UNKNOTTING AND ASCENDING NUMBERS OF KNOTS AND THEIR FAMILIES

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Abstract

Ascending numbers are determined for 64 knots with at most $n = 10$ crossings. After proving the theorem about the signature of alternating knot families, we distinguished all families of knots obtained from generating alternating knots with at most 10 crossings, for which the unknotting number can be confirmed by using the general formulae for signatures. For 11 families of knots general formulae are obtained for their ascending numbers.

Keywords: Conway notation, knot family, signature, unknotting number, ascending number.

1. Introduction

About Conway notation of knots the reader can consult the seminal paper by J. Conway [2], where this notation is introduced, the paper by A. Caudron [3], and books [4, 5]. In particular, drawings of all knots up to $n = 10$ crossings according to Conway notation, where every knot is represented by a single diagram, are given in the Appendix C of the book "Knots and links" by D. Rolfsen [4].

In Sections 2,3 and 4 we compute ascending numbers for 64 knots with at most $n = 10$ crossings and determine upper and lower bounds of ascending numbers for all knots up to $n = 10$ crossings. For twist knots, i.e., knots of the family $p\,2$ ($p \geq 1$) in the Conway notation, the ascending number is one, and for all other knots $a(K) \geq 2$, i.e., $a(K) \geq max(u(K), 2)$. This means, that if there is a diagram $\widetilde{K}$ of a K with $a(\widetilde{K}) = u(K)$, then $a(K) = a(\widetilde{K}) = u(K)$. Except for several knots, the unknotting numbers of knots with at most $n = 10$ crossings are known, and they are given in "Tables of knot invariants" by C. Livingston and J.C. Cha [6]. Bridge numbers of knots with $n \leq 10$ crossings are given in the same tables, but they are not useful for our purpose because for all knots with $n \leq 10$ the bridge number is 2 or 3. In order to improve upper bound given by the inequality (1) following from minimal crossing numbers, we computed ascending numbers of all minimal diagrams. As an additional improvement, for some knots we obtained upper bounds from ascending numbers of some of their non-minimal diagrams. For all computations we used the program "LinKnot" [5].

In Section 5 we prove the theorem on signature enabling computation of general formulae for the signature of alternating knot families given by their Conway symbols. These general formulae enabled us to recognize the families of knots obtained from alternating generating knots with at most $n = 10$ crossings for which unknotting numbers are determined by signatures computed in Section 6.

In Section 7 we consider some families of knots with ascending numbers that coincides with the unknotting number.

The ascending number of a link L is described in the paper "Ascending number of knots and links" by M. Ozawa [1]. In our paper we restrict the consideration of ascending numbers to knots, so we repeat the definitions of basic terms from [1]. A knot diagram is *based* if a base point (different from the crossing points) is specified on the diagram, and *oriented* if an orientation is assigned to it. Let K be a knot and $\widetilde{K}$ be a based oriented diagram of K. The *descending diagram* of $\widetilde{K}$, denoted by $d(\widetilde{K})$, is obtained as follows: beginning at the basepoint of $\widetilde{K}$ and proceeding in the direction specified by the orientation, change the crossings as necessary so that each crossing is first encountered as an over-crossing. Note that $d(\widetilde{K})$ is the diagram of a trivial knot.

Definition 1. Let K be a knot and let $\widetilde{K}$ be a based oriented diagram of K. The *ascending number* of K is defined as the number of different crossings between $\widetilde{K}$ and $d(\widetilde{K})$ and denoted by $a(\widetilde{K})$. The ascending number of K is defined as the minimum number of $a(\widetilde{K})$ over all based oriented knot diagrams $\widetilde{K}$ of K, and denoted by $a(K)$ [1].

Among theorems proved in [1], we relate four of them giving upper and lower bounds for ascending numbers of knots:

1. for a non-trivial knot K, we have

$$a(K) \leq \lfloor \frac{c(K)-1}{2} \rfloor \tag{1}$$

   where $c(K)$ is the minimum crossing number of K, and $\lfloor x \rfloor$ integer part of x;

2. for every non-trivial knot K, we have

$$a(K) \geq u(K)$$

   where $u(K)$ is the unknotting number of K;

3. the ascending number of a knot K is one *iff* K is a twist knot;

4. for a knot K, we have

$$a(K) \geq b(K) - 1$$

   where $b(K)$ is the bridge number of K.

2. Ascending numbers of knots up to 8 crossings

Ascending numbers of knots up to $n = 8$ crossings are given in the tables from paper [1] and illustrated by the corresponding based oriented knot diagrams giving the minimal ascending number, where the knot diagrams, which are the same as minimal crossing diagrams, are omitted. Among knots with the minimal diagram giving the ascending number we recognized two more knots: $7_6 = 2\,2\,1\,2$ and $8_{12} = 2\,2\,2\,2$, illustrated in Fig. 1. For knots $8_{16} = .2.2\,0$ and $8_{17} = .2.2$ we succeeded to find their non-minimal diagrams with diagram ascending number equal 2, so $a(8_{16}) = 2$ and $a(8_{17}) = 2$. In the corresponding tables every knot is given in classical Conway notation *Con* [4], followed by unknotting number u [6], upper bound for ascending number a_d (obtained mostly from minimal diagrams), and ascending number a. For knots with unknown ascending numbers a sequence is given, beginning with lower bounds and ending with the best known upper bound (e.g., $[2, 3]$). For knots up to $n = 8$ crossings, the computation of ascending numbers corresponding to all minimal diagrams gives no improvement of the upper bound obtained from the crossing number, but for many knots with $n = 9$ or $n = 10$ crossings it results in the upper bound equals 3, instead of the upper bound 4 obtained from the crossing number.

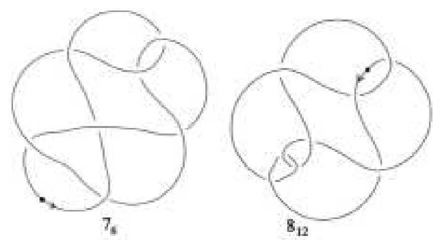

Figure 1: (a) Knot 7_6; (b) knot 8_{12}.

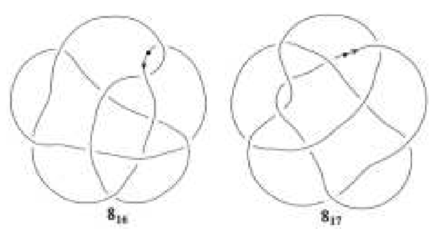

Figure 2: (a) Knot 8_{16}; (b) knot 8_{17}.

K	Con	u	a_d	a	K	Con	u	a_d	a
3_1	3	1	1	1	7_1	7	3	3	3
4_1	2 2	1	1	1	7_2	5 2	1	3	1
5_1	5	2	2	2	7_3	4 3	2	3	2
5_2	3 2	1	2	1	7_4	3 1 3	2	3	2
6_1	4 2	1	2	1	7_5	3 2 2	2	3	2
6_2	3 1 2	1	2	2	7_6	2 2 1 2	1	2	2
6_3	2 1 1 2	1	2	2	7_7	2 1 1 1 2	1	2	2

K	Con	u	a_d	a	K	Con	u	a_d	a
8_1	6 2	1	3	1	8_{12}	2 2 2 2	2	2	2
8_2	5 1 2	2	3	[2, 3]	8_{13}	3 1 1 1 2	1	3	2
8_3	4 4	2	3	2	8_{14}	2 2 1 1 2	1	3	2
8_4	4 1 3	2	3	2	8_{15}	2 1, 2 1, 2	2	3	2
8_5	3, 3, 2	2	3	[2, 3]	8_{16}	.2.2 0	2	3	2
8_6	3 3 2	2	3	2	8_{17}	.2.2	1	3	2
8_7	4 1 1 2	1	3	[2, 3]	8_{18}	8^*	2	2	2
8_8	2 3 1 2	2	3	2	8_{19}	3, 3, −2	3	3	3
8_9	3 1 1 3	1	3	[2, 3]	8_{20}	3, 2 1, −2	1	2	2
8_{10}	2 1, 3, 2	2	3	[2, 3]	8_{21}	2 1, 2 1 − 2	1	2	2
8_{11}	3 2 1 2	1	3	2					

3. Ascending numbers of knots with 9 crossings

According to [1], for knots with $n = 9$ crossings ascending numbers are known only for six knots: $a(9_3) = 3$, $a(9_4) = 2$, $a(9_6) = 3$, $a(9_7) = 2$, $a(9_{47}) = 2$, and $a(9_{48} = 2$, and they were determined by M. Okuda. For knots 9_{47} and 9_{48} they can be determined from their minimal diagrams. Hence, for 23 new non-trivial knots* with $n = 9$ crossings we obtained their ascending numbers. Based oriented diagrams corresponding to these knots are illustrated in Figs. 3-10. All these alternating knots with $n = 9$ crossings are given by their non-minimal based oriented diagrams giving their ascending numbers. For all remaining knots with $n = 9$ crossings, except for the knot 9_{40}, by computing diagram ascending numbers for all minimal crossing diagrams (or for some non-alternating diagrams in the case of knots 9_{29} and 9_{39}) we succeeded to reduce the set of possible values of the ascending number to $[2, 3]$ (meaning, 2 or 3).

K	Con	u	a_d	a	K	Con	u	a_d	a
9_1	9	4	4	4	9_{26}	3 1 1 1 1 2	1	3	[2, 3]
9_2	7 2	1	4	1	9_{27}	2 1 2 1 1 2	1	3	[2, 3]
9_3	6 3	3	4	3	9_{28}	2 1, 2 1, 2+	1	3	[2, 3]
9_4	5 4	2	4	2	9_{29}	.2.2 0.2	2	4	[2, 3]
9_5	5 1 3	2	4	2	9_{30}	2 1 1, 2 1, 2	1	3	[2, 3]
9_6	5 2 2	3	4	3	9_{31}	2 1 1 1 1 1 2	2	3	[2, 3]
9_7	3 4 2	2	4	2	9_{32}	.2 1.2 0	2	3	[2, 3]
9_8	2 4 1 2	2	3	2	9_{33}	.2 1.2	1	3	[2, 3]
9_9	4 2 3	3	4	3	9_{34}	8^*2 0	1	3	2
9_{10}	3 3 3	3	4	3	9_{35}	3, 3, 3	3	4	3
9_{11}	4 1 2 2	2	3	[2, 3]	9_{36}	2 2, 3, 2	2	3	[2, 3]
9_{12}	4 2 1 2	1	3	2	9_{37}	3, 2 1, 2 1	2	3	2
9_{13}	3 2 1 3	3	4	3	9_{38}	.2.2.2	3	4	3
9_{14}	4 1 1 1 2	1	3	2	9_{39}	2 : 2 : 2 0	1	4	[2, 3]
9_{15}	2 3 2 2	2	3	2	9_{40}	9^*	2	4	[2, 3, 4]
9_{16}	3, 3, 2+	3	4	3	9_{41}	2 0 : 2 0 : 2 0	2	3	[2, 3]
9_{17}	2 1 3 1 2	2	3	[2, 3]	9_{42}	2 2, 3, −2	1	2	2
9_{18}	3 2 2 2	2	4	2	9_{43}	2 1 1, 3, −2	2	3	[2, 3]
9_{19}	2 3 1 1 2	1	3	2	9_{44}	2 2, 2 1, −2	1	2	2
9_{20}	3 1 2 1 2	2	3	[2, 3]	9_{45}	2 1 1, 2 1, −2	1	2	2
9_{21}	3 1 1 2 2	1	3	2	9_{46}	3, 3, −3	2	2	2
9_{22}	2 1 1, 3, 2	1	3	[2, 3]	9_{47}	$8^* - 2 0$	2	2	2
9_{23}	2 2 1 2 2	2	4	2	9_{48}	2 1, 2 1, −3	2	2	2
9_{24}	3, 2 1, 2	1	3	[2, 3]	9_{49}	−2 0 : −2 0 : −2 0	3	3	3
9_{25}	2 2, 2 1, 2	2	3	2					

*For knots 9_1 and 9_2 is trivial to conclude that $a(9_1) = 4$, and $a(9_2) = 1$.

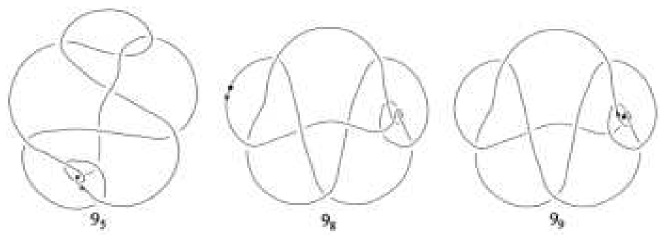

Figure 3: (a) Knot 9_5; (b) knot 9_8; (c) knot 9_9.

