

# APOCHROMATIC TWISS PARAMETERS OF DRIFT-QUADRUPOLE SYSTEMS WITH SYMMETRIES

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## Abstract

In this article we continue the study of the apochromatic Twiss parameters of straight drift-quadrupole systems with special attention given to the properties of these parameters for beamlines with symmetries.

## INTRODUCTION

A straight drift-quadrupole system can not be designed in such a way that a particle transport through it will not depend on the difference in particle energies and this dependence can not be removed even in first order with respect to the energy deviations. Nevertheless, the situation will change if instead of comparing the dynamics of individual particles one will compare the results of tracking of monoenergetic particle ensembles through the system or will look at chromatic distortions of the betatron functions appearing after their transport through the system. From this point of view, as it was proven in [1], for every drift-quadrupole system which is not a pure drift space there exists a unique set of Twiss parameters (apochromatic Twiss parameters), which will be transported through that system without first order chromatic distortions<sup>1</sup>. In this paper we continue the study of the apochromatic Twiss parameters of straight drift-quadrupole systems with special attention given to the properties of these parameters for beamlines with symmetries.

## TRANSFER MAPS AND APOCHROMATIC TWISS PARAMETERS

Because we are interested in the lowest order chromatic effects and because the map of the drift-quadrupole system does not have second order geometric aberrations and transverse coupling terms, we will restrict our further consideration to the motion in one degree of freedom, let's say, horizontal. As usual, we will take the longitudinal particle position  $\tau$  to be the independent variable and will use the variables  $z = (x, p_x)^\top$  for the description of the horizontal beam oscillations. Here  $x$  is the horizontal coordinate and  $p_x$  is the horizontal canonical momentum scaled with the constant kinetic momentum of the reference particle. To take into account energy dependence we will use the variable  $\varepsilon$ , which is proportional to the relative energy deviation, and we will treat this variable as a parameter. With the assumptions made and with the precision needed the horizontal map  $\mathcal{M}$  of the drift-quadrupole system can be represented through a Lie factorization as follows

$$: \mathcal{M} :=_2 \exp(-(\varepsilon/2) \cdot \mathcal{Q}(x, p_x)) : : M :, \quad (1)$$

where  $M$  is  $2 \times 2$  linear transfer matrix, the symbol  $=_m$  denotes equality up to order  $m$  (inclusive) with respect to the variables  $x, p_x$  and  $\varepsilon$  when maps on both sides of (1) are applied to the phase space vector  $z$ , and

$$\mathcal{Q}(x, p_x) = c_{20} x^2 + 2c_{11} x p_x + c_{02} p_x^2. \quad (2)$$

As it was shown in [1], for every drift-quadrupole system which is not a pure drift space the quadratic form  $\mathcal{Q}$  is negative-definite and the apochromatic Twiss parameters at the system entrance can be calculated using the coefficients of this quadratic form according to the following formulas

$$\beta_a = -\frac{c_{02}}{\sqrt{c_{20}c_{02} - c_{11}^2}}, \quad \alpha_a = -\frac{c_{11}}{\sqrt{c_{20}c_{02} - c_{11}^2}}. \quad (3)$$

## CHROMATIC LATTICE FUNCTIONS

The equations of particle motion in a straight drift-quadrupole channel, when linearized with respect to the transverse variables, are uncoupled between the horizontal and vertical degrees of freedom, and we again can limit our considerations to the energy dependent dynamics in one transverse dimension. The  $2 \times 2$  horizontal fundamental matrix  $M_\varepsilon$  can be expressed with the help of the energy dependent lattice functions  $\beta_\varepsilon, \alpha_\varepsilon, \gamma_\varepsilon$  and  $\mu_\varepsilon$ , if they are known, in the familiar form

$$M_\varepsilon(l) = T_\varepsilon^{-1}(l) \cdot R(\mu_\varepsilon(l)) \cdot T_\varepsilon(0), \quad (4)$$

where  $R(\mu_\varepsilon(l))$  is a 2 by 2 rotation matrix and

$$T_\varepsilon(\tau) = \begin{pmatrix} 1/\sqrt{\beta_\varepsilon(\tau)} & 0 \\ \alpha_\varepsilon(\tau)/\sqrt{\beta_\varepsilon(\tau)} & \sqrt{\beta_\varepsilon(\tau)} \end{pmatrix}. \quad (5)$$

