [SW2] the same behavior for (3) was established without assuming symmetry of the initial data. Thus the case of (2) without symmetry assumption was still open.

Although we have limited our discussion here to the case of \mathbb{P}^2 blown up at one point, the results of [SW1] and especially [SW2] apply to a much larger class of manifolds. Moreover, these results fit into a general program of understanding the Kähler-Ricci flow on algebraic varieties and the analytic minimal model program [Ts, SoT3, T2] (see also [EGZ, SoT1, SoT2, TZ, Zha1, Zha2]).

Our Theorem 1.1 is concerned with the setting of (2) for a general projective bundle over an algebraic variety B . Even in the case of a product $B \times \mathbb{P}^{r-1}$ with arbitrary initial metric ω_0 , the result of Theorem 1.1 is new.

Of course, the [FIK] conjectures predict that the convergence should occur without taking subsequences. A difficulty in (2) compared to the setting (3) is that the metric $\omega(t)$ becomes singular at *every* point of the manifold X .

A further interesting and difficult question is to analyze the singularity at time T by a rescaling procedure (see the discussion in [FIK]). In case (1) above, rescaling so that X has fixed volume, one obtains a compact Kähler-Ricci soliton [WZ, Zhu, TZhu]. For more general manifolds with $c_1(X) > 0$ this is related to a question of Yau [Y2] regarding stability in the sense of geometric invariant theory (see for example [CW, D, T1, PS, PSSW, Sz, T1, To]). In case (2), rescaling so that the fibers have fixed volume, one would expect to obtain a product $\mathbb{P}^1 \times \mathbb{C}$ (see the recent preprint [Fo] in the case of $U(2)$ symmetry). In case (3) the appropriate rescaling should yield the shrinking soliton constructed in [FIK].

In Section 2, we establish the key estimates (i) and (ii) mentioned above. In Section 3 we prove Theorem 1.1.

2 The main estimates

In this section we establish the main estimates. Assume X has complex dimension n and $X = \mathbb{P}(E)$ where E is a rank r holomorphic vector bundle over B . We will often write g for the Kähler metric with Kähler form ω . We use C , C' or C_i to denote a uniform constant, which may differ from line to line.

First, we have a lower bound for $\omega(t)$ from the parabolic Schwarz Lemma, as in [SoT1].

Lemma 2.1 *There exists a uniform constant $c > 0$ such that*

$$\omega \geq c\pi^*\omega_B. \quad (2.4)$$

Proof This estimate is well-known to hold. Indeed the argument is almost identical to the proof of Lemma 2.2 in [SW2] (see also [TZ]). We provide a brief sketch for the reader's convenience. First, we reformulate (1.1) as a parabolic complex Monge-Ampère equation. Define a family of reference metrics $\hat{\omega}_t \in [\omega(t)]$ by

$$\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t\pi^*\omega_B), \quad (2.5)$$

and let Ω be the unique volume form on X satisfying

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega = \frac{\partial}{\partial t}\hat{\omega}_t = \frac{1}{T}(\pi^*\omega_B - \omega_0) \in c_1(K_X), \quad \int_X \Omega = 1. \quad (2.6)$$

If $\varphi = \varphi(t)$ solves the parabolic complex Monge-Ampère equation

$$\frac{\partial\varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi)^n}{\Omega}, \quad \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0, \quad \varphi|_{t=0} = 0, \quad (2.7)$$

then $\omega = \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ solves (1.1). Conversely, given the solution $\omega = \omega(t)$ of (1.1) on $[0, T)$, we can extract the unique solution $\varphi = \varphi(t)$ of (2.7) for $t \in [0, T)$.

