SECOND-ORDER ACHROMATS WITH ARBITRARY LINEAR TRANSFER MATRICES

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Abstract

In this article we consider a system where a bend magnet block arranged in an achromat-like fashion is followed by a straight drift-quadrupole cell which is not a pure drift space. We formulate the necessary and sufficient conditions for this system to be a second-order achromat and show that it can be achieved using six, four or even only two sextupole families.

INTRODUCTION

As a second-order achromat we will understand a particle transport system whose linear transfer matrix is dispersion free and whose transfer map does not have transverse second-order aberrations. The first practical solution for the second-order achromat was presented at the end of 1970s in the paper [1], where the theory of achromats based on repetitive symmetry was developed, and quickly becomes part of many accelerator designs. Unfortunately, the overall transfer matrix of this achromat is always equal to the identity matrix (except, possibly, for the r_{56} element) and variety of transfer matrices of all other known second-order achromats is also very limited. The most natural way to satisfy a need for a second-order achromat with an arbitrary linear transfer matrix, as it seems at first sight, is to take a bend magnet system arranged in an achromat-like fashion with the total transfer matrix equal to the identity matrix, attach a drift-quadrupole block with the desired linear transfer matrix and then adjust the sextupoles installed in the first part in such a way that all transverse second-order aberrations of the total system are canceled. In this paper, using the group-theoretical point of view for the design of magnetic optical achromats developed in [2], we formulate the necessary and sufficient conditions for this system to be a second-order achromat and show that it can be achieved using six, four or two sextupole families. We also show that if one uses less than six sextupole families, then the linear transport in the achromat-like part cannot be designed independently from the properties of the attached straight drift-quadrupole cell.

DYNAMICAL VARIABLES AND MAPS

We will consider the beam dynamics in a mid-plane symmetric magnetostatic system and will use a complete set of symplectic variables $\boldsymbol{z} = (x, p_x, y, p_y, \sigma, \varepsilon)^\top$ as particle coordinates. In this set the variables $\hat{\boldsymbol{z}} = (x, p_x, y, p_y)^\top$ describe the transverse particle motion and the variables σ and ε characterize the longitudinal dynamics [2, 3]. We will represent particle transport from one longitudinal location to another by a symplectic map and we will assume that for arbitrary two longitudinal positions the point $\boldsymbol{z} = \boldsymbol{0}$ is the fixed point and that the corresponding map can be Taylor expanded in its neighborhood. We will use that up to any predefined order m the aberrations of a map $\mathcal{M}$ can be represented through a Lie factorization as

$$: \mathcal{M} : =_m \exp(: \mathcal{F}_{m+1} + \ldots + \mathcal{F}_3 :) : M :, \tag{1}$$

where each of the functions $\mathcal{F}_k$ is a homogeneous polynomial of degree k in the variables $\boldsymbol{z}$ and the symbol $=_m$ denotes equality up to order m (inclusive) when maps on both sides of (1) are applied to the phase space vector $\boldsymbol{z}$. We will also use that for the map $\mathcal{M}$ of a magnetostatic system which is symmetric about the horizontal midplane $y = 0$ all polynomials $\mathcal{F}_k$ in (1) do not depend on the variable σ and are even functions of the variables y and p_y.

SECOND-ORDER ABERRATIONS OF REPETITIVE SYSTEM ARRANGED IN ACHROMAT-LIKE FASHION

In this section we will consider a system constructed by a repetition of n identical cells ($n > 1$) with the cell map $\mathcal{M}_c$ given by the following Lie factorization

$$: \mathcal{M}_c : =_2 \exp(: \mathcal{F}_3^c(\boldsymbol{z}) :) : M_c : . \tag{2}$$

Let two by two symplectic matrices M_{cx} and M_{cy} be the horizontal and vertical focusing blocks of the six by six cell transfer matrix $M_c = (r_{mk}^c)$ and let us define the four by four cell transverse focusing matrix $\hat{M}_c$ as follows

$$\hat{M}_c \,=\, \mathrm{diag}\,(M_{cx},\, M_{cy})\,. \tag{3}$$

We will say that a repetitive n-cell system is arranged in achromat-like fashion if its linear transfer matrix M_c^n is dispersion free and if the cell transverse focusing matrix $\hat{M}_c$ generates a cyclic group of order n, which means that

$$\hat{M}_c^n = I \quad \text{and} \quad \hat{M}_c^m \neq I \quad \text{for} \quad m = 1, \ldots, n-1. \tag{4}$$

