Derivation of Invariant Varieties of Periodic Points from Singularity Confinement

Tsukasa YUMIBAYASHI,[1,*] Satoru SAITO,[2,†] and Yuki WAKIMOTO[1,‡]

[1]*Department of Physics, Tokyo Metropolitan University, Minamiohsawa 1-1, Hachiohji, Tokyo, 192-0397 Japan*

[2]*Hakusan 4-19-10, Midori-ku, Yokohama 226-0006 Japan*

It was shown, in our previous works, that periodic points of d dimensional rational integrable maps form varieties if the number of invariants p is greater than $d/2$. In the case of 3 dimensional Lotka-Volterra map with 2 invariants the varieties are generated iteratively by the map after it is recovered from the singularity confinement. We present many other examples of this phenomenon in this paper when $p = d - 1$, and discuss how to generalize our algorithm when p is less than $d - 1$.

PACS numbers: 02.0.+p, 05.45.-a, 45.05.+x, 02.30.Ik

Keywords: integrable maps, singularity confinement, invariant variety of periodic points

I. INTRODUCTION

We study in this paper rational maps

$$F : x = (x_1, x_2, ..., x_d) \to X = (X_1, X_2, ..., X_d), \qquad x, X \in \hat{\mathbb{C}}^d \tag{1}$$

in d dimensional complex space where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Since we consider rational maps it is convenient to write them as

$$X_j(x) = \frac{NX_j}{DX_j}, \qquad (j = 1, 2, ..., d)$$

where $NX_j(x), DX_j(x) \in \hat{\mathbb{C}}[x_1, x_2, ..., x_d]$ are polynomials irreducible from each other.

*email : yumibayashi-tsukasa@ed.tmu.ac.jp

†email : saito˙ru@nifty.com

‡email : wakimoto-yuki@ed.tmu.ac.jp

Let i be one of $\{1, 2, ..., d\}$ and denote by Σ_i the variety of zero set of DX_i. The points on Σ_i are mapped to $F(\Sigma_i)$, which is divergent. But, unless the infinity is a fixed point of the map, there is a possibility that it returns to a finite point after some steps of the map. This is the phenomenon known as singularity confinement (SC). If the points return to a finite region after m iteration of the map, we call this number m, the 'steps of SC'. This means that none of $DX_j^{(m+1)}(\Sigma_i),\ j = 1, 2, ..., d$ in $F^{(m+1)}(\Sigma_i)$ is identically zero, while $F^{(m)}(\Sigma_i)$ is divergent.

It is not difficult to see how this phenomenon takes place. Suppose the map has an inverse which is again rational. If Σ_- is the zero set of the denominators of the inverse map F^{-1}, it is mapped to $F^{-1}(\Sigma_-)$, which is divergent, by the inverse map. Conversely the points at infinity are mapped back to Σ_- by the forward map F. From this observation it is clear that when $F^{(m)}(\Sigma_i) \in F^{-1}(\Sigma_-)$, it is mapped to $F^{(m+1)}(\Sigma_i) \in \Sigma_-$, which is finite. This is the mechanism that the SC phenomenon undergoes.

Now we assume that the map has p invariants. It was proved in [1] that, when $p \geq d/2$, periodic points of all periods form varieties of non zero dimension, provided periodic points of one period form a variety. The existence of such a variety characterizes integrability of the map. Since these varieties are specified only by the invariants, we called them invariant varieties of periodic points, or IVPP for short. We shall refer this theorem as 'IVPP theorem' in this paper. Moreover, in [3], we have shown, in the 3 dimensional Lotka-Volterra (3dLV) map case, that the IVPPs of all periods can be derived from SC.

Since there exist two invariants in the 3dLV map, they are sufficient to parameterize the surface Σ_i by themselves. After the recovery from the singularity, $F^{(m+1)}(\Sigma_i)$ is on Σ_- by the reason we explained above. One can repeat the map further. Notice that all such images of Σ_i are also parametrized by the invariants alone. This means that $F^{(n+1)}(\Sigma_i)$ for all $n \geq m$ are sets of rational functions of the invariants. In other words the numerators and denominators of $X_j^{(n+1)}(\Sigma_i)$ are polynomial functions of the invariants.

