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Free Vibration Analysis of Functionally Graded Plates with Multiple Circular and Non-circular Cutouts

JANGHORBAN Maziar *, ROSTAMSOWLAT Iman

Department of Mechanical Engineering, Shiraz Branch, Islamic Azad University, Shiraz, Iran

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Abstract: Cutouts are inevitable in structures due to practical consideration. In order to investigate the free vibration of functionally graded plates with multiple circular and non-circular cutouts, finite element method is used. The volume fraction of the material constituents is assumed to follow a simple power law distribution. The parameters considered in this paper are as follows: cutout size, cutout location, number of cutouts and different boundary conditions. It should be mentioned that free vibration for FG plates(such as rectangular/skew/trapezoidal/circular plates) with multiple cutouts has not been studied yet and hence the results out coming from this paper may be used as bench marks for future works.

Key words: graded materials, rectangular/skew/trapezoidal/circular plates, multiple circular/non-circular cutouts

1 Introduction

Functionally graded materials are those, in which the volume fraction of the two or more materials is varied, as a power-law distribution, continuously as a function of position along certain dimension(s) of the structure. The functionally graded materials are microscopically heterogeneous and made from isotropic materials such as metals and ceramics. Due to their advantages of being able to withstand severe high-temperature gradient while maintaining structural integrity, FGMs are considered to be advanced composite materials in high temperature and vibration environment. The use of FG materials in advanced engineering structures exposed to high temperature, were first reported in 1984 in Japan[1]. The analyses of the FG plates/panels have received considerable attention of the researchers in recent past. PRAVEEN and REDDY[2] studied the static and dynamic responses of functionally graded ceramic-metal plate accounting for the transverse shear deformation in which effect of imposed temperature field on the response of the FG plate was discussed in detail. NG, et al[3], dealt with the parametric resonance of FG rectangular plates under harmonic in-plane loading. ROQUE, et al[4], used the asymmetric collocation method with multiquadrics basis functions and a higher-order shear deformation theory (HSDT) to find static deformations and natural frequencies of square FG plates of various aspect rations. WU, et al[5], obtained an explicit solution for the nonlinear static and dynamic responses of the FG rectangular plates. Their formulation is based on first-order shear deformation theory and Von-Karman nonlinear kinematics. FERREIRA and BATRA, et al[6], provided a global collocation method for natural frequencies of FG plates by a mesh less method with first order shear deformation theory. Skew and trapezoidal plates have quite a good number of applications in modern structures. Skew plate structures can be found frequently in modern construction in the form of reinforced slabs or stiffened plates. Such structures are widely used as floors in bridges, ship hulls, buildings, etc. Several researchers have addressed the linear and nonlinear static and dynamic problems of skew and trapezoidal plates[7–12]. For plates with cutout, CHAI[13] presented finite element and some experimental results on the free vibration of symmetric composite plates with central hole. HUANG and SAKIYAMA[14] proposed an approximate method for analyzing the free vibration of rectangular plates with different cutouts. LIU, et al[15], studied static and free vibration analysis of laminated composite plates using the conforming radial point interpolation method. They investigated circular and non-circular cutouts. Plates with multiple circular and non-circular cutouts are widely used in engineering structures (KHURASIA and RAWTANI[16]). LEE, et al[17, 18], proposed a semi-analytical approach to the free vibration analysis of a circular plate with multiple holes by using the indirect boundary integral method and the null field integral equation method, respectively. LEE and CHEN[19] also presented free vibration analysis of circular holes by using the multipole Trefftz method. From the review of available literature it is observed that the free vibration analysis of functionally graded arbitrary plates (such as rectangular/skew/trapezoidal/circular plates) with multiple circular and non-circular cutouts has not been studied yet. So it may be necessary to analyze this kind of problem based on finite element method.

* Corresponding author. E-mail: maziar_jany@yahoo.com

2 Material Properties

Material properties of the plates were assumed to vary across their thickness. In the case of thermal environment, to compute the results for FG plates, the properties are as follows[20]:

Based on the power law distribution, a typical material property P of the FG plate is obtained as

$$P(z) = P_{\mathbf{m}} + (P_{\mathbf{c}} - P_{\mathbf{m}})(v_f) \quad (1)$$

Where $0 < z < h$ and $v_f = (z/h)^p$, v_f is the volume fraction and p is the power law index which is assumed that the index parameter is $p = 1$ unless otherwise specified. Subscripts m and c refer to the metal and ceramic constituents which denote the material property of the bottom and top surface of plate, respectively.

The material properties used in the present study are as follows:

E_m= 70 GPa, ν_m=0.3,
α_m=23×10^{-6} 1/K, K_m=233 W/(m • K).