Dispersion Decomposition of the Cell Matrix

If the n-cell system is arranged in the achromat-like fashion, then its linear transfer matrix M_c^n is equal to the identity matrix (except, possibly, for the r_{56} element) and, as a consequence of this, the equations

$$M_{cx}^n \,=\, I, \tag{5}$$

$$\left(I + M_{cx} + \ldots + M_{cx}^{n-1}\right) \cdot \left(r_{16}^c,\, r_{26}^c\right)^\top = (0,\, 0)^\top \tag{6}$$

must be satisfied. There are two possibilities regarding solutions of these equations. Either $r_{11}^c + r_{22}^c \neq 2$, $M_{cx}^n = I$, and r_{16}^c and r_{26}^c are arbitrary, or $M_{cx} = I$ and $r_{16}^c = r_{26}^c = 0$. In both cases the cell matrix M_c can be represented in the form

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$$M_c \,=\, D_c\, N_c\, D_c^{-1}, \tag{7}$$

where the matrix

$$N_c \,=\, \begin{pmatrix} r_{11}^c & r_{12}^c & 0 & 0 & 0 & 0 \\ r_{21}^c & r_{22}^c & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{33}^c & r_{34}^c & 0 & 0 \\ 0 & 0 & r_{43}^c & r_{44}^c & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & C \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

is dispersion-free and the matrix D_c can be expressed in the form of a Lie operator as follows

$$: D_c := \,\exp(:\varepsilon\,(B\,x \,-\, A\,p_x)\,:). \tag{9}$$

If $r_{11}^c + r_{22}^c \neq 2$, then the decomposition (7) is unique,

$$A \,=\, \frac{r_{16}^c - r_{52}^c}{2 - r_{11}^c - r_{22}^c} \quad \text{and} \quad B \,=\, \frac{r_{26}^c + r_{51}^c}{2 - r_{11}^c - r_{22}^c} \tag{10}$$

are the initial conditions for the periodic (matched) cell dispersion and its derivative, and

$$C \,=\, r_{56}^c \,+\, \frac{r_{16}^c r_{51}^c + r_{26}^c r_{52}^c}{2 - r_{11}^c - r_{22}^c}. \tag{11}$$

And in the second case, when $M_{cx} = I$ and $r_{16}^c = r_{26}^c = 0$, the matrix N_c is equal to the cell matrix M_c and A and B can be chosen arbitrarily (for example, $A = B = 0$).

Representation of Second-Order Aberrations in the Form of a Single Lie Exponent

Using (7) the cell transfer map can be written as

$$: \mathcal{M}_c :=_2 : D_c :^{-1} \exp(: \mathcal{P}_3^c(\boldsymbol{z})\,:) \,: N_c :: D_c :, \tag{12}$$

where $\mathcal{P}_3^c(\boldsymbol{z}) = \mathcal{F}_3^c(x + A\varepsilon,\, p_x + B\varepsilon,\, y,\, p_y,\, \varepsilon)$, and for the map of the repetitive n-cell system $\mathcal{M}_{nc}$ we obtain after some straightforward manipulations

$$: \mathcal{M}_{nc} :=_2 \,\exp(:n\,\mathcal{S}_3(D_c^{-1}\boldsymbol{z}) \,-\, n\,C\varepsilon^2/2\,:). \tag{13}$$

In this representation the aberration function $\mathcal{S}_3$ is given by

$$\mathcal{S}_3(\boldsymbol{z}) \,=\, \frac{1}{n}\sum_{m=1}^{n-1} \mathcal{P}_3^c(\hat{M}_c^m\,\hat{\boldsymbol{z}},\,\varepsilon) \tag{14}$$

and is not an arbitrary polynomial anymore. It is the result of the application of the Reynolds (averaging) operator of the cyclic group C_n generated by the matrix $\hat{M}_c$ to the polynomial $\mathcal{P}_3^c$ and for an arbitrary $\mathcal{P}_3^c$ is a polynomial which remains invariant under the group action.

As an abstract object the group C_n is unique and for all possible matrices $\hat{M}_c$ satisfying (4) we have groups which are isomorphic each other, but not all of them are conjugate. So that as groups of symmetries they can be distinct and can have different number of invariant homogeneous polynomials (remaining aberrations in (14)), and this depends on the choice of the periodic cell phase advances μ_x^c and μ_y^c. For the mid-plane symmetric system the polynomial $\mathcal{F}_3^c$ (and therefore the polynomial $\mathcal{P}_3^c$) can have as much as 18 nonzero monomials responsible for the independent transverse aberrations, while with the proper selection of the cell phase advances the number of independent transverse aberrations of the n-cell system can be reduced to six for $n = 2, 3$ and to two for $n \geq 4$ [1].