The parametrization (4) is very widely used in accelerator physics, because the plot of the Twiss parameters along the beam line shows in the clear visual form the dynamics of the beam envelopes or, depending on interpretation, the dynamics of the second order moments of the beam distribution. The energy dependent lattice functions can be either calculated numerically for any given value of the energy offset or one can find them perturbatively using Taylor expansion with respect to the variable  $\varepsilon$ . The quantities which are typically used for the description of the first-order chromatic effects are chromaticity, which for the drift-quadrupole system of the length  $l$  can be determined as the integral

$$\xi(\beta_0) = -\frac{1}{2} \int_0^l \gamma_0(\tau) d\tau, \quad (6)$$

and two betatron amplitude difference functions

$$\hat{b}(\beta_\varepsilon) = \frac{1}{2} \cdot \left( \frac{1}{\beta_\varepsilon} \frac{d\beta_\varepsilon}{d\varepsilon} \right) \Big|_{\varepsilon=0}, \quad (7)$$

$$\hat{a}(\beta_\varepsilon) = \frac{1}{2} \cdot \left( \frac{d\alpha_\varepsilon}{d\varepsilon} - \frac{\alpha_\varepsilon}{\beta_\varepsilon} \frac{d\beta_\varepsilon}{d\varepsilon} \right) \Big|_{\varepsilon=0}, \quad (8)$$

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<sup>1</sup>For a drift space the apochromatic Twiss parameters do not exist.

which we denote with the hat on the top in comparison with their usual notations [2] because for us it is convenient to scale them with the factor  $1/2$ . If the values of the chromatic variables  $\hat{b}$ ,  $\hat{a}$  and  $\xi$  are known, then one can calculate the first (linear with respect to  $\varepsilon$ ) correction terms for the Twiss parameters and for the phase advance one finds

$$\left. \frac{d\mu_\varepsilon(l)}{d\varepsilon} \right|_{\varepsilon=0} = \xi(\beta_0) + \hat{a}(\beta_\varepsilon(0)) - \hat{a}(\beta_\varepsilon(l)). \quad (9)$$

One sees that three variables (6)-(8) are, in general, sufficient for the description of all first order chromatic effects. Nevertheless, there is still a need for additional variables to make this description more detailed. It is connected with the fact that the betatron amplitude difference functions  $\hat{b}$  and  $\hat{a}$  play a twofold role. They describe both, chromatic properties of the incoming Twiss parameters and interaction of the incoming Twiss parameters with the chromatic properties of the given beam line. So in the paper [1] we have introduced two additional (supplementing) chromatic variables  $\eta$  and  $\zeta$ , which we call apochromaticities and which (together with the chromaticity  $\xi$ ) can be obtained as coefficients of the expansion of the quadratic form (2) with respect to three independent quadratic invariants of linear motion. These variables satisfy the important equality

$$\xi^2(\beta_0) = \eta^2(\beta_0) + \zeta^2(\beta_0) + (c_{20} c_{02} - c_{11}^2), \quad (10)$$

and with their help the change in the betatron amplitude difference functions after the beam line passage can be calculated as follows

$$\hat{b}(\beta_\varepsilon(l)) + i \hat{a}(\beta_\varepsilon(l)) = \exp(i2\mu_0(l)) \cdot$$

$$\left[ \left( \hat{b}(\beta_\varepsilon(0)) + i \hat{a}(\beta_\varepsilon(0)) \right) + \left( \eta(\beta_0) + i \zeta(\beta_0) \right) \right]. \quad (11)$$

Taking into account the formula for the mismatch between the on and the off energy Twiss parameters

$$m_p(\beta_\varepsilon, \beta_0) = (\beta_\varepsilon \gamma_0 - 2\alpha_\varepsilon \alpha_0 + \gamma_\varepsilon \beta_0) / 2 =$$

$$1 + 2\varepsilon^2 \cdot \left( \hat{a}^2(\beta_\varepsilon) + \hat{b}^2(\beta_\varepsilon) \right) + O(\varepsilon^3), \quad (12)$$

one sees from (11) that if on the system entrance the incoming Twiss parameters for the particles with the nominal energy were apochromatic with  $\eta(\beta_0) = \zeta(\beta_0) = 0$ , then the mismatch at the beam line exit will be the same as at the beam line entrance (at least in the lowest order with respect to the energy deviation), which (once more) shows the importance of the apochromatic Twiss parameters for practical accelerator designs.