An important observation in [3] is that $DX_i^{(n+1)}(\Sigma_i)$ must vanish at the period n points, because $F^{(n+1)}(\Sigma_i)$ is the nth image of $F(\Sigma_i)$, which was divergent. This is possible only if the zero set of one of the irreducible factors of the polynomial function $DX_i^{(n+1)}(\Sigma_i)$ decides the period n points. Moreover this polynomial factor of the invariants must be the one which determines the IVPP of period n, because this map has a single polynomial for each period. In this way we could derive IVPPs of all periods simply by repeating the map.

We would like to emphasize that a direct derivation of an IVPP from the periodicity conditions by using algebraic method becomes more and more difficult as the period and/or the dimension of the map increases. On the other hand it is remarkable that our algorithm explained above enables us to derive a series of IVPPs by iteration of the map.

We can directly apply this algorithm when the number of invariants is $d-1$. One of the purpose of this paper is to show that this algorithm works, not only in the 3dLV map, but in general when $p = d-1$. Some examples of this type will be presented in the next section to derive many IVPPs of various integrable maps.

When the number of invariants p is less than $d-1$, a single polynomial of the invariants is not sufficient to determine IVPP of each period. Since we have not studied such cases so far, we must develop new methods. Our second purpose of this paper is to solve this problem. To this end we must introduce $d-p-1$ additional conditions by hand so that Σ_i is parameterized in terms of the invariants. One of the polynomials which determine IVPP can be derived by the same procedure of the $p = d-1$ case. We must determine other functions by some consistency relations. We show in §3 how this new algorithm works in the case of 3 point Toda map with $d = 6$ and $p = 4$. In the final section we summarize the mechanism of SC studied in this paper and argue that the varieties formed by indeterminate points of the map play an important part in this phenomenon.

II. GENERATION OF IVPP'S WHEN $p = d-1$

In this section we study the simplest case of the derivation of IVPPs from SC. Namely we assume that the map has p invariants,

$$H_1(x), H_2(x), ..., H_p(x), \tag{2}$$

and consider the case $p = d-1$. According to the IVPP theorem, the set of all period n points are on a variety specified by the invariants

$$v^{(n)} = \left\{ x \in \hat{\mathbb{C}}^d \middle| \gamma^{(n)}(H_1(x), ..., H_{d-1}(x)) = 0 \right\}. \tag{3}$$

Here $\gamma^{(n)}$ is an irreducible polynomial function of the invariants, when $p = d-1$, and specifies the IVPP of period n. Hence the dimension of the variety $v^{(n)}$ is $d-1$. Generally speaking the function $\gamma^{(n)}$ could be decided from the periodicity conditions of the map. It is, however, quite difficult task to find them directly in an algebraic method [1].

In order to derive IVPPs from SC, which we explained in the previous section, let us first summarize the algorithm in the case $p = d - 1$.

Algorithm 1

Let F be a rational map (1) which has $d-1$ invariants (2). We assume SC of this map, such that the points of Σ_i, which are mapped once to infinity, come back to a finite region after m steps. In other words $F^{(m+1)}(\Sigma_i)$ is finite, while $F^{(m)}(\Sigma_i)$ is not.

1. If we fix the values of the invariants, we can always solve the d set of equations

$$\{DX_i(x) = 0,\ H_1(x) = h_1,\ H_2(x) = h_2, ..., H_p(x) = h_p\} \tag{4}$$

   for $x = (x_1, x_2, ..., x_d)$. This enables us to parameterize the points on Σ_i in terms of the invariants $h := (h_1, h_2, ..., h_{d-1})$, which we denote as

$$p^{(0)} := \Big(x_1(h), x_2(h), ..., x_d(h)\Big).$$

2. In the second step we iterate the map to see how SC works. It will be done by the substitution of $p^{(0)}$ to $F^{(l)}(x),\ l \leq m$, which must be divergent in some components.