SECOND-ORDER ABERRATIONS OF STRAIGHT DRIFT-QUADRUPOLE CELL

The map of the straight drift-quadrupole cell $\mathcal{M}_s$ does not have second order geometric aberrations, does not generate second order dispersions and the transverse motion still remains uncoupled with the first nonlinear correction terms taken into account. Thus it can be written as

$$: \mathcal{M}_s :=_2 \,\exp(: \mathcal{F}_3^s(\boldsymbol{z})\,:) : M_s :, \tag{15}$$

where

$$\mathcal{F}_3^s(\boldsymbol{z}) = -\frac{\varepsilon}{2}\cdot\left(\mathcal{Q}_x(x, p_x) + \mathcal{Q}_y(y, p_y) \,-\, l_s\,\frac{\varepsilon^2}{\gamma_0^2}\right), \tag{16}$$

$\mathcal{Q}_x$ and $\mathcal{Q}_y$ are quadratic forms, l_s is the cell length, and γ_0 is the Lorentz factor of the reference particle.

The structure of the second-order aberrations (16) can be further clarified using that for every drift-quadrupole system (which is not a pure drift space) there exists an unique set of Twiss parameters (apochromatic Twiss parameters), which will be transported through that system without first order chromatic distortions [3]. Let $\beta_{x,y}^a$, $\alpha_{x,y}^a$ and $\gamma_{x,y}^a$ be these apochromatic Twiss parameters and

$$\begin{cases} I_x^a(\tau) \,=\, \gamma_x^a(\tau)\,x^2 + 2\alpha_x^a(\tau)\,x\,p_x + \beta_x^a(\tau)\,p_x^2 \\ I_y^a(\tau) \,=\, \gamma_y^a(\tau)\,y^2 \,+ 2\alpha_y^a(\tau)\,y\,p_y + \beta_y^a(\tau)\,p_y^2 \end{cases} \tag{17}$$

the corresponding Courant-Snyder quadratic forms. Then, as it was shown in [3], the quadratic forms $\mathcal{Q}_{x,y}$ can be expressed through these Courant-Snyder quadratic forms taken at the cell entrance as follows

$$\mathcal{Q}_{x,y} \,=\, \xi_{x,y}(\beta_{x,y}^a)\cdot I_{x,y}^a(0), \tag{18}$$

where $\xi_x(\beta_x^a)$ and $\xi_y(\beta_y^a)$ are the cell chromaticities calculated for the apochromatic Twiss parameters.

COMBINED SYSTEM AS SECOND-ORDER ACHROMAT

We now turn our attention to the main subject of this paper. Let us consider a system where a bend magnet block arranged in an achromat-like fashion is followed by a straight drift-quadrupole cell. The formulas (13) and (15) tell us that the map of the combined system $\mathcal{M}_s(\mathcal{M}_{nc})$ will not have transverse second-order aberrations, if and only if

$$\begin{aligned} & n\cdot\mathcal{S}_3(x - A\varepsilon,\, p_x - B\varepsilon,\, y,\, p_y,\, \varepsilon) \,- \\ & n\cdot\mathcal{S}_3(-A\varepsilon,\, -B\varepsilon,\, 0,\, 0,\, \varepsilon) \,= \\ & \frac{\varepsilon}{2}\cdot\left(\xi_x(\beta_x^a)\cdot I_x^a(0) \,+\, \xi_y(\beta_y^a)\cdot I_y^a(0)\right). \end{aligned} \tag{19}$$

This equation gives the necessary and sufficient conditions for the map of the combined system to be a second-order achromat and will be analyzed in more detail. As the first step, let us rewrite the function $\mathcal{S}_3$ in the form

$$\mathcal{S}_3(\boldsymbol{z}) \,=\, \sum_{m=0}^{3} \varepsilon^m \cdot \mathcal{S}_{3,3-m}(\hat{\boldsymbol{z}}), \tag{20}$$

where each of the functions $\mathcal{S}_{3,m}$ is a homogeneous polynomial of degree m in the transverse variables $\hat{\boldsymbol{z}}$. According to the mid-plane symmetry we have, additionally, that

$$\mathcal{S}_{3,1}(\hat{\boldsymbol{z}}) \,=\, \mathcal{S}_{3,1}(x,\, p_x) \tag{21}$$

does not depend on the variables y and p_y, and that

$$\mathcal{S}_{3,2}(\hat{\boldsymbol{z}}) \,=\, \mathcal{S}_{3,2}^{x}(x,\, p_x) \,+\, \mathcal{S}_{3,2}^{y}(y,\, p_y). \tag{22}$$

