The Witten-Reshetikhin-Turaev invariants of finite order mapping tori II

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Abstract

We identify the leading order term of the asymptotic expansion of the Witten-Reshetikhin-Turaev invariants for finite order mapping tori with classical invariants for all simple and simply-connected compact Lie groups. The square root of the Reidemeister torsion is used as a density on the moduli space of flat connections and the leading order term is identified with the integral over this moduli space of this density weighted by a certain phase for each component of the moduli space. We also identify this phase in terms of classical invariants such as Chern-Simons invariants, eta invariants, spectral flow and the rho invariant. As a result, we show agreement with the semiclassical approximation as predicted by the method of stationary phase.

1 Introduction

The Witten-Reshetikhin-Turaev quantum invariants were first proposed by Witten in his seminal paper [48], where he studied the Chern-Simons quantum field theory for a simple, simply connected compact Lie group G. He did so using path integral techniques, which let him to propose a combinatorial surgery formula for the invariants.

Shortly thereafter Reshetikhin and Turaev gave a rigorous construction of these quantum invariants using the representation theory of quantum groups. In fact, they subsequently constructed the whole topological quantum field theory (TQFT) $Z_G^{(k)}$ in [38, 37, 47] for $G = \mathrm{SU}(2)$. The other classical groups were treated in [44, 46]. The TQFT for $G = \mathrm{SU}(2)$ were also constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [16, 17]. Since then these constructions have been extended to other Lie groups G through the effort of many people. For a complete list we refer to the references in [47].

Witten also analyzed the Chern-Simons path integral from a perturbative point of view. The identification of the leading order asymptotics of the invariants in terms of classical topological invariants in the case of an isolated, irreducible flat connection, was proposed by Witten in [48]. There has been subsequent proposals for refinements and generalizations to this, for example by Freed and Gompf [24, Equation (1.3)] and by Jeffrey [29, Equation (5.1)], partially supported by computations of the quantum invariants, as well as solely from path integral techniques Axelrod, Lawrence, Mariño, Rozansky, Singer, Zagier [12, 13, 40, 42, 31, 32].

Since we have explicit combinatorial expressions for the quantum invariants, it is sensible to extract the perturbation expansion from these exact formulas. To this end we need an ansatz for the kind of asymptotic expansion we can expect based on Witten's path integral formula for the invariants. This leads us to the asymptotic expansion conjecture [1, Conjecture 7.7], [6] and [8, Conjecture 1].

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Conjecture 1.1 (Asymptotic expansion conjecture)**.** *There exist constants (depending on X)* $d_j \in \frac{1}{2}\mathbf{Z}$ *and* $b_j \in \mathbf{C}$ *for* $j = 0, \dots, n$ *and* $a_j^l \in \mathbf{C}$ *for* $j = 0, 1, \dots, n$, $l = 1, 2, \dots$ *such that the asymptotic expansion of* $Z_G^{(k)}(X)$ *in the limit* $k \to \infty$ *is given by*

$$Z_G^{(k)}(X) \sim \sum_{j=0}^{n} e^{2\pi i k q_j} k^{d_j} b_j \left(1 + \sum_{j=1}^{\infty} a_j^l k^{-l/2} \right), \tag{1.1}$$

where $q_0 = 0, q_1, \dots, q_n$ *are finitely many different values of the Chern-Simons functional on the space of flat G–connections on X.*

Here $\sim$ denotes **asymptotic equivalence** in the Poincaré sense, which means the following: Let

$$d = \max\{d_0, \dots, d_n\}.$$

Then for any non-negative integer L, there is a $c_L \in \mathbf{R}$ such that

$$\left| Z_G^{(k)}(X) - \sum_{j=0}^{n} e^{2\pi i k q_j} k^{d_j} b_j \left(1 + \sum_{l=0}^{L} a_j^l k^{-l/2} \right) \right| \leq c_L k^{d-(L+1)/2}$$

for all levels k. Of course such a condition only puts limits on the large k behavior of $Z_G^{(k)}(X)$.

Through the previous definition we can make the following definition of the *leading order term* of the asymptotics.

Definition 1.2. If $Z_G^{(k)}(X)$ satisfies Conjecture 1.1, then we write

$$Z_G^{(k)}(X) \stackrel{.}{\sim} \sum_{j=0}^{n} e^{2\pi i k q_j} k^{d_j} b_j$$

and we call the sum on the right *the leading order term* of (the asymptotic expansion of) $Z_G^{(k)}(X)$.

It is this leading order term for which there conjecturally is a classical topological expression. In fact, let $\mathcal{M}(X)$ be the moduli space flat G-connections on X and let us write the component decomposition as

$$\mathcal{M}(X) = \bigcup_{c \in C_X} \mathcal{M}(X)_c.$$

One expects that a square root of the Reidemeister torsion produces a measure on $\mathcal{M}(X)$. Combining results from [12, 13, 40, 41, 42, 11, 31, 32] and the references therein we arrive at the following conjectured formula for the leading order term.

Conjecture 1.3. *The leading order term of* $Z_G^{(k)}(X)$ *with respect to the Atiyah 2–framing is given by*

$$Z_G^{(k)}(X) \stackrel{.}{\sim} \sum_{c \in C_X} \frac{1}{|Z(G)|} e^{\pi i \dim G (1 + b^1(X))/4} \int_{A \in \mathcal{M}(X)_c} \sqrt{\tau_X(A)} e^{2\pi i \operatorname{CS}_X(A)(k+h)} e^{2\pi i \left(\operatorname{SF}(\theta, A)/4 - (\dim(H^0(X, d_A)) + \dim(H^1(X, d_A)))/8 \right)} k^{d_c} \tag{1.2}$$

and

$$d_c = \frac{1}{2} \max_{A \in \mathcal{M}(X)_c} \left(\dim(H^1(X, d_A)) - \dim(H^0(X, d_A)) \right),$$

where max *here means the maximum value* $\dim(H^1(X, d_A)) - \dim(H^0(X, d_A))$ *attained on a Zariski open subset of* $\mathcal{M}(X)_c$.

In the appendix we review the heuristics by which the method of stationary phase applies to the Chern-Simons path integral and produces this conjecture. Note, that the appendix contains the only non-rigorous part of this paper, which we decided to keep for motivational purposes. In the conjectured formula (1.2) we see expressions for the constants b_j and d_j in terms of Reidemeister torsion, spectral flow and dimensions of twisted cohomology groups.

In this paper we consider the Witten-Reshetikhin-Turaev quantum invariants of finite order mapping tori. We take as our starting point the formula derived in [6, Theorem 8.2] for the quantum invariants of such three manifolds.

Let Σ be a closed oriented surface. Then the mapping torus Σ_f of a diffeomorphism $f\colon\ \Sigma \to \Sigma$ is defined to be

$$\Sigma_f = \Sigma \times I/(x,1) \sim (f(x),0)$$

with the orientation given by the product orientation with the standard orientation on the interval $I = [0,1]$. We assume that f is of finite order. Let $\mathcal{M}(\Sigma)$ be the moduli space of flat connections on Σ with its symplectic structure ω. Denote by $|\mathcal{M}(\Sigma)| \subset \mathcal{M}(\Sigma)$ the fixed point set of f^* on $\mathcal{M}(\Sigma)$ of the diffeomorphism. Denote by C an indexing set for the set of all connected components $\{|\mathcal{M}(\Sigma)|_c\}_{c\in C}$ of $|\mathcal{M}(\Sigma)|$. There is a map $r\colon\ \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ given by restricting a flat connection on Σ_f to $\Sigma \times \{0\}$. We will write $\mathcal{M}(\Sigma_f)_c = r^{-1}(|\mathcal{M}(\Sigma)|_c)$. Let the prime superscript denote the part which is irreducible in $\mathcal{M}(\Sigma)$: $\mathcal{M}(\Sigma)' \subset \mathcal{M}(\Sigma)$ denotes the irreducible subset, while $\mathcal{M}(\Sigma_f)'_c = r^{-1}(|\mathcal{M}(\Sigma)|'_c)$. As it is argued in [6, Lemma 7.2], $\mathrm{CS}_X(A)$ is constant for $A \in \mathcal{M}(\Sigma_f)'_c$, therefore we write $\mathrm{CS}_X(c) := \mathrm{CS}_X(A)$ for $A \in \mathcal{M}(\Sigma_f)_c$. The following is the main result of [6].

Theorem 1.4 ([6, Theorem 8.2])**.** *The Witten-Reshetikhin-Turaev invariants of Σ_f are given by*

$$Z_G^{(k)}(\Sigma_f) = \det(f)^{-\frac{1}{2}\zeta} \sum_{c\in C} \exp(2\pi i k\,\mathrm{CS}_{\Sigma_f}(c)) \exp(k\omega_c) \cap \tau_\cdot(L^c_\cdot(O_{\mathcal{M}(\Sigma)})), \tag{1.3}$$

where

$$\zeta = \frac{k \dim G}{k+h}$$

is the central charge of the theory, h is the dual Coxeter number of G, ω_c is the restriction of ω to $|\mathcal{M}(\Sigma)|_c$, $\det(f)^{-\frac{1}{2}\zeta}$ is the so called framing correction defined in [6, Equation (5)] and $\tau_\cdot(L^c_\cdot(O_{\mathcal{M}(\Sigma)}))$ denotes a certain homology class with complex coefficients of $|\mathcal{M}(\Sigma)|_c$ discussed in [6].

We see immediately that the Asymptotic expansion conjecture for finite order mapping tori follows from Theorem 1.4.

The main result of this paper is the following theorem, which applies to all finite order elements f of the mapping class group of Σ.

Theorem 1.5. *For each $c \in C$ such that $\mathcal{M}(\Sigma_f)'_c$ is nonempty we have*

$$\begin{aligned} &k^{-d_c} \det(f)^{-\frac{1}{2}\zeta} e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(c)} \frac{1}{d_c!} (\exp(k\omega_c) \cap \tau_\cdot(L^c_\cdot(O_{\mathcal{M}(\Sigma)}))) \\ &\qquad = \frac{1}{|Z(G)|} \int_{A\in\mathcal{M}(\Sigma_f)'_c} e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(A)} \sqrt{\tau_{\Sigma_f}(A)} e^{2\pi i \frac{\rho_A(\Sigma_f)}{8}} + O(\frac{1}{k}), \end{aligned} \tag{1.4}$$

where $\rho_A(\Sigma_f)$ is the classical rho-invariant.

In Section 8 we give a proof of a well-known formula relating the spectral flow, the ρ–invariant and the Chern-Simons invariant for an arbitrary Lie group G, since the only proof we have found in the literature is for SU(2) (see [30, Section 7]). The precise relation—stated in Theorem 8.1—shows in particular, that Theorem 1.5 has an equivalent formulation in terms of spectral flow, which has the following theorem as an immediate consequence, once combined with Theorem 1.4.

Theorem 1.6. *If $\mathcal{M}(\Sigma)'_c$ is nonempty for every $c \in C$, then the above conjecture for the leading order term is correct, i.e.*

$$Z_G^{(k)}(\Sigma_f) \doteq \sum_{c\in C} \frac{1}{|Z(G)|} e^{\pi i \dim G(1+b^1(\Sigma_f))/4} \int_{A\in\mathcal{M}(\Sigma_f)'_c} \sqrt{\tau_{\Sigma_f}(A)} e^{2\pi i \,\mathrm{CS}(A)(k+h)} e^{2\pi i\left(\mathrm{SF}(\theta,A)/4-(\dim(H^0(\Sigma_f,d_A))+\dim(H^1(\Sigma_f,d_A)))/8\right)} k^{d_c} \tag{1.5}$$

and

$$d_c = \frac{1}{2} \max_{A\in\mathcal{M}(\Sigma_f)_c} \left(\dim(H^1(\Sigma_f,d_A)) - \dim(H^0(\Sigma_f,d_A))\right),$$

where max *here means the maximum value* $\dim(H^1(\Sigma_f,d_A)) - \dim(H^0(\Sigma_f,d_A))$ *attained on a Zariski open subset of* $\mathcal{M}(\Sigma_f)_c$.