3. Finally by the substitution of $p^{(0)}$ to $F^{(n+1)}(x), n \geq m$ we must find a polynomial factor in $DX_i^{(n+1)}$ which vanishes only at the points of period n. Since the factor is given by the invariants it must coincide with $\gamma^{(n)}$ in (3).

Once the algorithm is fixed the manipulation is straightforward. In the following we show three examples of the derivation of IVPPs of integrable maps when $p = d - 1$. We will see that exactly the same mechanism of the generation of IVPPs, which we have found in the case of 3dLV map in [1], works in all examples.

A. 2 dimensional Möbius Map

The first example is the two dimensional map $(x, y) \to (X, Y)$, which is defined by (5). In the following we use the notation (x, y), instead of (x_1, x_2) so on, to represent independent variables.

Map and Invariants

$$(x, y) \quad \to \quad (X, Y) = \left(x\frac{1-y}{1-x}, y\frac{1-x}{1-y} \right), \tag{5}$$

with an invariant

$$r = xy.$$

Parametrization of Σ_x and SC

According to the first step of Algorithm 1 we find the initial point of the map which is parameterized by the invariant r:

$$p^{(0)} = (1, r) \in \Sigma_x.$$

Then the SC map undergoes as

$$p^{(0)} \to (\infty, 0) \to (-1, -r) \to \left(-\frac{1+r}{2},\ -\frac{2r}{1+r} \right)$$
$$\to \left(-\frac{1+3r}{3+r},\ -\frac{r(3+r)}{1+3r} \right) \to \left(-\frac{1+6r+r^2}{4(1+r)},\ -\frac{4r(1+r)}{1+6r+r^2} \right) \to \cdots$$

The steps of the SC is 2, in this map.

IVPP

From the expression of the denominator of $X^{(4)}$ in $p^{(4)}$, we find $\gamma^{(3)} = 3 + r$. Similarly from $p^{(5)}$, it is clear $\gamma^{(4)} = 1 + r$, and so on. We obtain

$$\begin{aligned}
\gamma^{(3)} &= 3 + r \\
\gamma^{(4)} &= 1 + r \\
\gamma^{(5)} &= 5 + 10r + r^2 \\
\gamma^{(6)} &= 1 + 3r \\
&\text{etc.}
\end{aligned}$$

Remarks

Our map (5) can be reduced to the usual Möbius map,

$$X = \frac{x-r}{1-x}$$

whose IVPPs were given in [2], by a different method, as

$$v^{(n)} = \left\{ \frac{1}{1-r} - \cos^2\left(\frac{\pi}{n}\right) = 0 \right\}$$

in agreement with our above results. Notice that the IVPP of period 2 does not exist as long as the invariant r is finite.

B. 3 dimensional Korteweg-de Vries map

The three dimensional KdV map was introduced in [4].

Map and Invariants

$$(X, Y, Z) = \left(x\frac{1+xz+xyz^2}{1+yx+yzx^2}, y\frac{1+yx+yzx^2}{1+zy+zxy^2}, z\frac{1+zy+zxy^2}{1+xz+xyz^2} \right) \tag{6}$$

There are two independent invariants:

$$\begin{cases} f &= xyz+1 \\ g &= (1+xy)(1+yz)(1+zx)+1. \end{cases}$$

Parametrization of Σ_x and SC

$$p^{(0)} = \left(-\frac{f^2-2f+g}{g(f-1)},\ -\frac{f(f-1)(f-2)}{f^2-2f+g},\ \frac{g(f-1)}{f(f-2)} \right) \in \Sigma_x$$

$$\begin{aligned} p^{(0)} &\to (\infty, 0, 1-f) \to (\infty, 1-f, 0) \\ &\qquad \to \left(-\frac{f^2-2f+g}{g(f-1)},\ \frac{g(f-1)}{f(f-2)},\ -\frac{f(f-1)(f-2)}{f^2-2f+g} \right) \to \cdots \end{aligned}$$

The steps of SC is 3.

$$
\begin{aligned}
\gamma^{(2)} &= g, \\
\gamma^{(3)} &= (f^2-2f+g)^2 - gf(f-2), \\
\gamma^{(4)} &= (f^2-2f+g)^3 - (g-1)f^3(f-2)^3, \\
\gamma^{(5)} &= (f^2-2f+g)^6 - gf^2(f-2)^2(f^2-2f+3)(f^2-2f+g)^3 \\
&\quad -3g^3f(f-2)(f-1)^2(f^2-2f+g)^2 - g^3f^3(f-2)^3(g-1)^2,
\end{aligned}
$$

etc.