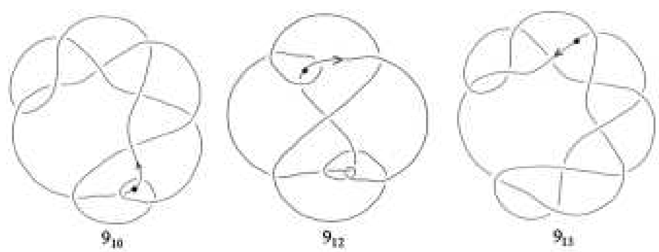

Figure 4: (a) Knot 9_{10}; (b) knot 9_{12}; (c) knot 9_{13}.

4. Ascending numbers of knots with 10 crossings

For knots with $n = 10$ crossings, none of ascending numbers (except for the twist knot $10_1 = 8\,2$ with $a(10_1) = 1$) were known. In this paper we computed ascending numbers for 39 knots with $n = 10$ crossings. For some of remaining knots, by using all minimal or some non-minimal diagrams we succeeded to improve upper and lower bound for ascending numbers to the set $[2, 3]$.

Because among 10-crossing knots there are some with unknown unknotting number ($[2, 3]$, meaning 2 or 3), the corresponding bounds for ascending number are denoted by $(2, 3)$ instead of $[2, 3]$, and $(2, 3, 4)$ instead of $[2, 3, 4]$. If in any of these cases unknotting number is equal to its lower bound, this will be a counterexample to the Bernhard-Jablan Conjecture [5, 7, 8].

Based oriented diagrams of 38 knots with $n = 10$ crossings for which we succeeded to compute their ascending numbers are illustrated in Figs. 11-23.

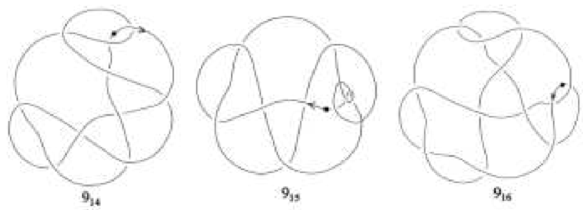

Figure 5: (a) Knot 9_{14}; (b) knot 9_{15}; (c) knot 9_{16}.

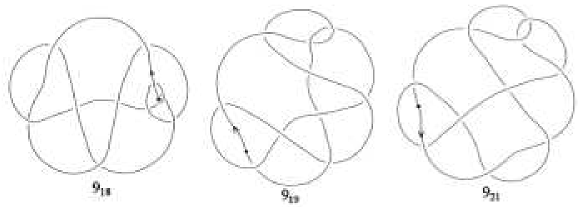

Figure 6: (a) Knot 9_{18}; (b) knot 9_{19}; (c) knot 9_{21}.

K	Con	u	a_d	a	K	Con	u	a_d	a
10_1	8 2	1	4	1	10_{84}	.2 2.2	1	4	[2, 3, 4]
10_2	7 1 2	3	4	[3, 4]	10_{85}	.4.2 0	2	4	[2, 3]
10_3	6 4	2	4	2	10_{86}	.3 1.2 0	2	4	[2, 3]
10_4	6 1 3	2	4	2	10_{87}	.2 2.2 0	2	4	[2, 3, 4]
10_5	6 1 1 2	2	4	[2, 3, 4]	10_{88}	.2 1.2 1	1	3	[2, 3]
10_6	5 3 2	3	4	3	10_{89}	2 1.2 1 0	2	3	[2, 3]
10_7	5 2 1 2	1	4	2	10_{90}	.3.2.2	2	4	[2, 3, 4]
10_8	5 1 4	2	3	[2, 3]	10_{91}	.3.2.2 0	1	4	[2, 3]
10_9	5 1 1 3	2	3	[2, 3]	10_{92}	.2 1.2.2 0	2	4	[2, 3, 4]
10_{10}	5 1 1 1 2	1	4	2	10_{93}	.3.2 0.2	2	4	[2, 3]
10_{11}	4 3 3	[2, 3]	4	(2, 3)	10_{94}	.3 0.2.2	2	4	[2, 3]
10_{12}	4 3 1 2	2	4	[2, 3]	10_{95}	.2 1 0.2.2	1	4	[2, 3, 4]
10_{13}	4 2 2 2	2	3	2	10_{96}	.2.2 1.2	2	4	[2, 3, 4]
10_{14}	4 2 1 1 2	2	4	[2, 3]	10_{97}	.2.2 1 0.2	2	4	[2, 3, 4]
10_{15}	4 1 3 2	2	4	[2, 3]	10_{98}	.2.2.2.2 0	2	4	[2, 3, 4]
10_{16}	4 1 2 3	2	4	[2, 3]	10_{99}	.2.2.2 0.2 0	2	4	[2, 3, 4]
10_{17}	4 1 1 4	1	4	[2, 3, 4]	10_{100}	3 : 2 : 2	[2, 3]	4	(2, 3)
10_{18}	4 1 1 2 2	1	4	2	10_{101}	2 1 : 2 : 2	3	4	[3, 4]
10_{19}	4 1 1 1 3	2	4	[2, 3]	10_{102}	3 : 2 : 2 0	1	4	[2, 3, 4]
10_{20}	3 5 2	2	4	2	10_{103}	3 0 : 2 : 2	3	4	3
10_{21}	3 4 1 2	2	4	[2, 3]	10_{104}	3 : 2 0 : 2 0	1	4	[2, 3, 4]
10_{22}	3 3 1 3	2	4	[2, 3]	10_{105}	2 1 : 2 0 : 2 0	2	4	[2, 3, 4]
10_{23}	3 3 1 1 2	1	4	[2, 3]	10_{106}	3 0 : 2 : 2 0	2	4	[2, 3]
10_{24}	3 2 3 2	2	4	2	10_{107}	2 1 0 : 2 : 2 0	1	4	[2, 3, 4]
10_{25}	3 2 2 1 2	2	4	[2, 3]	10_{108}	3 0 : 2 0 : 2 0	2	4	[2, 3]
10_{26}	3 2 1 1 3	1	4	[2, 3]	10_{109}	2.2.2.2	2	4	[2, 3, 4]
10_{27}	3 2 1 1 1 2	1	4	[2, 3]	10_{110}	2.2.2.2 0	2	4	[2, 3, 4]
10_{28}	3 1 3 1 2	2	4	[2, 3]	10_{111}	2.2.2 0.2	2	4	[2, 3, 4]
10_{29}	3 1 2 2 2	2	3	[2, 3]	10_{112}	8^*3	2	3	[2, 3]
10_{30}	3 1 2 1 1 2	1	4	[2, 3]	10_{113}	$8^*2\,1$	1	3	[2, 3]
10_{31}	3 1 1 3 2	1	4	2	10_{114}	$8^*3\,0$	1	3	[2, 3]
10_{32}	3 1 1 1 2 2	2	3	[2, 3]	10_{115}	$8^*2\,0.2\,0$	2	4	[2, 3, 4]
10_{33}	3 1 1 1 1 3	1	4	[2, 3]	10_{116}	$8^*2 : 2$	2	4	[2, 3]
10_{34}	2 5 1 2	2	4	2	10_{117}	$8^*2 : 2\,0$	2	4	[2, 3]
10_{35}	2 4 2 2	2	4	2	10_{118}	$8^*2 : .2$	1	4	[2, 3, 4]
10_{36}	2 4 1 1 2	2	4	2	10_{119}	$8^*2 : .2\,0$	1	4	[2, 3, 4]
10_{37}	2 3 3 2	2	4	2	10_{120}	$8^*2\,0 :: 2\,0$	3	4	[3, 4]
10_{38}	2 3 1 2 2	2	4	2	10_{121}	$9^*2\,0$	2	4	[2, 3, 4]
10_{39}	2 2 3 1 2	2	4	[2, 3]	10_{122}	$9^*.2\,0$	2	3	[2, 3]
10_{40}	2 2 2 1 1 2	2	4	[2, 3]	10_{123}	10^*	2	3	[2, 3]

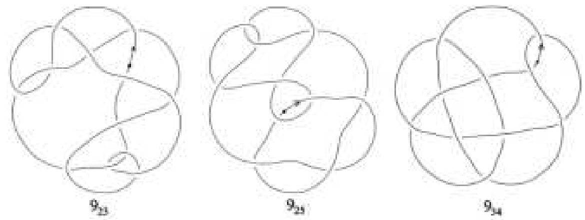

Figure 7: (a) Knot 9_{23}; (b) knot 9_{25}; (c) knot 9_{34}.

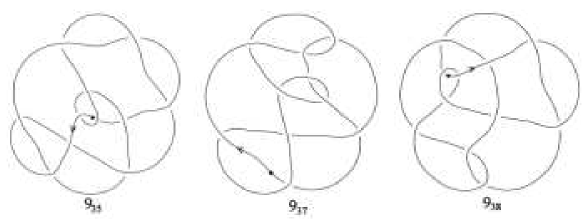

Figure 8: (a) Knot 9_{35}; (b) knot 9_{37}; (c) knot 9_{38}.

K	Con	u	a_d	a	K	Con	u	a_d	a
10_{41}	2 2 1 2 1 2	2	3	[2, 3]	10_{124}	5, 3, −2	4	4	4
10_{42}	2 2 1 1 1 1 2	1	3	[2, 3]	10_{125}	5, 2 1, −2	2	3	[2, 3]
10_{43}	2 1 2 2 1 2	2	3	[2, 3]	10_{126}	4 1, 3, −2	2	3	[2, 3]
10_{44}	2 1 2 1 1 1 2	1	3	[2, 3]	10_{127}	4 1, 2 1, −2	2	3	[2, 3]
10_{45}	2 1 1 1 1 1 1 2	2	3	[2, 3]	10_{128}	3 2, 3, −2	3	3	3
10_{46}	5, 3, 2	3	4	[3, 4]	10_{129}	3 2, 2 1, −2	1	3	[2, 3]
10_{47}	5, 2 1, 2	[2, 3]	4	(2, 3, 4)	10_{130}	3 1 1, 3, −2	1	3	[2, 3]
10_{48}	4 1, 3, 2	3	4	[2, 3, 4]	10_{131}	3 1 1, 2 1, −2	1	3	[2, 3]
10_{49}	4 1, 2 1, 2	3	4	3	10_{132}	2 3, 3, −2	1	3	2
10_{50}	3 2, 3, 2	2	4	[2, 3, 4]	10_{133}	2 3, 2 1 − 2	1	3	2
10_{51}	3 2, 2 1, 2	[2, 3]	4	(2, 3, 4)	10_{134}	2 2 1, 3, −2	3	3	3
10_{52}	3 1 1, 3, 2	2	4	[2, 3]	10_{135}	2 2 1, 2 1, −2	2	3	2
10_{53}	3 1 1, 2 1, 2	3	4	3	10_{136}	2 2, 2 2 − 2	1	2	2
10_{54}	2 3, 3, 2	[2, 3]	4	(2, 3, 4)	10_{137}	2 2, 2 1 1, −2	1	2	2
10_{55}	2 3, 2 1, 2	2	4	2	10_{138}	2 1 1, 2 1 1, −2	2	3	[2, 3]
10_{56}	2 2 1, 3, 2	2	4	[2, 3]	10_{139}	4, 3, −2 1	4	4	4
10_{57}	2 2 1, 2 1, 2	2	4	[2, 3]	10_{140}	4, 3, −3	2	3	[2, 3]
10_{58}	2 2, 2 2, 2	2	3	[2, 3]	10_{141}	4, 2 1, −3	1	3	[2, 3]
10_{59}	2 2, 2 1 1, 2	1	3	[2, 3]	10_{142}	3 1, 3, −2 1	3	4	3
10_{60}	2 1 1, 2 1 1, 2	1	3	[2, 3]	10_{143}	3 1, 3, −3	1	3	[2, 3]
10_{61}	4, 3, 3	[2, 3]	4	(2, 3, 4)	10_{144}	3 1, 2 1, −3	2	3	[2, 3]
10_{62}	4, 3, 2 1	2	4	[2, 3, 4]	10_{145}	2 2, 3, −2 1	2	3	2
10_{63}	4, 2 1 2 1	2	4	[2, 3]	10_{146}	2 2, 2 1, −3	1	2	2
10_{64}	3 1, 3, 3	2	4	[2, 3, 4]	10_{147}	2 1 1, 3, −3	1	2	2
10_{65}	3 1, 3, 2 1	2	4	[2, 3, 4]	10_{148}	(3, 2) (3, −2)	2	3	[2, 3]
10_{66}	3 1, 2 1, 2 1	3	4	[3, 4]	10_{149}	(3, 2) (2 1, −2)	2	3	[2, 3]
10_{67}	2 2, 3, 2 1	2	4	[2, 3, 4]	10_{150}	(2 1, 2) (3, −2)	2	3	[2, 3]
10_{68}	2 1 1, 3, 3	2	4	[2, 3, 4]	10_{151}	(2 1, 2) (2, 1, −2)	2	3	[2, 3]
10_{69}	2 1 1, 2 1, 2 1	2	3	[2, 3]	10_{152}	(3, 2) − (3, 2)	4	4	4
10_{70}	2 2, 3, 2+	2	3	[2, 3]	10_{153}	(3, 2) − (2 1, 2)	2	4	[2, 3, 4]
10_{71}	2 2, 2 1, 2+	1	3	[2, 3]	10_{154}	(2 1, 2) − (2 1, 2)	3	4	[3, 4]
10_{72}	2 1 1, 3, 2+	2	4	[2, 3, 4]	10_{155}	−3 : 2 : 2	2	3	[2, 3]
10_{73}	2 1 1, 2 1, 2+	1	3	[2, 3]	10_{156}	−3 : 2 : 2 0	1	3	[2, 3]
10_{74}	3, 3, 2 1+	2	4	[2, 3, 4]	10_{157}	−3 : 2 0 : 2 0	2	3	[2, 3]
10_{75}	2 1, 2 1, 2 1+	2	3	[2, 3]	10_{158}	−3 0 : 2 : 2	2	3	[2, 3]
10_{76}	3, 3, 2 + +	[2, 3]	4	(2, 3)	10_{159}	−3 0 : 2 : 2 0	1	2	2
10_{77}	3, 2 1, 2 + +	[2, 3]	4	(2, 3)	10_{160}	−3 0 : 2 0 : 2 0	2	2	2
10_{78}	2 1, 2 1, 2 + +	2	3	[2, 3]	10_{161}	3 : −2 0 : −2 0	3	3	3
10_{79}	(3, 2) (3, 2)	[2, 3]	4	(2, 3, 4)	10_{162}	−3 0 : −2 0 : −2 0	2	3	[2, 3]
10_{80}	(3, 2) (2 1, 2)	3	4	3	10_{163}	8* − 3 0	2	2	2
10_{81}	(2 1, 2) (2 1, 2)	2	4	[2, 3]	10_{164}	8*2 : −2 0	1	3	[2, 3]
10_{82}	.4.2	1	4	[2, 3]	10_{165}	8*2 : . − 2 0	2	3	[2, 3]
10_{83}	.3 1.2	2	4	[2, 3, 4]					