Recall that $\mathcal{M}(\Sigma_f)'_c$ consists of the irreducible representation whose restriction to Σ is irreducible. By Theorem 2.3 the hypothesis of Theorem 1.6 is satisfied in the case $G = \mathrm{SU}(n)$ and $g(\Sigma/\langle f\rangle) > 1$.

This paper is organized as follows. Section 2 contains a preliminary discussion about the Chern-Simons functional and the moduli spaces of flat connections. In Section 3 we express the leading order term of the Witten-Reshetikhin-Turaev invariants for each $c \in C$ as certain integrals of differential geometric data. In Section 4 and 5 we review Reidemeister torsion and compute it for mapping tori. In Section 6 we review the ρ–invariant and an essential result for finite order mapping tori by Bohn [18]. Section 7 combines the main results from Sections 3 and 5 to identify the classical invariants in the leading order term of the $Z_G^{(k)}(X)$ in the limit $k \to \infty$. Section 8 gives an equivalent formulation of this identification in Section 3 in terms of spectral flow. In the appendix we present the heuristics which lead to the conjectured identification of the leading order term with classical topological invariants.

The results of this paper relies on the results of [6], which was obtained by using the gauge theory approach to the Witten-Reshetikhin-Turaev TQFT. The first named author has obtained other results about this TQFT using the gauge theory approach, such as the asymptotic faithfulness of the quantum representations [3] and the determination of the Nielsen-Thurston classification via these same representations [4] (see also [9]). He has further related these quantum representations to deformation quantization of moduli spaces both in the abelian and in the non-abelian case, please see [2], [7] and [5]. The second named author has answered some open questions by Jeffrey [29] about this TQFT for torus-bundles over S^1 by using cut-and-paste methods to perform spectral flow computations [27].

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2 The Chern-Simons invariant and moduli spaces of flat connections

In this section we give some necessary definitions and make some remarks regarding normalizations before we consider the moduli space and recall its decomposition into connected components.

Normalizations for the Chern-Simons functional and Poincare duality

Let $\langle\cdot,\cdot\rangle$ be a multiple of the Killing form on the Lie Algebra $\mathfrak{g}$ of a simple and simply-connected compact Lie group G normalized so that $-\frac{1}{6}\langle\theta\wedge[\theta\wedge\theta]\rangle$ is a minimal integral generator of $H^3(G,\mathbf{R})$, where θ is the Maurer-Cartan form. A connection on a principal G–bundle P is a G–equivariant, horizontal Lie algebra valued 1–form on P. The group of gauge transformations $\mathcal{G}$ consists of all bundle automorphism $P\to P$, which acts on connections by pull-back. Let X be an oriented, closed 3–manifold. Since G is simply-connected, every principal G–bundle over X is trivializable; therefore let us fix a trivialization to simplify notation, which allows us to identify the affine space of connections with Lie algebra valued 1-forms $\mathcal{A}_X=\Omega^1(X;\mathfrak{g})$. Furthermore, the moduli space of flat G–connections on X, denoted by $\mathcal{M}(X)$, can be identified with the moduli space of flat connections in the trivial G–bundle. The Chern-Simons invariant is the map $\mathcal{A}_X\to\mathbf{R}$ given by

$$\mathrm{CS}_X(A)=\int_X\langle A\wedge dA+\frac{1}{3}A\wedge[A\wedge A]\rangle. \tag{2.1}$$

It is not difficult to see that—with our choice of normalization for the inner product on $\mathfrak{g}$—CS_X factors through $\mathcal{G}$ as an $\mathbf{R}/\mathbf{Z}$–valued map. It is also not difficult to see, that the map $\mathcal{G}\to\mathbf{Z}$ given by $\Phi\mapsto\mathrm{CS}_X(\Phi^*A)-\mathrm{CS}_X(A)$ is onto.

Let Σ be a closed oriented surface and consider the space of connections $\mathcal{A}_\Sigma$ in a trivial principal G–bundle over Σ. The symplectic structure on $\mathcal{A}_\Sigma$ is naturally given by

$$\omega(a,b)=-2\int_\Sigma\langle a\wedge b\rangle. \tag{2.2}$$

This gives a (stratified) symplectic structure on the moduli space $\mathcal{M}(\Sigma)$ of flat G–connections on Σ.

In order to view the square root of Reidemeister torsion as a density, we need to identify $H^2(\Sigma_f,d_A)$ with $(H^1(\Sigma_f,d_A))^*$ using Poincaré duality. We define

$$\mathrm{PD}(a)(b)=\int_{\Sigma_f}2\langle a\wedge b,\rangle$$

so that it agrees with the identification induced by $-\omega$ on $\mathcal{A}_\Sigma$ in the context of surface bundles.

Connected components of $\mathcal{M}(\Sigma_f)$ and $|\mathcal{M}(\Sigma)|$

Recall that $r\colon\mathcal{M}(\Sigma_f)'_c\to|\mathcal{M}(\Sigma)|'_c$ is a $|Z(G)|$–sheeted covering map (see [6, Section 7]). We get a complete description of the leading order term of the Witten-Reshetikhin-Turaev invariants in terms of a sum of integrals over the components $\mathcal{M}(\Sigma_f)'_c$ of $\mathcal{M}'(\Sigma_f)$, if every $|\mathcal{M}(\Sigma)|_c$ contains an irreducible representation.

For a chosen diffeomorphism $f\colon\Sigma\to\Sigma$ of order m consider the projection $\pi\colon\Sigma\to\tilde{\Sigma}$ to the quotient surface $\tilde{\Sigma}:=\Sigma/\langle f\rangle$. The components of $|\mathcal{M}(\Sigma)|$ have been studied in [6, Section 6]. Let us first review the main results.

Σ is an m–fold branched cover over $\tilde{\Sigma}$ with branch points $\tilde{p}_1,\dots,\tilde{p}_n$, for which

$$\pi^{-1}(\tilde{p}_i)=\{p_i,f(p_i),\dots,f^{m_i-1}(p_i)\}\quad\text{with}\quad m_i<m.$$

Choose small disjoint closed discs D_i around each p_i, $i=1,\dots,n$, such that $f^j(D_i)$, $j=0,\dots,m_i$, $i=0,\dots,n$, are disjoint. Let Σ' be the complement of the interior of all these discs and $\tilde{\Sigma}':=\Sigma'/\langle f\rangle$.

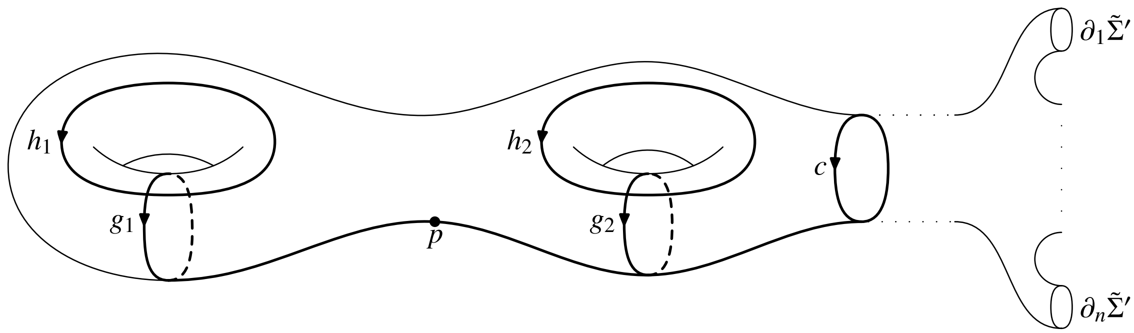

Figure 1: Images of the representation $\rho\colon\ \pi_1(\tilde{\Sigma}') \to G$

The indexing set for the connected components of $|\mathcal{M}'(\Sigma)|$ is shown in [6, Proposition 6.1] to be

$$\Delta := \{(z, c_1, \ldots, c_n) \in Z(G) \times \mathrm{Cl}^n \mid z \in c_i^{l_i}\}/Z(G),$$

where Cl is the set of conjugacy classes of G and $l_i := \frac{m}{m_i}$. We have a surjective map $\Delta \to C$ if $|\mathcal{M}(\Sigma)| = \overline{|\mathcal{M}'(\Sigma)|}$. By [6, Proposition 6.3] this is the case for $G = \mathrm{SU}(n)$. For $c(\delta) := (c_1^{-k_1}, \ldots, c_n^{-k_n})$ let $\mathcal{M}(\tilde{\Sigma}', c(\delta))$ be the moduli space of flat G–connections on $\tilde{\Sigma}'$ with holonomy around $\partial_i\tilde{\Sigma}'$ in $c_i^{k_i}$, $i = 1, \ldots, n$. By [6, Theorem 6.1], a component $|\mathcal{M}'(\Sigma)|_\delta$ can be described as the space $\mathcal{M}''(\tilde{\Sigma}', c(\delta))/Z_\delta$, where $\mathcal{M}''(\tilde{\Sigma}', c(\delta))$ consists of the flat G–connections in $\mathcal{M}'(\tilde{\Sigma}', c(\delta))$, which remain irreducible when pulled back via π.

Proposition 2.1. *Let G be a connected compact Lie group, $\tilde{\Sigma}'$ a genus two surface with one boundary circle $\partial_i\tilde{\Sigma}'$, and $\pi\colon\ \Sigma' \to \tilde{\Sigma}'$ a covering map. Then $\mathcal{M}''(\tilde{\Sigma}', c)$ is nonempty for every $c \in$* Cl.

Proof. Write

$$\pi_1(\tilde{\Sigma}') = \langle x_1, y_1, x_2, y_2\rangle,$$

then the moduli space $\mathcal{M}(\tilde{\Sigma}', c)$, $c \in$ Cl, consists of all conjugacy classes of ρ satisfying

$$\rho([x_1, y_1][x_2, y_2]) \in c.$$

By Auerbach's Generation Theorem [28, Theorem 6.82], we have $G = \overline{\langle g_1', h_1'\rangle}$ for some g_1', h_1'. Choose g_1, h_1 such that $g_1^m = g_1'$ and $h_1^m = h_1$. By Gotô's Commutator Theorem [28, Theorem 6.55], we find $g_2, h_2 \in G$ such that

$$[g_1, h_1][g_2, h_2] \in c.$$

Consider the representation (see Figure 1) determined on the generators by

$$\begin{aligned}\tilde{\rho}\colon\ \pi_1(\tilde{\Sigma}') &\to G\\ x_i &\mapsto g_i\\ y_i &\mapsto h_i.\end{aligned}$$

Clearly, $x_1^m, y_1^m \in \mathrm{im}(\pi\colon\ \pi_1(\Sigma') \to \pi_1(\tilde{\Sigma}'))$ and therefore $g_1', h_1' \in \mathrm{im}(\rho)$. We claim that $\rho := \pi^*\tilde{\rho}$ is irreducible, i.e. $\mathrm{Stab}_G(\rho) = Z(G)$. We automatically have $Z(G) \subset \mathrm{Stab}_G(\rho)$. Let $g \in \mathrm{Stab}_G(\rho)$. Then g is in particular in the centralizer $C_G(\{g_1', h_1'\}) = C_G(\langle g_1', h_1'\rangle)$. Therefore by continuity

$$g \in C_G(\overline{\langle g_1', h_1'\rangle}) = C_G(G) = Z(G).$$

Therefore ρ is irreducible. ◻

A glance at Figure 1 gives the following.

Corollary 2.2. *Let $\tilde{\Sigma}$ be a surface of genus greater than 1. Then $\mathcal{M}''(\tilde{\Sigma}', c(\delta))$ is nonempty for all $\delta \in \Delta$.*

We summarize the discussion of this section.