Since $c_0(T-t)^n\Omega \leq \hat{\omega}_t^n \leq C_0\Omega$ for uniform positive constants c_0 and C_0 , an elementary maximum principle argument shows that φ is uniformly bounded (see [TZ] or Lemma 2.1 of [SW2]). We then compute using the parabolic Schwarz lemma computation of [SoT1] that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_\omega(\pi^*\omega_B) \leq C \operatorname{tr}_\omega(\pi^*\omega_B), \quad (2.8)$$

for a uniform constant C . On the other hand, for a uniform constant $c' > 0$,

$$\Delta\varphi = \operatorname{tr}_\omega(\omega - \hat{\omega}_t) \leq n - c' \operatorname{tr}_\omega(\pi^*\omega_B). \quad (2.9)$$

The result follows by applying the maximum principle to the quantity

$$Q = \log \operatorname{tr}_\omega(\pi^*\omega_B) - A\varphi \quad (2.10)$$

for A sufficiently large and using the fact that φ is uniformly bounded. \square

We will make use of this in the key estimate:

Lemma 2.2 *There exists a constant $C > 0$ such that for t in $[0, T)$,*

$$\mathrm{tr}_{\omega_0} \omega(t) \leq C. \quad (2.11)$$

Proof For any line bundle L , $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$. Hence by replacing E by $E \otimes A^{-1}$ for some sufficiently ample line bundle A , we can assume that the dual bundle E^* is generated by global sections.

Fix an arbitrary $p \in B$. Choose r sections s_1, \dots, s_r of the dual bundle E^* , which are linearly independent at p . Write

$$f = s_1 \wedge s_2 \wedge \dots \wedge s_r$$

for the corresponding section of the line bundle $\bigwedge^r E^*$. Let $U \subset B$ be the set where f does not vanish. On this set s_1, \dots, s_r give a biholomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{P}^{r-1}$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{P}^{r-1} \\ & \searrow \pi & \swarrow \mathrm{pr}_1 \\ & & U \end{array} \quad (2.12)$$

commutes, where pr_1 is the projection map onto the first factor.

Let $\omega_{\mathrm{prod}} = (\mathrm{pr}_1)^* \omega_B + (\mathrm{pr}_2)^* \omega_{\mathrm{FS}}$ be the product metric on $U \times \mathbb{P}^{r-1}$, where ω_{FS} is the Fubini-Study metric on \mathbb{P}^{r-1} . Define $\omega_{\mathrm{sing}} = \Phi^* \omega_{\mathrm{prod}}$. Then ω_{sing} is a smooth Kähler metric on $\pi^{-1}(U)$, which we can also think of as a singular metric on X .

Fix a Hermitian metric h on E , and write h for the induced metric on E^* , $\bigwedge^r E^*$ as well. We claim that there is a constant C such that

$$\mathrm{tr}_{\omega_{\mathrm{sing}}} \omega_0 \leq C |f|_h^{-2}. \quad (2.13)$$

This can be seen as follows. The metric ω_0 is uniformly equivalent to the metric ω_{loc} , say, we obtain by locally, over a small ball $V \subset B$, taking (non-holomorphic) sections $\sigma_1, \dots, \sigma_r$ of E^* which are pointwise orthonormal with respect to h , and then using these sections to pull back the product metric on $V \times \mathbb{P}^{r-1}$. Thus to compare the metrics ω_{sing} and ω_{loc} along each fiber of $\mathbb{P}(E)$, we need to compare the Fubini-Study metrics on a given projective space constructed using two different Hermitian metrics. On a given fiber $\mathbb{P}(E_q)$ the metric ω_{loc} is constructed using the Hermitian metric h , whereas ω_{sing} is constructed using the metric on E_q in which s_1, \dots, s_n give an orthonormal basis of E_q^* . At each point q let us write A for the linear map $E_q \rightarrow \mathbb{C}^r$ given by s_1, \dots, s_r , and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ for the eigenvalues of $A^* A$, where the adjoint is formed by using the metric h on E_q and the standard metric on \mathbb{C}^r . Lemma 2.3 below then implies that $\omega_{\mathrm{sing}} \geq C \lambda_1 \lambda_2 \lambda_r^{-2} \omega_0$. Each eigenvalue is bounded above uniformly since s_1, \dots, s_r are defined over the whole of B , so $\lambda_1 \lambda_2$ is bounded below by the determinant. The determinant of $A^* A$ is given by $|f|_h^2$, so it follows that $\omega_{\mathrm{sing}} \geq C |f|_h^2 \omega_0$.