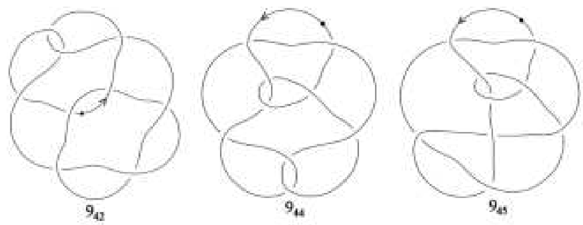

Figure 9: (a) Knot 9_{42}; (b) knot 9_{44}; (c) knot 9_{45}.

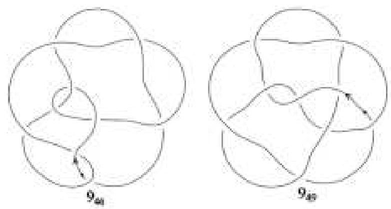

Figure 10: (a) Knot 9_{46}; (b) knot 9_{49}.

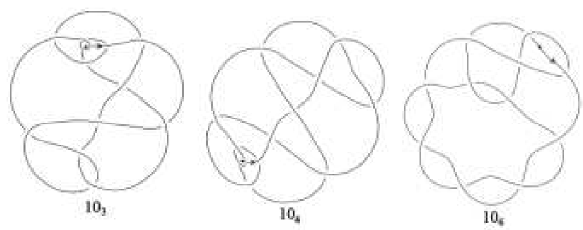

Figure 11: (a) Knot 10_3; (b) knot 10_4; (c) knot 10_6.

5. Signature and alternating knot families

Definition 2. Let S denote the set of numbers in the unreduced[†] Conway symbol $C(L)$ of a link L. Given $C(L)$ and an arbitrary (non-empty) subset $\tilde{S} = \{a_1, a_2, \ldots, a_m\}$ of S, the family $F_{\tilde{S}}(L)$ of knots or links derived from L is constructed by substituting each $a_i \in \tilde{S}$, $a_i \neq 1$ in $C(L)$ by $sgn(a)(|a|+n)$, for $n \in N^+$.

For even integers $n \geq 0$ this construction preserves the number of components, i.e., we obtain (sub)families of links with the same number of components. If all parameters in a Conway symbol of a knot or link are 1,2, or 3, such a link is called *generating*.

[†]The Conway notation is called *unreduced* if 1's denoting elementary tangles in vertices are not omitted in symbols of polyhedral links.

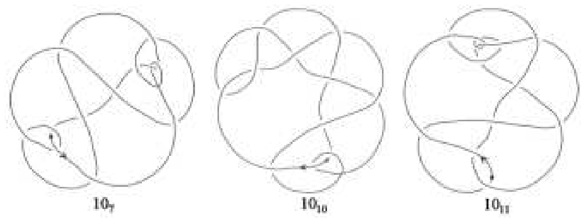

Figure 12: (a) Knot 10_7; (b) knot 10_{10}; (c) knot 10_{11}.

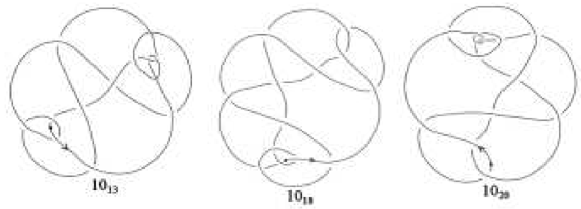

Figure 13: (a) Knot 10_{13}; (b) knot 10_{18}; (c) knot 10_{20}.

K. Murasugi [9] defined *signature* σ_K of a knot K as the signature of the matrix $S_K + {S_K}^T$, where ${S_K}^T$ is the transposed matrix of S_K, and S_K is the Seifert matrix of the knot K.

For alternating knots, signature can be computed by using a combinatorial formula derived by P. Traczyk [10]. We will use this formula, proved by J. Przytycki, in the following form, taken from [11], Theorem 7.8, Part (2):

Theorem 1. If D is a reduced alternating diagram of an oriented knot, then

$$\sigma_D = -\frac{1}{2}w + \frac{1}{2}(W - B) = -\frac{1}{2}w + \frac{1}{2}(|D_{s+}| - |D_{s-}|),$$

where w is the writhe of D, W is the number of white regions in the checkerboard coloring of D, which is for alternating minimal diagrams equal to the number of cycles $|D_{s+}|$ in the state $s+$, and B is the number of black regions in the checkerboard coloring of D equal to the number of the cycles $|D_{s-}|$ in the state $s-$.

Introducing orientation of a knot, every n-twist (chain of digons) becomes *parallel* or *anti-parallel*. For signs of crossings and checkerboard coloring we use the conventions shown in Fig. 24.

Lemma 1. By replacing n-twist ($n \geq 2$) by $(n+2)$-twist in the Conway symbol of an alternating knot K, the signature changes by -2 if the replacement is made in a parallel twist with positive crossings, the signature changes by $+2$ if the replacement is made in a parallel twist with negative crossings, and remains unchanged if the replacement is made in an anti-parallel twist.

Proof: According to the preceding theorem:

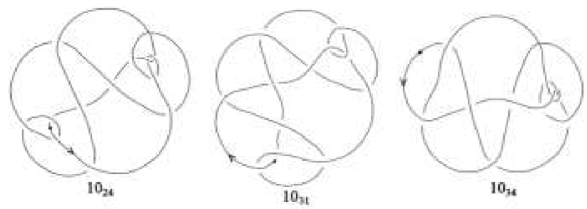

Figure 14: (a) Knot 10_{24}; (b) knot 10_{31}; (c) knot 10_{34}.

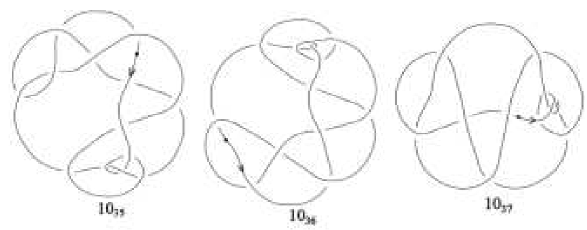

Figure 15: (a) Knot 10_{35}; (b) knot 10_{36}; (c) knot 10_{37}.

1. by adding a full twist in a parallel positive n-twist the writhe changes by $+2$, the number of the white regions W remains unchanged, the number of black regions B increases by $+2$, and the signature changes by -2;
2. by adding a full twist in a parallel negative n-twist the writhe changes by -2, the number of white regions W increases by 2, the number of black regions B remains unchanged, and the signature increases by 2;
3. by adding a full twist in an anti-parallel positive n-twist the writhe changes by $+2$, the number of white regions W increases by 2, the number of black regions B remains unchanged, and the signature remains unchanged;
4. by adding a full twist in an anti-parallel negative n-twist the writhe changes by -2, the number of white regions W remains unchanged, the number of black regions B increases by 2, and the signature remains unchanged.

Theorem 2. The signature σ_K of an alternating knot K given by its Conway symbol is

$$\sigma_K = \sum_P -2[\frac{n_i}{2}]c_i + 2c_0,$$

where the sum is taken over all parallel twists n_i, $c_i \in \{1, -1\}$ is the sign of crossings belonging to a parallel twist n_i, and $2c_0$ is an integer constant which can be computed from the signature of the generating knot.

The proof of this theorem follows directly form the preceding Lemma, claiming that only additions of twists in parallel twists in a Conway symbol result in the

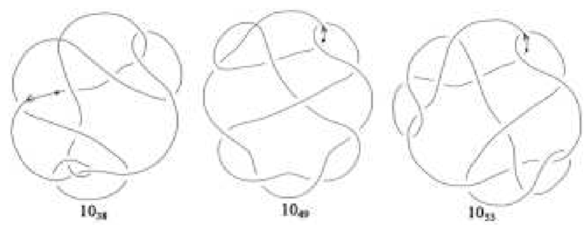

Figure 16: (a) Knot 10_{38}; (b) knot 10_{49}; (c) knot 10_{53}.

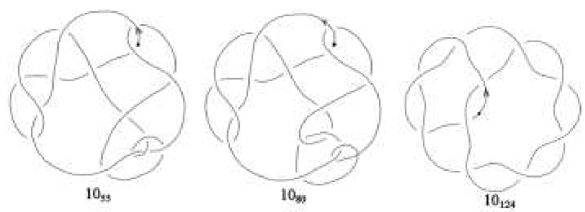

Figure 17: (a) Knot 10_{55}; (b) knot 10_{80}; (c) knot 10_{124}.

change of signature, and that by every such addition, signature changes by $-2c_i$. Notice that the condition that we are making twist replacements in the standard Conway symbols, i.e., Conway symbols with the maximal twists, is essential for computation of general formulae for the signature of alternating knot families.

Example 1: For the family of Montesinos knots with the Conway symbol of the form $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ (Fig. 25), beginning with the generating knot $3\,2,2\,2,3\,1,3$, the parallel twists with negative signs are $2p_2$ and $2p_6+1$, the parallel twist with positive signs is $2p_5+1$, and the remaining twists are anti-parallel. Hence, the signature is $\sigma = -2p_2+2p_5-2p_6+2c_0$. Since the writhe of the generating knot $G = 3\,2,2\,2,3\,1,3$ is $w = -4$ and its checkerboard coloring has $W = 9$ white and $B = 9$ black regions, its signature is 2. Evaluating the formula $\sigma = 2p_2-2p_5 2p_6+2c_0$ for $\sigma_G =$, $p_2 = 1$, $p_5 = 1$, and $p_6 = 1$, we obtain $c_0 = 0$. Hence, the general formula for the signature of knots belonging to the family $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ is $2p_2-2p_5+2p_6$.

Example 2: For the family of polyhedral knots with the Conway symbol of the form $(2p_1+1) : (2p_2) : (2p_3)$ ($p_1 \geq 1$, $p_2 \geq 1$, $p_3 \geq 1$), beginning with the knot $3 : 2 : 2$ (Fig. 26), all twists are parallel twists with positive crossings, and the formula for the signature is $-2p_1-2p_2-2p_3-2$, i.e., $c_0 = 2$. Constant c_0 is computed from the signature of the generating knot $3 : 2 : 2$ which is equal to -4.