Theorem 2.3. *If $\tilde{\Sigma}$ is a surface of genus greater than 1 and $G = \mathrm{SU}(N)$, then $\mathcal{M}(\Sigma_f)'_c = r^{-1}(|\mathcal{M}(\Sigma)|'_c)$ is nonempty for every $c \in C$.*

3 The leading order term of the Witten-Reshetikhin-Turaev invariant

Let us now identify the leading order term of the asymptotic expansion (1.3) of the Witten-Reshetikhin-Turaev invariants of finite order mapping tori as an integral of differential geometric terms.

We first consider the framing correction term as defined in [6, Equation (5)]

$$\det(f)^{\alpha} := \mathrm{tr}(\tilde{f} \colon \mathcal{L}^{\alpha}_{D,\sigma} \to \mathcal{L}^{\alpha}_{D,\sigma}),$$

where $\tilde{f}$ is a lift of f to the rigged mapping class group (see [6, Section 2]), which acts on any power, say α, of the determinant line bundle $\mathcal{L}_D$ over Teichmüller space and σ is a point in Teichmüller space preserved by f. For the rest of this paper we denote Σ with the complex structure σ simply by Σ. The framing correction term is obtained by setting $\alpha = \zeta$.

Proposition 3.1. *For a finite order automorphism $f \colon \Sigma \to \Sigma$ of a surface Σ we have*

$$\det(f)^{\alpha} = \exp\left(\sum_{0 \neq \tilde{\omega} \in (-\frac{1}{2}, \frac{1}{2})} -2\pi i \alpha \tilde{\omega}_i \right),$$

where $e^{2\pi i \tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2}, \frac{1}{2})$, are the eigenvalues of the pull-back $f^* \colon H^{1,0}(\Sigma, \bar{\partial}) \to H^{1,0}(\Sigma, \bar{\partial})$.

Proof. Let us identify $H^1(\Sigma, \mathbf{R})$ with $H^{1,0}(\Sigma, \bar{\partial})$ via

$$H^1(\Sigma, \mathbf{R}) \hookrightarrow H^1(\Sigma, \mathbf{C}) \xrightarrow{\mathrm{pr}} H^{1,0}(\Sigma, \bar{\partial}),$$

where pr is the projection to the subspace. We get that the diagram

$$\begin{array}{ccc} H^{1,0}(\Sigma, \mathbf{R}) & \xleftarrow{f^*} & H^{1,0}(\Sigma, \mathbf{R}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(\Sigma, \mathbf{R}) & \xleftarrow{f^*} & H^1(\Sigma, \mathbf{R}) \end{array}$$

commutes. By naturality of Poincaré duality, the diagram

$$\begin{array}{ccc} H^1(\Sigma, \mathbf{R}) & \xleftarrow{f^*} & H^1(\Sigma, \mathbf{R}) \\ \downarrow \mathrm{PD} & & \downarrow \mathrm{PD} \\ H_1(\Sigma, \mathbf{R}) & \xrightarrow{f_*} & H_1(\Sigma, \mathbf{R}) \end{array}$$

commutes. In particular, the eigenvalues of $\mathrm{PD}^{-1}\circ f_*\circ \mathrm{PD}$ and f^* are inverses of each other. In analogy to [6, Section 5] we get that

$$\det(f)^\alpha = \exp\left(\sum_{0\neq\omega\in(-\frac{1}{2},\frac{1}{2})} -2\pi i\alpha\tilde{\omega}_i\right)$$

where $e^{-2\pi i\tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2},\frac{1}{2})$, are the eigenvalues of $\mathrm{PD}^{-1}\circ f_*\circ \mathrm{PD}$, or equivalently, where $e^{2\pi i\tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2},\frac{1}{2})$, are the eigenvalues of f^*. □

We now turn to the contribution from each component of the fixed point variety which contains irreducible connections. Let $c \in C$ with $|\mathcal{M}(\Sigma)|'_c$ nonempty and consider $\omega_c^{d_c}\cap\tau_i(L^c_\cdot(\mathcal{O}_{\mathcal{M}(\Sigma)})) \in H_*(|\mathcal{M}(\Sigma)|'_c)$. In order to give a formula for the top degree term of this element, we need to fix a complex structure on Σ which is preserved by f. This induces the structure of an algebraic projective variety on $\mathcal{M}(\Sigma)$ and hence also on $|\mathcal{M}(\Sigma)|_c$. Algebraic varieties have fundamental classes and we denote the fundamental class of $|\mathcal{M}(\Sigma)|_c$ by $[|\mathcal{M}(\Sigma)|_c]$. From the Lefschetz-Riemann-Roch Theorem in [15, Section 0.6] we have that

$$\tau_i(L^c_\cdot(\mathcal{O}_{\mathcal{M}(\Sigma)})) = \mathrm{Ch}^\cdot(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1}\cap[|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}} \in H_*(|\mathcal{M}(\Sigma)|'_c),$$

where $[|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}}$ is the Todd fundamental class defined in [14] and $\lambda^c_{-1}\mathcal{M}(\Sigma)$ is a certain element in the K-theory of $|\mathcal{M}(\Sigma)|_c$ with complex coefficients also defined in [14] (see also [6, Section 8] for a computation of this element in the case at hand). We recall that the highest degree term of $[|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}}$ equals $[|\mathcal{M}(\Sigma)|_c]$. The top degree is $d_c = \dim_{\mathbf{C}}|\mathcal{M}(\Sigma)|'_c$, so the contribution from $\mathrm{Ch}(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1}$ to top degree term of $\omega_c^{d_c}\cap\tau_i(L^c_\cdot(\mathcal{O}_{\mathcal{M}(\Sigma)}))$ will simply be its degree zero part. Following [6, Section 8], we have $\lambda^c_{-1}\mathcal{M}(\Sigma) = L^\cdot\left(\sum(-1)^i[\Lambda^i\mathcal{N}_c^*]\right)$, where $\mathcal{N}_c^*$ is the conormal sheaf to $|\mathcal{M}(\Sigma)|_c$ (thought of as an f-equivariant sheaf) and $L^\cdot$ is the homomorphism determined by $L^\cdot(E_a) = [E_a]\otimes a \in K^0(|\mathcal{M}(\Sigma)|_c)\otimes\mathbf{C}$ for an a–eigensheaf E_a of f. Since f is finite order, $\mathcal{N}_c^*$ splits as the direct sum $\mathcal{N}_c^* = \bigoplus_j \mathcal{N}^*_{c,j}$ of a_j–eigensheaves $\mathcal{N}^*_{c,j}$ of f, where $a_j = e^{2\pi i\frac{j}{m}}$ and $j = 1,\ldots,m-1$, we then have

$$L^\cdot(\mathcal{N}_c^*) = \sum_{j=1}^{m-1}\mathcal{N}^*_{c,j}\otimes a_j.$$

Then the degree zero part of $\mathrm{Ch}(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1}$ is

$$\lambda_{-1}(\mathrm{Rank}\,\mathcal{N}_c^*)^{-1} = \prod_{i=1}^{m-1}(1-a_i)^{-r_i} = \frac{1}{\det(1-df|_{\mathcal{N}_c^*})}, \quad r_i = \mathrm{Rank}\,\mathcal{N}^*_{c,i}.$$

This shows the following.

Proposition 3.2.

$$\omega^{d_c}\cup\mathrm{Ch}^\cdot(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1} = \frac{\omega^{d_c}}{\det(1-df|_{\mathcal{N}_c^*})}.$$

Now the expression of the leading order term of the Witten-Reshetikhin-Turaev invariants for each $c \in C$ in differential geometric terms is an immediate consequence of Proposition 3.1 and 3.2.

Theorem 3.3. *Let $|\mathcal{M}(\Sigma)|_c$ be a connected component of $|\mathcal{M}(\Sigma)|$ containing irreducible connections, then*

$$k^{-d_c}\det(f)^{\frac{1}{2}\zeta}e^{2\pi ik\,\mathrm{CS}_{\Sigma_f}(c)}\frac{1}{d_c!}(\exp(k\omega_c)\cap\tau.(L^c_.(\mathcal{O}_{\mathcal{M}(\Sigma)})))$$
$$=\exp\left(i\pi\zeta\sum_{0\neq\tilde{\omega}_j\in(-\frac{1}{2},\frac{1}{2})}\tilde{\omega}_j\right)e^{2\pi ik\,\mathrm{CS}_{\Sigma_f}(c)}\int_{a\in|\mathcal{M}(\Sigma)|'_c}\frac{1}{d_c!}\frac{(\omega_c)^{d_c}_{[a]}}{\det(1-df|_{\mathcal{N}^*_{[a]}})}+O(\frac{1}{k}),$$

where $e^{2\pi i\tilde{\omega}_j}$, $\tilde{\omega}_j\in[-\frac{1}{2},\frac{1}{2})$, are the eigenvalues of $f^*\colon\ H^{1,0}(\Sigma,\bar{\partial})\to H^{1,0}(\Sigma,\bar{\partial})$ and $\mathcal{N}^*_{[a]}\coloneqq\mathcal{N}^*_{c,[a]}$ is the fiber over $[a]\in|\mathcal{M}(\Sigma)|_c$ of the conormal sheaf $\mathcal{N}^*_c$ of $|\mathcal{M}(\Sigma)|_c$.

Notice, that a connected component may contain more than one irreducible component (in the Zariski topology). These components can be of different dimensions, but only the components of dimension d_c will contribute to the integral.

4 Reidemeister torsion

In this section we will summarize some basic facts about Reidemeister torsion, which is a term in the asymptotic expansion of the Witten-Reshetikin-Turaev invariants. To keep the proofs less technical we will consider it as a density. Note that, it is possible and could be interesting to lead this discussion in the context of sign-determined Reidemeister torsion as defined in [45] (see for example [19]).

Torsion of a complex

The notation has been adapted from [22] and [29]. Let F be either $\mathbf{R}$ or $\mathbf{C}$. Let L be a 1–dimensional vector space over F. We will denote by L^{-1} the dual of a complex line L and by $l^{-1}\in L^{-1}$ the inverse of l given by $l^{-1}(l)=1$. By a density on L we mean a function

$$|\cdot|\colon\ L\to\mathbf{R}\quad\text{such that}\quad|c\omega|=|c||\omega|\text{ for }c\in F,\omega\in L.$$

We denote the densities on L by $|L^*|$. For an n–dimensional vector space V over F we let $\det V=\Lambda^nV$ and define a density on V to be an element of $|\det V^*|$. A density on a manifold M is a section of the density bundle $|\det T^*M|$. Every volume form ω on V gives a density $|\omega|$. If we choose an orientation, we can identify densities with volume forms.

Definition 4.1. Given a finite cochain complex $(C^\bullet,d)$ of finite-dimensional complex vector spaces, we denote

$$\det C^\bullet=\bigotimes_{j=0}^{n}(\det C^j)^{(-1)^j}.$$

Then the torsion

$$\tau_{C^\bullet,d}\in\left|(\det C^\bullet)^{-1}\otimes(\det H^\bullet(C,d))\right|$$

is given by

$$\tau_{C_\bullet,d}=\bigotimes_{j=0}^{n}\left(\left|ds^{j-1}\wedge s^j\wedge\hat{h}^j\right|^{(-1)^{j+1}}\otimes\left|h^j\right|^{(-1)^j}\right),$$

after an arbitrary choice of

- $s^j \in \bigwedge^{k_j} C^j$ with $ds^j \neq 0$, where k_j is the rank of $d\colon\ C^j \to C^{j+1}$,
- $h^j \in \det H^j(C)$ non-zero and
- a lift $\hat{h}^j \in \bigwedge^{l_j} C^j$ of h^j, where $l_j = \dim H^j(C^\bullet, d)$.

We will use the *Multiplicativity Lemma* as our main computational tool.

Lemma 4.2. *Let*

$$0 \to C_1^\bullet \xrightarrow{\nu^\bullet} C_2^\bullet \xrightarrow{\mu^\bullet} C_3^\bullet \to 0 \tag{4.1}$$

be a short exact sequence of cochain complexes, choose compatible volume elements $\omega_i^\bullet$ in $C_i^\bullet$—that is, $\omega_2^j = \nu^*(\omega_1^j) \wedge \omega'^j$ with $\mu^*(\omega'^j) = \omega_3^j$ for $\omega_i^j \in \det C_i^j$—, and let $H^\bullet$ be the long exact sequence associated to (4.1)*. Then*

$$\tau_{C_2^\bullet}(\omega_2) = \tau_{C_1^\bullet}(\omega_1) \cdot \tau_{C_3^\bullet}(\omega_3) \cdot \tau_{H^\bullet},$$

where $\omega_i = \prod (\omega_i^j)^{(-1)^j}$.