C. 4 dimensional Lotka-Volterra Map

The Lotka-Volterra map of an arbitrary dimension was defined in [4]. Since we have shown the derivation of IVPPs from the SC in the case of the 3 dimensional LV map in [3], we study here the four dimensional case.

Map and Invariants

$$
\begin{aligned}
(X,Y,Z,W) &= \left(x\frac{1-y-z+yz+zw}{1-z-w+zw+wx}, y\frac{1-z-w+zw+wx}{1-w-x+wx+xy} \right. \\
&\qquad \left. , z\frac{1-w-x+wx+xy}{1-x-y+xy+yz}, w\frac{1-x-y+xy+yz}{1-y-z+yz+zw} \right)
\end{aligned}
$$

$$
\begin{cases}
r = xyzw \\
f = xyzw - (1-x)(1-y)(1-z)(1-w) \\
g = (1-x-z)(1-y-w) + 1
\end{cases}
$$

$$
\begin{aligned}
p^{(0)} &= \Bigg(\frac{1}{2}\frac{(-g^2+g-gf+2f+gN)f}{gf-g^2r+f^2}, -\frac{1}{2}\frac{gf-2gf-f+f^2+fN}{f},\\
&\quad -\frac{1}{2}\frac{(-gf+2gr+f-f^2+fN)g}{-gf-g^2r+f^2+g^2f}, \frac{1}{2}\frac{-g-2f+g^2+gf+gN}{(-1+g)g}\Bigg)\\
&\in \Sigma_x
\end{aligned}
$$

where

$$
N = \sqrt{1-2g-2f+g^2+2gf+f^2+4r-4gr},
$$

$$
\begin{aligned}
p^{(0)} &\to \left(\infty, 0, -\frac{1}{2}\frac{-f+1-g+N}{-1+g}, 1\right) \to \left(1, \frac{1}{2}\frac{f+g-1+N}{-1+g}, 0, \infty\right)\\
&\to \Bigg(\frac{1}{2}\frac{-g-2f+g^2+gf+gN}{(-1+g)g}, -\frac{1}{2}\frac{(-gf+2gr+f-f^2+fN)g}{-gf-g^2r+f^2+g^2f},\\
&\quad -\frac{1}{2}\frac{gf-2gr-f+f^2+fN}{f}, \frac{1}{2}\frac{(-g^2+g-gf+2f+gN)f}{gf-g^2r+f^2}\Bigg)\\
&\to \cdots .
\end{aligned}
$$

The steps of SC is 3.

IVPP

$$
\begin{aligned}
\gamma^{(2)} &= g\\
\gamma^{(3)} &= f^2-g^2r+g^2f\\
\gamma^{(4)} &= -2g^2r+g^2f+2f^2\\
\gamma^{(5)} &= r^2fg^6+3f^5g^2+3g^4r^2f^2-3rf^4g^2-4g^4rf^3+f^6-r^3g^6+g^4f^4
\end{aligned}
$$

etc.

III. GENERALIZATION TO THE CASE $p < d-1$

Next we consider generalization of our argument in the previous section to the case $p < d-1$. For this purpose we would like to propose a new algorithm which replaces

Algorithm 1. We will apply this new algorithm to derive IVPPs of 3 point Toda map, in which $p = d - 2$ with $d = 6$.

Algorithm 2

Let F be a rational map of (1) which has p invariants (2). We assume that the steps of SC of this map is finite. Since the number of the invariants is not sufficient, we must introduce $d - p - 1$ additional conditions to (4), so that Σ_i is parameterized only by the invariants.