Example 3: Let us consider pretzel knots and links (Fig. 27) given by Conway symbol $p_1, \ldots, p_n$ ($n \geq 3$). We obtain knots if all p_i ($i = 1, ..., n$) are odd and n is an odd number, or if one twist is even, and all the others are odd. If all twists are odd and n is an odd number, all twists are anti-parallel, and the signature is $\sigma_K = n-1$ for every such knot. If $n = 3$, for the pretzel knots of the form

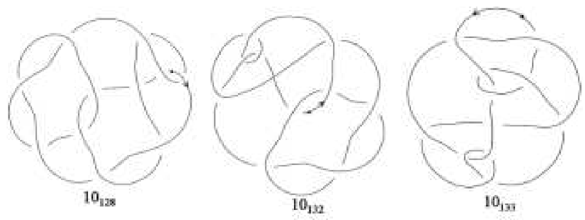

Figure 18: (a) Knot 10_{128}; (b) knot 10_{132}; (c) knot 10_{133}.

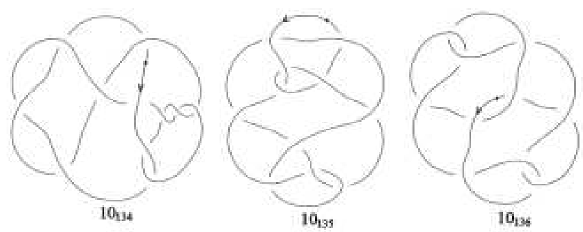

Figure 19: (a) Knot 10_{134}; (b) knot 10_{135}; (c) knot 10_{136}.

$(2p_1+1),(2p_2+1),(2q)$, the twists $2p_1+1$ and $2p_2+1$ are parallel with positive crossings, the twist $2q$ is antiparallel, and the signature is $\sigma_K = 2p_1+2p_2$. For $n \geq 4$, for pretzel knots consisting of an even number of odd twists and one even twist, $2p_1+1$, ..., $2p_{2k}+1$, $2q$, all odd twists are parallel with positive crossings, the even twist $2q$ is anti-parallel, and the signature is $\sigma_K = 2p_1+2p_2+\ldots+2p_{2k+1}$. For $n \geq 4$, for pretzel knots consisting of an odd number of odd twists and one even twist, $2p_1+1$, ..., $2p_{2k+1}+1$, $2q$, all twists are parallel with positive crossings, and the signature is $\sigma_K = 2p_1+2p_2+\ldots+2p_{2k+1}+2q$. Hence, for this class of pretzel knots we simply conclude that their unknotting number is given by the formula $u_K = p_1+p_2+\ldots+p_{2k+1}+q$.

Example 4: Let us consider knots of the form $t_1,\ldots t_n+t$ ($n \geq 3$), where t_i and t are twists (Fig. 28a). If the twists of an odd length are denoted by p, and twists of an even length by q, we have six possible cases:

1. if the tangle $t_1,\ldots,t_n$ consists of $2k$ odd twists $p_1,\ldots,p_{2k}$, and the tangle t is an odd twist p, the signature is given by the formula $2k+2[\frac{p}{2}]$

2. if the tangle $t_1,\ldots,t_n$ consists of $2k+1$ odd twists $p_1,\ldots,p_{2k}$, and the tangle t is an even twist q, the signature is given by the formula $2k+q$

3. if the tangle $t_1,\ldots,t_n$ consists of $2k+1$ odd twists $p_1,\ldots,p_{2k+1}$ and an even twist q_1, and the tangle t is an odd twist p, the signature is given by the formula $\sum_{i=1}^{2k+1} 2[\frac{p_i}{2}]$

4. if the tangle $t_1,\ldots,t_n$ consists of $2k+1$ odd twists $p_1,\ldots,p_{2k+1}$ and an even

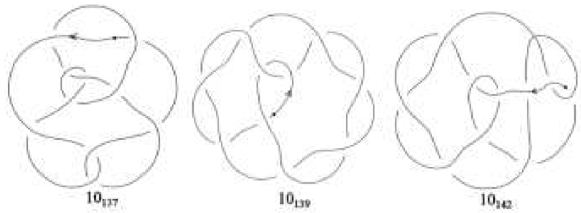

Figure 20: (a) Knot 10_{137}; (b) knot 10_{139}; (c) knot 10_{142}.

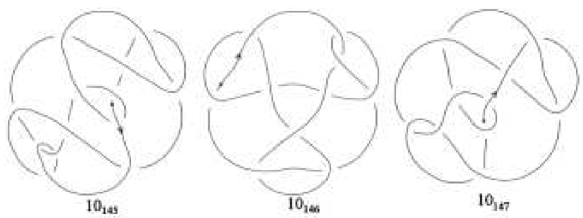

Figure 21: (a) Knot 10_{145}; (b) knot 10_{146}; (c) knot 10_{147}.

twist q_1, and the tangle t is an even twist q, the signature is given by the formula $\sum_{i=1}^{2k+1} 2[\frac{p_i}{2}] + q_1$

5. if the tangle $t_1, \ldots, t_n$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$ and an even twist q_1, and the tangle t is an odd twist p, the signature is given by the formula $\sum_{i=1}^{2k} 2[\frac{p_i}{2}] + q_1$

6. if the tangle $t_1, \ldots, t_n$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$ and an even twist q_1, and the tangle t is an even twist q, the signature is given by the formula $\sum_{i=1}^{2k} 2[\frac{p_i}{2}]$.

Example 5: As a more complex example, we provide general formulae for the signature of knots of the type $(t_1, \ldots, t_m)\,(t'_1, \ldots, t_n)$ $(m \geq 2,\ n \geq 2)$, where twists are denoted by t_i or t'_i (Fig. 28b). If the twists of an odd length are denoted by p, and twists of an even length by q, we have seven possible cases:

1. if the first tangle $t_1, \ldots, t_m$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$, and the second tangle $t'_1, \ldots, t'_n$ consists of $2r+1$ odd twists $p'_1, \ldots, p'_{2r+1}$, the signature is given by the formula

$$\sum_{i=1}^{2r+1} 2[\frac{p'_i}{2}] + 2k$$

2. if the first tangle $t_1, \ldots, t_m$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$, and the second tangle $t'_1, \ldots, t'_n$ consists of $2r$ odd twists $p'_1, \ldots, p'_{2r}$, the signature is

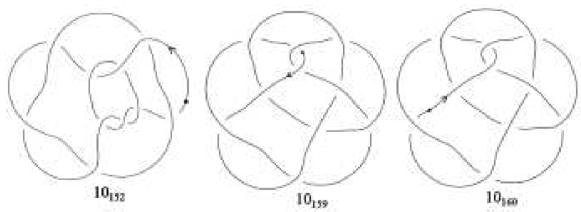

Figure 22: (a) Knot 10_{152}; (b) knot 10_{159}; (c) knot 10_{160}.

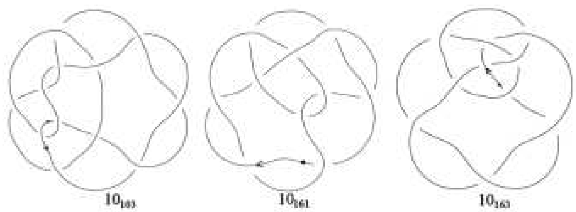

Figure 23: (a) Knot 10_{103}; (b) knot 10_{161}; (b) knot 10_{163}.

given by the formula

$$\sum_{i=1}^{2k} 2[\frac{p_i}{2}] - \sum_{i=1}^{2r} 2[\frac{p_i'}{2}]$$

3. if the first tangle $t_1, \ldots, t_m$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$ and one even twist q_1, and the second tangle $t_1', \ldots, t_n'$ consists of $2r+1$ odd twists $p_1', \ldots, p_{2r+1}'$, the signature is given by the formula

$$\sum_{i=1}^{2k} 2[\frac{p_i}{2}] + q_1 + 2r$$

4. if the first tangle $t_1, \ldots, t_m$ consists of $2k+1$ odd twists $p_1, \ldots, p_{2k+1}$ and one even twist q_1, and the second tangle $t_1', \ldots, t_n'$ consists of $2r+1$ odd twists $p_1', \ldots, p_{2r+1}'$, the signature is given by the formula

$$\sum_{i=1}^{2k+1} 2[\frac{p_i}{2}] + 2r$$

5. if the first tangle $t_1, \ldots, t_m$ consists of $2k$ odd twists $p_1, \ldots, p_{2k}$ and one even twist q_1, and the second tangle $t_1', \ldots, t_n'$ consists of $2r$ odd twists $p_1', \ldots, p_{2r}'$ and one even twist q_1', the signature is given by the formula

$$\sum_{i=1}^{2k} 2[\frac{p_i}{2}] - \sum_{i=1}^{2r} 2[\frac{p_i'}{2}]$$

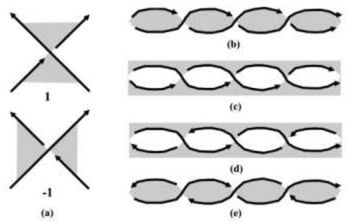

Figure 24: (a) Positive crossing and negative crossing (b) parallel positive twist; (c) parallel negative twist; (d) antiparallel positive twist; (e) antiparallel negative twist.

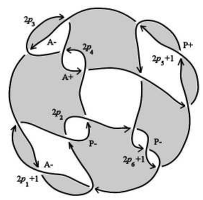

Figure 25: Knot family $(2p_1+1)\,(2p_2),(2p_3)\,(2p_4),(2p_5+1)\,1,(2p_6+1)$ beginning with knot $3\,2,2\,2,3\,1,3$.

6. if the first tangle $t_1,\ldots,t_m$ consists of $2k$ odd twists $p_1,\ldots,p_{2k}$ and one even twist q_1, and the second tangle $t'_1,\ldots,t'_n$ consists of $2r+1$ odd twists $p'_1,\ldots,p'_{2r+1}$ and one even twist q'_1, the signature is given by the formula

$$\sum_{i=1}^{2k} 2[\frac{p_i}{2}] - \sum_{i=1}^{2r+1} 2[\frac{p'_i}{2}] - q'_1$$

7. if the first tangle $t_1,\ldots,t_m$ consists of $2k+1$ odd twists $p_1,\ldots,p_{2k+1}$ and one even twist q_1, and the second tangle $t'_1,\ldots,t'_n$ consists of $2r+1$ odd twists $p'_1,\ldots,p'_{2r+1}$ and one even twist q'_1, the signature is given by the formula

$$\sum_{i=1}^{2k+1} 2[\frac{p_i}{2}] - \sum_{i=1}^{2r+1} 2[\frac{p'_i}{2}] + q_1 - q'_1.$$

6. Unknotting numbers of knot families

K. Murasugi [9] proved the lower bound for the unknotting number of knots, $u(K) \geq \frac{|\sigma_K|}{2}$. Using this criterion, for many (sub)families of knots we can confirm that their BJ-unknotting numbers, i.e., unknotting numbers computed according to Bernhard-Jablan Conjecture [5] represent the actual unknotting numbers of these

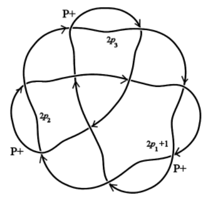

Figure 26: Knot family $(2p_1+1):(2p_2):(2p_3)$ beginning with knot $3:2:2$.

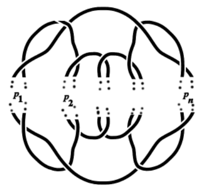

Figure 27: Pretzel knot $p_1, p_2, \ldots, p_n$.

(sub)families. In the following table is given the list of (sub)families with this property obtained from knots with at most $n = 8$ crossings, where in the first column is given the first knot belonging to the family, in the second its Conway symbol, in the third the general Conway symbol, in the fourth the general formula for the signature, in the fifth the unknotting number confirmed by the signature, and in the sixth the conditions for this unknotting number[‡].