E_m =70 GPa, ν_n=0.3,
α_n=23×10^{-6} 1/K, K_n=233 W/(m • K).

The density for metal and ceramic can assume 2 700 kg / m^3 and 3 800 kg / m^3, respectively. The thickness of plates are assumed to be h = 0.05 m.

3 Modeling Functionally Graded Plates

3.1 Elements

The finite element software (ANSYS) is used with the aim of analyzing in addition SOLID 45 is used for the 3-D modeling of solid structures (Fig. 1). The element is defined by eight nodes having three degrees of freedom at each node. More than 2 000 nodes might be used in the work for calculating the results.

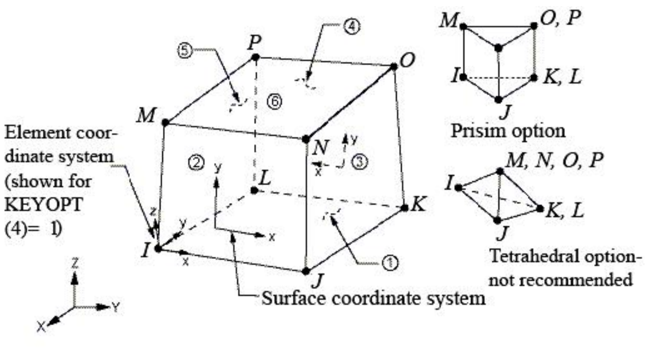

Fig. 1. SOLID 45 Geometry

3.2 Methods

It is mentioned that for free vibration analysis, subspace method is used. The subspace iteration method is described in detail by BATHE[21]. Enhancements as suggested by WILSON and ITOH[22] are also included as outlined subsequently.

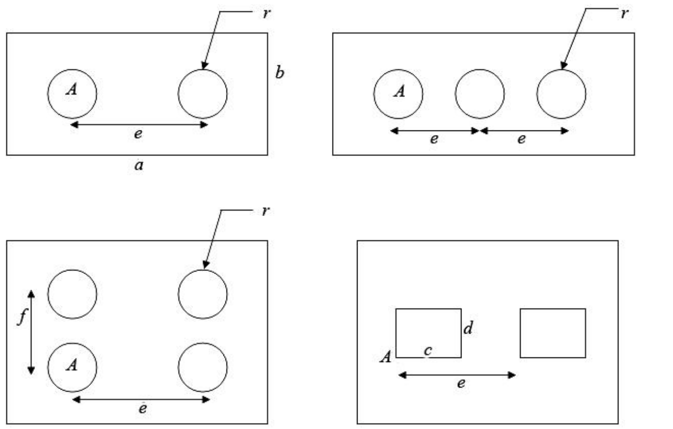

Fig. 2. Rectangular plates with multiple cutouts

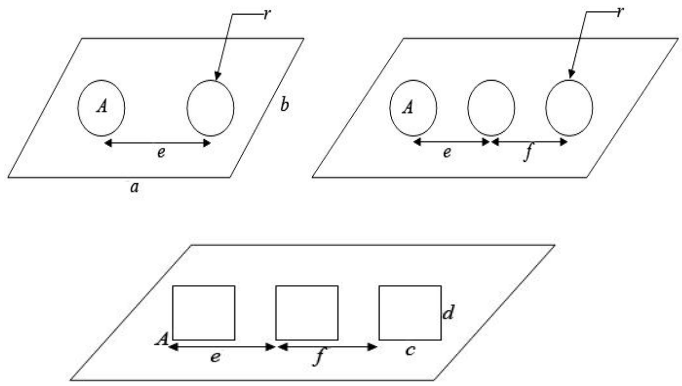

Fig. 3. Skew plates with multiple cutouts

4 Numerical Results

In order to demonstrate the accuracy of methodology for free vibration analyses of plates, several different plate examples problems such as rectangular plates, skew plates, trapezoidal plates and circular plates with multiple cutouts under different boundary conditions will be studied. In the following examples, free vibrations of various plates are analyzed. The results in this study were compared with the results of isotropic plates with a circular and non-circular cutout. Then, results were obtained for FG plates with multiple cutouts.

4.1 Isotropic plates

Free vibration of a simply supported square plate with a square cutout at the center, is analyzed. The geometry and material parameters are length $a = 10$, size ratio

$c/a = 0.5$, thickness ratio $h/a = 0.01$, density $\rho = 8000$ kg / m^3, Young modulus $E = 200$ **GPa**. The present results are compared with solution given by HUANG and SAKIYAMA[14] and LIU, et al[15], in Table 1. It is observed that good agreements are attained with their results.

