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ON 'A CHARACTERIZATION OF $\mathbb{R}$-FUCHSIAN GROUPS ACTING ON THE COMPLEX HYPERBOLIC PLANE'

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Abstract. We indicate a $\mathbb{C}$-Fuchsian counter-example to the result with the above title announced at http://www.maths.dur.ac.uk/events/Meetings/LMS/2011/GAL11/program.pdf and prove a stronger statement.

1. Introduction

The following result

'*We prove that a complex hyperbolic non-elementary Kleinian group G acting on two-dimensional complex hyperbolic space $\mathbf{H}^2_\mathbb{C}$ is $\mathbb{R}$-Fuchsian, that is, G leaves invariant a totally real plane in $\mathbf{H}^2_\mathbb{C}$, if and only if every loxodromic element of G is either hyperbolic or loxodromic whose elliptic part is of order* 2.'

is announced at http://www.maths.dur.ac.uk/events/Meetings/LMS/2011/GAL11/program.pdf as a content of a one-hour talk.

The assertion as it stands is wrong (see a $\mathbb{C}$-Fuchsian counter-example in Section 3). The following theorem directly implies a corrected statement.

Theorem. *Let V be a $\mathbb{C}$-linear space equipped with a hermitian form $\langle -,-\rangle$ of signature $++-$ and let $G \le \mathrm{SU}\,V$ be a subgroup such that the trace $\operatorname{tr} g$ of every loxodromic element $g \in G$ belongs to $\mathbb{R}\delta_g$, where $\delta_g^3 = 1$. Suppose that G contains a loxodromic element. Then either there exists a* 1*-dimensional G-stable $\mathbb{C}$-subspace in V or there exists a totally real* 3*-dimensional G-stable $\mathbb{R}$-subspace in V.*

2. Proof of Theorem

We assume that there is no 1-dimensional G-stable $\mathbb{C}$-subspace in V.

2.1. First, suppose that $\operatorname{tr} G \subset \mathbb{R}$.

Let $W \le V$ be a G-stable $\mathbb{R}$-subspace in V. Then the $\mathbb{C}$-subspaces $\mathbb{C}W$, $W \cap iW$, and $W^\perp := \big\{v \in V \mid \langle v, W\rangle = 0\big\}$ are obviously G-stable. It follows that $\dim_\mathbb{R} W$ cannot equal

- 1 because, otherwise, $\mathbb{C}W$ is a 1-dimensional G-stable $\mathbb{C}$-subspace in V;
- 2 because, otherwise, $\dim_\mathbb{C} \mathbb{C}W$ equals 1 or 2 and, in the latter case, $\dim_\mathbb{C} W^\perp = 1$;
- 4 because, otherwise, either W is a complex subspace with $\dim_\mathbb{C} W^\perp = 1$ or $W + iW = \mathbb{C}W = V$ and $\dim_\mathbb{R}(W \cap iW) = 2$, that is, $\dim_\mathbb{C}(W \cap iW) = 1$;
- 5 because, otherwise, $W + iW = \mathbb{C}W = V$ and $\dim_\mathbb{R}(W \cap iW) = 4$, that is, $\dim_\mathbb{C}(W \cap iW) = 2$.

Suppose that $\dim_\mathbb{R} W = 3$. Let $g \in G$ be loxodromic. The eigenvalues of g are $1, r^{-1}, r$, where $0, \pm 1 \ne r \in \mathbb{R}$. Denote by $e_0, e_1, e_2 \in V$ the corresponding eigenvectors, where e_0 is positive and orthogonal to the isotropic e_1, e_2 such that $c := \langle e_1, e_2\rangle \ne 0$. Since $W \cap iW = 0$, there is no $\mathbb{C}$-subspace in W. Therefore, $\dim_\mathbb{R}(W \cap \mathbb{C}e_i) \le 1$. On the other hand, since the characteristic polynomial of g equals $(x-1)(x-r^{-1})(x-r)$, there is a basis of eigenvectors of g in W. Thus, we can assume that $e_0, e_1, e_2 \in W$. Clearly, W is totally real if $c \in \mathbb{R}$. Suppose that $c \notin \mathbb{R}$. Then $\operatorname{Im}\langle W, w\rangle = 0$ for

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$w\in W$ is equivalent to $w\in\mathbb{R}e_0$. For any $h\in G$, we have $0=\operatorname{Im}\langle W,e_0\rangle=\operatorname{Im}\langle hW,he_0\rangle=\operatorname{Im}\langle W,he_0\rangle$. So, $Ge_0\subset\mathbb{R}e_0$. A contradiction.

