Network Extreme Eigenvalue - from Mutimodal to Scale-free Network

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Abstract

The extreme eigenvalues of adjacency matrices are important indicators on the influences of topological structures to collective dynamical behavior of complex networks. Recent findings on the ensemble averageability of the extreme eigenvalue further authenticate its sensibility in the study of network dynamics. Here we determine the ensemble average of the extreme eigenvalue and characterize the deviation across the ensemble through the discrete form of random scale-free network. Remarkably, the analytical approximation derived from the discrete form shows significant improvement over the previous results. This has also led us to the same conclusion as [Phys. Rev. Lett. 98, 248701 (2007)] that deviation in the reduced extreme eigenvalues vanishes as the network size grows. In addition, we found that the extreme eigenvalue of individual network can be better represented by the ensemble average when the networks are sparse.

Many concepts in network science have been well recognized as fundamental tools for exploring the dynamics of complex systems. In particular, scale-free networks are used widely to describe and model social, biological and economic systems [1, 2, 3, 4, 5]. In an ensemble of scale-free networks, although the degree distribution of the nodes remains the same, the topological structure of each individual network can be diverse with different connections introduced between the nodes. Such structural diversity can lead to discrepancy in dynamics of the individual network. Since the structural influences on certain dynamical processes are governed by the extreme eigenvalues of the network adjacency matrices [6, 7, 8, 9, 10, 11, 12, 13], deviations in the extreme eigenvalues in network ensembles are of increasing interest. Recently, it is found that the extreme eigenvalues of adjacency matrices, despite fluctuate widely in an ensemble of scale-free networks, are well characterized by the ensemble average after normalized by functions of the maximum degrees [14]. Specifically, it has been proven that the probability of having greatly deviated extreme eigenvalues in the ensemble diminishes as the size of the network increases. Considering the rich assortment of possible structural configurations for scale-free networks in an ensemble, this averageability is significant as it implies that dynamical processes which are governed by the extreme eigenvalues can be described simply using the ensemble average without the need of incorporating the connection details of the individual network. In particular, the average of network synchronization ability and epidemic spreading threshold are shown to be well approximated by functions of the ensemble average of the eigenvalues. Therefore, finding a way to determine the ensemble average of the extreme eigenvalues becomes crucial to uncover the topological influences of the network structure on a number of network dynamical processes.

To the best of our knowledge, the extreme eigenvalue of adjacency matrix of random, undirected scale-free network has been analytically approximated up to the second order correction as $\lambda_H^2 \approx k_H + k_H^{(1)} - 1$ which gives better precision over the previous result $\lambda_H^2 \approx k_H$ [15, 16, 17, 18]. Note that k_H is the largest degree of the network and $k_H^{(1)}$ denotes the average degree of the first nearest neighbors of node H. The probability distribution of the largest degree $P_d(k_H)$ is given by the Fréchet distribution and the ensemble average of k_H can be calculated from $P_d(k_H)$. This approximation provides, however, limited precision to $\langle \lambda_H \rangle$. The second order correction, on the other hand, introduce another fluctuation on λ_H which depends on the local connection of the largest degree node. In addition, $\langle k_H^{(1)} \rangle$ will need to be solved before a second order correction can be obtained for $\langle \lambda_H \rangle$.

In this paper, we investigate the extreme eigenvalue of the scale-free network through its discrete form, the multimodal network. Note that unlike the network studied in [19, 20], we consider only scale-free networks which are uncorrelated and undirected. Benefited from the mathematical properties of multimodal network that are more tractable, we found a way to analytically determine the ensemble average of the extreme eigenvalues while investigating the circumstances under which individual network can be better represented by the ensemble average. In addition, for bimodal networks which are shown to be more robust than the scale-free networks against both random and target removal of nodes [21, 22], we study the difference between them and scale-free networks in terms of the extreme eigenvalue and its ensemble averageability.

A multimodal network [23] with m modes contains m distinct peaks in the degree distribution: $P(k) = \sum_{i=1}^{m} r_i \delta(k - k_i)$. Note that $\delta(x)$ is the Dirac's delta function, $r_i = r_1 a^{-(i-1)}$ and $k_i = k_1 b^{-(i-1)}$ for $i = 1, 2, \cdots, m$. It is assumed that $a > 1$ and $0 < b < 1$ such that the degree distribution of the multimodal network follows a power law $P(k_i) = r_i \propto k_i^{-\beta}$, and hence $r_1 > r_2 > \cdots > r_m$ for $k_1 < k_2 < \cdots < k_m$. As $m \to \infty$, the multimodal network converges to a scale-free network. For a given network's size N with average degree $\langle k \rangle$, $k_m = \sqrt{\langle k \rangle N}$. By further fixing the value of the smallest possible degree k_1 between 1 and $\langle k \rangle$, and thus $b = (k_1/k_m)^{\frac{1}{m-1}}$, the rest of the parameters can be determined through the following equations:

$$\sum_{i=1}^{m} r_i = r_1 \sum_{i=1}^{m} a^{-(i-1)} = 1 \,, \tag{1}$$

$$\sum_{i=1}^{m} k_i r_i = k_1 r_1 \sum_{i=1}^{m} (ab)^{-(i-1)} = \langle k \rangle \,. \tag{2}$$