Table 1. Non-dimensional frequencies of isotropic square plate with square cut out at the center (simply support for external boundaries, $\varpi = [\rho h \omega^2 a^4 / [D(1-\nu^2)]]$, $h/a = 0.01$)

Mode	Present	LIU, et al[15]	HUANG, et al[14]
1	4.921 4	4.971 7	4.839
2	6.434 7	6.481 0	6.435
3	6.434 7	6.482 1	6.440
4	8.585 9	8.550 9	8.492
5	8.686 3	8.865 6	8.875
6	10.734 0	10.720 0	10.810
7	10.734 0	10.767 0	10.830
8	12.235 0	12.045 0	12.290
9	13.329 0	13.370 0	13.530
10	14.399 0	14.180 0	14.110

Consider a square plate with a circular cutout at the center, the length of plate is $a = 10$, the ratio of the radius to length is $r/a = 0.1$, and the thickness ratio is $h/a = 0.01$. The material properties in above example are used. Table 2 shows the comparison of present results with solution given by HUANG and SAKIYAMA[14] and LIU, et al[15], for a clamped plate.

Table 2. Non-dimensional frequencies of isotropic square plate with circular cut out at the center (simply support for external boundaries, $\varpi = (\rho h \omega^2 a^4 / [D(1-\nu^2)])$, $h/a = 0.01$, $r/a = 0.1$)

Mode	Present	LIU, et al[15]	HUANG, et al[14]
1	6.166 6	6.149	6.240
2	8.619 7	8.577	8.457
3	8.620 0	8.634	8.462
4	10.478 0	10.42 0	10.23 0
5	11.521 0	11.41 0	11.72 0
6	12.017 0	11.84 0	12.30 0
7	12.964 0	12.83 0	13.04 0
8	12.964 0	12.84 0	13.04 0

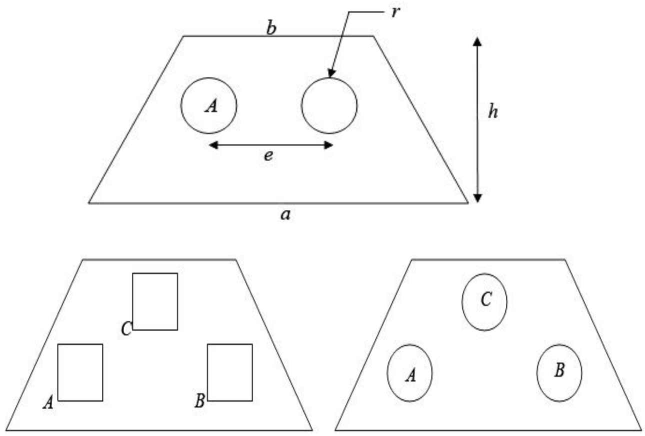

Fig. 4. Trapezoidal plates with multiple cutouts

4.2 FG Plates

Results were obtained for FG plates with multiple cutouts. Free vibration of different geometries with different types of boundary condition was investigated. First we analyzed rectangular plate with both multiple circular and non-circular cutouts. Then skew and trapezoidal plates with different cutouts were considered and finally, disks with multiple circular cutouts were mentioned.

4.2.1 Rectangular plates

Functionally graded rectangular plate with two circular cutouts(Fig. 2) at location A($x/b = 0.65$, $y/b = 1.35$) with radius ratio $r/b = 0.15$ is presented. Rectangular plate with length $b = 1$, center to center distance ratio $e/b = 0.7$ and length ratio $a/b = 2$ was used(Table 3).

Table 3. Natural frequencies of FG rectangular plate with two holes (fully clamped for external boundaries, $e/b = 0.7$)

Mode	Frequency	Mode	Frequency
1	503.900	6	1 269.400
2	663.070	7	1 519.300
3	832.460	8	1 725.600
4	1 157.300	9	1 817.400
5	1 183.600	10	2 056.700

In Table 4, same geometry with different cutout locations is investigated. In this example one can see the effect of the relative center to center distance on the natural frequencies. It is shown that with increasing the relative distance, for the first mode, the frequencies will decrease. Free vibration analysis of functionally graded rectangular plates with three circular cutouts (Fig. 2) at location A($x/b = 0.15$, $y/b = 0.5$) with radius ratio $r/b = 0.15$ and center to center distance ratio $e/b = 0.7$ was investigated(Table 5).

The geometry of the plate is the same as above.