Suppose that V has no proper G-stable $\mathbb{R}$-subspaces. Let $A:=\mathbb{R}G$ denote the real span of G and $D:=\operatorname{End}_A V$ denote the division $\mathbb{R}$-algebra of endomorphisms of the simple A-module V (Schur's lemma). By Artin-Wedderburn theorem, a quotient algebra of A is isomorphic to $\operatorname{End}V_D$. Since $\dim_{\mathbb{R}}V=6$, we have $D=\mathbb{R}$ and $\dim_{\mathbb{R}}\operatorname{End}V_D=36$ or $D=\mathbb{C}$ and $\dim_{\mathbb{C}}\operatorname{End}V_D=9$. On the other hand, $A\le\operatorname{End}_{\mathbb{C}}V$ and $\dim_{\mathbb{C}}\operatorname{End}_{\mathbb{C}}V=9$. Hence, $A=\operatorname{End}_{\mathbb{C}}V$, which contradicts $\operatorname{tr}A\subset\mathbb{R}$.

2.2. Without loss of generality, we can assume that G contains a nontrivial cubic root of unity. Then there exists a loxodromic $g\in G$ with $\operatorname{tr}g\in\mathbb{R}$. In a suitable basis e_0,e_1,e_2 with the Gram matrix $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, such a g has the form $g:=\begin{bmatrix}1&0&0\\0&r^{-1}&0\\0&0&r\end{bmatrix}$, where $0,\pm1\ne r\in\mathbb{R}$.

2.3. Remark. *Let $g\in G$ be loxodromic with $\operatorname{tr}g\in\mathbb{R}$ and let $e_0,e_1,e_2\in V$ be eigenvectors of g with the Gram matrix $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$. Then, for every $h\in G$, there exists a cubic root of unity δ such that $\langle he_0,e_0\rangle,\langle he_1,e_2\rangle,\langle he_2,e_1\rangle,\operatorname{tr}(g^nh)\in\mathbb{R}\delta$ for all $n\in\mathbb{Z}$.*

Proof. It is easy to see that $\operatorname{tr}(g^nh)=\langle he_0,e_0\rangle+r^{-n}\langle he_1,e_2\rangle+r^n\langle he_2,e_1\rangle$. If $\langle he_1,e_2\rangle\ne0$ or $\langle he_2,e_1\rangle\ne0$, then g^nh is loxodromic for sufficiently large $|n|$. Therefore, $\langle he_0,e_0\rangle,\langle he_1,e_2\rangle,\langle he_2,e_1\rangle\in\mathbb{R}\delta$ for a suitable cubic root of unity δ. If $\langle he_1,e_2\rangle=\langle he_2,e_1\rangle=0$, then $h=\begin{bmatrix}-\varepsilon^{-2}&0&0\\0&0&a\varepsilon\\0&a^{-1}\varepsilon&0\end{bmatrix}$ with $a>0$ and $|\varepsilon|=1$. Since $h^2=\begin{bmatrix}\varepsilon^{-4}&0&0\\0&\varepsilon^2&0\\0&0&\varepsilon^2\end{bmatrix}$ and $\langle h^2e_1,e_2\rangle=\langle h^2e_2,e_1\rangle=\varepsilon^2\ne0$, we obtain $\varepsilon^2\in\mathbb{R}\overline{\delta}$, where $\delta^3=1$. Again, we get $\langle he_0,e_0\rangle,\langle he_1,e_2\rangle,\langle he_2,e_1\rangle\in\mathbb{R}\delta$ ■

2.4. Lemma. *Let $g,h\in G$ be loxodromic with $\operatorname{tr}g,\operatorname{tr}h\in\mathbb{R}$. Then $\operatorname{tr}(gh)\in\mathbb{R}$.*

Proof. In some bases e_0,e_1,e_2 and f_0,f_1,f_2 with Gram matrix $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, we respectively have $g=\begin{bmatrix}1&0&0\\0&r^{-1}&0\\0&0&r\end{bmatrix}$ and $h=\begin{bmatrix}1&0&0\\0&s^{-1}&0\\0&0&s\end{bmatrix}$, where $0,\pm1\ne r,s\in\mathbb{R}$. Let $g_{ij}:=\langle e_i,f_j\rangle$. Then $e_i=g_{i0}f_0+g_{i2}f_1+g_{i1}f_2$ for $i=0,1,2$. By Remark 2.3, for every $n\in\mathbb{Z}$, there exists some cubic root of unity δ_n such that $\langle h^ne_0,e_0\rangle,\langle h^ne_1,e_2\rangle,\langle h^ne_2,e_1\rangle\in\mathbb{R}\delta_n$. Taking δ such that $\delta_n=\delta$ for infinitely many n's, from