1. We propose to replace (4) by

$$\Big\{ DX_i(x) = 0, DX_i^{(2)}(x) = 0, ..., DX_i^{(d-p)}(x) = 0,$$
$$H_1(x) = h_1, H_2(x) = h_2, ..., H_p(x) = h_p \Big\} \tag{7}$$

   to solve for $x = (x_1, x_2, ..., x_d)$. This enables us to parameterize Σ_i in terms of the values h of the invariants.

2. In the second step we iterate the map by the substitution of $p^{(0)}$ to $F^{(l)}(x)$, $l \leq m$, which must be divergent. The addition of new conditions in (7) shifts the point of recovery from the divergences and increases the steps m of the SC. Otherwise it does not change the mechanism of the SC.

3. By the substitution of $p^{(0)}$ to $F^{(n+1)}(x), n \geq m$ we will find a single polynomial in $DX_i^{(n+1)}$ which vanishes only at the points of n period. It must be one of the independent set of functions which determine the IVPP of period n.

4. Finally, to find other elements of the IVPP set of period n, we identify the other components of $F^{(n+1)}(\Sigma_i)$ with those of $F(\Sigma_i)$, which are all written by the invariants.

A. 3 point Toda map

We show one example obtained by the application of the new Algorithm 2. Although the Toda map of arbitrary dimension was given again in [4], we present here only 3 dimensional case. We mention that the analysis using computer algebra becomes harder as the number of freedom increases, although the algorithm itself is the same.

$$
\begin{aligned}
&(X, Y, Z, U, V, W) \\
= &\left(y\frac{zu+zx+wu}{yw+yz+vw}, z\frac{xv+xy+uv}{zu+zx+wu}, x\frac{yw+yz+vw}{xv+xy+uv}, \right. \\
&\quad \left. u\frac{yw+yz+vw}{zu+zx+wu}, v\frac{zu+zx+wu}{xv+xy+uv}, w\frac{xv+xy+uv}{yw+yz+vw} \right)
\end{aligned} \tag{8}
$$

$$
\begin{cases}
r &= xyz \\
t &= x+y+z+u+v+w \\
f &= xy+yz+zx+uv+vw+wu+xv+yw+zu \\
g &= uvw-xyz
\end{cases} \tag{9}
$$

Parameterization of Σ_x and SC

In order to parametrize Σ_x we use $DX^{(2)}(x)$ in (7), and obtain

$$
\begin{aligned}
p^{(0)} = &\left(\frac{r(-g^2t+gf^2+f^2r)}{g^3}, \frac{g^2 f}{-g^2t+f^2r}, \frac{(-g^2t+f^2r)g}{(-g^2t+gf^2+f^2r)f}, \right. \\
&\left. -\frac{(r+g)(-g^2t+f^2r)}{g^3}, -\frac{g(-g^2t+gf^2+f^2r)}{(-g^2t+f^2r)f}, \frac{g^2 f}{-g^2t+gf^2+f^2r} \right)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
p^{(0)} &\to \left(\infty, -\frac{g}{f}, 0, 0, \frac{g}{f}, \infty\right) \to \left(\frac{0}{0}, 0, \frac{0}{0}, 0, \frac{0}{0}, \frac{0}{0}\right) \\
&\to \left(0, \frac{0}{0}, \frac{0}{0}, 0, \frac{0}{0}, \frac{0}{0}\right) \to \left(-\frac{g}{f}, \infty, 0, 0, \infty, \frac{g}{f}\right) \\
&\to \left(\frac{g^2 f}{-g^2t+f^2r}, \frac{r(-g^2t+gf^2+f^2r)}{g^3}, \frac{(-g^2t+f^2r)g}{(-g^2t+gf^2+f^2r)f}, \right. \\
&\qquad \left. -\frac{(r+g)(-g^2t+f^2r)}{g^3}, \frac{g^2 f}{-g^2t+gf^2+f^2r}, -\frac{g(-g^2t+gf^2+f^2r)}{(-g^2t+f^2r)f} \right) \\
&\to \dots
\end{aligned} \tag{11}
$$

Here $0/0$ means that the denominator and the numerator of the component become zero separately, so that we can not determine its value. In other words the point is indeterminate.