We shall follow the method outlined in [15] to find λ_H of the multimodal network. Let A be the adjacency matrix of a network, then $(A^n)_{ji}$ will be the number of walks of length n from node j to node i, denoted by $y_{j\to i}(n)$. In the eigen-decomposition form, we have $A^n = vD^n v'$, where v is the square matrix whose columns are the eigenvectors of A, v' denotes the inverse of v and D is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e. $D_{ll} = \lambda_l$. Hence,

$$y_{j\to i}(n) = (A^n)_{ji} = \sum_{l} \lambda_l^n \, v_{j,l} \, v'_{l,i} \,. \tag{3}$$

Note that Eq.(3) gives a summation over the nth power of the eigenvalues. When n is sufficiently large, the nth power of λ_H will be much larger than the nth power of the rest of the eigenvalues. Therefore, $y_{j\to i}(n)$ can be approximated in terms of only the maximum eigenvalue λ_H as

$$y_{j\to i}(n) \approx \lambda_H^n v_{j,H} v'_{H,i} \,. \tag{4}$$

Now if we consider the number of walks of length $n+2$ which start and terminate at node H,

$$y_{H\to H}(n+2) = y_{H\to H}(n)\, y_{H\to H}(2) + \sum_{j\neq H} y_{H\to j}(n) y_{j\to H}(2) \,, \tag{5}$$

then according to Eq. (4),

$$\lambda_H^2 \approx y_{H\to H}(2) + \sum_{j\neq H} \frac{y_{H\to j}(n)\, y_{j\to H}(2)}{y_{H\to H}(n)} \,. \tag{6}$$

The first term on the right hand side of Eq. (6) corresponds to the number of the nearest neighbors of node H, i.e. the largest degree of the network k_H. In [15], the second term on the right hand side of Eq. (6) is shown numerically to be very small for scale-free networks and is hence neglected. Since we are interested in finding a better approximation to the ensemble average of the maximum eigenvalues, we shall include the second term on the right hand side of Eq. (6) in the calculation of λ_H through a statistical approach. To start from node H, the possible number of walk of length n is

$$W = k_H \, k_H^{(1)} \, k_H^{(2)} \cdots k_H^{(n-1)} \,, \tag{7}$$

where $k_H^{(i)}$ is the average degree of the ith nearest neighbors from node H. Since out of the $N\langle k\rangle$ total number of in-links and out-links, k_j are directing into node j, hence, the probability of these walks to end up at node j is $\frac{k_j}{N\langle k\rangle}$. Indeed, we have verified from numerically generated networks that $\frac{y_{H\to j}(n)}{y_H(n)} \to \frac{k_j}{N\langle k\rangle}$ for large N (see Fig. 1). Therefore,

$$\begin{aligned} y_{H\to j}(n) &= k_H \, k_H^{(1)} \, k_H^{(2)} \cdots k_H^{(n-1)} \, \frac{k_j}{N\langle k\rangle} \,, & (8) \\ y_{H\to H}(n) &= k_H \, k_H^{(1)} \, k_H^{(2)} \cdots k_H^{(n-1)} \, \frac{k_H}{N\langle k\rangle} \,, & (9) \end{aligned}$$

and

$$y_{j\to H}(2) = k_j \, k_j^{(1)} \, \frac{k_H}{N\langle k\rangle} \,. \tag{10}$$

For multimodal scale-free network, there is a finite number, m of distinct degrees k_i, each with probability r_i. Thus,

$$\lambda_H^2 \approx k_m + \sum_{i=1}^{m} R_i k_i^2 \frac{k_i^{(1)}}{\langle k\rangle} \,, \tag{11}$$

where

$$R_i = \begin{cases} r_i & \text{for } 1 < i < m-1 \,, \\ r_i - 1/N & \text{for } i = m \,. \end{cases} \tag{12}$$