To prove the lemma we will apply the maximum principle to the quantity

$$H = \log (|f|_h^3 \mathrm{tr}_{\omega_{\mathrm{sing}}} \omega(t)), \quad (2.14)$$

on the set $\pi^{-1}(U)$. By the claim above, H tends to negative infinity along $X \setminus \pi^{-1}(U)$ and hence a maximum must occur in $\pi^{-1}(U)$ at each fixed time.

We recall the well-known evolution inequality [C] (see also [Y, A]) for $\omega = \omega(t)$ solving (1.1):

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\hat{\omega}} \omega \leq -\frac{1}{\operatorname{tr}_{\hat{\omega}} \omega} g^{k\bar{\ell}} \hat{R}_{k\bar{\ell}}^{\bar{j}i} g_{i\bar{j}}, \quad (2.15)$$

where $\hat{\omega}$ is any fixed Kähler metric on X , and $\hat{R}_{k\bar{\ell}}^{\bar{j}i}$ are its curvature components. We will apply this with $\hat{\omega} = \omega_{\text{sing}}$.

First note that the curvature tensor of ω_{prod} has a lower bound

$$\begin{aligned} R_{i\bar{j}k\bar{\ell}}(\omega_{\text{prod}}) &\geq (\operatorname{pr}_2^* g_{\text{FS}})_{i\bar{j}} (\operatorname{pr}_2^* g_{\text{FS}})_{k\bar{\ell}} + (\operatorname{pr}_2^* g_{\text{FS}})_{i\bar{\ell}} (\operatorname{pr}_2^* g_{\text{FS}})_{k\bar{j}} \\ &\quad - c(\operatorname{pr}_1^* g_B)_{i\bar{j}} (\operatorname{pr}_1^* g_B)_{k\bar{\ell}} \end{aligned} \quad (2.16)$$

for a constant c depending only on the curvature of g_B , where the inequality of tensors is meant in the sense of Griffiths. It follows that

$$g^{i\bar{j}} g_{k\bar{\ell}} (R_{i\bar{j}}^{\bar{\ell}k}(\omega_{\text{sing}})) \geq -C(\operatorname{tr}_{\omega_{\text{sing}}} \omega)(\operatorname{tr}_{\omega} \pi^* \omega_B), \quad (2.17)$$

for a uniform constant C . Then compute, using (2.15), (2.17) and the fact that the curvature of the metric h on $\bigwedge^r E^*$ is bounded by some multiple of $\pi^* \omega_B$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) H \leq C \operatorname{tr}_{\omega} \pi^* \omega_B \leq C'. \quad (2.18)$$

where we used Lemma 2.1 for the second inequality.

It follows from the maximum principle that H is bounded from above. This gives the bound

$$\operatorname{tr}_{\omega_{\text{sing}}} \omega \leq C |f|_h^{-3}. \quad (2.19)$$

It follows that for some open subset $U' \subset\subset U$ containing p , we have a bound $\operatorname{tr}_{\omega_0} \omega \leq C$ on $\pi^{-1}(U')$, where C depends on p . We can repeat the argument at each point of B , and by compactness of B , a finite number of the open sets U' cover B . Taking the largest of the corresponding constants C we obtain the required result. \square

We have used the following lemma in the proof:

Lemma 2.3 *Let $A : \mathbb{C}^r \rightarrow \mathbb{C}^r$ be an invertible linear map, and write $\psi : \mathbb{P}^{r-1} \rightarrow \mathbb{P}^{r-1}$ for the induced map of projective spaces. Let ω_{FS} be the standard Fubini-Study metric on \mathbb{P}^{r-1} . Then*

$$\psi^* \omega_{\text{FS}} \geq \frac{\lambda_1 \lambda_2}{\lambda_r^2} \omega_{\text{FS}},$$

where $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ are the eigenvalues of the matrix $A^* A$.

Proof Let ξ be a tangent vector of type $T^{1,0}$ at $x \in \mathbb{P}^{r-1}$ with $|\xi|_{\omega_{\text{FS}}}^2 = 1$. Replacing A by UA for U a unitary transformation we may assume that x is the point $[1 : 0 : \dots : 0]$. Choose holomorphic coordinates $z^i = Z_{i+1}/Z_1$ for $i = 1, \dots, r-1$ where Z_1, \dots, Z_r are the homogeneous coordinates on \mathbb{P}^{r-1} . Applying another unitary

transformation (fixing $[1 : 0 : \dots : 0]$), we may assume without loss of generality that $\xi = \partial/\partial z^1$.