Table 4. The effect of the relative center to center distance on the natural frequencies of FG rectangular plate with two holes(fully clamped boundary condition for external boundaries)

Mode	e / b		
	0.5	0.6	0.7
1	516.590	510.710	503.900
2	642.850	654.290	663.070
3	826.800	823.050	832.460
4	1 147.600	1 153.400	1 157.300
5	1 287.400	1 247.100	1 183.600
6	1 288.800	1 278.800	1 269.400
7	1 526.900	1 525.700	1 519.300
8	1 598.100	1 671.500	1 725.600
9	1 817.700	1 818.600	1 817.400

Table 5. Natural frequencies of FG rectangular plate with three holes (fully clamped for external boundaries)

Mode	*Frequency*	Mode	*Frequency*
1	500.450	6	1 286.000
2	614.970	7	1 458.800
3	931.330	8	1 777.200
4	1 161.600	9	1 800.500
5	1 249.900	10	2 103.700

Table 6. Natural frequencies of FG rectangular plate with four holes (fully clamped for external boundaries)

Mode	Frequency	Mode	Frequency
1	486.400	6	1 435.400
2	633.550	7	1 489.500
3	801.230	8	1 679.300
4	1136.300	9	1 890.000
5	1251.000	10	2 035.200

Four circular cutouts in functionally graded rectangular plate(Fig. 2) which is demonstrated above, are also mentioned here. Cutouts are at location A($x/b=0.65$, $y/b=0.25$) with radius ratio $r/b=0.1$ and center to center distance ratios $e/b=0.7$ and $f/b=0.4$ (Table 6).

Natural frequencies of functionally graded rectangular plates with two square cutouts are calculated in Table 7. Same rectangular plate with cutouts at location A($x/b=0.4$, $y/b=0.3$) with length ratios $c/b=0.1$, $d/b=0.1$ and center to center distance ratio $e/b=0.7$ is presented.

Table 7. Natural frequencies of FG rectangular plate with two square cutout (clamped-free boundary condition for external boundaries)

Mode	Frequency	Mode	Frequency
1	470.740	6	1 087.400
2	476.140	7	1 095.600
3	541.350	8	1 136.300
4	837.440	9	1 245.300
5	922.060	10	1 375.900

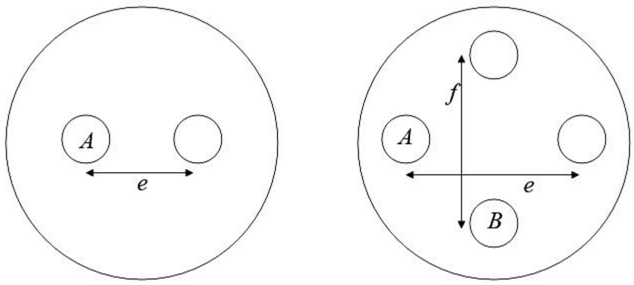

Fig. 5. Circular plates with multiple cutouts

4.2.2 Skew plates

In this section functionally graded skew plates with multiple cutouts were discussed. Frequencies were calculated for first ten modes of vibration for different boundary conditions. Skew plates with length $b=1$, and length ratio $a/b=1$ with skew angle $\theta=45$ is investigated(Tables 8-11). Functionally graded skew plates with two circular cutouts(Fig.3) at location A($x/b=0.65$, $y/b=0.45$) with radius ratio $r/b=0.1$ and center to center distance ratio $e/b=0.3$ is presented(Table 8).

In Table 9, natural frequencies of FG skew plate with three circular cutouts(Fig. 3) at the location A($x/b=0.45$, $y/b=0.45$) with radius ratio $r/b=0.1$ and center to center distance ratios $e/b=0.25$, $f/b=0.3$ are discussed.

In Table 10, the effect of different boundary conditions on the natural frequencies of FG skew plate with three square cutouts is studied. One can see that for fully clamped(CCCC) and clamped-simply(CSCS) boundary conditions the results are close to each other but for clamped-free(CFCF) boundary conditions the results are of much more difference.

Table 8. Natural frequencies of FG skew plate with two holes (fully clamped for external boundaries, $\theta=45$)

Mode	Frequency	Mode	Frequency
1	871.850	6	2 635.700
2	1 312.900	7	3 153.100
3	1 847.000	8	3 324.000
4	1 966.100	9	3 479.300
5	2 555.100	10	3 904.700

Table 9. Natural frequencies of FG skew plate with three holes (clamped-free boundary condition for external boundaries, $\theta=45$)

Mode	Frequency	Mode	Frequency
1	392.790	6	1 287.200
2	466.160	7	1 704.000
3	812.390	8	1 810.300
4	982.220	9	2 056.400
5	1 104.200	10	2 104.200

Table 10. Effect of different boundary conditions on the natural frequencies of FG skew plate with three square cutouts ($\theta=45$)

Mode	CCCC	CSCS	CFCF
1	2060.6	2027.2	922.76
2	2124.2	2102.6	1457.2
3	2586.9	2505.7	1549.1
4	2685.5	2616.4	1783.9
5	3618.3	3499.2	1963.1
6	3682.6	3548.7	2094.0
7	4543.8	4412.1	2186.6
8	4784.2	4667.6	2579.1
9	4892.2	4729.6	2707.3