For a proof see [22, Corollary 1.20] or [33, Theorem 3.2].

The Wang exact sequence

In order to compute the Reidemeister torsion, we will employ the Wang exact sequence. Due to some confusion in the literature and since we are dealing with the cohomological formulation of Reidemeister torsion, we would like to carefully review and state this useful tool and our main application.

Let $(C^\bullet, d) = \bigoplus_{i=0}^n (C^i, d^i)$ be a chain complex and $f^\bullet = \{f^i\colon\ (C^i, d^i) \to (C^i, d^i)\}$ be a chain map. Then the algebraic mapping torus $(T^\bullet(f^\bullet), d_f)$ is the cochain complex with $T^i(f) := C^i \oplus C^{i-1}$ and boundary operator $d_f^i(x, y) := (d^i(x), -d^{i-1}(y) + \mu^i(x))$, where $\mu^\bullet = \mathrm{Id}^\bullet - f^\bullet\colon\ C^\bullet \to C^\bullet$. It is not difficult to confirm, that we get a short exact sequence

$$0 \to (C^{\bullet-1}, -d) \xrightarrow{\nu^{\bullet-1}} (T^\bullet(f^\bullet), d_f) \xrightarrow{\pi^\bullet} (C^\bullet, d) \to 0 \tag{4.2}$$

of chain complexes, where $\nu^\bullet$ is the inclusion into first summand and $\pi^\bullet$ is the projection onto the second summand. Observe that $(C^\bullet, -d)$ and $(C^\bullet, d)$ are isomorphic chain complexes and that $H^i(C^{\bullet-1}, d) = H^{i-1}(C^\bullet, d)$. This yields a long exact sequence $H_W^\bullet$ by the name Wang exact sequence

$$\cdots \to H_W^i(C^\bullet) \xrightarrow{\mu^i} H_W^i(C^\bullet) \xrightarrow{\nu^i} H_W^{i+1}(T^\bullet(f)) \xrightarrow{\pi^{i+1}} H_W^{i+1}(C^\bullet) \to \cdots .$$

It is easy to check, that the boundary map is indeed induced by $\mu^\bullet$. Together with the Multiplicativity Lemma 4.2 we get the following useful result.

Corollary 4.3. *Let $\omega^j \in \det C^j$ be a volume form for all j and let $\omega := \prod (\omega_i^j)^{(-1)^j}$. Then we have $\tau_{C^\bullet(M)}(\omega) = \tau_{C^{\bullet-1}}(\omega^{-1})$ and therefore for $\omega_f = \nu^*(\omega) \wedge \omega'$ with $\pi^*(\omega') = \omega^{-1}$*

$$\tau_{C^\bullet(M_f)}(\omega_f) = \tau_{H_W^\bullet}.$$

In particular, this is independent of the choice of ω.

Reidemeister torsion

If each C^j comes equipped with a volume form, then the torsion is an element of $|\det H^\bullet(C^\bullet, d)|$. If X is a smooth manifold, W an inner product space and $\rho\colon\ \pi \to \mathrm{GL}(W)$ a representation of $\pi = \pi_1(X)$, then we can consider the cellular chain complex with local coefficients in W twisted by ρ given by

$$C^\bullet(X, W_\rho) = \mathrm{Hom}_{\mathbf{Z}\pi}(C_\bullet(\widetilde{X}), W),$$

where $\tilde{X}$ is the universal cover of X. Note that $C_\bullet(\widetilde{X})$ has a natural inner product, by which the cells are orthonormal. If furthermore ρ preserves the inner product on W, then $C^\bullet(X, W_\rho)$ carries an induced inner product and therefore volume forms. Then the *Reidemeister torsion of X* is a density given by

$$\tau_X(W_\rho) = \tau_{(C^\bullet(X,W_\rho),d)} \in \left|\det H^\bullet(C^\bullet, d)\right|$$

and is independent of the choice of the cell decomposition of X. The use of cochain complexes rather than chain complexes in defining Reidemeister torsion simplifies the notation in our arguments considerably when interpreting the torsion in terms of twisted de Rham cohomology. Even though we need to choose a multiple of the killing form as a metric on $\mathfrak{g}$ in order to identify Reidemeister torsion defined through chains and Reidemeister torsion defined through cochains, it is not difficult to see that the identification is independent of this choice. If A is a G-connection and the representation $\mathrm{ad} \circ \mathrm{hol}(A) = \rho$ is associated to a flat G–connection A via the adjoint representation

$$\mathrm{ad}\colon\ G \to \mathrm{O}(\mathfrak{g}^{\mathbf{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbf{C}}),$$

which takes values in the orthogonal group with respect to the killing form on $\mathfrak{g}$, we define

$$\tau_X(A) \coloneqq \tau_X(\mathfrak{g}_\rho).$$

Note that we can also consider the complexified adjoint representation

$$\mathrm{Ad}\colon\ G \to \mathrm{U}(\mathfrak{g}^{\mathbf{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbf{C}}),$$

where we have extended the killing form to a sesquilinear form on $\mathfrak{g}^{\mathbf{C}}$. We then also have

$$\tau_X(A) = \tau_X(\mathfrak{g}^{\mathbf{C}}_{\mathrm{Ad}\circ\mathrm{hol}(A)}).$$

5 Reidemeister torsion of mapping tori

We will see in this section that for $c \in C$

$$\int_{\mathcal{M}(\Sigma_f)'_c} \tau_{\Sigma_f}(A)^{\frac{1}{2}} = |Z(G)| \int_{|\mathcal{M}(\Sigma)|'_c} \frac{|\omega_c^{d_c}|}{|\det(1 - df|_{\mathcal{N}^*_{[a]}})|}, \tag{5.1}$$

where $\mathcal{N}^*_{[a]} = \mathcal{N}^*_{c,[a]}$ and the conormal sheaf $\mathcal{N}^*_c$ is the dual of the normal sheaf

$$\mathcal{N}_c = \frac{T\mathcal{M}(\Sigma)|_{|\mathcal{M}(\Sigma)|_c}}{T|\mathcal{M}(\Sigma)|_c}.$$

In the above equation we identified $H^2(\Sigma_f, d_A)$ with $H^1(\Sigma_f, d_A)^*$ via PD. In fact, we will even show an equality for irreducible components on the level of densities. The factor $|Z(G)|$ then stems from the fact that $r\colon\ \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ is a $|Z(G)|$–sheeted covering map (see [6, Section 7]).

Notice, that $T|\mathcal{M}(\Sigma)|$ is simply the kernel of the bundle map

$$1 - df^*\colon\ T\mathcal{M}(\Sigma) \to T\mathcal{M}(\Sigma)$$

and is therefore isomorphic to the bundle of 1–eigenspaces of

$$f^*\colon\ T_{[a]}\mathcal{M}(\Sigma) \to T_{[a]}\mathcal{M}(\Sigma),\ \text{where } [a] \in |\mathcal{M}(\Sigma)|_c.$$

We can fix an isomorphism $H^{0,1}(\Sigma, \bar{\partial}_a) \cong T_{[a]}\mathcal{M}(\Sigma)$ to get an equivalent statement for $H^{0,1}(\Sigma, \bar{\partial}_a)$. Also, note that the eigenvalues of $1 - df^*\colon\ \mathcal{N}_{[a]} \to \mathcal{N}_{[a]}$ and of $1 - df\colon\ \mathcal{N}^*_{[a]} \to \mathcal{N}^*_{[a]}$ are the same, where df is short for $(df^*)^*$.

General mapping tori

Consider a CW complex M and an orientation preserving simplicial homeomorphism $f\colon\ M \to M$. The torsion for the mapping torus M_f of f has been computed in [25, Proposition 3] (see also [20, Section 6.2] and [34, Example 2.17]) only when M_f is an acyclic CW complex. In this section we will give a generalization of the computation for mapping tori to the non-acyclic case. The computations in [19] of sign-determined Reidemeister torsion for fibered knots for the local coefficient systems $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbf{C})$ use the same basic tools, namely the Wang exact sequence and the Multiplicity Lemma.

Let $\rho\colon\ \pi_1 M_f \to G$ be a G–representation of $\pi_1 M_f$ acting on $\mathfrak{g}$ by the adjoint representation. If we denote by $C_g\colon\ G \to G$ the conjugation action, then ρ is determined by a representation $\rho'\colon\ \pi_M \to G$ satisfying $\rho' = C_g \circ (f^*\rho')$ for some $g \in G$. The choice of g induces a chain map $f^\bullet = f^\bullet_g\colon\ C^\bullet(M, \mathfrak{g}_{\rho'}) \to C^\bullet(M, \mathfrak{g}_{\rho'})$. It is easy to check that the algebraic mapping torus $T(f^\bullet)$ is isomorphic—in fact, isometric—to $C^\bullet(M_f, \mathfrak{g}_\rho)$ induced by the cell decomposition of S^1 into two cells and $C^\bullet(M)$.

In this section let us from now on drop the coefficients in the cohomology and cochain groups entirely with the understanding that we consider coefficients twisted by representations compatible with the restriction. Instead of μ^i and π^i we will sometimes use the more familiar notation μ^* and π^*, when the grading is clear. Consider the diagram in cohomology induced by the Wang exact sequence and a positive multiple Θ of Poincaré duality on M

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{n-i}(M_f) & \xrightarrow{\pi^*} & H^{n-i}(M) & \xrightarrow{\mu^*} & H^{n-i}(M) & \xrightarrow{\nu^*} & H^{n-i+1}(M_f) & \longrightarrow \cdots \\
& \Theta \vdots & & \Theta\downarrow & & \Theta\downarrow & & \Theta\vdots & \\
\cdots \longrightarrow & \big(H^{i+1}(M_f)\big)^* & \xrightarrow{\nu} & \big(H^i(M)\big)^* & \xrightarrow{\mu'} & \big(H^i(M)\big)^* & \xrightarrow{\pi} & \big(H^i(M_f)\big)^* & \longrightarrow \cdots,
\end{array}$$

where $\mu' = 1 - f^{-1}$ and we write $f^{-1} = ((f^{-1})^*)^*$. It is easy to check that the middle square commutes. Furthermore, since

$$\mu' = (1 - f) \circ (-f^{-1}) = \mu \circ (-f^{-1}) = (-f^{-1}) \circ \mu,$$

and $(-f^{-1})$ is an isomorphism, the exactness of the above sequence implies the exactness of the lower sequence. We can define isomorphisms $\Theta\colon\ \operatorname{im}\nu^* \to \operatorname{im}\pi$ so that the above diagram commutes, and we can extend these maps arbitrarily to isomorphisms $\Theta\colon\ H^{n-i}(M_f) \to H^{i+1}(M_f)$. We extend Θ to the exterior algebra by setting $\Theta(a \wedge b) = \Theta(a) \wedge \Theta(b)$.

Before we can compute Reidemeister torsion of a general mapping torus, we need a few technical facts. For finite order mapping tori the situation simplifies considerably and the result is more pleasing.

Lemma 5.1. *Let* $0 \neq h^{i+1} \in \det(\operatorname{im}(\pi^i))$. *Then we can find* $h^i_+ \wedge h^i_- \in \det(H^i(M))$ *such that* $\nu^*(h^i_-) = h^{i+1}$ *and* $\mu^*(h^i_+) \wedge h^i_- \neq 0$.