Equation (11) implies that λ_H depends on the specific way the nodes within the network are connected, which can deviate broadly across the ensemble. For an ensemble with fixed k_1 and $k_m = \sqrt{\langle k\rangle N}$, the degree distribution of multimodal networks vary with different values of $\langle k\rangle$. Specifically, they are more heavy-tailed for large $\langle k\rangle$ and less heavy-tailed for small $\langle k\rangle$. When the network size is large, $\langle k\rangle$ has to be small for a fixed value of k_m, and this results in a smaller variation in the distribution of $k_i^{(1)}$ in the multimodal network ensemble. Hence, the values of λ_H in an ensemble of multimodal networks deviate less as the networks become more sparse. In an ensemble of sparse networks, the individual network can thus be well represented by the ensemble average. We approximate the ensemble average of multimodal networks as

$$\langle \lambda_H \rangle = \sqrt{k_m + \sum_{i=1}^{m} R_i k_i^2}\,. \tag{13}$$

In order to determine the ensemble averages of scale-free networks, we find $\langle \lambda_H \rangle$ in the limit of large m. Here, we set $k_1 = \langle k \rangle/2$ so that the multimodal networks converge to scale-free networks generated from the Barabási-Albert (BA) model as m increases. As shown in Fig. 2, $\langle \lambda_H \rangle$ is the highest for the bimodal network and it decreases gradually as the number of modes increases until it finally converges to the ensemble average of the BA networks. This indicates that the ensemble averages of the extreme eigenvalues of scale-free networks can be determined through their discrete form by letting m large. In Figs. 3, we show the dependence of $\langle \lambda_H \rangle$ on $\langle k \rangle$ and N. By having $m = 30$, we compare the ensemble averages given by Eq. (13) and those given by $\langle \lambda_H \rangle = \sqrt{\langle k_H \rangle}$ with numerical results averaged over 50 realizations of the BA networks. It is assumed that $\langle k_H \rangle = \langle k \rangle \sqrt{N}/2$ [2] for the BA networks. As shown in Fig. 3, our results give values of λ_H that are closer to the numerical results compared to approximation from the previous results especially when the network is sparse.

In [14], the smallest degrees are fixed while the largest degrees are allowed to fluctuate in the network ensemble. For multimodal networks, the ensembles have however fixed value of k_m but varying values of k_1. The choice of different values of k_1 can lead to deviation in λ_H. Specifically, $k_1 = 1$ gives the extreme eigenvalue that is the largest, and λ_H decreases as k_1 increases (see Fig. 4). In other words, the deviation in λ_H can become larger in ensembles with different k_1. Furthermore, as shown in Fig. 5(a), the discrepancy between the largest and smallest extreme eigenvalues, $\Delta\lambda_H = \lambda_H(k_1 = 1) - \lambda_H(k_1 = 5)$ increases as N grows. The extreme eigenvalues in ensembles of multimodal networks with erratic k_1 are thus not ensemble averageable. Nonetheless, if we normalize the extreme eigenvalue with the largest degree, the reduced extreme eigenvalue $\hat{\lambda}_H = \lambda_H/k_m$ is ensemble averageable. In Fig. 5(b), for multimodal networks with $\langle k \rangle = 6$, we show the dependence of the deviations of the normalized extreme eigenvalues, $\Delta\hat{\lambda}_H$ on the network size. It decreases as N increases for both $m = 2$ and 30. Compare to multimodal networks with $m = 30$, bimodal networks show larger extreme eigenvalues with greater deviations. Hence, for a bimodal network, although the tolerance against both random and targeted removal of node is optimal, epidemic spreading is less controllable.

In conclusion, the ensemble averages of the extreme eigenvalues of scale-free networks can be determined more precisely through the multimodal networks with a large number of modes. Previous approximations give much lower values on ensemble average of the extreme eigenvalues, and this can cause an overestimation of epidemic threshold. When dealing with network dynamics such as the epidemic spreading of the community-acquired meticilin-resistant Staphylococcus aureus (CA-MRSA) superbugs that are resistant to many antibiotics

[24], over-estimating the epidemic threshold can lead to serious consequences. In view of this, the analytical solution derived from the multimodal network which is able to provide significantly closer approximation to the ensemble average of extreme eigenvalue of scale-free network is important. In addition, we found that this approximation of ensemble average can characterize the individual network more appropriately when the networks are sparse. For dense network, one will have to expect a higher value for the ensemble average from that given by Eq. (13).

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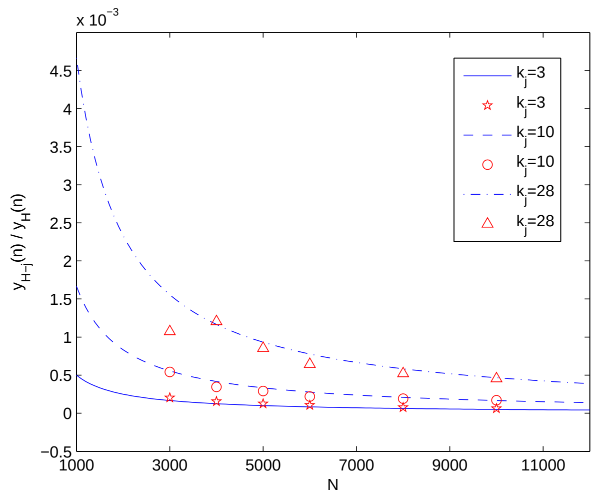

Figure 1: Convergence of $\frac{y_{H\to j}(n)}{y_H(n)}$ to $\frac{k_j}{N\langle k\rangle}$ for large N. Analytical results are shown in lines while numerical results are shown as stars, circles and triangles. Note that $\langle k\rangle = 6$ and $n = 100$.