Substituting (20) into (19) we obtain that the equation (19) is equivalent to the following system of four equations: Condition for the absence of geometric aberrations

$$\mathcal{S}_{3,3}(\hat{\boldsymbol{z}}) = 0, \tag{23}$$

two conditions for the absence of chromatic focusing terms

$$2n \cdot \mathcal{S}_{3,2}^{x}(x,\, p_x) \,=\, \xi_x(\beta_x^a) \cdot I_x^a(0), \tag{24}$$

$$2n \cdot \mathcal{S}_{3,2}^{y}(y,\, p_y) \,=\, \xi_y(\beta_y^a) \cdot I_y^a(0), \tag{25}$$

and condition for the absence of second-order dispersions

$$n \cdot \mathcal{S}_{3,1}(x, p_x) = \xi_x(\beta_x^a) \cdot ([\gamma_x^a(0) A + \alpha_x^a(0) B] \cdot x + [\alpha_x^a(0)\, A \,+\, \beta_x^a(0)\, B] \cdot p_x)\,. \tag{26}$$

The first important observation from the equations (23)-(26) is that the functions in their left hand sides are invariants of the group generated by the matrix $\hat{M}_c$, and therefore so must be the functions in the right hand sides. Otherwise these equations cannot be satisfied whatever number of sextupole families we will use in the first dispersive part of the system.

If the matrix M_{cx} is equal to the identity matrix, then for $n = 2$ the function $\mathcal{S}_3$ has the same 18 independent transverse aberrations as the function $\mathcal{F}_3^c$ and for $n \geq 3$ this number is reduced to 12. That is not much and, if we want to use automatic cancellation of some aberrations in the achromat-like part of the system efficiently, we have to assume that $M_{cx} \neq I$. But from this assumption it follows that linear in x and p_x functions can not be invariants (i.e. $\mathcal{S}_{3,1}(x, p_x) = 0$) and thus the right hand side of the equation (26) must be equal to zero. There is only one possibility to satisfy this condition, namely one has to make the basic cell of the achromat-like part of our system to be free from the linear dispersions, i.e. r_{16}^c and r_{26}^c must be equal to zero.

The equations (24) and (25) are similar to each other and we consider only the first. The way of satisfying the equation (24) depends on the choice of the horizontal phase advance. If this phase advance is in the second order resonance (i.e. if μ_x^c is multiple of π), then monomials x^2, xp_x and p_x^2 are invariants and the equation (24) can be solved with the three sextupole families properly arranged in the dispersive regions of the achromat-like part of the system. If not, then there is only one functionally independent invariant of the group generated by the matrix $\hat{M}_c$ which is quadratic in x, p_x, and this invariant can be chosen equal to the Courant-Snyder quadratic form corresponding to the periodic Twiss parameters of the matrix M_{cx}. So in this situation the equation (24) can be satisfied with one sextupole family, but if, and only if, the periodic Twiss parameters of the matrix M_{cx} coincide with the horizontal apochromatic Twiss parameters of the straight drift-quadrupole cell.

And finally, the equation (23) can be satisfied either by sextupoles or one can select such phase advances $\mu_{x,y}^c$ that none of the combinations $3\mu_x^c$ and $\mu_x^c \pm 2\mu_y^c$ is multiple of 2π and use automatic cancellation, which excludes the case $n = 3$ from considerations [1].

SUMMARY

In this paper we have considered a mid-plane symmetric magnetostatic beamline where a bend magnet n-cell repetitive system with the overall linear transfer matrix equal to the identity matrix is followed by a straight drift-quadrupole block (which is not a pure drift space) and presented the necessary and sufficient conditions for this beamline to be a second-order achromat in the form of the four equations (23)-(26). Besides that, we have shown that these equations can be satisfied using only six, four or two sextupole families. In doing so one has to select such periodic cell phase advances $\mu_{x,y}^c$ that none of the combinations $3\mu_x^c$ and $\mu_x^c \pm 2\mu_y^c$ is multiple of 2π (which excludes from considerations the case $n = 3$) and then decide, for each transverse plane separately, if the phase advance for this plane (μ_x^c or μ_y^c) will be multiple of π or not. If it will be multiple of π, then one simply uses three sextupole families for this plane, and if not, then one sextupole family is sufficient, but one has to make the periodic Twiss parameters of the matrix M_c for this plane to coincide with the corresponding apochromatic Twiss parameters of the straight drift-quadrupole cell. And in all cases the linear cell transport matrix M_c must be free from dispersions.

For completeness, let us note that if the attached drift-quadrupole block is a drift space with $\mathcal{Q}_{x,y} = -l_s p_{x,y}^2$, then the solution with the minimal number of sextupole families requires μ_x^c be odd multiple of π and μ_y^c be multiple of π (from this it follows that $n = 2$), B in (10) equal to zero, and six sextupole families.

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