IVPP

The IVPPs of the Toda map are given by intersections of two functions in the form $v^{(n)} = \{\gamma_1^{(n)} = 0,\ \gamma_2^{(n)} = 0\}$. We have found

$$\begin{aligned}
\gamma_1^{(3)} &= f, \qquad \gamma_2^{(3)} = tg^2 \\
\gamma_1^{(4)} &= g^2t - f^2r, \qquad \gamma_2^{(4)} = tf + g \\
\gamma_1^{(5)} &= -2f^3r^2 + 2frtg^2 - f^3rg - g^4, \quad \gamma_2^{(5)} = 4f^3rg^3 + f^6r^2 - g^6 \\
\gamma_1^{(6)} &= 5f^4r^2 + 5gf^4r - 6g^2f^2tr + g^2f^4 - 3g^3f^2t + 2g^4f + g^4t^2, \\
\gamma_2^{(6)} &= 4f^6 + 3t^2f^2g^2 - 4f^3g^2 + g^4
\end{aligned}$$

etc.

Remarks

Since we have chosen the function $DX^{(2)}$ as the additional condition for the parametrization of Σ_x, the number of steps of the SC increased from 3 to 5. This change makes us unable to decide the IVPPs of period 2 and period 3. From our point of view, however, this does not lower the value of our method. We can always impose periodicity conditions to find the IVPP of each period separately. For instance, in the map (11), period 3 map requires $p^{(1)} = p^{(4)}$, or writing it explicitly

$$\left(\infty, -\frac{g}{f}, 0, 0, \frac{g}{f}, \infty\right) = \left(-\frac{g}{f}, \infty, 0, 0, \infty, \frac{g}{f}\right),$$

from which we find $f = 0$, while $g \neq 0$. The comparison of $p^{(5)}$ with $p^{(2)}$ shows that they can be the same only if $tg^2 = 0$. In this way $\gamma_{1,2}^{(3)}$ are obtained.

As for the period 2 case, we can manipulate the periodicity conditions directly. We find that the points on the intersection of the following three surfaces

$$\begin{aligned}
xyu^3 - 3xyz(x + y - z + u)u - z(x^2 + yz)(y^2 + zx) &= 0 \\
yzv^3 - 3xyz(y + z - x + v)v - x(y^2 + zx)(z^2 + xy) &= 0 \\
zxw^3 - 3xyz(z + x - y + w)w - y(z^2 + xy)(x^2 + yz) &= 0
\end{aligned}$$

satisfy the period 2 conditions.

IV. CONCLUSION

In conclusion we would like to summarize the mechanism of the SC in the cases we have studied in this paper.

We choose the starting point $p^{(0)}$ of the map on the variety

$$\Sigma_+ := \bigcup_{j=1}^{d} \Sigma_j, \qquad \Sigma_j := \{x | DX_j(x) = 0\}.$$

A component of the image $p^{(1)} \in F(\Sigma_+)$ of the point $p^{(0)}$ is certainly divergent. But, at the same time, some other components must vanish, so that the invariants remain finite. This means $p^{(0)} \in \Sigma_+ \cap \Sigma'_+$, where $\Sigma'_+ = \cup_j \{x | NX_j = 0\}$ is the zero set of the numerators of the map. Since Σ'_+ coincides with Σ_+ in all our examples studied in this paper, we simply write it Σ_+.

We would like to know the steps of SC. For this purpose let us denote the zero set of the denominators of the inverse map by

$$\Sigma_- = \bigcup_{j=1}^{d} \Sigma_j^{-1}, \qquad \Sigma_j^{-1} := \{x | DX_j^{(-1)}(x) = 0\}.$$

We also use the notation to specify the set of divergent points of the map

$$\Lambda_\mp(\infty) := F^{\pm 1}(\Sigma_\pm).$$

Then one can expect that the map of the SC proceed according to

$$\Sigma_+ \to \Lambda_-(\infty) \to \Lambda_+(\infty) \to \Sigma_- \to \ \cdots \ . \tag{12}$$

