KNOT POLYNOMIALS: MYTHS AND REALITY

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Abstract

This article provides an overview of relative strengths of polynomial invariants of knots and links, such as the Alexander, Jones, Homflypt, and Kaufman two-variable polynomial, Khovanov homology, factorizability of the polynomials, and knot primeness detection.

1 Introduction

In some sources the end of the 19^{th} century is called the "dark age of the knot theory", because knots and links (KLs) are recognized "by hand" or some other "non-exact methods". However, first knot tables were created at this time by P.G. Tait, T.P. Kirkman and C.N. Little, after more than five years of a hard work. In knot tabulation, almost nothing important happened almost a century, until the computer derivation of knot and link tables by M. Thistlethwaite and his collaborators [1], and now computations have reached the limit even with the use of supercomputers. Let us give the overview of the polynomial invariants we have at hand.

The first knot polynomial introduced by J.W. Alexander was used by K. Reidemeister in his book *Knotentheorie* in 1932 to distinguish knots up to $n = 9$ crossings. A new series of invariants, beginning with the Jones polynomial, is recently extended by using categorifications to the more powerful invariants.

Appearance of every knot invariant is usually connected with the progress in different fields of mathematics (e.g., the Alexander polynomial and Fox calculus, Khovanov homology and categorifications in different fields of algebra) and its connections with other sciences, in particular with physics (e.g., the Jones polynomial and its relation with the Potts model). In this paper we will not discuss the impact of knot polynomials to the development of mathematics or other fields of science, but only their ability to distinguish different KLs.

One of the first things we learn in knot theory is the computation of polynomial knot invariants, mostly those that can be computed by using skein relations. After learning that the Alexander polynomial is not able to distinguish a left trefoil from the right, that it cannot recognize unknot, that the Jones polynomial can distinguish left and right trefoil and (maybe) recognizes unknots, we believe that we have in our hand a very powerful tool for knot recognition, despite of the fact (usually illustrated by a few standard examples) that for every polynomial invariant exist KLs (not only mutants) that it cannot distinguish.

For all computations we used the program *LinKnot* [2], combined with the programs [1,3,4].

2 Distinction of knots and links by polynomial invariants

In order to compare different polynomial invariants and their ability to distinguish different KLs we computed different KL polynomials for all KLs up to $n = 12$ crossings and the number of KLs sharing the same polynomial with some other KL. Because there are 4684 alternating KLs with $n \leq 12$ crossings, consisting of 1851 knots and 2833 links, and 3993 non-alternating KLs with $n \leq 12$ crossings consisting of 1126 knots and 2867 links, i.e., 8677 KLs in total, we believe that this is a large enough sample from which we can make some conclusions.

In the following tables is given the name of the corresponding polynomial, number of knots sharing the same polynomial with some other knot, their percent among all knots, the same results for links, and the total number and percent of KLs that cannot be distinguished by the corresponding polynomial. The Table 1 contains the data about alternating, Table 2 about non-alternating KLs, the Table 3 is the sum of Table 1 and Table 2, and Table 4 shows the results of computations for all KLs, where alternating KLs are not separated from non-alternating ones.

Table 1

Alternating	Knots		Links		Total	
Alexander	846	46%	1732	61%	2578	55%
Jones	601	32%	672	24%	1273	27%
Khovanov	599	32%	406	14%	1005	21%
Homflypt	274	15%	285	10%	559	12%
Kauffman	93	5%	243	9%	336	7%

In our computation are not included some very powerful KL invariants: colored Jones polynomials and Links-Gould invariant, which cannot be computed for so large amount of KLs in a reasonable time. In the recognition of KLs, odd Khovanov homology gives the same results as the Khovanov homology[i].