$$\langle h^ne_0,e_0\rangle=g_{00}\overline{g}_{00}+s^{-n}g_{02}\overline{g}_{01}+s^ng_{01}\overline{g}_{02},$$

$$\langle h^ne_1,e_2\rangle=g_{10}\overline{g}_{20}+s^{-n}g_{12}\overline{g}_{21}+s^ng_{11}\overline{g}_{22},\qquad\langle h^ne_2,e_1\rangle=g_{20}\overline{g}_{10}+s^{-n}g_{22}\overline{g}_{11}+s^ng_{21}\overline{g}_{12},$$

we obtain

$$g_{00}\overline{g}_{00},g_{02}\overline{g}_{01},g_{01}\overline{g}_{02},g_{10}\overline{g}_{20},g_{12}\overline{g}_{21},g_{11}\overline{g}_{22},g_{20}\overline{g}_{10},g_{22}\overline{g}_{11},g_{21}\overline{g}_{12}\in\mathbb{R}\delta.$$

If $g_{11}\overline{g}_{22}=0$, then $e_2\ne e_1=f_1\ne f_2$ or $e_1\ne e_2=f_2\ne f_1$ (the equalities and inequalities are meant in the projective sense). Hence, $g_{12}\overline{g}_{21}\ne0$. We conclude that $\delta=1$ ■

2.5. Lemma. *Let $g,h_1,h_2\in G$ be such that g is loxodromic and $\operatorname{tr}g,\operatorname{tr}(g^nh_1),\operatorname{tr}(g^nh_2)\in\mathbb{R}$ for all $n\in\mathbb{Z}$. Then $\operatorname{tr}(g^nh_1^{-1}h_2)\in\mathbb{R}$ for all $n\in\mathbb{Z}$.*

Proof. Using the symmetry between h_1,h_2 and replacing h_1,h_2 by g^kh_1,g^kh_2, if necessary, we can assume (as in the proof of Remark 2.3) that h_2 is loxodromic unless both h_1,h_2 have the type $\begin{bmatrix}\pm1&0&0\\0&0&a\varepsilon\\0&a^{-1}\varepsilon&0\end{bmatrix}$ in the basis related to g, where $a>0$ and $\varepsilon^2=\mp1$. In this particular case, $h_1^{-1}h_2$ is diagonal with coefficients in $\mathbb{R}\cup\mathbb{R}i$. By Remark 2.3, for some cubic root of unity δ, we have $\operatorname{tr}(g^nh_1^{-1}h_2)\in\mathbb{R}\delta$ for all $n\in\mathbb{Z}$. Therefore, the mentioned coefficients have to be real.

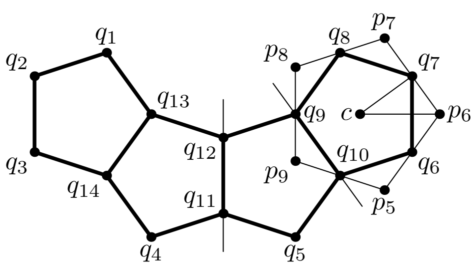
So, we assume that h_2 is loxodromic with $\operatorname{tr} h_2 \in \mathbb{R}$. Suppose that $\operatorname{tr}(g^n h_1^{-1} h_2) \in \mathbb{R}\delta$ for all $n \in \mathbb{Z}$, where $\delta^3 = 1$ and $\delta \neq 1$. For some $m \in \mathbb{Z}$, we have $0 \neq \operatorname{tr}(g^m h_1^{-1} h_2) \in \mathbb{R}\delta$ as, otherwise, we are done. Hence, by Remark 2.3, $\operatorname{tr}(g^m h_1^{-1} h_2^n) \in \mathbb{R}\delta$ for all $n \in \mathbb{Z}$. In particular, $\operatorname{tr}(g^m h_1^{-1}) \in \mathbb{R}\delta$, which implies $\operatorname{tr}(g^m h_1^{-1}) = 0$.

Suppose that $g^k h_1^{-1}$ is loxodromic for some $k \in \mathbb{Z}$. As in the proof of Remark 2.3, we conclude that $g^n h_1^{-1}$ is loxodromic for all sufficiently large/small n. By Lemma 2.4, $\operatorname{tr}(g^n h_1^{-1} h_2) \in \mathbb{R}$ for all such n's, implying $\operatorname{tr}(g^n h_1^{-1} h_2) = 0$, a contradiction.