In fact points on Σ_+ are mapped mostly in this way. It will be worthwhile to present some examples of $\Lambda_\pm(\infty)$. In the 3dKdV map case

$$\begin{aligned}
\Lambda_-(\infty) &= \Big[(\infty, 0, 1-f), \quad (1-f, \infty, 0), \quad (0, 1-f, \infty)\Big], \\
\Lambda_+(\infty) &= \Big[(\infty, 1-f, 0), \quad (0, \infty, 1-f), \quad (1-f, 0, \infty)\Big],
\end{aligned} \tag{13}$$

whereas those of the 3 point Toda map we have

$$\begin{aligned}\Lambda_-(\infty) &= \Bigg[\left(\infty, y+v, 0, 0, \frac{zv}{y+v}, \infty\right), \left(0, \infty, z+w, \infty, 0, \frac{xw}{z+w}\right) \\ &\qquad , \left(x+u, 0, \infty, \frac{yu}{x+u}, \infty, 0\right)\Bigg] \\ \Lambda_+(\infty) &= \Bigg[\left(0, y+u, \infty, \frac{xu}{y+u}, 0, \infty\right), \left(\infty, 0, z+v, \infty, \frac{yv}{z+v}, 0\right) \\ &\qquad , \left(x+w, \infty, 0, 0, \infty, \frac{zw}{x+w}\right)\Bigg] . \end{aligned} \tag{14}$$

In the 2 dimensional Möbius map, $\Lambda_\pm(\infty)$ are the same and given by

$$\Lambda_+(\infty) = \Lambda_-(\infty) = \Big[(\infty,\ 0),\ \ (0, \infty)\Big].$$

Therefore we see that the number of steps of SC is three unless $\Lambda_+(\infty)$ and $\Lambda_-(\infty)$ coincide.

This simple rule is true, but not always. We have already encountered an exception in the Toda map (11). An additional condition changed the number of steps of SC. In the case (11) the point on $\Lambda_-(\infty)$ is mapped twice before it is moved to $\Lambda_+(\infty)$. We have noticed that these two extra points are indeterminate. In fact we can show that the same parameterization as (10) of the initial point could be obtained if we used the conditions $(NW^{(2)} = 0,\ DW^{(2)} = 0)$, instead of $(DX = 0,\ DX^{(2)} = 0)$ in (7).

Now there arises a question. Since the indeterminate points are on a subset of Σ_+, they must play some role in SC. Indeed they are on the intersection of the zero set of the denominators $\Sigma_\pm$ and those of the numerators $\Sigma'_\pm$, which we denote

$$\Lambda_\pm := \Sigma_\pm \cap \Sigma'_\pm.$$

In order to clarify the nature of the map on $\Lambda_\pm$, we first recall that, in the 3dLV map, IVPPs of all periods intersect along the lines which are specified by the conditions [3]

$$f = 0, \quad g = 0. \tag{15}$$

It is remarkable that this is true in all our examples studied in this paper, except for the 2 dimensional Möbius map, which has only one invariant. We can see it directly from the expression of $\gamma^{(n)}$'s in §2 and §3. Since every point on the lines (15) is occupied by periodic points of all periods simultaneously, the map must behave quite singular on these lines. In fact we can convince ourselves that the conditions (15) exactly agree with the intersections

$\Lambda_\pm$, where both numerators and denominators of the map vanish altogether. In other words IVPPs of different periods can stay together only at the indeterminate points of the map.

Next, to clarify the behavior of the map on $\Lambda_\pm$, let us study the 3dKdV map (6) a little more carefully. A point on $\Lambda_\pm$ can be expressed as

$$p_+ = \left(x,\ 1-\frac{1}{x},\ \frac{1}{1-x}\right), \quad p_- = \left(x, \frac{1}{1-x}, 1-\frac{1}{x}\right), \tag{16}$$

in general. The substitution of p_+ to F yields

$$F(p_+) = \left(\frac{0}{0}, \frac{0}{0}, \frac{0}{0}\right),$$

which has no information. Instead of starting from Λ_+ one can approach it by shifting p_+ to

$$p_{+a} := \left(x,\ 1-\frac{1}{x}+a\varphi(x),\ \frac{1}{1-x}+a\psi(x)\right)$$