Proof. Let $h^i_+ \wedge h^i_- \in \det(H^i(M))$ such that $\nu^*(h^i_-) = h^{i+1}$. If $\mu^*(h^i_+) \wedge h^i_- = 0$, then let $k^i \in \Lambda(H^i(M))$ with $0 \neq \mu^*(k^i) \wedge h^i_- \in \det(H^i(M))$. Now choose $\lambda > 0$ small enough that for $\tilde{h}^i_+ := h^i_+ + \lambda k^i$

$$\tilde{h}^i_+ \wedge h^i_- \neq 0.$$

Then we also have

$$\mu(\tilde{h}^i_+) \wedge h^i_- = \lambda\mu(k^i) \wedge h^i_- \neq 0. \qquad \square$$

Proposition 5.2. *Let M_f be a mapping torus of a homeomorphism $f\colon M \to M$,* $\dim M = n$*. Then we may choose $h^i \in \Lambda(H^i(M_f))$ and $h^i_-, h^i_+ \in \Lambda(H^i(M))$ for all i with*

$$\begin{aligned} &0 \neq \nu^*(h^{i-1}_-) \wedge h^i \in \det(H^i(M_f)),\\ &0 \neq \pi^*(h^i) \wedge h^i_+ \in \det(H^i(M)), \\ \textit{and}\quad &0 \neq \mu^*(h^i_+) \wedge h^i_- \in \det(H^i(M)). \end{aligned} \tag{5.2}$$

so that they satisfy

$$|\Theta(\nu^*(h^{n-i}_-))(h^i)| = 1 \quad \textit{and} \quad |h^i_- \wedge h^i_+| = |\pi^*(h^i) \wedge h^i_+|, \tag{5.3}$$

Furthermore, the Reidemeister torsion is

$$\tau(M_f) = \left|\bigotimes_{i=0}^{n+1}(\nu^*(h^{i-1}_-) \wedge h^i)^{(-1)^i}\right| \prod_{i=0}^{n} |\det(\tilde{\mu}^i)|^{(-1)^{i+1}}.$$

where $\tilde{\mu}^i$ is determined by

$$\tilde{\mu}^i(h^i_- \wedge h^i_+) = h^i_- \wedge \mu^*(h^i_+).$$

Proof. The Wang exact sequence and Lemma 5.1 allow us to choose $h^i \in \Lambda(H^i(M_f))$ and $h^i_-, h^i_+ \in \Lambda(H^i(M))$ for all i with

$$\begin{aligned} &0 \neq \nu^*(h^{i-1}_-) \wedge h^i \in \det(H^i(M_f)),\\ &0 \neq \pi^*(h^i) \wedge h^i_+ \in \det(H^i(M)),\\ &0 \neq \mu^*(h^i_+) \wedge h^i_- \in \det(H^i(M)),\\ \text{and}\quad &0 \neq h^i_+ \wedge h^i_-. \end{aligned}$$

By rescaling we can assume $|\Theta(\nu^*(h^{n-i}_-))(h^i)| = 1$. Notice that, if h^i and h^{n-i}_- satisfy this condition, so do λh^i and $\frac{1}{\lambda}h^{n-i}_-$ for $\lambda > 0$. By choosing λ appropriately we may therefore assume that

$$|h^i_- \wedge h^i_+| = |\pi^*(h^i) \wedge h^i_+|.$$

Then

$$|\det\tilde{\mu}^i| \cdot |\pi^*(h^i) \wedge h^i_+| = |\det\tilde{\mu}^i| \cdot |h^i_- \wedge h^i_+| = |\tilde{\mu}^i(h^i_- \wedge h^i_+)| = |h^i_- \wedge \mu^*(h^i_+)|,$$

and therefore

$$\bigotimes_{i=0}^{n} |\pi^*(h^i) \wedge h^i_+|^{(-1)^i} \bigotimes_{i=0}^{n} |\mu^*(h^i_+) \wedge h^i_-|^{(-1)^{i+1}} = |\det\tilde{\mu}^i|^{i+1}.$$

By Corollary 4.3, the theorem follows. $\square$

Note, that even though the system of equations (5.3) seems to be overdetermined, half of them are equivalent to the other half, since the above diagram is commutative. Also observe that, even though our result seems to depend on $\Theta\colon H^{n-i}(M) \to H^{i+1}(M)$, we can use a different multiple of Poincaré duality without changing Reidemeister torsion: This can be easily verified by the sceptical reader by considering the cases n odd and n even separately.

Finite order mapping tori

We can enhance Theorem 5.2 and make it more useful for finite order mapping tori, if we put some restrictions on μ^*. Before we do that, let us state and prove a simple fact from linear algebra.

Lemma 5.3. *For a linear map $T\colon\ V \to V$ between finite-dimensional vector spaces, the following are equivalent*

1. *$\bar{T}\colon\ V/\ker T \to V/\ker T$ induced by T is an isomorphism.*

2. *$\hat{T} = T|_{\operatorname{im} T}\colon\ \operatorname{im} T \to \operatorname{im} T$ is an isomorphism.*

Furthermore $\det \bar{T} = \det \hat{T}$.

Proof. Clearly, $\bar{T}$ is an isomorphism if and only if $\operatorname{im} T \hookrightarrow V \to V/\ker T$ is an isomorphism. The last statement is equivalent to $\operatorname{im} T \cap \ker T = 0$. This implies that $\hat{T}\colon\ \operatorname{im} T \to \operatorname{im} T$ is an isomorphism. On the other hand, if $0 \neq v \in \operatorname{im} T \cap \ker T$, then $\hat{T}$ is not injective, because $T(v) = 0$.

Furthermore, if $\{b_i\}_i$ is a basis of $\operatorname{im} T$, then $\{[b_i]\}_i$ is a basis of $V/\ker T$. It follows immediately, that $\det \bar{T} = \det \hat{T}$. □

Proposition 5.4. *Assume that*

$$\bar{\mu}^i\colon\ H^i(M)/\ker(\mu^i) \to H^i(M)/\ker(\mu^i)$$

is an isomorphism. Then we can choose h^i with $0 \neq \pi^*(h^i) \in \det(\operatorname{im} \pi^i)$ satisfying

$$\Theta(\nu^*(\pi^*(h^{n-i})))(h^i) = 1. \tag{5.4}$$

Furthermore, we have $\det(\bar{\mu}^i) = \det(\tilde{\mu}^i)$, where $\tilde{\mu}^i$ is the map from Theorem 5.2. In particular, if $\bar{\mu}^i$ is an isomorphism for all i—for example for finite order mapping tori—we have

$$\tau(M_f) = \left|\bigotimes_{i=0}^{n+1}(\nu^*(\pi^*(h^{i-1})) \wedge h^i)^{(-1)^i}\right| \prod_{i=0}^{n} |\det(\bar{\mu}^i)|^{(-1)^{i+1}}.$$

Proof. Suppose that $\bar{\mu}^i$ is an isomorphism. Choose h^i with $0 \neq \pi^*(h^i) \in \det(\operatorname{im} \pi^i)$. In view of Lemma 5.3 we can find $h^i_+ \in \det \operatorname{im} \mu^i$ with $0 \neq \pi^*(h^i) \wedge h^i_+ \in \det H^i(M)$. Since $h^i_+ \in \det \ker \nu^i$, we deduce $0 \neq \nu^* \circ \pi^*(h^i) \in \det \operatorname{im} \nu^i$, which allows us to rescale h^i so that it satisfies (5.4) above. If we set $h^i_- := \pi^*(h^i)$, it is straightforward to see that (5.3) is satisfied and that

$$\det \tilde{\mu}^i = \det \hat{\mu}^i.$$ □

Finite order mapping tori of surfaces

We will now focus on the case of a mapping torus Σ_f of finite order for a closed surface Σ. The goal of this section is to identify integral of the square root of Reidemeister torsion with the leading order term in formula (1.3) as predicted by the semiclassical approximation of the path integral. To this end, we will only do this for the case, when $c \in C$ contains an open, dense submanifold $|\mathcal{M}(\Sigma)|'_c$ of irreducible connections of $|\mathcal{M}(\Sigma)|$. More specifically, we will establish an identification on the level of densities for $|\mathcal{M}(\Sigma)|'_c$. Notice, that while the square root of Reidemeister torsion is a density on a submanifold of the irreducible connections of $\mathcal{M}(\Sigma_f)$, ω^{d_c} is a density on top-dimensional component of $|\mathcal{M}(\Sigma)|'_c$.

Therefore the density on $|\mathcal{M}(\Sigma)|$ needs to be pulled back to a density on $\mathcal{M}(\Sigma)$ via the natural restriction and $|Z(G)|$–sheeted covering map $r\colon\ \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ before we can relate it to Reidemeister torsion. We also need to point out, that by treating Reidemeister torsion as a density, we chose to identify $H^2(\Sigma_f, d_A)$ with $(H^1(\Sigma_f, d_A))^*$ for $A \in \mathcal{A}_{\Sigma_f}$ via PD.

Before we prove the main theorem, we would like to mention the following simple fact.

Lemma 5.5. *Let (V^{2n}, ω) be a symplectic vector space. We can identify V with V^* by*

$$\Theta(v)(w) := -\omega(v, w)$$

and extend this map to the exterior algebra by $\Theta(v \wedge w) := \Theta(v) \wedge \Theta(w)$. Then the volume form $\mathrm{vol} = \frac{1}{n!}\omega^n \in \det V^*$ *on V satisfies*

$$\Theta(\mathrm{vol}^{-1})(\mathrm{vol}^{-1}) = 1,$$

where $\mathrm{vol}^{-1} \in \det V$ *is given by* $\mathrm{vol}(\mathrm{vol}^{-1}) = 1$.

Proof. Form a symplectic basis $\{a_i, b_i\}_{i=1,\dots,n}$ of V, that is, $\omega(a_i, b_j) = -\omega(b_j, a_i) = \delta_{ij}$. Then we have

$$\omega = -\sum_{i=1}^{n} \Theta(b_i) \wedge \Theta(a_i) = \sum_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i).$$

Then

$$\mathrm{vol} = \frac{1}{n!}\omega^n = \frac{1}{n!} \bigwedge^{n} \sum_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i) = \bigwedge_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i)$$

as well as

$$vol^{-1} = (-1)^n \bigwedge_{i=1}^{n} b_i \wedge a_i = \bigwedge_{i=1}^{n} a_i \wedge b_i.$$

Therefore we get the desired equation

$$\Theta(\mathrm{vol}^{-1})(\mathrm{vol}^{-1}) = \left(\bigwedge_{i=1}^{n}(\Theta(a_i) \wedge \Theta(b_i))\right)\mathrm{vol}^{-1} = \mathrm{vol}(\mathrm{vol}^{-1}) = 1. \qquad \square$$

Theorem 5.6. *Let A be an irreducible flat connection on Σ_f such that $a := r(A)$ is irreducible on Σ and $c \subset |\mathcal{M}(\Sigma)|$ is a connected component containing a. Let ω be the usual symplectic form on $H^1(\Sigma, d_a)$ given by* (2.2) *and identify $H^2(\Sigma_f, d_A)$ with $H^1(\Sigma_f, d_A)^*$ via* PD. *Then we have*

$$\tau_{\Sigma_f}(A)^{\frac{1}{2}} = \frac{1}{d_c!} \frac{|r^*(\omega_c)^{d_c}|}{\sqrt{|\det(1 - f^1)|}},$$

where $d_c = \frac{1}{2}(\dim_{\mathbf{R}} H^1(\Sigma_f, d_A) - \dim_{\mathbf{R}} H^0(\Sigma_f, d_A))$ and the restriction ω_c of ω to $\ker(1 - f^1)$ *is a symplectic form on* $\ker(1 - f^1)$.

Proof. The Reidemeister torsion

$$\tau_{\Sigma_f}(A) \in \det H^0(\Sigma_f, d_A) \otimes \det H^1(\Sigma_f, d_A)^* \otimes \det H^2(\Sigma_f, d_A) \otimes \det H^3(\Sigma_f, d_A)^*$$

has been computed in Theorem 5.4. To this end, we are only interested in A irreducible. Then we have $H^0(\Sigma_f, d_A) = H^3(\Sigma_f, d_A) = 0$ and possibly connected components $c \subset |\mathcal{M}(\Sigma)|$, which are not just isolated points. This is in contrast to [29, Proposition 5.6], where f has isolated fixed points on $\mathcal{M}(\Sigma)$.