K	Con	Fam	σ	u	$Cond$
3_1	3	$(2p_1+1)$	$2p_1$	p_1	
4_1	2 2	$(2p_1)\,(2p_2)$	0		
5_2	3 2	$(2p_1+1)\,(2p_2)$	$2p_2$	p_2	
6_2	3 1 2	$(2p_1+1)\,1\,(2p_2)$	$2p_1$	p_1	$p_1 \geq p_2$
6_3	2 1 1 2	$(2p_1)\,1\,1\,(2p_2)$	$2p_1-2p_2$	$\lvert p_1-p_2\rvert$	$p_1 \neq p_2$
7_4	3 1 3	$(2p_1+1)\,1\,(2p_2+1)$	2		
7_5	3 2 2	$(2p_1+1)\,(2p_2)\,(2p_3)$	$2p_1+2p_3$	p_1+p_3	
7_6	2 2 1 2	$(2p_1)\,(2p_2)\,1\,(2p_3)$	$2p_3$	p_3	$p_2 \leq p_3$
7_7	2 1 1 1 2	$(2p_1)\,1\,1\,1\,(2p_2)$	0		
8_5	3, 3, 2	$(2p_1+1),(2p_2+1),(2p_3)$	$2p_1+2p_2$	p_1+p_2	$p_1 \geq p_3$ or $p_2 \geq p_3$
8_6	3 3 2	$(2p_1+1)\,(2p_2+1)\,(2p_3)$	$2p_1$		
8_8	2 3 1 2	$(2p_1)\,(2p_2+1)\,1\,(2p_3)$	$2p_1-2p_3$	p_3-p_1	$p_3-p_1>p_2$
8_9	3 1 1 3	$(2p_1+1)\,1\,1\,(2p_2+1)$	$2p_1-2p_2$	$\lvert p_1-p_2\rvert$	$p_1 \neq p_2$
8_{10}	3, 2 1, 2	$(2p_1+1),(2p_2)\,1,(2p_3)$	$2p_1-2p_2+2p_3$	$p_1-p_2+p_3$	$p_3>p_2$
8_{11}	3 2 1 2	$(2p_1+1)\,2p_2)\,1\,(2p_3)$	$2p_2$	p_2	$p_2 \geq p_3$
8_{12}	2 2 2 2	$(2p_1)\,(2p_2)\,(2p_3)\,(2p_4)$	0		
8_{13}	3 1 1 1 2	$(2p_1+1)\,1\,1\,1\,(2p_2)$	$2p_2-2$	p_2-1	$p_2-1>p_1$
8_{14}	2 2 1 1 2	$(2p_1)\,(2p_2)\,1\,1\,(2p_3)$	$2p_1$	p_1	$p_2 \leq p_3$
8_{15}	2 1, 2 1, 2	$(2p_1)\,1,(2p_2)\,1,(2p_3)$	$2p_1+2p_2$	p_1+p_2	
8_{16}	.2.2 0	$.(2p_1).(2p_2)\,0$	$2p_1+2p_2-2$	p_1+p_2-1	
8_{17}	.2.2	$.(2p_1).(2p_2)$	$2p_1-2p_2$	$\lvert p_1-p_2\rvert$	$p_1=1, p_2>1$ or $p_2=1, p_1>1$

[‡]Conditions for unknotting numbers are determined from the experimental results obtained for knots up to $n = 20$ crossings.

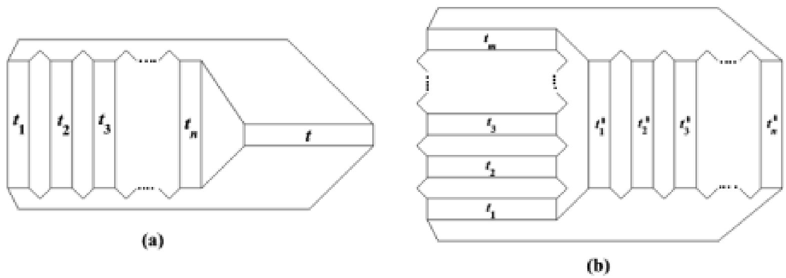

Figure 28: (a) Knot $t_1, \ldots t_n + t$; (b) knot $(t_1, \ldots, t_m)\,(t'_1, \ldots, t'_n)$.

K	Con	Fam	σ	u	$Cond$
9_{10}	3 3 3	$(2p_1+1)\,(2p_2+1)\,(2p_3+1)$	$2p_2+2$		
9_{13}	3 2 1 3	$(2p_1)\,(2p_2)\,1\,(2p_3+1)$	$2p_1+2$		
9_{15}	2 3 2 2	$(2p_1)\,(2p_2+1)\,(2p_3)\,(2p_4)$	$2p_1$		
9_{16}	3, 3, 2+	$(2p_1+1),(2p_2+1),(2p_3)+$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
9_{17}	2 1 3 1 2	$(2p_1)\,1\,(2p_2+1)\,1\,(2p_3)$	$-2p_2$	p_2	$p_1+p_3 \leq p_2$
9_{18}	3 2 2 2	$(2p_1+1)\,(2p_2)\,(2p_3)\,(2p_4)$	$2p_2+2p_4$	p_2+p_4	
9_{19}	2 3 1 1 2	$(2p_1)\,(2p_2+1)\,1\,1\,(2p_3)$	0		
9_{20}	3 1 2 1 2	$(2p_1+1)\,1\,(2p_2)\,1\,(2p_3)$	$2p_1+2p_3$	p_1+p_3	$p_1+p_3 \geq p_2$
9_{21}	3 1 1 2 2	$(2p_1+1)\,1\,1\,(2p_2)\,(2p_3)$	2		
9_{22}	2 1 1, 3, 2	$(2p_1)\,1\,1,(2p_2+1),(2p_3)$	$-2p_2$	p_2	$p_1+p_3-1 \leq p_2$
9_{23}	2 2 1 2 2	$(2p_1)\,(2p_2)\,1\,(2p_3)\,(2p_4)$	$2p_1+2p_4$	p_1+p_4	
9_{24}	2 1, 3, 2+	$(2p_1)\,1,(2p_2+1),(2p_3)+$	$2p_1-2p_2$		
9_{25}	2 2, 2 1, 2	$(2p_1)\,(2p_2),(2p_3)\,1,(2p_4)$	$-2p_3$		
9_{26}	3 1 1 1 1 2	$(2p_1+1)\,1\,1\,1\,1\,(2p_2)$	$2p_1$	p_1	
9_{27}	2 1 2 1 1 2	$(2p_1)\,1\,(2p_2)\,1\,1\,(2p_3)$	$2p_3-2p_2$	p_3-p_2	$p_2<p_3$
9_{28}	2 1, 2 1, 2+	$(2p_1)\,1,(2p_2)\,1,(2p_3)+$	$2p_1+2p_2-2p_3$	$p_1+p_2-p_3$	$p_3 \leq p_1$ or $p_3 \leq p_2$
9_{29}	.2.2 0.2	$.(2p_1).(2p_2)\,0.(2p_3)$	$-2p_2$		
9_{30}	2 1 1, 2 1, 2	$(2p_1)\,1\,1,(2p_2)\,1,(2p_3)$	$2p_2-2p_3$	p_3-p_2	$p_1+p_2 \leq p_3$
9_{31}	2 1 1 1 1 1 2	$(2p_1)\,1\,1\,1\,1\,1\,(2p_2)$	$2p_1-2p_2-2$	p_1+p_2-1	$p_1+p_2>2$
9_{32}	.2 1.2 0	$.(2p_1)\,1.(2p_2)\,0$	$2p_2$		
9_{33}	.2 1.2	$.(2p_1)\,1.(2p_2)$	$-2p_2+2$	p_2-1	$p_2-p_1 \geq 2$
9_{34}	$8^*2\,0$	$8^8(2p_1)\,0$	0		
9_{35}	3, 3, 3	$(2p_1+1),(2p_2+1),(2p_3+1)$	2		
9_{36}	2 2, 3, 2	$(2p_1)\,(2p_2),(2p_3)\,1,(2p_4)$	$2p_3+2p_4$	p_3+p_4	$p_2 \leq p_4$
9_{37}	2 1, 2 1, 3	$(2p_1)\,1,(2p_2)\,1,(2p_3+1)$	0		
9_{38}	.2.2.2	$.(2p_1).(2p_2).(2p_3)$	$2p_2+2$		
9_{39}	2 : 2 : 2 0	$(2p_1):(2p_2):(2p_3)\,0$	2		
9_{41}	2 0 : 2 0 : 2 0	$(2p_1)\,0:(2p_2)\,0:(2p_3)\,0$	0		

K	Con	Fam	σ	u	$Cond$
10_{22}	3 3 1 3	$(2p_1+1)\,(2p_2+1)\,1\,(2p_1+1)$	$2p_1-2p_3$	p_3-p_1	$p_3-p_1>p_2$
10_{23}	3 3 1 1 2	$(2p_1+1)\,(2p_2+1)\,1\,1,(2p_3)$	$2p_2-2p_3+2$	p_2-p_3+1	$p_3\le p_2$
10_{24}	3 2 3 2	$(2p_1+1)\,(2p_2)\,(2p_3+1)\,(2p_4)$	$2p_2$		
10_{25}	3 2 2 1 2	$(2p_1+1)\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_4\le p_3$
10_{26}	3 2 1 1 3	$(2p_1+1)\,(2p_2)\,1\,1\,(2p_3+1)$	$2p_2-2p_3$	$\|p_2-p_3\|$	$p_3-p_2>p_1$ or $p_2>p_3$
10_{27}	3 2 1 1 1 2	$(2p_1+1)\,(2p_2)\,1\,1\,1(2p_3)$	$2p_1-2p_3+2$	$\|p_1-p_3+1\|$	$p_3\le p_1, p_2=2$ or $p_3-p_1>p_2+1$
10_{28}	3 1 3 1 2	$(2p_1+1)\,1\,(2p_2+1)\,1\,(2p_3)$	$-2p_3+2$	p_3-1	$p_1+p_2+1<p_3$
10_{29}	3 1 2 2 2	$(2p_1+1)\,1\,(2p_2)\,(2p_3)\,(2p_4)$	$2p_1$		
10_{30}	3 1 2 1 1 2	$(2p_1+1)\,1\,(2p_2)\,1\,1\,(2p_3)$	2		
10_{31}	3 1 1 3 2	$(2p_1+1)\,1\,1\,(2p_2+1)\,(2p_3)$	$-2p_3+2$		
10_{32}	3 1 1 1 2 2	$(2p_1+1)\,1\,1\,1\,(2p_2)\,(2p_3)$	$2p_1-2p_3$	$\|p_1-p_3\|$	$p_1>p_2+p_3$ or $p_2=2, p_3>p_1$
10_{33}	3 1 1 1 1 3	$(2p_1+1)\,1\,1\,1\,1\,(2p_2+1)$	0		
10_{37}	2 3 3 2	$(2p_1)\,(2p_2+1)\,(2p_3+1)\,(2p_4)$	$2p_1-2p_4$		
10_{38}	2 3 1 2 2	$(2p_1)\,(2p_2+1)\,1\,(2p_3)\,(2p_4)$	$-2p_4$		
10_{39}	2 2 3 1 2	$(2p_1)\,(2p_2)\,(2p_3+1)\,1\,(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_1+p_2+p_3\le p_4$
10_{40}	2 2 2 1 1 2	$(2p_1)\,(2p_2)\,(2p_3)\,1\,1\,(2p_4)$	$2p_1+2p_3-2p_4$	$\|p_1+p_3-p_4\|$	$p_4<p_3$ or $p_4>p_1+p_2+p_3$
10_{41}	2 2 1 2 1 2	$(2p_1)\,(2p_2)\,1\,(2p_3)\,1\,(2p_4)$	$2p_3$	p_3	$p_2+p_4\le p_3$
10_{42}	2 2 1 1 1 1 2	$(2p_1)\,(2p_2)\,1\,1\,1\,1\,(2p_3)$	$-2p_3+2$		
10_{43}	2 1 2 2 1 2	$(2p_1)\,1\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$2p_1-2p_4$		
10_{44}	2 1 2 1 1 1 2	$(2p_1)\,1\,(2p_2)\,1\,1\,1\,(2p_3)$	$2p_1$	p_1	$p_2+p_3\le p_1+1$
10_{45}	2 1 1 1 1 1 1 2	$(2p_1)\,1\,1\,1\,1\,1\,1\,(2p_2)$	0		
10_{50}	3 2, 3, 2	$(2p_1+1)\,(2p_2),(2p_3+1),(2p_4)$	$2p_2+2p_3$	p_2+p_3	$p_4\le p_2$
10_{51}	3 2, 2 1, 2	$(2p_1+1)\,(2p_2),(2p_3)\,1,(2p_4)$	$2p_2+2p_4-2p_3$		
10_{52}	3 1 1, 3, 2	$(2p_1+1)\,1\,1,(2p_2+1),(2p_3)$	$-2p_2-2p_3+2$	p_2+p_3-1	$p_3-p_1\ge 2$
10_{53}	3 1 1, 2 1, 2	$(2p_1+1)\,1\,1,(2p_2)\,1,(2p_3)$	$2p_2+2$		
10_{54}	2 3, 3, 2	$(2p_1)\,(2p_2+1),(2p_3+1),(2p_4)$	$2p_1-2p_3-2p_4$	$p_3+p_4-p_1$	$p_4>p_1+p_2$
10_{55}	2 3, 2 1, 2	$(2p_1)\,(2p_2+1),(2p_3)\,1,(2p_4)$	$2p_1+2p_3$	p_1+p_3	
10_{56}	2 2 1, 3, 2	$(2p_1)\,(2p_2)\,1,(2p_3+1),(2p_4)$	$2p_1+2p_3$	p_1+p_3	$p_4\le p_1+p_2$
10_{57}	2 2 1, 2 1, 2	$(2p_1)\,(2p_2)\,1,(2p_3)\,1,(2p_4)$	$2p_1+2p_4-2p_3$	$p_1+p_4-p_3$	$p_3<p_4$
10_{58}	2 1 1, 2 1 1, 2	$(2p_1)\,1\,1,(2p_2)\,1\,1,(2p_3)$	0		
10_{59}	2 2, 2 1 1, 2	$(2p_1)\,(2p_2),(2p_3)\,1\,1,(2p_4)$	$2p_4$	p_4	$p_2+p_3-1\le p_4$
10_{60}	2 1 1, 2 1 1, 2	$(2p_1)\,1\,1,(2p_2)\,1\,1,(2p_3)$	0		
10_{64}	3 1, 3, 3	$(2p_1+1)\,1,(2p_2+1),(2p_3+1)$	$2p_1-2p_2-2p_3$	$p_2+p_3-p_1$	$\max(p_2,p_3)>p_1$