and taking the limit $a \to 0$. Here $\varphi(x)$ and $\psi(x)$ are some functions of x. In the $a = 0$ limit the substitutions of p_{+a} to $F^{(n)}$ yield

$$\begin{aligned}
F(p_{+0}) &= \left(\frac{x+(x-2)R}{(1-x)(1-R)}, \frac{(1-x)(1-R)}{(1-2x)+R}, -\frac{(1-2x)+R}{x+(x-2)R}\right), \\
F^{(2)}(p_{+0}) &= \left(\frac{(1-2x)+R}{(1-x)(1-R)}, -\frac{x+(x-2)R}{(1-2x)+R}, \frac{(1-x)(1-R)}{x+(x-2)R}\right), \\
&\text{etc.,}
\end{aligned} \tag{17}$$

where $R := (1-x)^2\psi/\varphi$. Since $\varphi(x)$ and $\psi(x)$ are arbitrary it is apparent that the image of the map is indeterminate. Nevertheless one can show, in general,

$$F(p_{+0}) \in \Lambda_\mp, \quad \text{if} \quad p_{+0} \in \Lambda_\pm. \tag{18}$$

Namely the points are mapped alternatively between two intersections Λ_+ and Λ_-.

We notice that, when $f = 0$, the points of $\Lambda_\pm(\infty)$ in (13) are reproduced by $p_\pm$ at $x = \infty, 0, 1$, respectively. Hence the SC takes place also on $\Lambda_\pm$, but only at some limited points. From this point of view the lines specified by $p_\pm$ are nothing but the blow up of $\Lambda_\pm(\infty)$ at $f = 0$, in the 3dKdV map case.

Similarly in the 3 point Toda map case a point on $\Lambda_\pm$, which is again characterized by (15), is mapped to indeterminate points $\left(\frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}\right)$. We can parameterize these points on $\Lambda_\pm$ as

$$\begin{aligned}
p_+ &= \left(x, y, -\frac{uw}{x+u}, u, -\frac{xy}{x+u}, w\right) \in \Lambda_+ \\
p_- &= \left(w, u, -\frac{xy}{x+u}, x, y, -\frac{uw}{x+u}\right) \in \Lambda_-
\end{aligned} \tag{19}$$

which are further mapped according to

$$F(p_\pm) \in \Lambda_\mp,$$

respectively. Therefore a point which is trapped into $\Lambda_\pm$ remains there. It should be compared with the SC map (12) of a generic point on Σ_+.

On the other hand, in our 3 point Toda map (11), there appeared different type of indeterminate points, which was obtained for a particular set of initial points on Σ_+. If we had adopted a different additional function in (7) of the Algorithm 2, we would have another route of SC map. As we denote by $\tilde{\Lambda}_\pm$ such sets of indeterminate points, the general SC map will undergo as

$$\Sigma_+ \to \Lambda_-(\infty) \to \tilde{\Lambda}_+ \to \tilde{\Lambda}_- \to \Lambda_+(\infty) \to \Sigma_- \to \cdots .$$

Details will be discussed in our forthcoming paper.

---

[1] S. Saito and N. Saitoh, J. Phys. Soc. Jpn., **76** No.2 p.024006 (2007),
http://jpsj.ipap.jp/link?JPSJ/76/024006
S. Saito and N. Saitoh, J. Phys. A: Math. Theor. **40** 12775-12787 (2007)
S. Saito and N. Saitoh, SIGMA, **2**, Paper 098 (2006)
http://www.emis.de/journals/SIGMA.

[2] N. Saitoh and S. Saito, J. Phys. Soc. Jpn., **177** 024001 (2008).

[3] S. Saito and N. Saitoh J. Math. Phys. **51** 063501 (2010).

[4] R. Hirota, S. Tsujimoto and T. Imai, "Difference Scheme of Soliton Equations", in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, ed. by P.L.Christiansen at al., p.7 (Plenum Press, New York, 1993).
R. Hirota, and S. Tsujimoto, J.Phys.Soc.Jpn. **64** 3125-3127 (1995).