Furthermore, PD identifies $H^2(\Sigma_f, d_A)$ with the dual of $H^1(\Sigma_f, d_A)$ and $\dim H^1(\Sigma_f, d_A) = \dim \mathcal{M}(\Sigma_f)_c = \dim |\mathcal{M}(\Sigma)|_c$. In summary we get

$$0 \neq \sqrt{\tau_{\Sigma_f}(A)} \in |\det H^1(\Sigma_f, d_A)^*|.$$

Since we also assume irreducibility of $a = r(A)$, $\ker(\pi^1) = \mathrm{im}(\nu^0) = 0$. Furthermore,

$$0 \neq \omega_c^{d_c} \in \det(E_1(f^1))^* = \det(\ker(1 - f^1))^* = \det(\ker(\mu^1))^* = \det(\mathrm{im}(\pi^1))^*.$$

Since π^1 is injective, we can define an element $h^1 \in H^1(\Sigma)$ by requiring

$$\pi^*(h^1) = (\omega_c^{d_c})^{-1} \in \det(\mathrm{im}(\pi^1)).$$

All that is left to complete the proof of the theorem is that this choice of h^1 indeed satisfies condition (5.4). Since $H^2(\Sigma, d_a) = 0$ we have $\det(\mathrm{im}(\nu^1)) = \det(H^2(\Sigma_f, d_A))$. Since the map $\bar{\mu}^1$ from Theorem 5.4 is an isomorphism and $0 \neq \pi^*(h^1) \in \ker \mu^1$, we have

$$0 \neq \nu^*(\pi^*(h^1)) \in \det(H^2(\Sigma_f, d_A)).$$

We would like to apply Theorem 5.4. Since we chose PD to identify $H^2(\Sigma_f, d_A) = (H^1(\Sigma_f, d_A))^*$ we need to check that $\Theta = \mathrm{PD}$ indeed satisfies condition (5.4). We see, that condition (5.4) is equivalent to

$$\mathrm{PD}((\omega_c^{d_c})^{-1})((\omega_c^{d_c})^{-1}) = \Theta(\pi^*(h^1))(\pi^*(h^1)) = \pi(\Theta(\pi^*(h^1))(h^1) = \Theta(\nu^*(\pi^*(h^1)))(h^1) = 1,$$

which is satisfied by Lemma 5.5. □

Geometrically, $T_{[a]}\mathcal{M}(\Sigma) \cong H^{0,1}(\Sigma, \bar{\partial}_a)$, and therefore $|\det(1 - f^1)| = |\det(1 - df_{\mathcal{N}^*_{r(A)}})|^2$, where we again understand df as $(d(f^*))^*$.

Theorem 5.7. *Let A be an irreducible flat connection on Σ_f such that $r(A)$ is irreducible on Σ. If we identify densities with volume forms using the orientation induced by $r^*(\omega_c)_A^{d_c}$, we have*

$$\sqrt{\tau_{\Sigma_f}(A)} = \frac{1}{d_c!} \frac{r^*(\omega_c)_A^{d_c}}{|\det(1 - df|_{\mathcal{N}^*_{r(A)}})|}.$$

This shows that over the moduli space of irreducible flat connections A with $r(A)$ irreducible we indeed have the identity (5.1).

6 The ρ–invariant

Another classical topological invariant, which appears in the expansion of the Witten-Reshetikin-Turaev invariants, is the ρ–invariant. We will briefly review the definition for 3–manifolds in the context of the adjoint representation and relate it to the original definition using the defining representation before we state the result from [18], which will be relevant for us.

The Definition

For a formally self-adjoint, elliptic differential operator D of first order, acting on sections of a vector bundle over a closed manifold X, one defines the η–function

$$\eta(D,s) := \sum_{0\neq\lambda\in\mathrm{Spec}(D)} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}, \quad \mathrm{Re}(s) \text{ large.} \tag{6.1}$$

The function $\eta(D,s)$ admits a meromorphic continuation to the whole s-plane with no pole at the origin. Then $\eta(D) := \eta(D,0)$ is called the η–invariant of D.

As a special case, let G be a compact, simple, simply-connected Lie group and A a G–connection on a Riemannian 3–manifold X. Then the odd signature operator coupled to A is the formally self-adjoint, elliptic, first order differential operator

$$\begin{aligned} D_A\colon\ \Omega^0(X;\mathfrak{g})\oplus\Omega^1(X;\mathfrak{g}) &\longrightarrow \Omega^0(X;\mathfrak{g})\oplus\Omega^1(X;\mathfrak{g}) \\ (\alpha,\beta) &\longmapsto (d_A^*\beta, d_A\alpha + *d_A\beta), \end{aligned} \tag{6.2}$$

where $d_A\colon\ \Omega^p(X;\mathfrak{g}) \to \Omega^{p+1}(X;\mathfrak{g})$ is the covariant derivative associated to A and G acts on $\mathfrak{g}$ via the adjoint action. If A is flat, the ρ–invariant is given by

$$\rho_A(X) := \eta(D_A) - \eta(D_\theta), \tag{6.3}$$

where θ is the trivial connection. The ρ–invariant is metric-independent and gauge-invariant. We write $\rho_{\mathrm{hol}(A)} = \rho_A$, where the representation $\mathrm{hol}(A)\colon\ \pi_1 X \to G$ is the holonomy of A.

In the original definition [10] by Atiyah, Patodi and Singer, their ρ–invariant has been similarly defined for a $\mathrm{U}(n)$–representation, where $\mathrm{U}(n)$ acts on $\mathbf{C}^n$ by the defining representation. We will briefly describe its relationship to our definition of the ρ–invariant in (6.3). With respect to an ad-invariant metric on $\mathfrak{g}$—for example the killing form—on $\mathfrak{g}$, the adjoint representation takes values in the orthogonal endomorphisms of $\mathfrak{g}$

$$\mathrm{ad}\colon\ G \to \mathrm{SO}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g}).$$

We can consider the complexified adjoint representation

$$\mathrm{Ad}\colon\ G \to \mathrm{SU}(\mathfrak{g}^{\mathbf{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbf{C}}).$$

Then $\rho_{\mathrm{hol}(A)}$ is equal to the Atiyah-Patodi-Singer ρ– invariant of $\mathrm{Ad}\circ\mathrm{hol}(A)$.

The Rho–invariant of finite order mapping tori

Let Σ be a surface and P a principal G–bundle. In order to make use of the results in [18], we consider the bundle $\mathrm{Ad}\,P$ associated to the complexified adjoint representation, which is a Hermitian vector bundle of rank $\dim G$.

The chirality operator τ_Σ on $\Omega^p(\Sigma)$ is given by

$$\tau_\Sigma = (-1)^{\frac{p(p-1)}{2}+2p} i *_p,$$

where $*_p$ is the Hodge star operator on $\Omega^p(\Sigma)$. Note that the splitting into ±1–eigenspaces of τ_Σ restricted to the harmonic forms $\mathcal{H}_a^\bullet(\Sigma;\mathrm{Ad}\,P) = \ker\Delta_a$ of $\Delta_a := d_a d_a^* + d_a^* d_a$

$$\mathcal{H}_a^\bullet(\Sigma;\mathrm{Ad}\,P) = \mathcal{H}_a^+(\Sigma;\mathrm{Ad}\,P) \oplus \mathcal{H}_a^-(\Sigma;\mathrm{Ad}\,P)$$

is invariant under Φf^* for any gauge transformation $\Phi\colon P \to P$ satisfying $\Phi f^* a = a$. Since the unitary structure on $\operatorname{Ad} P$ arises from $\operatorname{ad}\colon G \to \mathrm{O}(\mathfrak{g})$ (see [18, Remark (ii) on page 136]), we have

$$\operatorname{tr}\log[\Phi f^*|_{\mathcal{H}_a^+(\Sigma;\operatorname{Ad}P)\cap\Omega^1}] = \operatorname{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^-(\Sigma;\operatorname{Ad}P)\cap\Omega^1}] - \operatorname{tr}\log[\Phi f^*|_{\mathcal{H}_a^-(\Sigma;\operatorname{Ad}P)\cap\Omega^1}],$$

$$\operatorname{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^+(\Sigma;\operatorname{Ad}P)\cap\Omega^1} = \operatorname{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^-(\Sigma;\operatorname{Ad}P)\cap\Omega^1},$$

and $$\operatorname{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^+(\Sigma)\cap\Omega^1}] = \operatorname{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^-(\Sigma)\cap\Omega^1}]$$

where tr log is defined for a diagonalizable map T as

$$\operatorname{tr}\log T := \sum_{j=1}^{n} \theta_j \in \mathbf{R},$$

where $e^{2\pi i\theta_j}$ are the eigenvalues of T, and where we require $\theta_j \in [0,1)$. Also note that

$$\mathcal{H}_a^+(\Sigma;\operatorname{Ad}P) \cap \Omega^1 = H^{1,0}(\Sigma,\bar\partial_a) \qquad \text{and} \qquad \mathcal{H}_a^-(\Sigma;\operatorname{Ad}P) \cap \Omega^1 = H^{0,1}(\Sigma,\bar\partial_a).$$

A fixed isomorphism $H^{0,1}(\Sigma,\bar\partial_a) \cong T_{[a]}\mathcal{M}(\Sigma)$ gives the commutative diagram

$$\begin{array}{ccc} H^{0,1}(\Sigma,\bar\partial_a) & \xrightarrow{\Phi f^*} & H^{0,1}(\Sigma,\bar\partial_a) \\ \cong\downarrow & & \downarrow\cong \\ T_{[a]}\mathcal{M}(\Sigma) & \xrightarrow{df^*} & T_{[a]}\mathcal{M}(\Sigma), \end{array}$$

and we have $\operatorname{rk}[(df^* - \mathrm{Id})|_{T_{[a]}\mathcal{M}(\Sigma)}] = \operatorname{rk}\mathcal{N}_{[a]}$. We simplify [18, Theorem 4.2.4] as follows.

Theorem 6.1. *Let $f\colon \Sigma \to \Sigma$ be a finite order homeomorphism. Let A be a flat G–connection over Σ_f with $r([A]) = [a]$. Then*

$$\begin{aligned}\rho_A(\Sigma_f) = {}& -4\operatorname{tr}\log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}] + 2\operatorname{rk}\mathcal{N}_{[a]} \\ & - 4\dim G\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] + 2\dim G\operatorname{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}].\end{aligned} \tag{6.4}$$

Remark 6.2. It follows from the proof of [18, Theorem 4.2.4], that

$$\eta(D_A) = -4\operatorname{tr}\log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}] + 2\operatorname{rk}\mathcal{N}_{[a]},$$

and $$\eta(D_\theta) = -4\dim G\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] + 2\dim G\operatorname{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}].$$

7 Identifying the classical invariants

In this section we identify the classical invariants in the leading order term of the Witten-Reshetikhin-Turaev invariants (1.3) of a finite order mapping torus $X = \Sigma_f$ as conjectured by the stationary phase approximation (A.5). More precisely, since the leading order term of

$$\zeta = \frac{k\dim G}{k+h} = \dim G - \frac{h\dim G}{k+h}$$

is simply $\dim G$, we identify the classical invariants in the leading order term

$$\det(f)^{-\frac{1}{2}\dim G} e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(c)} \frac{1}{d_c!}(\omega_c^{d_c} \cap \tau_{d_c}(L^c_\cdot(O_{\mathcal{M}(\Sigma)})))k^{d_c} \tag{7.1}$$

of (1.3) corresponding to an irreducible component $|\mathcal{M}(\Sigma)|_c$ of the variety $|\mathcal{M}(\Sigma)|$ containing an irreducible connection. Theorem 3.3 gives an expression of (7.1) in terms of an integral over $|\mathcal{M}(\Sigma)|_c'$. We reformulate this as an integration of classical invariants of 3–manifolds over $\mathcal{M}(\Sigma)_c'$.