K	Con	Fam	σ	u	$Cond$
10_{65}	$3\,1, 3, 2\,1$	$(2p_1+1)\,1, (2p_2+1), (2p_3)\,1$	$2p_2-2p_3+2$		
10_{66}	$3\,1, 2\,1, 2\,1$	$(2p_1+1)\,1, (2p_2)\,1, (2p_3)\,1$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
10_{68}	$2\,1\,1, 3, 3$	$(2p_1)\,1\,1, (2p_2+1), (2p_3+1)$	$2p_1-2$		
10_{67}	$2\,2, 3, 2\,1$	$(2p_1)\,(2p_2), (2p_3+1), (2p_4)\,1$	$2p_1$		
10_{69}	$2\,1\,1, 2\,1, 2\,1$	$(2p_1)\,1\,1, (2p_2)\,1, (2p_3)\,1$	$2p_1$		
10_{70}	$2\,2, 3, 2+$	$(2p_1)\,(2p_2), (2p_3+1), (2p_4)+$	$2p_3$		
10_{71}	$2\,2, 2\,1, 2+$	$(2p_1)\,(2p_2), (2p_3)\,1, (2p_4)+$	$2p_4-2p_3$		
10_{72}	$2\,1\,1, 3, 2+$	$(2p_1)\,1\,1, (2p_2+1), (2p_3)+$	$-2p_2-2p_3$	p_2+p_3	$p_1-1\leq p_2+p_3$
10_{73}	$2\,1\,1, 2\,1, 2+$	$(2p_1)\,1\,1, (2p_2)\,1, (2p_3)+$	$2p_2$	p_2	$p_3=1$
10_{74}	$3, 3, 2\,1+$	$(2p_1+1), (2p_2+1), (2p_3)\,1+$	2		
10_{75}	$2\,1, 2\,1, 2\,1+$	$(2p_1)\,1, (2p_2)\,1, (2p_3)\,1+$	0		
10_{76}	$3, 3, 2+2$	$(2p_1+1), (2p_2+1), (2p_3)+(2p_4)$	$2p_1+2p_2$		
10_{77}	$3, 2\,1, 2+2$	$(2p_1+1), (2p_2)\,1, (2p_3)+(2p_4)$	$2p_1+2p_3-2p_2$		
10_{78}	$2\,1, 2\,1, 2+2$	$(2p_1)\,1, (2p_2)\,1, (2p_3)+(2p_4)$	$2p_1+2p_2$	p_1+p_2	$p_4\leq p_1+p_2$
10_{79}	$(3,2)\,(3,2)$	$((2p_1+1),(2p_2))\,((2p_3+1),(2p_4))$	$2p_1+2p_2-2p_3-2p_4$		
10_{80}	$(3,2)\,(2\,1,2)$	$((2p_1+1),(2p_2))\,((2p_3)\,1,(2p_4))$	$2p_1+2p_2+2p_3$	$p_1+p_2+p_3$	
10_{81}	$(2\,1,2)\,(2\,1,2)$	$((2p_1)\,1,(2p_2))\,((2p_3)\,1,(2p_4))$	$2p_1-2p_3$		
10_{83}	$.3\,1.2$	$.(2p_1+1)\,1.(2p_2)$	$-2p_2+2$	p_2-1	$p_2>p_1+1$
10_{84}	$.2\,2.2$	$.(2p_1)\,(2p_2).(2p_3)$	$2p_1+2p_3$	p_1+p_3	$p_2=1$ or $p_3\geq 2$
10_{86}	$.3\,1.2\,0$	$.(2p_1+1)\,1.(2p_2)\,0$	$2p_2$		
10_{87}	$.2\,2.2\,0$	$.(2p_1)\,(2p_2).(2p_3)\,0$	$2p_1-2p_3$		
10_{88}	$.2\,1.2\,1$	$.(2p_1)\,1.(2p_2)\,1$	0		
10_{89}	$.2\,1.2\,1\,0$	$.(2p_1)\,1.(2p_2)\,1\,0$	2		
10_{90}	$.3.2.2$	$.(2p_1+1).(2p_2).(2p_3)$	$2p_2-2p_3$	p_3-p_2	$p_3>p_1+p_2$
10_{91}	$.3.2.2\,0$	$.(2p_1+1).(2p_2).(2p_3)\,0$	$2p_1-2p_2-2p_3+2$	$p_2+p_3-p_1-1$	$p_1\leq p_3, p_2>1$
10_{92}	$.2\,1.2.2\,0$	$.(2p_1)\,1.(2p_2).(2p_3)\,0$	$2p_1+2p_2$	p_1+p_2	$p_3-1\leq p_1$
10_{93}	$.3.2\,0.2$	$.(2p_1+1).(2p_2)\,0.(2p_3)$	$-2p_2-2p_3$	p_2+p_3	$p_1<p_2$ or $p_3>p_1+1$
10_{94}	$.3\,0.2.2$	$.(2p_1+1)\,0.(2p_2).(2p_3)$	$2p_1+2p_2-2p_3$	$p_1+p_2-p_3$	$p_1-p_3\geq 1$ or $p_2-p_3\geq 1$
10_{95}	$.2\,1\,0.2.2$	$(2p_1)\,1\,0.(2p_2).(2p_3)$	$2p_1-2p_2-2$	p_2-p_1+1	$p_1=p_3=1$
10_{96}	$.2.2\,1.2$	$.(2p_1).(2p_2)\,1.(2p_3)$	0		
10_{97}	$.2.2\,1\,0.2$	$.(2p_1).(2p_2)\,1\,0.(2p_3)$	2		
10_{98}	$.2.2.2.2\,0$	$.(2p_1).(2p_2).(2p_3).(2p_4)\,0$	$2p_1+2p_3$	p_1+p_3	$p_4\leq p_1$ or $p_4\leq p_3$
10_{99}	$.2.2.2\,0.2\,0$	$.(2p_1).(2p_2).(2p_3)\,0.(2p_4)\,0$	$2p_1+2p_4-2p_2-2p_3$		
10_{100}	$3:2:2$	$(2p_1+1):(2p_2):(2p_3)$	$2p_1+2p_2+2p_3-2$	$p_1+p_2+p_3-1$	$p_2>1$ or $p_3>1$
10_{101}	$2\,1:2:2$	$(2p_1)\,1:(2p_2):(2p_3)$	$2p_1+2$		
10_{102}	$3:2:2\,0$	$(2p_1+1):(2p_2):(2p-3)\,0$	$2p_3-2p_2$	p_2-p_3	$p_3=1, p_2-p_1>1$
10_{103}	$3\,0:2:2$	$(2p_1+1)\,0:(2p_2):(2p_3)$	$2p_2+2p_3-2$		
10_{104}	$3:2\,0:2\,0$	$(2p_1+1):(2p_2)\,0:(2p_3)\,0$	$2p_1-2p_2-2p_3+2$	$p_1-p_2-p_3+1$	$p_1>p_2, p_3=1$ or $p_1>p_3, p_2=1$

K	Con	Fam	σ	u	$Cond$
10_{105}	$2\,1:2\,0:2\,0$	$(2p_1)\,1:(2p_2)\,0:(2p_3)\,0$	$2p_1$	p_1	$p_1>p_2+p_3$
10_{106}	$3\,0:2:2\,0$	$(2p_1+1)\,0:(2p_2):(2p_3)\,0$	$2p_1+2p_3-2p_2$	$p_1+p_3-p_2$	$p_2+1\le p_1$
10_{107}	$2\,1\,0:2:2\,0$	$(2p_1)\,1\,0:(2p_2):(2p_3)\,0$	$2p_1-2$		
10_{108}	$3\,0:2\,0:2\,0$	$(2p_1+1)\,0:(2p_2)\,0:(2p_3)\,0$	$-2p_2-2p_3+2$	p_2+p_3-1	$p_2>p_1+2$ or $p_3>p_1+2$
10_{109}	$2.2.2.2$	$(2p_1).(2p_2).(2p_3).(2p_4)$	$2p_1+2p_3-2p_2-2p_4$	$p_1+p_3-p_2-p_4$	$p_1\ge p_2+p_4$
10_{110}	$2.2.2.2\,0$	$(2p_1).(2p_2).(2p_3).(2p_4)\,0$	$-2p_4$	p_4	$p_4\ge p_1+p_3$
10_{111}	$2.2.2\,0.2$	$(2p_1).(2p_2).(2p_3)\,0.(2p_4)$	$2p_2+2p_3$	p_2+p_3	$p_2+p_3\ge p_4\ge p_1$
10_{112}	8^*3	$8^*(2p_1+1)$	$2p_1$	p_1	$p_1\ge 2$
10_{113}	$8^*2\,1$	$8^*(2p_1)\,1$	$2p_1$	p_1	$p_1\ge 2$
10_{114}	$8^*3\,0$	$8^*(2p_1+1)\,0$	0		
10_{115}	$8^*2\,0.2\,0$	$8^*(2p_1)\,0.(2p_2)\,0$	0		
10_{116}	$8^*2:2$	$8^*(2p_1):(2p_2)$	$2p_1+2p_2-2$	p_1+p_2-1	$p_1\ge 2$ or $p_2\ge 2$
10_{117}	$8^*2:2\,0$	$8^*(2p_1):(2p_2)\,0$	$2p_2$		
10_{118}	$8^*2:.2$	$8^*(2p_1):.(2p_2)\,0$	$2p_1-2p_2$	$\lvert p_1-p_2\rvert$	$p_1\ge 2, p_2=1$ $p_2\ge 2, p_1=1$ or $\lvert p_1-p_2\rvert\ge 2$
10_{119}	$8^*2:.2\,0$	$8^*(2p_1):.(2p_2)\,0$	$-2p_2+2$	p_2-1	$p_2-p_1\ge 2$
10_{120}	$8^*2\,0::2\,0$	$8^*(2p_1)\,0::(2p_2)\,0$	4		
10_{121}	$9^*2\,0$	$9^8(2p_1)\,0$	2		
10_{122}	$9^*.2\,0$	$9^*.(2p_1)\,0$	0		

In the following table we provide the same results for link families obtained from generating links with at most $n=9$ crossings.