So, h_1^{-1} is of the type $\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & a\varepsilon \\ 0 & a^{-1}\varepsilon & 0 \end{bmatrix}$. This contradicts $\operatorname{tr}(g^m h_1^{-1}) = 0$ ■

2.6. By Lemma 2.5, $H := \{h \in G \mid \operatorname{tr}(g^n h) \in \mathbb{R} \text{ for all } n \in \mathbb{Z}\}$ is a subgroup in G. Obviously, G is generated by H and the cubic roots of unity. It suffices to deal with H in place of G. In other words, we can assume that $\operatorname{tr} G \subset \mathbb{R}$.

3. Counter-example



Let $\Delta(c, p_6, q_7)$ be a geodesic triangle in the hyperbolic plane with the corresponding interior angles $\frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}$. The area of this triangle equals $\frac{\pi}{10}$. Taking 10 congruent triangles with common vertex c, we obtain a pentagon with area $\operatorname{area}(p_5, p_6, p_7, p_8, p_9) = \pi$. By [ABG], the reflections $R(q_i)$ in the middle points q_i, $i = 6, 7, 8, 9, 10$, of the sides of the pentagon satisfy the relation $R(q_{10})R(q_9)R(q_8)R(q_7)R(q_6) = \pm 1$ in $\mathrm{SU}(1,1)$ and provide a discrete group H_5. Note that, by the definition from [ABG], we have $R(q)x := i\Big(x - 2\frac{\langle x, q\rangle}{\langle q, q\rangle} q\Big)$.

Denote $Q(q) := -iR(q)$ (in the complex hyperbolic plane, $Q(q) \in \mathrm{SU}\,V$). We consider 3 more copies of the pentagon $P(q_6, q_7, q_8, q_9, q_{10})$, namely: $P(q_5, q_{10}, q_9, q_{12}, q_{11})$, $P(q_4, q_{11}, q_{12}, q_{13}, q_{14})$, and $P(q_1, q_2, q_3, q_{14}, q_{13})$. The geodesics $\mathrm{G}\prec q_9, q_{10}\succ$ and $\mathrm{G}\prec q_{11}, q_{12}\succ$ are ultraparallel (this can be shown with the help of SEs; see [ABG]). The geodesics of this type separate the four pentagons, so that we have exactly what is drawn on the picture. Since $Q(q_{10})Q(q_9)Q(q_8)Q(q_7)Q(q_6) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \pm(-i)^5 & 0 \\ 0 & 0 & \pm(-i)^5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \mp i & 0 \\ 0 & 0 & \mp i \end{bmatrix}$ and $Q(q)Q(q) = 1$ in $\mathrm{SU}\,V$, we have

$$1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \mp i & 0 \\ 0 & 0 & \mp i \end{bmatrix}^4 = \big(Q(q_8)Q(q_7)Q(q_6)Q(q_{10})Q(q_9)\big) \cdot \big(Q(q_9)Q(q_{10})Q(q_5)Q(q_{11})Q(q_{12})\big) \cdot$$

$$\cdot \big(Q(q_{12})Q(q_{11})Q(q_4)Q(q_{14})Q(q_{13})\big) \cdot \big(Q(q_{13})Q(q_{14})Q(q_3)Q(q_2)Q(q_1)\big) =$$

$$= Q(q_8)Q(q_7)Q(q_6)Q(q_5)Q(q_4)Q(q_3)Q(q_2)Q(q_1).$$

By [ABG], we obtain a $\mathbb{C}$-Fuchsian faithful and discrete representation of H_8 and, hence, a $\mathbb{C}$-Fuchsian faithful and discrete representation of the fundamental group G_8 of a surface of genus 3. As G_8 consists of all words of even length in the $Q(q_i)$'s, $i = 1, 2, 3, 4, 5, 6, 7, 8$, every element $I \in G_8$ has the form $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-1}\alpha & 0 \\ 0 & 0 & r\alpha \end{bmatrix}$ in a suitable basis e_0, e_1, e_2, where e_1, e_2 are isotropic points in the complex geodesic, e_0 is its polar point, $r > 0$, and $|\alpha| = 1$. Since $I \in \mathrm{SU}\,V$, we obtain $\alpha = \pm 1$.

4. References

[ABG] S. Anan′in, E. C. Bento Gonçalves, *A hyperelliptic view on Teichmüller space.* I, preprint http://arxiv.org/abs/0709.1711