By Theorem 5.7 we have for $A \in \mathcal{M}(\Sigma_f)_c$

$$\sqrt{\tau_{\Sigma_f}(A)} = \frac{1}{d_c!}\frac{r^*(\omega_c)_A^{d_c}}{|\det(1-df|_{\mathcal{N}^*_{r(A)}})|},$$

keeping in mind, that we have identified densities with volume forms in the orientation induced by $r^*(\omega_c)_A^{d_c}$. Notice, that for a complex root of unity $\xi = e^{2\pi i\theta}$ with $\theta \in (0,1)$, we have $1-\xi = \xi(\xi^{-1}-1) = \xi(\bar{\xi}-1)$. Therefore

$$\left(\frac{1-\xi}{|1-\xi|}\right)^2 = \frac{1-\xi}{1-\bar{\xi}} = -\xi.$$

By observing that the real part of $1-\xi$ is always positive, we see that

$$\frac{1}{1-\xi} = \frac{1}{|1-\xi|}e^{2\pi i(\frac{1}{4}-\frac{\theta}{2})} = \frac{1}{|1-\xi|}e^{-2\pi i\frac{\theta}{2}}i. \tag{7.2}$$

For $a := r(A)$ the maps $df|_{T^*_{[a]}\mathcal{M}(\Sigma)}$ and $df^*|_{T_{[a]}\mathcal{M}(\Sigma)}$ have the same eigenvalues, and we have $\operatorname{rk}\mathcal{N}_{[a]} = \operatorname{rk}\mathcal{N}^*_{[a]}$. Therefore by Proposition 3.2, Equation (7.2) and Remark 6.2 we get

$$\begin{aligned}\frac{1}{d_c!}r^*\left(\omega_c^{d_c}\cup \operatorname{Ch}^{\cdot}(\lambda_{-1}^c\mathcal{M}(\Sigma))^{-1}\right)_A &= \frac{1}{d_c!}\frac{r^*(\omega_c)_A^{d_c}}{\det(1-df)|_{\mathcal{N}^*_{[a]}}}\\ &= \tau_{\Sigma_f}(A)^{\frac{1}{2}}\exp\left(-2\pi i\frac{\operatorname{tr}\log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}]}{2}\right)i^{\operatorname{rk}\mathcal{N}_{[a]}}\\ &= \tau_{\Sigma_f}(A)^{\frac{1}{2}}e^{\frac{\pi i}{4}\eta(D_A)}.\end{aligned}$$

In particular, we get

$$\begin{aligned}\frac{1}{d_c!}(\omega_c^{d_c}\cap\tau_{d_c}(L^c_{\cdot}(\mathcal{O}_{\mathcal{M}(\Sigma)}))) &= \int_{|\mathcal{M}(\Sigma)|_c'}\frac{1}{d_c!}\omega_c^{d_c}\cup\operatorname{Ch}^{\cdot}(\lambda_{-1}^c\mathcal{M}(\Sigma))^{-1}\\ &= \int_{A\in\mathcal{M}(\Sigma_f)_c'}\tau_{\Sigma_f}(A)^{\frac{1}{2}}e^{\frac{\pi i}{4}\eta(D_A)}\end{aligned} \tag{7.3}$$

Observe, that we can rewrite

$$\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] - \frac{\operatorname{rk}[(f^*-\mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}]}{2} = \sum_{0\neq\tilde\omega_i\in(-\frac{1}{2},\frac{1}{2})}\tilde\omega_i,$$

where $e^{2\pi i\tilde\omega_i} = \omega_i$, $\tilde\omega_i \in [-\frac{1}{2},\frac{1}{2})$, are the eigenvalues of the pull-back $f^*\colon \mathcal{H}^{1,0}(\Sigma,\bar\partial)\to\mathcal{H}^{1,0}(\Sigma,\bar\partial)$. By Proposition 3.1 we therefore have

$$\det(f)^{-\frac{1}{2}\dim G} = \exp\left(i\pi\dim G\left(\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] - \frac{\operatorname{rk}[(f^*-\mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}]}{2}\right)\right).$$

Therefore, it is easy to see from Remark 6.2 that the leading order term of $\det(f)^{-\frac{1}{2}\zeta}$ is given by

$$\det(f)^{-\frac{1}{2}\dim G} = e^{-\frac{\pi i}{4}\eta(D_\theta)}. \tag{7.4}$$

Together, (7.3) and (7.4) prove Theorem 1.5. In particular, we have the following.

Theorem 7.1. *Let each $\mathcal{M}(\Sigma_f)'_c$ be nonempty for every $c \in C$. Then*

$$(7.5) \qquad Z_G^{(k)}(\Sigma_f) \dot{\sim} \frac{1}{|Z(G)|} \sum_{c\in C} \int_{A\in\mathcal{M}(\Sigma_f)'_c} k^{d_c} e^{2\pi i k \, \mathrm{CS}_{\Sigma_f}(A)} \sqrt{\tau_{\Sigma_f}(A)} e^{2\pi i \frac{\rho_A(\Sigma_f)}{8}},$$

and each factor of the integrand gets identified in the leading term of the Witten-Reshetikhin-Turaev invariants.

8 Spectral flow

The spectral flow along a path of formally self-adjoint, elliptic differential operators D_t is the algebraic intersection number in $[0,1] \times \mathbf{R}$ of the track of the spectrum

$$\{(t, \lambda) \mid t \in [0,1], \lambda \in \mathrm{Spec}(D_t)\}$$

and the line segment from $(0, -\varepsilon)$ to $(1, -\varepsilon)$. We choose the $(-\varepsilon, -\varepsilon)$–convention, which makes the spectral flow additive under concatenation of paths of connections.[1]

The main statement of this section relating spectral flow, the Chern-Simons invariant and the ρ–invariant for a compact Lie group seems to be well-known. Since it depends on several conventions and we have not found a general proof anywhere in the literature, we decided to provide a proof in this paper in the hope that it may be a useful reference. With slightly different conventions, this has been proven in [30, Section 7] for SU(2). Even though the main proof is completely analogous, we give a detailed exposition for the convenience of the reader.

The dual Coxeter number

Let G be a simple Lie group of dimension n and rank r. Consider any positive definite normalization $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ of the Killing form on $\mathfrak{g}$. Given a basis $\{X_i\}_{i=1,\dots,n}$ of $\mathfrak{g}$ and its dual basis $\{X^i\}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, the quadratic Casimir is the element

$$\Omega = \sum_i X_i \otimes X^i \in \mathfrak{g} \otimes \mathfrak{g}.$$

As an element of the universal enveloping algebra it commutes with all elements of $\mathfrak{g}$. The Casimir invariant in the adjoint representation is given by

$$\mathrm{ad}_*(\Omega) = \sum_i \mathrm{ad}_*(X_i)\,\mathrm{ad}_*(X^i) \in \mathrm{End}(\mathfrak{g}).$$

By Schur's Lemma we know, that it is proportional to the identity with factor—by definition—the Casimir eigenvalue C_{ad} in the adjoint representation with respect to the normalization $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

Therefore, we have for all $X, Y \in \mathfrak{g}$

$$\mathrm{tr}(\mathrm{ad}_*(X)\,\mathrm{ad}_*(Y)) = K\langle X, Y\rangle_{\mathfrak{g}},$$

where K is determined by

$$K = K\frac{1}{n}\sum_{i=1}^n \langle X_i, X^i\rangle_{\mathfrak{g}} = \frac{1}{n}\sum_{i=1}^n \mathrm{tr}(\mathrm{ad}_*(X_i)\,\mathrm{ad}_*(X^i)) = \frac{1}{n} C_{\mathrm{ad}}\,\mathrm{tr}(\mathrm{Id}) = C_{\mathrm{ad}}.$$

[1]In the literature one also frequently finds the $(-\varepsilon, \varepsilon)$–convention (see for example [23, 30]), so we need to be careful when relating to formulas found elsewhere.

The inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ gives rise to the identification $\mathfrak{g}^* \to \mathfrak{g}, \beta \to X_\beta$, where $\beta(X) = \langle X_\beta, X\rangle_{\mathfrak{g}}$ for all $X \in \mathfrak{g}$. We also have an induced inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}^*$ given by $\langle\beta,\gamma\rangle_{\mathfrak{g}} := \langle X_\beta, X_\gamma\rangle_{\mathfrak{g}}$. Then we have $C_{\mathrm{ad}} = \langle\theta,\theta\rangle_{\mathfrak{g}} \cdot h$ for the maximal root θ (see for example [26, Equation (1.6.51)]), where the dual Coxeter number h is independent of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

Notice that for the inner product on $\mathrm{SU}(n)$ given by $\langle X, Y\rangle_{\mathfrak{su}(n)} = -\operatorname{tr}(XY)$, the maximal root $\tilde\theta$ satisfies $\langle\tilde\theta,\tilde\theta\rangle_{\mathfrak{su}(n)} = 2$. We therefore get

$$-\operatorname{tr}(\mathrm{ad}_*(X)\,\mathrm{ad}_*(Y)) = -2n\operatorname{tr}(XY) \qquad \text{for } X, Y \in \mathfrak{su}(n). \tag{8.1}$$

Relating Chern classes via the adjoint representation

We have seen in Section 6 that with respect to an ad-invariant metric on $\mathfrak{g}$, we can consider the complexified adjoint representation

$$\mathrm{Ad}\colon\ G \to \mathrm{SU}(n) \subset \mathrm{End}(\mathfrak{g}^{\mathbf{C}}),$$

and its differential

$$\mathrm{Ad}_*\colon\ \mathfrak{g} \to \mathfrak{su}(n) \subset \mathrm{End}(\mathfrak{g}^{\mathbf{C}}).$$

It is easy to see, that $C_{\mathrm{Ad}} = C_{\mathrm{ad}}$.

We can define the second Chern form $c_2(B)$ of a connection B in a principal G–bundle P over a 4–manifold Z by

$$c_2(B) := \langle F_B \wedge F_B\rangle_{\mathfrak{g}},$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is the normalization of the Killing form on $\mathfrak{g}$ introduced in Section 2. This normalization is given in terms of the Killing form by

$$\langle X, Y\rangle_{\mathfrak{g}} = \frac{1}{16\pi^2 h}\operatorname{tr}(\mathrm{Ad}_X, \mathrm{Ad}_Y), \tag{8.2}$$

which is shown in [21, page 242] together with a list of the dual Coxeter numbers h. Note that in this normalization $c_2(B)$ represents an integral generator of the second cohomology.

$\mathrm{Ad}\,P$, the complexified adjoint bundle of P, is a Hermitian vector bundle, which we view as a principal $\mathrm{SU}(n)$–bundle via its frame bundle. Therefore, it makes sense to consider the adjoint bundle $\mathrm{Ad}(\mathrm{Ad}\,P)$ of $\mathrm{Ad}\,P$, whose fiber is $\mathfrak{u}(n)$. The connection B in P induces a connection $\mathrm{Ad}\,B$ in $\mathrm{Ad}\,P$ as follows. Given a section $s\colon U \to P$ for $U \subset M$ open, then s^*B is a $\mathfrak{g}$–valued 1–forms on U. B is uniquely determined by the family of 1–forms $B^s_g := (sg)^*B$, where $g \in C^\infty(U;G)$. In this way, $\mathrm{Ad}\,B$ is determined by $\{\mathrm{Ad}_* \circ B^s_g\}_{g\in C^\infty(U;G)}$. Similarly we get $F_{\mathrm{Ad}\,B} = \mathrm{Ad}\,F_B$, where $\mathrm{Ad}\,F_B \in \Omega^2(\mathrm{Ad}(\mathrm{Ad}\,P))$.