K	Con	Fam	σ	u	$Cond$
2^2_1	2	$(2p_1)$	$-2p_1+1$	p_1	
5^2_1	$2\,1\,2$	$(2p_1)\,1\,(2p_2)$	$-2p_1+1$	p_1	$p_1>p_2$
6^2_2	$3\,3$	$(2p_1+1)\,(2p_2+1)$	$-2p_1-1$		
6^2_3	$2\,2\,2$	$(2p_1)\,(2p_2)\,(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	
7^2_2	$3\,1\,1\,2$	$(2p_1+1)\,1\,1\,(2p_2)$	$-2p_1+2p_2$	$\lvert p_1-p_2\rvert$	
7^2_3	$2\,3\,2$	$(2p_1)\,(2p_2+1)\,(2p_3)$	$-2p_1+1$		
7^2_4	$3,2,2$	$(2p_1+1),(2p_2),(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	$p_2=1$
7^2_5	$2\,1,2,2$	$(2p_1)\,1,(2p_2),(2p_3)$	$2p_2+2p_3-2p_1+1$		
7^2_6	$.2$	$.(2p_1)$	$-2p_1+1$		
8^2_4	$3\,2\,3$	$(2p_1+1)\,(2p_2)\,(2p_3+1)$	$-2p_1-2p_3-1$		
8^2_5	$3\,1\,2\,2$	$(2p_1+1)\,1\,(2p_2)\,(2p_3)$	$-2p_1-1$		
8^2_7	$2\,1\,2\,1\,2$	$(2p_1)\,1\,(2p_2)\,1\,(2p_3)$	$-2p_1-2p_3+1$	p_1+p_3	$p_1+p_3\ge p_2$
8^2_9	$2\,2,2,2$	$(2p_1)\,(2p_2),(2p_3),(2p_4)$	$-2p_3-2p_4+1$		
8^2_{10}	$2\,1\,1,2,2$	$(2p_1)\,1\,1,(2p_2),(2p_3)$	$2p_3-1$	p_3	$p_1=p_2=1$
8^2_{11}	$3,2,2+$	$(2p_1+1),(2p_2),(2p_3)+$	$-2p_1+1$		
8^2_{12}	$2\,1,2,2+$	$(2p_1)\,1,(2p_2),(2p_3)+$	$-2p_1+2p_2-1$	p_1-p_2+1	$p_2=1$

K	Con	Fam	σ	u	$Cond$
8^2_{13}	$.2\,1$	$.(2p_1)\,1$	-1		
8^2_{14}	$.2:2$	$.(2p_1):(2p_2)$	1		
9^2_6	$3\,3\,1\,2$	$(2p_1+1)\,(2p_2+1)\,1\,(2p_3)$	$-2p_1+2p_3-1$	p_3-p_1	$p_3 \geq p_1+p_2$
9^2_7	$3\,2\,1\,1\,2$	$(2p_1+1)\,(2p_2)\,1\,1\,(2p_3)$	$-2p_1+1$	p_1+1	$p_2=1, p_3 \leq p_1+2$
9^2_8	$3\,1\,3\,2$	$(2p_1+1)\,1\,(2p_2+1)\,(2p_3)$	$-2p_1+2p_3-1$	p_1-1	$p_1 \geq p_2+p_3$
9^2_9	$3\,1\,1\,1\,3$	$(2P-1+1)\,1\,1\,1\,(2p_2+1)$	$-2p_1+1$	p_1	$p_1-p_2 \geq 1$
9^2_{11}	$2\,2\,2\,1\,2$	$(2p_1)\,(2p_2)\,(2p_3)\,1\,(2p_4)$	$-2p_1-2p_3+1$	p_1+p_3	$p_3 \geq p_4$
9^2_{12}	$2\,2\,1\,1\,1\,2$	$(2p_1)\,(2p_2)\,1\,1\,1\,(2p_3)$	$-2p_1+2p_3-1$	p_3-p_1 p_1-p_3+1	$p_3 > p_1+p_2$ $p_2=1, p_1 > p_3$
9^2_{15}	$3\,2,2,2$	$(2p_1+1)\,(2p_2),(2p_3),(2p_4)$	$-2p_2-2p_4+1$		
9^2_{16}	$3\,1\,1,2,2$	$(2p_1+1)\,1\,1,(2p_2),(2p_3)$	$2p_2+2p_3-3$		
9_{17}	$2\,3,2,2$	$(2p_1)\,(2p_2)\,1,(2p_3),(2p_4)$	$-2p_2+2p_3+2p_4-1$		
9^2_{18}	$2\,2\,1,2,2$	$(2p_1)\,(2p_2)\,1,(2p_3),(2p_4)$	$-2p_1-2p_4+1$	p_1+p_4	$p_3=1$
9^2_{21}	$3\,1,3,2$	$(2p_1+1)\,1,(2p_2+1),(2p_3)$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_3 > p_1$
9^2_{22}	$3\,1,2\,1,2$	$(2p_1+1)\,1,(2p_2)\,1,(2p_3)$	$-2p_1-2p_2-1$		
9^2_{23}	$3,3,2\,1$	$(2p_1+1),(2p_2+1)\,(2p_3)\,1$	$-2p_1-2p_2+2p_3-1$	$p_1+p_2-p_3+1$	$p_1 \geq p_3$ or $p_2 \geq p_3$
9^2_{24}	$2\,1,2\,1,2\,1$	$(2p_1)\,1,(2p_2)\,1,(2p_3)\,1$	$-2p_1-2p_2-2p_3-1$	$p_1+p_2+p_3$	
9^2_{25}	$2\,2,2,2+$	$(2p_1)\,(2p_2),(2p_3),(2p_4)+$	$-2p_2+1$		
9^2_{26}	$2\,1\,1,2,2+$	$(2p_1)\,1\,1,(2p_2),(2p_3)+$	-1		
9^2_{27}	$3,2,2+2$	$(2p_1+1),(2p_2),(2p_3)+(2p_4)$	$-2p_1-2p_3+1$		
9^2_{28}	$2\,1,2,2,2+2$	$(2p_1)\,1,(2p_2),(2p_3)+(2p_4)$	$-2p_1+2p_2+2p_3-1$		
9^2_{29}	$(3,2)\,(2,2)$	$(2p_1+1),(2p_2))\,((2p_3),(2p_4))$	$2p_1+2p_2-2p_3-2p_4-1$		
9^2_{30}	$(2\,1,2)\,(2,2)$	$((2p_1)\,1,(2p_2))\,((2p_3),(2p_4))$	$-2p_1-2p_3-2p_4+1$	$p_1+p_3+p_4$	
9^2_{32}	$.3\,1$	$.(2p_1+1)\,1$	-1		
9^2_{33}	$.2\,2$	$.(2p_1)\,(2p_2)$	$-2p_1+1$		
9^2_{34}	$.3.2$	$.(2p_1+1).(2p_2)$	$-2p_1+2p_2-1$	p_1	$p_1 > 1, p_2=1$
9^2_{35}	$.3.2\,0$	$.(2p_1+1).(2p_2)\,0$	$-2p_1-2p_2+1$	p_1+p_2	
9^2_{36}	$.3:2$	$.(2p_1+1):(2p_2)$	$-2p_1+2p_2-1$	p_1-p_2+1	$p_1 > 1, p_2=1$
9^2_{37}	$.3:2\,0$	$.(2p_1+1):(2p_2)\,0$	$-2p_1+2p_2-1$		
9^2_{38}	$.2\,1:2\,0$	$.(2p_1)\,1:(2p_2)\,0$	$-2p_1+3$		
9^2_{39}	$.2.2.2\,0$	$.(2p_1).(2p_2).(2p_3)\,0$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_3 > p_1$
9^2_{40}	$2:2:2$	$(2p_1):(2p_2):(2p_3)$	$-2p_1-2p_2-2p_3+3$		
9^2_{41}	$2:2\,0:2\,0$	$(2p_1):(2p_2):(2p_3)\,0$	$-2p_1+2p_2+2p_3-1$	$p_2+p_3-p_1$	$p_2 > p_1$ or $p_3 > p_1$
9^2_{42}	8^*2	$8^*(2p_1)$	$-2p_1+1$	p_1	$p_1 \geq 2$
6^3_1	$2,2,2$	$(2p_1),(2p_2),(2p_3)$	$-2p_1+2p_2-2p_3$		
7^3_1	$2,2,2+$	$(2p_1),(2p_2),(2p_3)+$	$-2p_1+2$		
8^2_3	$3\,1,2,2$	$(2p_1+1)\,1,(2p_2),(2p_3)$	$-2p_1+2p_2+2p_3-2$		
8^3_3	$2,2,2+2$	$(2p_1),(2p_2),(2p_3)+(2p_4)$	$-2p_1-2p_3+2$		
8^3_4	$(2,2)\,(2,2)$	$((2p_1),(2p_2))\,((2p_3),(2p_4))$	$-2p_1-2p_2+2p_3+2p_4$		
8^3_5	$.3$	$.(2p_1+1)$	$-2p_1$		
8^3_6	$.2:2\,0$	$.(2p_1):(2p_2)\,0$	$-2p_1+2p_2$		

9^3_1	$2\,1\,2,2,2$	$(2p_1)\,1\,(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_3-2p_4+2$		
9^3_2	$2\,1\,1\,1,2,2$	$(2p_1)\,1\,1\,1,(2p_2),(2p_3)$	$-2p_1+2p_3$		
9^3_3	$3,2,2,2$	$(2p_1+1),(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_4+2$		
9^3_4	$2\,1,2,2,2$	$(2p_1)\,1,(2p_2),(2p_3),(2p_4)$	$-2p_1+2p_2+2p_4-2$		
9^3_6	$3\,1,2,2$	$(2p_1+1)\,1,(2p_2),(2p_3)+$	$-2p_1+2p_2-2$		
9^3_7	$2,2,2+3$	$(2p_1),(2p_2),(2p_3)+(2p_4+1)$	$-2p_1+2$		
9^3_8	$(2,2+)\,(2,2)$	$((2p_1),(2p_2)+)\,((2p_3),(2p_4))$	$-2p_1+2p_3+2p_4$		
9^3_9	$(2,2)\,1\,(2,2)$	$((2p_1),(2p_2)\,1\,((2p_3),(2p_4))$	$-2p_1-2p_4+2$		
9^3_{10}	$.2\,1\,1$	$.(2p_1)\,1\,1$	$-2p_1$		
9^3_{11}	$.2\,1:2$	$(-2p_1)\,1:2$	$-2p_1+2$		
9^3_{12}	$.(2,2)$	$.((2p_1),(2p_2))$	$-2p_1+2$		
8^4_1	$2,2,2,2$	$(2p_1),(2p_2),(2p_3),(2p_4)$	$-2p_1-2p_4+3$		

7. Ascending numbers of alternating knot families

Our next goal was to compute ascending numbers of alternating knots belonging to some families with known unknotting numbers and to find their based oriented diagrams showing the ascending number. These results for the families beginning with knots with $n \leq 8$ crossings are described in Theorems 5.1-5.4, for the families beginning with knots with $n = 9$ crossings in Theorems 5.5-5.9, and for the families beginning with knots with $n = 10$ crossings in Theorems 5.10-5.15.

For all these families, except for the first and the last family, ascending numbers are computable only from non-minimal diagrams. Hence, in Figs. 29-38 every family is represented by its minimal diagram (a), and non-minimal based oriented diagram (b) giving the corresponding ascending number.

Theorem 3. For knots $3_1 = 3$, $5_1 = 5$, $7_1 = 7$, $9_1 = 9$, ... of the family $2p + 1$ ($p \geq 1$), the minimal ascending number is $a(K) = u(K) = p$, and it is realized on the minimal diagrams (Fig. 29).

Theorem 4. For knots $7_3 = 4\,3$, $9_3 = 6\,3$, ... of the family $(2p)\,3$ ($p \geq 2$) the minimal ascending number is $a(K) = u(K) = p$, and it is realized on the non-minimal diagrams of the form $((((1,(-1,(((1^{2p-2}),-1),-1))),1),-1),-1,-1)$ (Fig. 30b), where by 1^{2p-2} is denoted the sequence $1,\ldots,1$ of the length $2p-2$.

Theorem 5. For knots $7_5 = 3\,2\,2$, $9_6 = 5\,2\,2$, $9_9 = 4\,2\,3$, ... of the family $(2p + 1)\,2\,(2q)$ ($p \geq 1$, $q \geq 1$), the minimal ascending number is $a(K) = u(K) = p + q$, and it is itself on the non-minimal diagrams of the form $(2p + 1), -2\,1, (2q)$ (Fig. 31b).

Theorem 6. For knots $8_{15} = 2\,1, 2\,1, 2$, $10_{49} = 4\,1, 2\,1, 2$, ... of the family $(2p)\,1$, $(2q)\,1, 2$ ($p \geq 1$, $q \geq 1$) minimal ascending number is $a(K) = u(K) = p + q$, and it is itself on the non-minimal diagrams of the form $(2p)\,1, (2q)\,1, -2, 1$ (Fig. 32b).

For the knot family $2p + 1$ ($p \geq 1$) the absolute value of the signature of a knot $2p + 1$ ($p \geq 1$) is $2p$. For the knot family $(2p)\,3$ ($p \geq 2$) the absolute value of the signature is $2p$. For the family of knots $(2p + 1)\,2\,(2q)$ ($p \geq 1$, $q \geq 1$) the absolute value of the signature is $2p + 2q$. For the family of knots $(2p)\,1, (2q)\,1, 2$ ($p \geq 1$, $q \geq 1$) the absolute value of the signature is $2p + 2q$.