To finish this section, let us note that if the apochromatic Twiss parameters  $\beta_a$  and  $\alpha_a$  and the chromaticity  $\xi(\beta_a)$  are known, then for an arbitrary Twiss parameters  $\beta_0$  and  $\alpha_0$  the following relations hold

$$\xi(\beta_0) = \xi(\beta_a) \cdot m_p, \quad (13)$$

$$\eta(\beta_0) = -\xi(\beta_a) \cdot \sqrt{m_p^2 - 1} \cdot \sin(2\theta). \quad (14)$$

$$\zeta(\beta_0) = -\xi(\beta_a) \cdot \sqrt{m_p^2 - 1} \cdot \cos(2\theta), \quad (15)$$

Here  $m_p = m_p(\beta_0, \beta_a)$  and  $\theta$  is the mismatch phase with

$$\sin(2\theta) = \frac{1}{\sqrt{m_p^2 - 1}} \cdot \left( \alpha_0(0) \frac{\beta_a(0)}{\beta_0(0)} - \alpha_a(0) \right), \quad (16)$$

$$\cos(2\theta) = \frac{1}{\sqrt{m_p^2 - 1}} \cdot \left( \frac{\beta_a(0)}{\beta_0(0)} - m_p \right). \quad (17)$$

## PERIODIC SYSTEMS

In this section we will consider a system constructed by a repetition of  $n$  identical cells ( $n > 1$ ) with the assumption that the linear (energy independent) cell transfer matrix allows periodic beam transport with the periodic Twiss parameters  $\beta_0^p$  and  $\alpha_0^p$ , and with the periodic cell phase advance  $\mu_0^p$  which is not a multiple of  $\pi$ . Let  $\eta(\beta_0^p)$  and  $\zeta(\beta_0^p)$  be the one cell apochromaticities. Iterating the propagation formula (11)  $n$  times, we obtain

$$\hat{b}(\beta_\varepsilon^p(nl)) + i \hat{a}(\beta_\varepsilon^p(nl)) =$$

$$\exp(i2n\mu_0^p) \cdot \left[ \left( \hat{b}(\beta_\varepsilon^p(0)) + i \hat{a}(\beta_\varepsilon^p(0)) \right) +$$

$$\frac{\sin(n\mu_0^p)}{\sin(\mu_0^p)} \cdot \exp(-i(n-1)\mu_0^p) \cdot \left( \eta(\beta_0^p) + i \zeta(\beta_0^p) \right) \right]. \quad (18)$$

The second addend inside the square brackets in the right hand side of this formula gives us the  $n$ -cell apochromaticities and one sees that if the  $n$ -cell phase advance  $n\mu_0^p$  is multiple of  $\pi$ , then these apochromaticities are equal to zero. So we came to the following statement, which is valid for each transverse plane separately:

For any system built out of  $n$  identical drift-quadrupole cells with the overall transfer matrix equal to the identity or to the minus identity matrix and with the cell matrix not equal to the identity or to the minus identity matrix, the cell periodic Twiss parameters (which are unique under assumptions made) are unique apochromatic Twiss parameters of the  $n$ -cell system.

Note that, though not in such general form and without addressing the question of uniqueness, this statement cannot be considered as a completely new result. For example in [3] the same was shown for the sequence of FODO cells by making explicit calculations involving thin-lens model for the quadrupole focusing.

## SCALING AND TELESCOPES

Scaling of lattice parameters is a procedure which allows to adapt known optics solutions to the new geometrical dimensions. For a drift-quadrupole system scaling by a factor  $\lambda > 0$  consists of the elongation of the system length by the factor  $\lambda$  with the simultaneous reduction of the quadrupole coefficient  $k_1(\tau)$  by the factor  $\lambda^2$ . The usefulness of this procedure is determined by the fact that any betatron function of the original system, when elongated to the length of

the new system and multiplied by the factor  $\lambda$ , becomes the betatron function of the scaled system.

The map of the scaled system

$$: \mathcal{M}_\lambda := \exp(-(\varepsilon/2) \cdot \mathcal{Q}_\lambda(x, p_x)) : M_\lambda :, \quad (19)$$

is connected to the map of the original system (1) by the relation

$$: \mathcal{M}_\lambda := \Lambda(\lambda) :^{-1} : \mathcal{M} : \Lambda(\lambda) :, \quad (20)$$

where  $\Lambda(\lambda) = \text{diag}(\sqrt{\lambda}, 1/\sqrt{\lambda})$  is the scaling matrix. In particular, the quadratic form  $\mathcal{Q}_\lambda$  is given by the formula

$$\mathcal{Q}_\lambda(x, p_x) = \lambda^{-1} c_{20} x^2 + 2c_{11} x p_x + \lambda c_{02} p_x^2, \quad (21)$$

and, applying formulas (3) to the coefficients of this quadratic form, one sees that the apochromatic Twiss parameters have the same scaling properties as any other betatron functions, as could be expected.

The scaling is not only important by itself, it also gives a systematic way for the design of beam magnification (or demagnification) telescopes. Let us consider a system constructed by a repetition of  $n$  cells and let us assume that the cell with the index  $m$  ( $m = 2, \dots, n$ ) is the copy of the first cell scaled by the factor  $\lambda^{m-1}$ . The overall transfer matrix of this system is equal to the matrix

$$\Lambda^n(\lambda) \cdot (\Lambda^{-1}(\lambda) M)^n, \quad (22)$$

and, if the second multiplier in the product (22) is equal to the plus or minus identity matrix, then the system becomes a telescope.