4.2.3 Trapezoidal plates

Natural frequencies of FG trapezoidal plate with length $a=1$, and length ratios $b/a=0.7$ and $h/a=1$ are studied here(Tables 11-15). In Table 11, FG trapezoidal plate with two circular cutouts(Fig. 4) at location A($x/a=0.25$, $y/a=0.45$) with radius ratio $r/a=0.1$ and center to center distance ratio $e/a=0.45$ is analyzed. Parameters study of this plate is investigated in Table 12. One can see that with increasing the radius ratio of the left hole, the natural frequencies of the plate will decrease. Same plate is studied with three circular cutouts(Fig. 4) at locations A($x/a=0.25, y/a=0.35$), B($x/a=0.65$, $y/a=0.35$) and C($x/a=0.45$, $y/a=0.75$) with radius ratio $r/a=0.1$ in Table 13. In Table 14, the effects of radius ratio of three holes on the natural frequencies of FG trapezoidal plate with three holes are presented. Again, it is shown that with increasing the radius ratio, the natural frequencies of the plate will decrease.

Table 11. Natural frequencies of FG trapezoidal plate with two holes (fully clamped for external boundaries, $h/a=1$, $b/a=0.7$, $r/a=0.1$)

Mode	Frequency	Mode	Frequency
1	840.270	6	3 245.100
2	1 469.600	7	3 549.300
3	1 869.800	8	3 710.700
4	2 367.400	9	3 951.800
5	2 501.300	10	4 441.100

Table 12. Effect of radius to length ratio of the left hole on the natural frequencies of FG trapezoidal plate with two holes(fully clamped for external boundaries, $h/a=1$, $b/a=0.7$, $r/a=0.1$)

Mode	r/a		
	0.05	0.10	0.15
1	852.840	840.270	817.180
2	1 489.500	1 469.600	1 444.300
3	1 867.000	1 869.800	1 897.700
4	2 405.100	2 367.400	2 332.400
5	2 536.400	2 501.300	2 463.600
6	3 156.600	3 245.100	3 277.300
7	3 542.100	3 549.300	3 648.800
8	3 775.300	3 710.700	3 736.300
9	3 945.000	3 951.800	3 982.100

Table 13. Natural frequencies of FG trapezoidal plate with three holes (clamped-free boundary condition for external boundaries, $h/a=1$, $b/a=0.7$, $r/a=0.1$)

Mode	Frequency	Mode	Frequency
1	447.660	6	1 769.700
2	537.260	7	1 874.700
3	936.570	8	2 132.000
4	1 167.400	9	2 135.400
5	1 301.800	10	2 282.500

Table 14. The effect of radius ratio of three holes on the natural frequencies of FG trapezoidal plate with three holes(fully clamped for external boundaries, $h/a=1$, $b/a=0.7$, $r/a=0.1$)

Mode	r/a		
	0.05	0.10	0.15
1	438.870	447.660	462.490
2	541.990	537.260	534.930
3	985.510	936.570	887.020
4	1 166.800	1 167.400	1 206.300
5	1 320.700	1 301.800	1 301.600
6	1 829.200	1 769.700	1 686.200
7	1 899.500	1 874.700	1 805.100
8	2 237.900	2 132.000	1 892.900
9	2 279.400	2 135.400	2 122.000

Table 15. Natural frequencies of FG trapezoidal plate with three square cutouts (fully clamped for external and internal boundaries, $h/a=1$, $b/a=0.7$, $c/a=0.15$)

Mode	Frequency	Mode	Frequency
1	3 055.500	6	5 525.600
2	3 515.800	7	5 862.800
3	3 939.200	8	5 937.800
4	4 162.500	9	6 357.700
5	4 907.900	10	6 564.500

FG trapezoidal plate with three square cutouts at locations A($x/a=0.2$, $y/a=0.25$), B($x/a=0.65$, $y/a=0.25$) and C($x/a=0.4$, $y/a=0.7$) with length ratio $c/b=0.1$ was investigated(Table 15). In Figs. 6 and 7, first and second mode shapes for FG trapezoidal plate with three square cutouts are shown.

4.2.4 *Circular plates*

Natural frequencies of FG circular plate with the center at location (0,0) with radius $R=1$ is studied here (Tables 16, 17). Here, an FG circular plate with two circular cutouts (Fig.5) at location A($x/R=-0.55$, $y/R=0$) with radius ratio $r/R=0.1$ and center to center distance ratio $e/R=1.1$ is analyzed. Parameters studies of this plate are investigated in Table 16. In figures 8 and 9, First and tenth mode shapes for FG circular plate with two circular cutouts are shown.