The metric $\psi^*\omega_{FS}$ is given by

$$\psi^*\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log\left(\sum_{i,j=1}^n a_{ij}Z_j\bar{Z}_i\right),$$

where a_{ij} are the entries of A^*A . At the point $[1 : 0 : \dots : 0]$ we then have

$$\psi^*\omega_{FS} = \frac{\sum a_{ij}dZ_j \wedge d\bar{Z}_i}{a_{11}|Z_1|^2} - \frac{\sum a_{1j}a_{i1}dZ_j \wedge d\bar{Z}_i}{a_{11}^2|Z_1|^2},$$

so

$$|\xi|_{\psi^*\omega_{FS}}^2 = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}^2} \geq \frac{\lambda_1\lambda_2}{\lambda_r^2},$$

since the 2×2 minor $a_{11}a_{22} - a_{12}a_{21}$ is bounded below by the product of the two smallest eigenvalues. Since ξ was an arbitrary unit vector of type $T^{1,0}$, the result follows. \square

Note that Lemma 2.2 shows in particular that the diameter of each fiber is uniformly bounded with respect to the evolving metric ω . In fact, we can show that the diameters tend to zero as $t \rightarrow T$. For $b \in B$, write F_b for the fiber $\pi^{-1}(b)$.

Lemma 2.4 *There exists a constant C such that for all $b \in B$ and t in $[0, T)$,*

$$\text{diam}_{g(t)}F_b \leq C(T - t)^{1/3},$$

where we are writing $g(t)$ for the evolving metric $g(t)$ restricted to X_s .

Proof The argument is similar to that of Lemma 3.2 in [SW2], but we include the proof for the sake of completeness. Let p, q be two points in the fiber F_b . Under an identification $F_b \cong \mathbb{P}^{r-1}$ we can choose a line $\mathbb{P}^1 \subset F_b$ containing p, q . First note that from (2.5) we have

$$\int_{\mathbb{P}^1} \omega(t) = \frac{1}{T} \int_{\mathbb{P}^1} (T - t)\omega_0 \leq C(T - t). \quad (2.20)$$

Fix a background Kähler metric \hat{g} on \mathbb{P}^1 and write g for the metric $g(t)$ restricted to \mathbb{P}^1 . Then (2.20) gives us

$$\int_{\mathbb{P}^1} (\text{tr}_{\hat{g}} g) \hat{\omega} \leq C(T - t), \quad (2.21)$$

while from Lemma 2.2 we have

$$\text{tr}_{\hat{g}} g \leq C. \quad (2.22)$$

We wish to show that

$$d_g(p, q) \leq C(T - t)^{1/3}.$$

Let $\varepsilon = (T - t)^{1/3}$. Without loss of generality, we may assume that p and q lie in a fixed coordinate chart U with holomorphic coordinate $z = x + iy$. Moreover,

we may assume that p corresponds to the origin in \mathbb{C} , and q to the point $(x_0, 0)$ with $x_0 > 0$. We also assume that the rectangle $\mathcal{R} = [0, x_0] \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^2 = \mathbb{C}$ is contained in U . Since the fixed metric \hat{g} is uniformly equivalent to the Euclidean metric in \mathcal{R} , we have from (2.21),

$$\int_{-\varepsilon}^{\varepsilon} \left(\int_0^{x_0} (\text{tr}_{\hat{g}} g) dx \right) dy = \int_{\mathcal{R}} (\text{tr}_{\hat{g}} g) dx dy \leq C(T - t).$$

It follows that for some $y' \in (-\varepsilon, \varepsilon)$,

$$\int_0^{x_0} (\text{tr}_{\hat{g}} g)(x, y') dx \leq \frac{C}{\varepsilon}(T - t) = C(T - t)^{2/3}.$$