Table 2

Non-alternating	Knots		Links		Total	
Alexander	697	62%	2123	74%	2820	71%
Jones	459	41%	797	28%	1256	31%
Khovanov	398	35%	459	16%	857	21%
Homflypt	254	23%	400	14%	654	16%
Kauffman	146	13%	327	11%	473	12%

Table 3

Sum	Knots		Links		Total	
Alexander	1543	52%	3855	68%	5398	62%
Jones	1060	36%	1469	26%	2529	29%
Khovanov	997	33%	865	15%	1862	21%
Homflypt	528	18%	685	12%	1213	14%
Kauffman	239	8%	570	10%	809	9%

Table 4

All	Knots		Links		Total	
Alexander	1832	62%	4169	73%	6001	69%
Jones	1213	41%	1565	27%	2778	32%
Khovanov	1117	38%	921	16%	2038	23%
Homflypt	600	20%	707	12%	1307	15%
Kauffman	239	8%	570	10%	809	9%

Definition 1 *For a link L given in an unreduced* [ii] *Conway notation $C(L)$, let S denote a set of numbers in the Conway symbol, excluding numbers denoting basic polyhedra and zeros (marking the position of tangles in the vertices of polyhedra), and S_f the set obtained by substituting every positive number from S different from 1 by 2, and every negative number from S different from -1 by -2. For $C(L)$ and an arbitrary (non-empty) subset $\tilde{S}$ of S the family $F_{\tilde{S}}(L)$ of knots or links derived from L is constructed by substituting each $a \in S_f$, $a \neq 1$, by $sgn(a)(|a| + k_a)$ for $k_a \in N$.*

[i] The authors are thankful to Krzystof Putyra for noticing the errors in the computations of Khovanov and odd Khovanov homology, that appeared in the first version of this paper.

[ii] The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

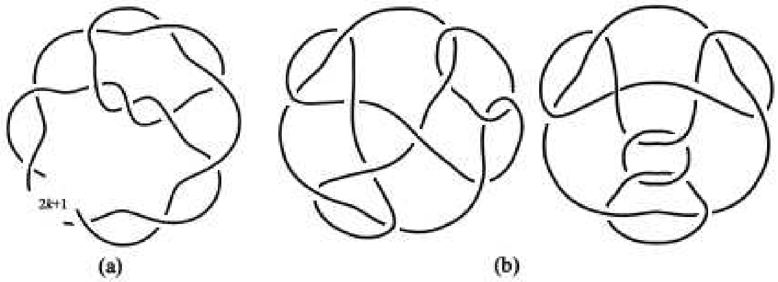

Figure 1: (a) Knot family $(2k+1), 3, -3$; (b) 2-component link $(2\,1, 2\,1)\,1\,(2, 2+)$; (c) 4-component link $(2, 2, 2)\,(2\,1, 2\,1)$.

If k_a is an even number ($k_a \in N$), the number of components is preserved inside a family, i.e., we obtain families of knots or links with the same number of components.

For the Alexander polynomial, there are even families of knots that can not be distinguished one from another. For example, for all knots of the family of nonalternating pretzel knots $(2k+1), 3, -3$ (Fig. 1a), the Alexander polynomial is $2 - 5x + 2x^2$.

All polynomials distinguish knots from links, but Alexander polynomial cannot distinguish links according to the number of components, and all the other polynomials distinguish them. For example, 2-component link $(2\,1, 2\,1)\,1\,(2, 2+)$ with $n = 12$ crossings and 4-component link $(2, 2, 2)\,(2\,1, 2\,1)$ with $n = 12$ crossings (Fig. 1b) have the same Alexander polynomial $1 - 9x + 34x^2 - 64x^3 + 64x^4 - 34x^5 + 9x^6 - x^7$, and 3-component link $6^*2\,2 : .(2, -2)\,0$ with $n = 12$ crossings and 5-component link $2, 2, 2, 2, 2+$ with $n = 11$ crossings have the same Alexander polynomial $1 - 9x + 27x^2 - 38x^3 + 27x^4 - 9x^5 + x^6$. However, up to $n = 12$ crossings the Alexander polynomial distinguishes links with an odd number of components from links with even number of components. Up to $n = 12$ crossings all the remaining polynomials completely distinguish links according to the number of components.