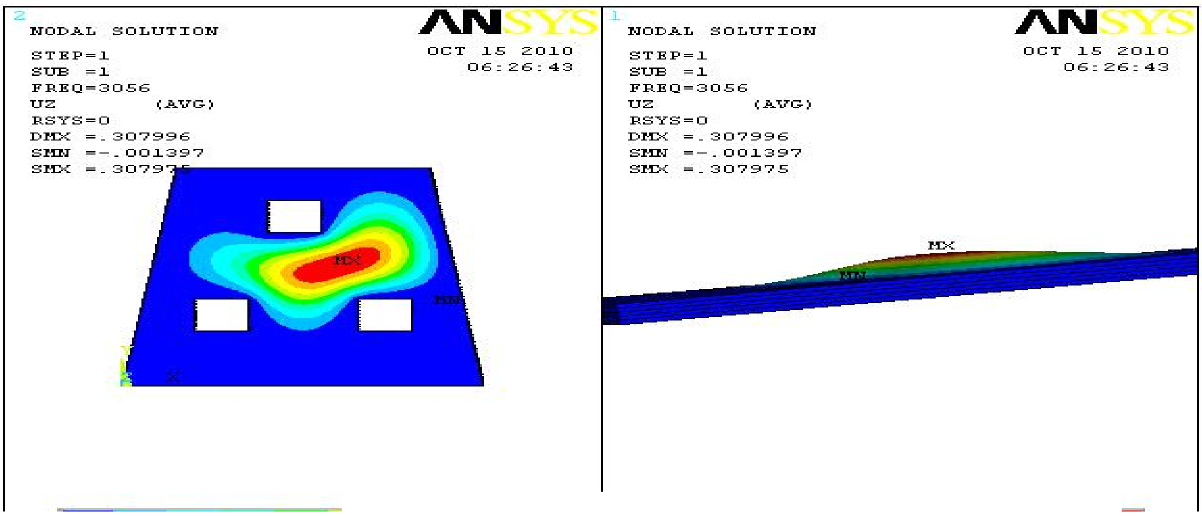

Fig. 6. First mode shape for FG trapezoidal plate with cutouts

In Table 17, natural frequencies of FG circular plate with four circular cutouts (Fig.5) at the location A($x/b=-0.7$, $y/b=0$) and B($x/b=0$, $y/b=-0.7$) with radius ratio $r/R=0.1$ and center to center distance ratios $e/b=1.4$, $f/b=1.4$ are discussed.

5 Conclusions

This analysis has been presented for free vibration of functionally graded plates with multiple circular and non-circular cutouts. The accuracy of the method is demonstrated by comparing the results with those of the existing solutions. Parameter studies were also performed to show the effects of cutout size, cutout location, number of cutouts and boundary condition for the plate on natural frequencies. Some of the results of present work are as follows:

(1) With increasing the relative distance in FG rectangular plate with two circular cutouts, for the first mode, the frequencies will decrease.

(2) For fully clamped (CCCC) and clamped-simply (CSCS) boundary conditions the results for FG skew plate with three square cutouts, are close to each other but for clamped-free (CFCF) boundary conditions the results are of much more difference.

(3) With increasing the radius ratio of the left hole for FG trapezoidal plate with two holes, the natural frequencies of the plate will decrease.

(4) With increasing the radius ratio of FG trapezoidal plate with three holes, the natural frequencies of the plate will decrease.

Table 16. Parameter study of natural frequencies of FG disk with two holes(fully clamped for external boundaries)

Mode	r/R		
	0.1	0.15	0.2
1	206.010	207.640	204.840
2	420.340	420.050	411.340
3	427.100	437.290	444.640
4	684.400	678.390	659.380
5	694.610	720.890	727.090
6	782.630	795.190	823.530
7	991.480	977.630	946.890
8	1 009.400	1 051.300	1 069.600
9	1 173.500	1 161.200	1 176.000

Table 17. Natural frequencies of FG disk with four holes (fully clamped for external boundaries, $\theta=45$)

Mode	Frequency	Mode	Frequency
1	201.000	6	780.020
2	418.020	7	987.200
3	418.720	8	988.240
4	671.820	9	1 180.600
5	688.250	10	1 184.800