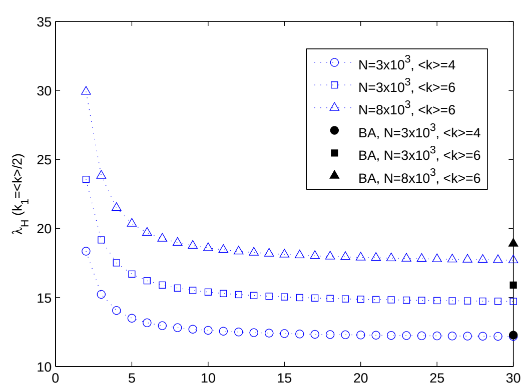

Figure 2: Dependence of the ensemble average of the extreme eigenvalues, $\langle \lambda_H \rangle$ on the number of mode, m of the multimodal network for $k_1 = \langle k\rangle/2$. Note that analytical results and numerical results averaged over 50 realizations of network are shown respectively as unfilled and filled triangles, squares and circles.

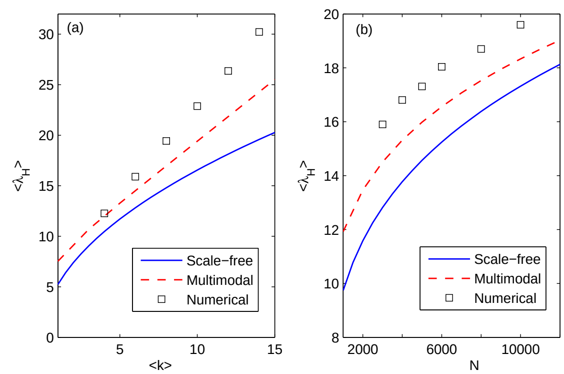

Figure 3: Dependence of $\langle\lambda_H\rangle$ on (a) $\langle k\rangle$ and (b) N for ensembles of scale-free networks with $k_1 = \langle k\rangle/2$. Note that $N = 3\times 10^3$ for (a) and $\langle k\rangle = 6$ for (b). $\langle\lambda_H\rangle = \sqrt{k_H}$ are shown in solid curves, analytical results from Eq. (13) are shown in dashed curves and numerical results of the BA model averaged over 50 realizations of network are shown as squares.

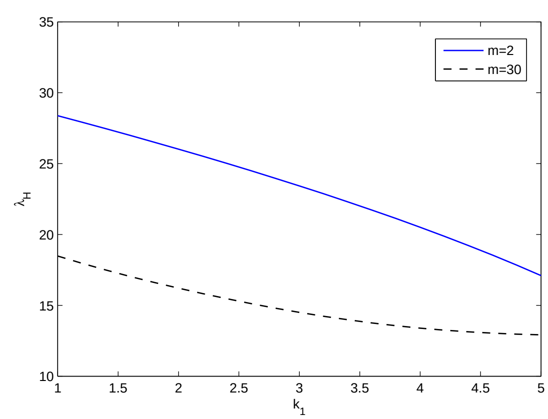

Figure 4: Dependence of the extreme eigenvalue (λ_H) on the smallest degree (k_1) of the multimodal network with $\langle k\rangle = 6$, $N = 3\times 10^3$ and $m = 2$ (solid line) or 30 (dashed line).

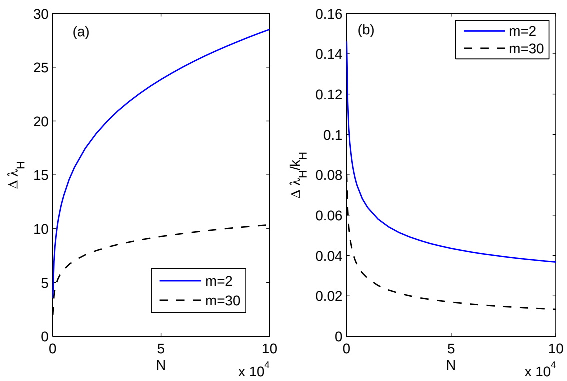

Figure 5: Dependence of (a) $\Delta\lambda_H$ and (b) $\Delta\hat{\lambda}$ on N of the multimodal network with $\langle k\rangle = 6$ and $m = 2$ (solid lines) or 30 (dashed lines).