If p' and q' are the points represented by $(0, y')$ and (x_0, y') then

$$\begin{aligned} d_g(p', q') &\leq \int_0^{x_0} \left(\sqrt{g(\partial_x, \partial_x)} \right) (x, y') dx \\ &= \int_0^{x_0} \left(\sqrt{\text{tr}_{\hat{g}} g} \sqrt{\hat{g}(\partial_x, \partial_x)} \right) (x, y') dx \\ &\leq \left(\int_0^{x_0} (\text{tr}_{\hat{g}} g)(x, y') dx \right)^{1/2} \left(\int_0^{x_0} (\hat{g}(\partial_x, \partial_x))(x, y') dx \right)^{1/2} \\ &\leq C(T - t)^{1/3}. \end{aligned}$$

But from (2.22), we have

$$d_g(p, p') \leq C d_{\hat{g}}(p, p') \leq C' \varepsilon = C'(T - t)^{1/3},$$

and similarly for $d_g(q, q')$. Then

$$d_g(p, q) \leq d_g(p, p') + d_g(p', q') + d_g(q', q) \leq C(T - t)^{1/3},$$

as required. □

It is natural to expect that in fact $\text{diam}_{g(t)} F_b$ decays at the faster rate of $(T - t)^{1/2}$, but we will not need this for our application.

3 Gromov-Hausdorff convergence

Using the estimates from Section 2 we now prove Theorem 1.1.

Lemma 3.1 *Write $d_t : X \times X \rightarrow \mathbb{R}$ for the distance function induced by the metric $\omega(t)$. There exists a sequence of times $t_i \rightarrow T$, such that the functions d_{t_i} converge uniformly to a function $d_{\infty} : X \times X \rightarrow \mathbb{R}$.*

Moreover if for $p, q \in B$ we let $d_{B, \infty}(p, q) = d_{\infty}(\tilde{p}, \tilde{q})$, where $\tilde{p} \in F_p$ and $\tilde{q} \in F_q$, then $d_{B, \infty}$ defines a distance function on B , which is uniformly equivalent to that induced by ω_B .

Proof First note that the functions $d_t : X \times X \rightarrow \mathbb{R}$ are uniformly bounded. Indeed by Lemma 2.2 we have a constant C such that $d_t(x, y) < \sqrt{C}d_0(x, y) < C'$ for any $t < T$ and $x, y \in X$. In addition the functions $d_t : X \times X \rightarrow \mathbb{R}$ are equicontinuous with respect to the metric on $X \times X$ induced by d_0 , since for $x, x', y, y' \in X$ we have

$$\begin{aligned} |d_t(x, y) - d_t(x', y')| &\leq |d_t(x, y) - d_t(x, y')| + |d_t(x, y') - d_t(x', y')| \\ &\leq d_t(y, y') + d_t(x, x') \\ &\leq \sqrt{C}(d_0(y, y') + d_0(x, x')). \end{aligned}$$

By the Arzela-Ascoli theorem there is a sequence of times $t_i \rightarrow T$ such that the functions d_{t_i} converge uniformly to a continuous function $d_\infty : X \times X \rightarrow \mathbb{R}$. It follows that d_∞ is non-negative, symmetric and satisfies the triangle inequality.

Let $d_B : B \times B \rightarrow \mathbb{R}$ be the distance function on B induced by the metric ω_B . From Lemma 2.1 we have a constant $c > 0$ such that $d_t(x, y) \geq \sqrt{c}d_B(\pi(x), \pi(y))$. It follows that the limit d_∞ satisfies

$$d_\infty(x, y) \geq \sqrt{c}d_B(\pi(x), \pi(y)). \quad (3.23)$$

At the same time there is a constant C_1 such that for any $p, q \in B$ we have $d_0(F_p, F_q) < C_1d_B(p, q)$, so using both Lemma 2.2 and Lemma 2.4 we have

$$\begin{aligned} d_t(x, y) &\leq \text{diam}_{g(t)}F_{\pi(x)} + \text{diam}_{g(t)}F_{\pi(y)} + \sqrt{C}d_0(F_{\pi(x)}, F_{\pi(y)}) \\ &\leq 2C(T - t)^{1/3} + C_1\sqrt{C}d_B(\pi(x), \pi(y)). \end{aligned}$$