So let us assume that

$$(\Lambda^{-1}(\lambda) M)^n = \delta \cdot I, \quad \delta = \pm 1. \quad (23)$$

Then the map of the total system  $\mathcal{M}_t$  takes the form

$$: \mathcal{M}_t := \exp(-(\varepsilon/2) \cdot \mathcal{S}_\lambda(x, p_x)) : \delta \Lambda^n(\lambda) :, \quad (24)$$

where the quadratic form

$$\mathcal{S}_\lambda(x, p_x) = \sum_{m=0}^{n-1} : \Lambda^{-1}(\lambda) M :^m \mathcal{Q}(x, p_x) \quad (25)$$

is negative-definite (as for any drift-quadrupole system) and, clearly, is the invariant of the cyclic group generated by the matrix  $\Lambda^{-1}(\lambda) M$ . If this cyclic group has more than two elements, then  $\mathcal{S}_\lambda$  must be proportional to the Courant-Snyder quadratic form which corresponds to the (unique in this case) periodic Twiss parameters of the matrix  $\Lambda^{-1}(\lambda) M$ . Thus we have proven the following:

If the condition (23) is satisfied and if the matrix  $\Lambda^{-1}(\lambda) M$  is not equal to the identity or to the minus identity matrix, then the apochromatic Twiss parameters of the considered  $n$ -cell telescope coincide with the periodic Twiss parameters of the matrix  $\Lambda^{-1}(\lambda) M$ .

Note that for  $\lambda = 1$  this result recovers the statement of the previous section concerning  $n$ -cell periodic systems. Note also that apochromatic properties of some telescopes constructed by scaling on the basis of the doublet cell were studied in [4] by using thin-lens approximation, and the possibility to make the scaled  $n$ -cell system a second order achromat by introducing bending of the central trajectory and sextupole fields was addressed in [5].

## MIRROR SYMMETRY

Let us consider a mirror symmetric system of the length  $2l$  where the drift-quadrupole cell which is not a pure drift space is followed by its reversed image and let us assume that the map of the first part is given by the expression (1). Then the map of the second (reversed) part can be calculated as follows

$$: \mathcal{M}_r := \exp(-(\varepsilon/2) \cdot \mathcal{Q}_r(\mathbf{z})) : T_r M^{-1} T_r :, \quad (26)$$

where  $\mathcal{Q}_r(\mathbf{z}) = \mathcal{Q}(M^{-1} T_r \mathbf{z})$  and  $T_r = \text{diag}(1, -1)$  is the reversal symmetry matrix. And for the map of the total system one finds

$$: \mathcal{M}_t := \exp(-(\varepsilon/2) \cdot \mathcal{Q}_t(\mathbf{z})) : T_r M^{-1} T_r M :, \quad (27)$$

where

$$\mathcal{Q}_t(\mathbf{z}) = \mathcal{Q}(\mathbf{z}) + \mathcal{Q}(M^{-1} T_r M \mathbf{z}). \quad (28)$$

The quadratic form  $\mathcal{Q}_t$  uniquely determines the apochromatic Twiss parameters of the total system at the system entrance and, instead of the study of the evolution of the apochromatic Twiss parameters along the beamline, one can equivalently analyze the propagation of this quadratic form along the system length. Let us look at the images of the quadratic form  $\mathcal{Q}_t$  at the system symmetry point (in the middle of the system)

$$: M :^{-1} \mathcal{Q}_t(\mathbf{z}) = \mathcal{Q}(M^{-1} \mathbf{z}) + \mathcal{Q}(M^{-1} T_r \mathbf{z}) \quad (29)$$

and at the system exit

$$: T_r M^{-1} T_r M :^{-1} \mathcal{Q}_t(\mathbf{z}) = \mathcal{Q}_t(T_r \mathbf{z}). \quad (30)$$

The formulas (29) and (30) tell us that the apochromatic Twiss parameters of the mirror symmetric system are mirror symmetric by itself, i.e.

$$\beta_a(2l) = \beta_a(0), \quad \alpha_a(2l) = -\alpha_a(0), \quad \alpha_a(l) = 0, \quad (31)$$

as (like for the scaling) also could be expected.

Let us finish this paper with the remark that if one will consider the both parts of the system separately, then one will find that the entrance apochromatic Twiss parameters of the second (reversed) part are the reversed image of the exit apochromatic Twiss parameters of the first part, and that the apochromatic Twiss parameters of the total system will coincide with the apochromatic Twiss parameters of both parts if, and only if, the exit apochromatic alpha of the first part is equal to zero.

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