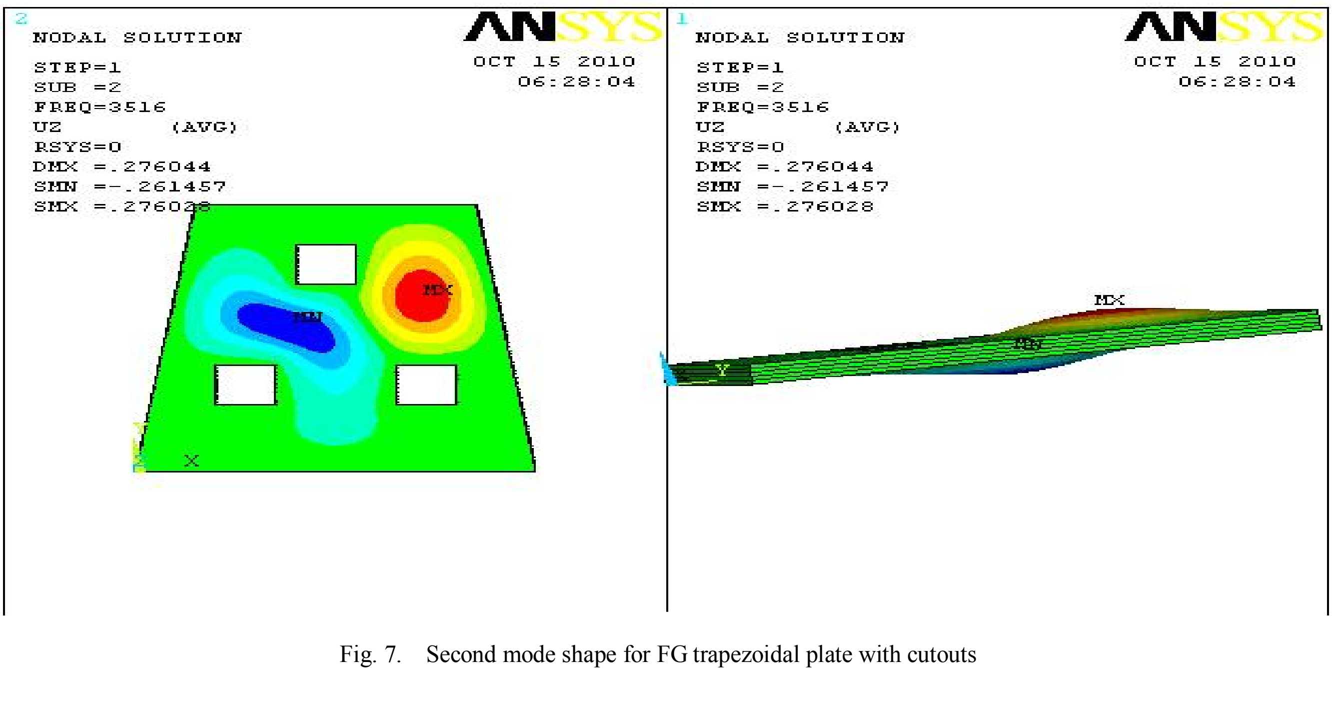

Fig. 7. Second mode shape for FG trapezoidal plate with cutouts

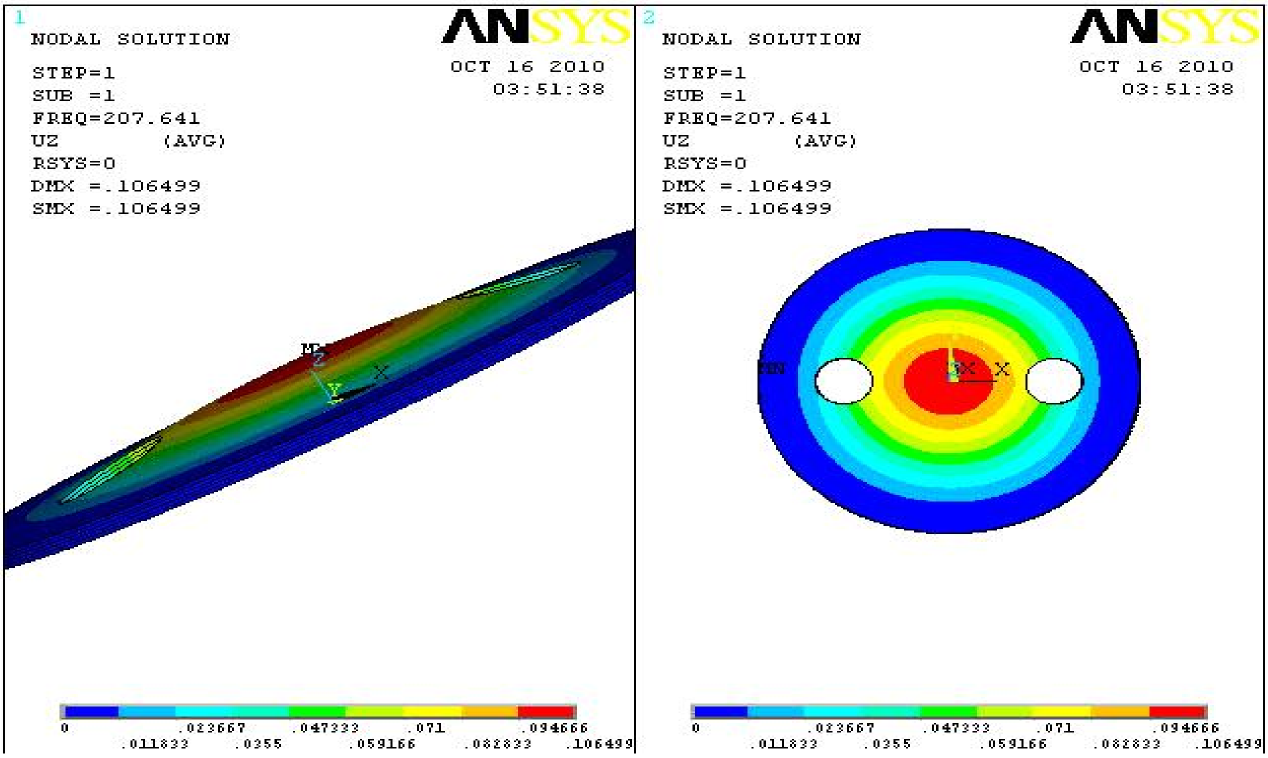

Fig. 8. First mode shape for FG circular plate with cutouts