This implies that

$$d_\infty(x, y) \leq C_2d_B(\pi(x), \pi(y)). \quad (3.24)$$

For $p, q \in B$ we now define $d_{B,\infty}(p, q) = d_\infty(\tilde{p}, \tilde{q})$, where $\tilde{p} \in F_p$ and $\tilde{q} \in F_q$. This is independent of the choice of lifts \tilde{p}, \tilde{q} since if say \tilde{p}' is a different lift of p , then by (3.24) and the triangle inequality we have

$$d_\infty(\tilde{p}', \tilde{q}) \leq d_\infty(\tilde{p}, \tilde{q}) + d_\infty(\tilde{p}, \tilde{p}') = d_\infty(\tilde{p}, \tilde{q}),$$

and by switching \tilde{p}, \tilde{p}' we get the reverse inequality. Moreover it follows from (3.23) and (3.24) that $d_{B,\infty}$ is uniformly equivalent to d_B . This concludes the proof. \square

Theorem 3.1 *In the notation of Lemma 3.1 we have $(X, d_{t_i}) \rightarrow (B, d_{B,\infty})$ in the Gromov-Hausdorff sense, where we recall that d_{t_i} is the distance function induced by the metric $\omega(t_i)$.*

Proof We use the characterization of Gromov-Hausdorff convergence given in, for example, [F]. Given metric spaces (X, d_X) and (Y, d_Y) , the Gromov-Hausdorff distance $d_{\text{GH}}(X, Y)$ is the infimum of all $\varepsilon > 0$ such that the following holds. There exist maps $F : X \rightarrow Y$ and $G : Y \rightarrow X$ such that

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| &< \varepsilon, & \text{for all } x_1, x_2 \in X, \\ d_X(x, G \circ F(x)) &< \varepsilon, & \text{for all } x \in X, \end{aligned}$$

and the two symmetric properties for Y also hold.

Now define $F : X \rightarrow B$ to be the projection $F = \pi$, and $G : B \rightarrow X$ to be any (possibly discontinuous) map such that $F \circ G$ is the identity on B . Then for any $x_1, x_2, x \in X$ we have

$$\begin{aligned} |d_{t_i}(x_1, x_2) - d_{B, \infty}(F(x_1), F(x_2))| &= |d_{t_i}(x_1, x_2) - d_{\infty}(x_1, x_2)| \\ d_{t_i}(x, G \circ F(x)) &\leq \text{diam}_{g(t_i)} F_{\pi(x)}, \end{aligned}$$

and both of these quantities converge to zero uniformly in x_1, x_2 as $t_i \rightarrow T$, because $d_{t_i} \rightarrow d_{\infty}$ uniformly, and because of Lemma 2.4. At the same time for any $p_1, p_2, p \in B$ we have

$$\begin{aligned} |d_{B, \infty}(p_1, p_2) - d_{t_i}(G(p_1), G(p_2))| &= |d_{\infty}(G(p_1), G(p_2)) - d_{t_i}(G(p_1), G(p_2))| \\ d_{B, \infty}(p, F \circ G(p)) &= 0, \end{aligned}$$

and again the first quantity goes to zero as $t_i \rightarrow T$ since $d_{t_i} \rightarrow d_{\infty}$ uniformly.

This shows that as $i \rightarrow \infty$ we have $d_{\text{GH}}((X, d_{t_i}), (B, d_{B, \infty})) \rightarrow 0$, which is what we wanted to prove. \square

Remark 3.1 Throughout the paper we have focused on manifolds of the form $\mathbb{P}(E)$, where E is a vector bundle over a projective manifold B . It is likely that the behavior of the Kähler-Ricci flow is similar for more general fibrations $X \rightarrow B$ over a Kähler manifold B , where the Kähler class of the metric at time t tends to the pull-back of a Kähler class from B as $t \rightarrow T$, just as in Equation (1.3). The main difficulty in extending Theorem 1.1 in this way is generalizing Lemma 2.2. Indeed it is crucial in the maximum principle argument that the fibers admit a metric with non-negative bisectional curvature. On the other it is likely that the arguments do extend without any additional difficulties for fibrations where the fibers admit such a metric.

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* Department of Mathematics
Rutgers University, Piscataway, NJ 08854

** Department of Mathematics
Columbia University, NY 10027

*** Department of Mathematics
University of California San Diego, La Jolla, CA 92093