In the book [2], for families of alternating KLs we proposed the following conjecture:

Conjecture 2 *For every two alternating nonisotopic KLs L_1 and L_2 belonging to the same family F, $P(L_1) \neq P(L_2)$ for every polynomial invariant P.*

From the obtained results we conclude that amount of all KLs with $n \leq 12$ crossings that cannot be detected by the mentioned polynomial invariants is between 69% (Alexander polynomial) and 9% (Kauffman two-variable polynomial). In this amount are included mutant KLs that can not be distinguished by any polynomial invariant.

Comparing the results from Table 3 and Table 4 we conclude that for all polynomials, except for the Kauffman polynomial the results are worst if alternating

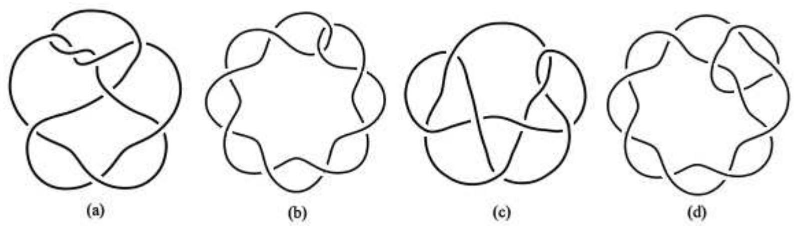

Figure 2: (a) Knot 3 1 1 3; (b) knot 7 2; (c) 2-component link 2 1 1 1 1 2; (d) 3-component link $6, 2, -2$.

and non-alternating KLs are not separated before the computations, i.e., that for all polynomials, except for the Kauffman polynomial there exist pairs (or groups) of KLs with the same polynomial, which contain alternating and non-alternating KLs. Up to $n = 12$ crossings, every two KLs with the same Kauffman polynomial have the same number of crossings. Hence, we have the following open problem:

Open problem 1: *Find an alternating KL with the same Kauffman polynomial as some other non-alternating KL.*

3 Factorizability of KL polynomials and KL primeness detection

The other test we made is the factorization, i.e., the ability of an invariant to detect primeness of KLs. For all polynomial invariants P, except the Khovanov polynomial $P(L_1 \# L_2) = P(L_1)P(L_2)$. However, the mentioned polynomials are factorizable for some prime KLs as well. For example, the Jones polynomial is factorizable for the link family 6, 10, …, $4k + 2$, and for the rational knots 3 1 1 3 (8_9) (Fig. 2a), 7 2 (9_2) (Fig. 2b), Homflypt polynomial is factorizable for the 2-component link 2 1 1 1 1 2 (8^2_8) (Fig. 2c), and for knot 4 2 1 2 (9_{12}) 2-colored and 3-colored Jones polynomials are factorizable for the 3-component link $6, 2, -2$ (Fig. 2d), *etc.* The only exceptions we found are Tutte polynomial[iii] and Kauffman two-variable polynomial.

Conjecture 3 *The Tutte polynomial and Kauffman two-variable polynomial detect primeness, i.e, they are not factorizable for prime KLs.*

We expect that the conjecture about Tutte polynomial can be proved on the basis of the irreducibility of the Tutte polynomial of connected matroids (Brylawski theorem) [5]. Trying to find the counterexample to the conjecture about Kauffman polynomial we checked without success all rational KLs up to $n = 19$ crossings,

[iii]Tutte polynomial is not KL invariant, because it is not invariant under Reidemeister moves, but it can be considered as the invariant of particular minimal diagrams of alternating KLs.

all Montesinos KLs up to $n = 18$ crossings, all knots from *Knotscape* tables up to $n = 16$ crossings, and all links up to $n = 12$ crossings.

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