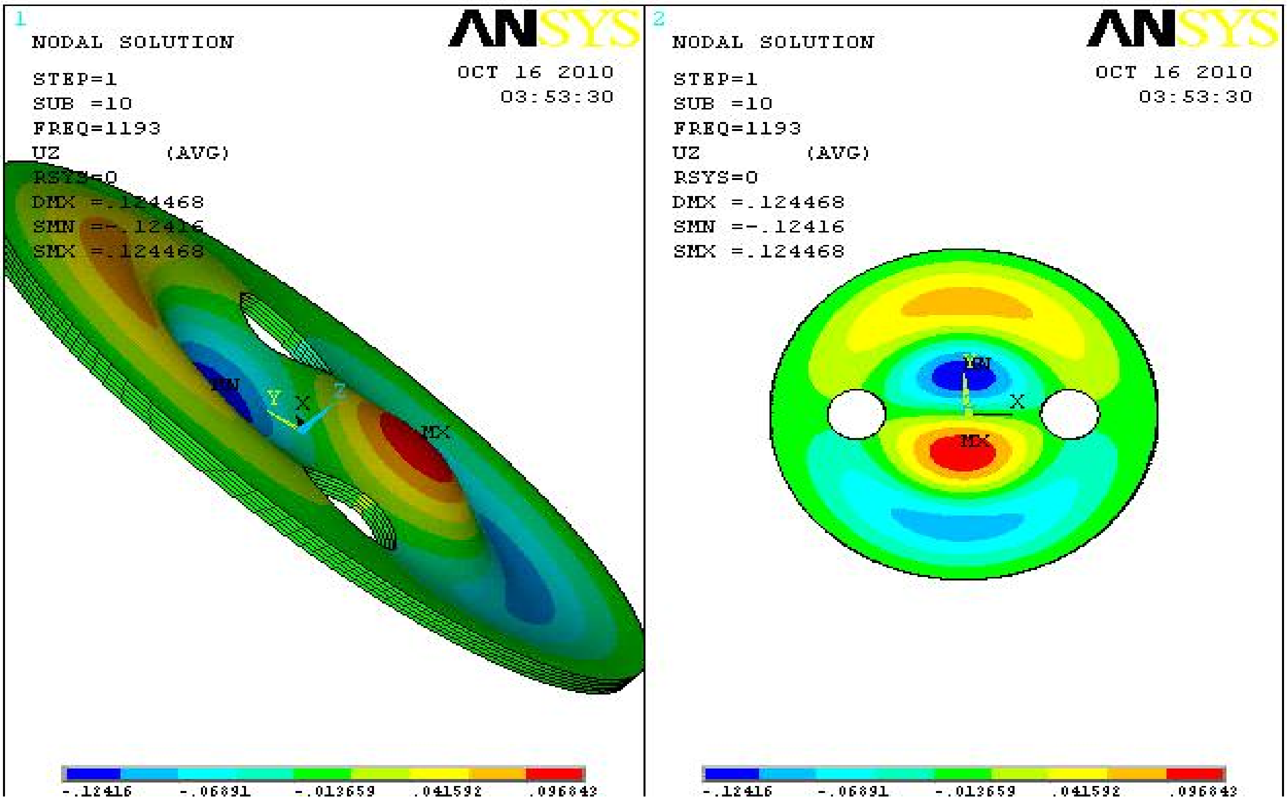

Fig. 9. Tenth mode shape for FG circular plate with cutouts

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