K. Murasugi [9] proved the lower bound for the unknotting number for knots $u(K) \geq \frac{|\sigma_K|}{2}$. For the family of knots $2p+1$ ($p \geq 1$), from Murasugi's Theorem 1.1, half of the signature is p, for the family of knots $(2p)\,3$ ($p \geq 2$) from Theorem 1.2

half of the signature is p, for the family of knots $(2p+1)\,2\,(2q)$ $(p \geq 1,\ q \geq 1)$ from Theorem 1.3 it is $p+q$, and for the family of knots $(2p+1)\,2\,(2q)$ $(p \geq 1,\ q \geq 1)$ from Theorem 1.4 half of the signature is $p+q$. The proof of the Theorems 1.1-1.4 follows from the fact that in each of the families the unknotting number is equal to a half of the absolute value of the signature, and it is realized on the minimal diagrams of the knots belonging to the families from Theorems 1.1-1.4, respectively. Hence, unknotting number for these families is equal to the minimal diagram unknotting number. For all these families, we effectively constructed corresponding diagrams giving minimal ascending number equal to the unknotting number, so it follows that for each of the knots from these families $a(K) = u(K) = \frac{|\sigma_K|}{2}$.

We provide similar theorems for certain families beginning with nine and ten crossing knots. Their proof is analogous to the proof of Theorems 5.1-5.4. For rational knot family $5\,(2p)$ $(p \geq 2)$ from Theorem 5.5 the absolute value of the signature is $2p$; for rational knot family $(2p+1)\,4\,(2q)$ $(p \geq 1,\ q \geq 1)$ from Theorem 5.6 the absolute value of the signature is $2p+2q$; for knot family $(2p+1)\,4\,(2q)$ $(p \geq 1,$ $q \geq 1)$ from Theorem 5.7 the absolute value of the signature is $2p+4$; for rational knot family $3\,2\,2\,(2p)$ $(p \geq 1)$ from Theorem 5.8 the absolute value of the signature is $2p+2$; for rational knot family $(2p)\,2\,1\,2\,(2q)$ $(p \geq 1,\ q \geq 1)$ from Theorem 5.9 the absolute value of the signature is $2p+2q$; for knot family $(2p)\,3,(2q)\,1,2$ $(p \geq 1,$ $q \geq 1)$ from Theorem 5.10 the absolute value of the signature is $2p+2q$; for knot family $(2p)\,3,(2q)\,1,2$ $(p \geq 1,\ q \geq 1)$ from Theorem 5.11 the absolute value of the signature is $2p+4$. By similar arguments, applied to non-alternating knots we can conclude that for knot family $(2p)\,2\,1,(2q+1),-2$ $(p \geq 1,\ q \geq 1)$ from Theorem 5.12 the absolute value of the signature is $2p+2q$, and for knot family $-3\,0:(2p)\,0:2\,0$ $(p \geq 1)$ from Theorem 5.13 the absolute value of the signature is $2p+2$.

Theorem 7. For knots $9_4 = 5\,4, \ldots$ of the family $5\,(2p)$ $(p \geq 2)$, the minimal ascending number is $a(K) = u(K) = p$, and it is realized on the non-minimal diagrams of the form $((1,(-1,(((((1,(1,(-1,-1))),1),-1),-1),-1))),1^{2p-2})$ (Fig. 33b), where by 1^{2p-2} is denoted the sequence $1,\ldots,1$ of the length $2p-2$.

Theorem 8. For knots $9_7 = 3\,4\,2, \ldots$ of the family $(2p+1)\,4\,(2q)$ $(p \geq 1,\ q \geq 1)$ the minimal ascending number is $a(K) = u(K) = p+q$, and it is realized on the non-minimal diagrams of the form $(((-1,(1,((((-1,((1^{2p}),1)),-1),1),1))),-1),(-1)^{2q})$ (Fig. 34b), where by 1^{2p-2} and $(-1)^{2q-2}$ are denoted the sequence $1,\ldots,1$ of the length $2p-2$, and $-1,\ldots,-1$ of the length $2q-2$, respectively.

Theorem 9. For knots $9_{16} = 3,3,2+, \ldots$ of the family $3,3,(2p)+$ $(p \geq 1)$ the minimal ascending number is $a(K) = u(K) = p+2$, and it is realized on the non-minimal diagrams of the form $-(1,1)\,1\,1,-(1,1)\,1\,1,((-1)^{2p+1})\,1$ (Fig. 35b), where by 1^{2p+1} is denoted the sequence $1,\ldots,1$ of the length $2p+1$.

Theorem 10. For knots $9_{18} = 3\,2\,2\,2, \ldots$ of the family $3\,2\,2\,(2p)$ $(p \geq 1)$, the minimal ascending number is $a(K) = u(K) = p+1$, and it is realized on non-minimal diagrams of the form $(((((1,(-1,(((-1,(1,1)),-1),-1))),1),-1),-1),(-1)^{2p})$ (Fig. 36b), where by 1^{2p} is denoted the sequence $-1,\ldots,-1$ of the length $2p$.

Theorem 11. For knots $9_{23} = 2\,2\,1\,2\,2, \ldots$ of the family $(2p)\,2\,1\,2\,(2q)$ $(p \geq 1,$ $q \geq 1)$, the minimal ascending number is $a(K) = u(K) = p+q$, and it is realized on the non-minimal diagrams of the form $(((((((1,(1,(1^{2p}))),-1),-1),1),1),-1),-1),$

$(-1)^{2q})$ (Fig. 37b), where by 1^{2p} and $(-1)^{2q}$ are denoted the sequence $1,\ldots,1$ of the length $2p$, and $-1,\ldots,-1$ of the length $2q$, respectively.

Theorem 12. For knots $10_{50} = 2\,3, 2\,1, 2, \ldots$ of the family $(2p)\,3,(2q)\,1,2$ ($p \geq 1$, $q \geq 1$), the minimal ascending number is $a(K) = u(K) = p+q$, and it is realized on the non-minimal diagrams of the form $(((-1,(-1,-(1^{2p}))),1),1),((((-(1^{2q}),1),-1),1),1),-(1,1),1$ (Fig. 38b), where by 1^{2p} and 1^{2q} are denoted the sequence $1,\ldots,1$ of the length $2p$, and $1,\ldots,1$ of the length $2q$, respectively.

Theorem 13. For knots $10_{80} = (3,2)\,(2\,1,2),\ldots$ of the family $(3,2)\,((2p)\,1,2)$ ($p \geq 1$), the minimal ascending number is $a(K) = u(K) = p+2$, and it is realized on the non-minimal diagrams of the form $(((1,-(1,1)),1),-(1,1),1)\,(((((-(1^{2p}),1),-1),1),1),-(1,1),1)$ (Fig. 39b), where by 1^{2p} is denoted the sequence $1,\ldots,1$ of the length $2p$.

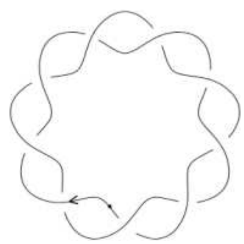

Figure 29: Family $2p+1$ ($p \geq 1$) and its corresponding minimal based oriented diagram giving the ascending number p.

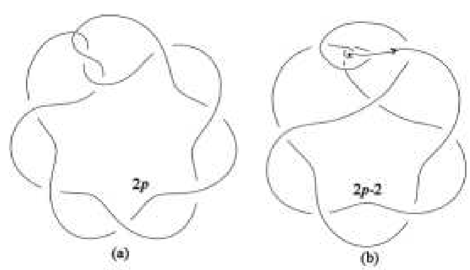

Figure 30: (a) Family $(2p)\,3$ ($p \geq 2$) with the ascending number p (b).

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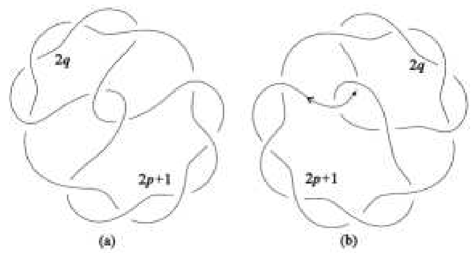

Figure 31: (a) Family $(2p+1)\,2\,(2q)$ $(p \geq 1,\ q \geq 1)$ with the ascending number $p+q$ (b) .

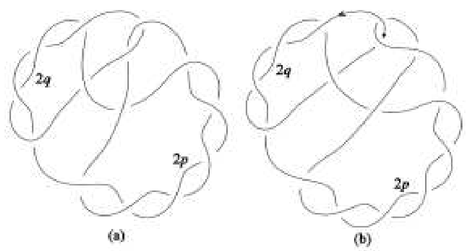

Figure 32: (a) Family $(2p)\,1,(2q)\,1,2$ $(p \geq 1,\ q \geq 1)$ with the ascending number $p+q$ (b) .

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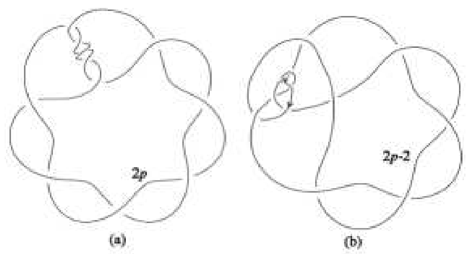

Figure 33: (a) Family $5\,(2p)$ $(p \geq 2)$ with the ascending number p (b).

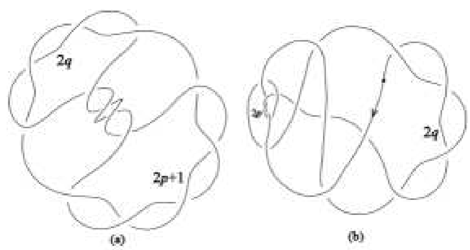

Figure 34: (a) Family $(2p+1)\,4\,(2q)$ $(p \geq 1,\, q \geq 1)$ with the ascending number $p+q$ (b).

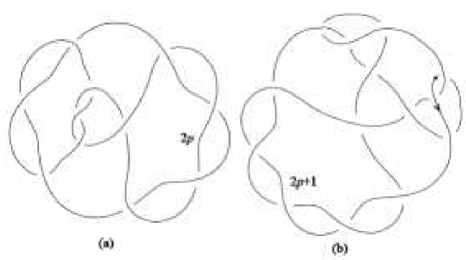

Figure 35: (a) Family $3, 3, (2p)+$ $(p \geq 1)$ with the ascending number $p+2$ (b).

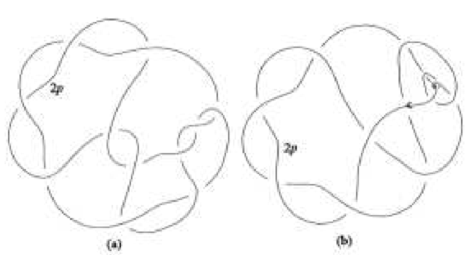

Figure 36: (a) Family $3\,2\,2\,(2p)$ $(p \geq 1)$ with the ascending number $p+1$ (b).

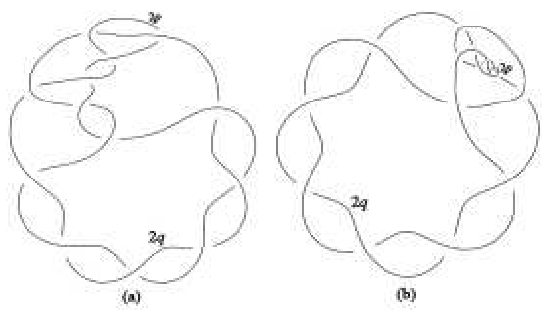

Figure 37: (a) Family $(2p)\,2\,1\,2\,(2q)$ $(p \geq 1,\ q \geq 1)$ with the ascending number $p+q$ (b).

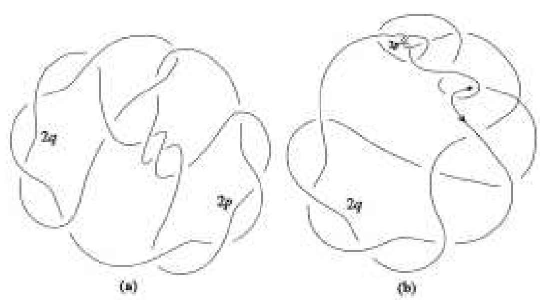

Figure 38: (a) Family $(2p)\,3,(2q)\,1,2$ $(p \geq 1,\ q \geq 1)$ with the ascending number $p+q$ (b).

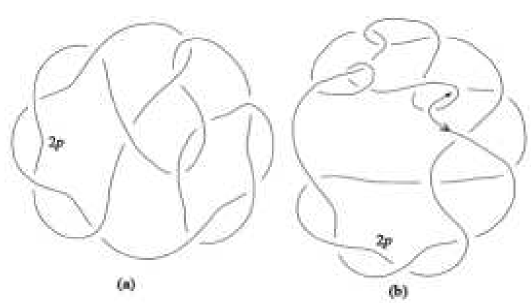

Figure 39: (a) Family $(3,2)\,((2p)\,1,2)$ $(p \geq 1)$ with the ascending number $p+2$ (b).