By the previous paragraph we can consider the second Chern form

$$c_2(\mathrm{Ad}\,B) = \langle F_{\mathrm{Ad}\,B} \wedge F_{\mathrm{Ad}\,B}\rangle_{\mathfrak{su}(n)} = \langle \mathrm{Ad}\,F_B \wedge \mathrm{Ad}\,F_B\rangle_{\mathfrak{su}(n)}$$

of $\mathrm{Ad}\,B$ in $\mathrm{Ad}\,P$. By Equation (8.2) and (8.1) we get

$$\begin{aligned} c_2(\mathrm{Ad}\,B) &= \frac{1}{16\pi^2 n}\operatorname{tr}(\mathrm{Ad}(\mathrm{Ad}\,F_B) \wedge \mathrm{Ad}(\mathrm{Ad}\,F_B)) = \frac{2n}{16\pi^2 n}\operatorname{tr}(\mathrm{Ad}\,F_B \wedge \mathrm{Ad}\,F_B) \\ &= 2n\frac{16\pi^2 h}{16\pi^2 n}\langle F_B \wedge F_B\rangle_{\mathfrak{g}} = 2hc_2(B). \end{aligned}$$

Notice that $c_1(\mathrm{Ad}\,B) = 0$, because $\mathrm{Ad}\,P$ is the complexification of $\mathrm{ad}\,P$, and therefore $\mathrm{ch}_2(\mathrm{Ad}\,B) = \frac{1}{2}c_1^2(\mathrm{Ad}\,B) - c_2(\mathrm{Ad}\,B) = -c_2(\mathrm{Ad}\,B)$. This gives

$$-\mathrm{ch}_2(\mathrm{Ad}\,B) = c_2(\mathrm{Ad}\,B) = 2h\,c_2(B) \tag{8.3}$$

The relationship to the ρ–invariant and the Chern-Simons function

We are ultimately interested in the spectral flow $\mathrm{SF}(D_{A_t})$ of the odd signature operator coupled to a path of connections A_t from the trivial connection θ to another flat connection A. Since the spectral flow only depends on the endpoints, we will call this the spectral flow from θ to A

$$\mathrm{SF}(\theta, A) := \mathrm{SF}(D_{A_t}).$$

Theorem 8.1. *Let G be a simple Lie group and A a flat G–connection, then we get*

$$\mathrm{SF}(\theta, A) = -4h\,\mathrm{CS}(A) + \frac{\rho_A(X)}{2} - \frac{\dim G(1 + b^1(X))}{2} + \frac{\dim(H^0(X, d_A)) + \dim(H^1(X, d_A))}{2}.$$

We note that this Theorem combined with Theorem 7.1 implies Theorem 1.6 from the introduction.

Proof. The proof is analogous to the argument in [30, Section 7]. Notice, that because we have $\langle X, Y\rangle = -\frac{1}{8\pi^2}\,\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{su}(n)$, our Chern-Simons function has a different sign than the Chern-Simons function used for example in [30, 35]. Let $S_B\colon\ \Omega^1 \to \Omega^0 \oplus \Omega^2_-$ be the self-duality operator on $Z = X \times I$ defined by $\omega \mapsto (d_B^*\omega, P_-(d_B\omega))$ for a connection B on Z, where P_- is the projection to the anti-self-dual 2–forms. We will use the "outward normal first" convention to orient Z, so that we do not have to introduce signs in Stokes' Theorem. Near the boundary we have $S_B \circ \Psi_2 = \Psi_1(D'_A + \frac{\partial}{\partial u})$, where $D'_A(a, b) = (-d_A^* b, *d_A b - da_A)$, $A = B|_{\partial Z}$, $\Psi_2(a, b) = a\,du + b$ and $\Psi_1(a, b) = (-a, P_-(b\,du))$. By the Atiyah-Patodi-Singer index theorem (see also [30, Theorem 7.1])) we get for the connection $B = A_t$ on Z

$$\mathrm{SF}(D_{A_t}) = \mathrm{Index} S_B = \int_Z \hat{A}(Z)\mathrm{ch}(V_-)\mathrm{ch}(\mathrm{Ad}\,B) + \frac{1}{2}(\eta(D_{A_1}) + \dim\ker D_{A_1}) - \frac{1}{2}(\eta(D_{A_0}) + \dim\ker D_{A_0}),$$

where $\mathrm{ch}(\mathrm{Ad}\,B)$ is the total Chern character form of the connection $\mathrm{Ad}\,B$ in the trivial bundle $Z \times \mathfrak{g}^{\mathbf{C}}$ induced by B and V_- is the complex spinor bundle of $-\frac{1}{2}$–spinors on Z, whose rank is 2 for a 4–manifold. Consider $c_2(B) = \langle F_B \wedge F_B\rangle$. Then by Stokes' theorem

$$\mathrm{CS}(A_1) - \mathrm{CS}(A_0) = \int_Z c_2(B).$$

By Equation (8.3) we have

$$2h(\mathrm{CS}(A_1) - \mathrm{CS}(A_0)) = 2h\int_Z c_2(B) = -\int_Z \mathrm{ch}(\mathrm{Ad}\,B).$$

We have

$$\hat{A}(Z) = 1 + \frac{1}{24}c_2(Z),$$

so that the integrand in the index theorem can be split up

$$\int_Z \hat{A}(Z)\mathrm{ch}(V_-)\mathrm{ch}(\mathrm{Ad}\,B) = \int_Z \left(\hat{A}(Z)\mathrm{ch}(V_-)\,\mathrm{rk}(\mathrm{Ad}\,B) + 2\,\mathrm{ch}_2(\mathrm{Ad}\,B)\right).$$

The first contribution can immediately be computed to be zero by applying the index theorem to a constant path at the trivial connection. The second contribution is precisely $-4h(\mathrm{CS}(A_1) - \mathrm{CS}(A_0))$. By definition, the difference of the η–invariants $\eta(D_A) - \eta(D_\theta)$ is the ρ–invariant. After identifying the cohomology with the kernel of the odd signature operator, the theorem follows. □

A A heuristic discussion of the path integral

As a disclaimer, we would like to mention that this appendix reviews parts of [48] and is the only non-rigorous part in the paper, which we decided to include for motivational purposes. See Rozansky's work [40] and in particular [41] for a detailed account.

For a function $f\colon \mathbf{R}^n \to \mathbf{R}$ with finitely many non-degenerate critical points and a compactly supported function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$, we have the asymptotic behaviour

$$\int_{\mathbf{R}^n} e^{ikf(x)}\varphi(x)\,dx \sim_{k\to\infty} \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} \sum_{x\in\mathrm{Crit}(f)} e^{\frac{\pi i}{4}\,\mathrm{sign}\,\mathrm{Hess}_x(f)} \frac{e^{ikf(x)}\varphi(x)}{\sqrt{|\det \mathrm{Hess}_x(f)|}}$$

by the method of stationary phase. We may assume that $\varphi(x) = 1$ for $x \in \mathrm{Crit}(f)$ and $\varphi \equiv 0$ outside of a compact set. Therefore, we will abuse the notation and eliminate the function φ from the formulas entirely. Let G be a simple, simply-connected, compact Lie group. If $G/Z(G)$ acts freely from the right on $\mathbf{R}^n$ and $e^{ikf(x)}$ is G–invariant, then the Jacobian J of the G action on x induces the measure $d[x] = |\det J(x)|dx$ on $\mathbf{R}^n/G$ and we get the leading order asymptotic behaviour

$$\text{(A.1)}\quad \frac{\dim G}{|Z(G)|}\int_{\mathbf{R}^n/G} e^{ikf(x)}\,d[x] \sim_{k\to\infty} \left(\frac{2\pi}{k}\right)^{\frac{n-\dim G}{2}} \sum_{[x]\in\mathrm{Crit}(f)/G} e^{\frac{\pi i}{4}\,\mathrm{sign}\,\mathrm{Hess}_x(f)} \frac{e^{ikf(x)}}{\sqrt{|\det \mathrm{Hess}_x(f)|}} \frac{|\det J(x)|\dim G}{|Z(G)|}.$$

Also see [49, Section 2.2] and [41, Section 2.2] for the appearance of the factor $\frac{1}{|Z(G)|}$.

According to Witten [48], the invariants $Z_G^{(k)}(X)$ can be written as the path integral characterizing the Chern-Simons theory

$$Z_G^{(k)}(X) = \int_{A\in\mathcal{A}} e^{2\pi i k\,\mathrm{CS}(A)}\,dA,$$

where we have identified $\mathcal{A} = \Omega^1(X, \mathfrak{g})$. Even though the right-hand side is not mathematically rigorous, we would like to formally apply the above method of stationary phase to this path integral. This procedure in quantum field theory is known as the Faddeev-Popov method (see for example [36, 39] for more information). $\mathcal{G} = C^\infty(X, G)$ acts on $\mathcal{A}$. It can easily be seen that $|Z(\mathcal{G})| = |Z(G)|$, and we need to ignore $\dim G$. In our case, $\det D$ is the zeta-regularized determinant of a formally self-adjoint elliptic differential operator D

$$\det D = e^{-\zeta_k'(0)},\ \text{where } \zeta(s) = \sum_{\lambda_j\neq 0} \lambda_j^{-s},$$

where λ_j are the eigenvalues of D. The differential of the $\mathcal{G}$ action on A can be seen to be d_A. Observe that

$$\|d_A\varphi\|^2 = \langle \Delta_A^{(0)}\varphi, \varphi\rangle_{L^2} = \lambda\|\varphi\|^2,$$

where the L^2 inner product on $\Omega^k(X; \mathfrak{g})$ is given by

$$\langle a, b\rangle_{L^2} = \int_X \langle a \wedge *b\rangle,$$

$\Delta_A^{(k)}$ is the twisted Laplacian on $\Omega^k(X, \mathfrak{g})$ and φ is an eigenvector of $\Delta_A^{(0)}$ with (positive) eigenvalue λ. Therefore we have

$$\text{(A.2)}\quad |\det J(A)| = \sqrt{\det \Delta_A^{(0)}},$$

which is the Faddeev-Popov determinant in disguise.

On a finite-dimensional Riemannian manifold we have

$$\mathrm{Hess}_x(f)(X,Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle$$

for a critical point x of a Morse-function f, where ∇ is the Levi-Civita connection. We can view the L^2 inner product on the space of connections $\mathcal{A}$ as a metric on $\mathcal{A}$. We can use it to identify vectors and covectors of $T_{[A]}(\mathcal{B}) = \operatorname{coker} d_A \cong \ker d_A^*$. With respect to this the linearization of $\mathrm{CS} : \mathcal{B} \to \mathbf{R}/\mathbf{Z}$ is given by the gradient $\operatorname{grad} \mathrm{CS}|_A = *F_A\colon\ \ker d_A^* \to \ker d_A^*$. Consider now the odd signature operator coupled to a connection A, as defined in (6.2) Notice that $D_A^2 = \Delta_A^{(0)} \oplus \Delta_A^{(1)}$ and therefore $(\det D_A)^2 = \det \Delta_A^{(0)} \det \Delta_A^{(1)}$. Let A be flat, then we have under the decomposition $\Omega^1(X;\mathfrak{g}) = \operatorname{im} d_A \oplus \ker d_A^*$

$$D_A = H_A \oplus S_A$$

where

$$\begin{aligned} S_A\colon\ \Omega^0(X;\mathfrak{g}) \oplus \operatorname{im} d_A &\longrightarrow \Omega^0(X;\mathfrak{g}) \oplus \operatorname{im} d_A \\ (\alpha,\beta) &\longmapsto (d_A^*\beta, d_A\alpha) \end{aligned}$$

has symmetric spectrum and satisfies $|\det S_A| = \det \Delta_A^{(0)}$, while $H_A = \operatorname{proj}_{\ker d_A^*} *d_A\colon\ \ker d_A^* \to \ker d_A^*$ is the linearization of $\operatorname{grad}_A \mathrm{CS}$ satisfying $\langle H_A(a), b\rangle = \mathrm{Hess}_A \mathrm{CS}(a,b)$. Therefore we have

$$|\det \mathrm{Hess}_A \mathrm{CS}\,| = |\det H_A| = \frac{|\det D_A|}{|\det S_A|} = \frac{|\det D_A|}{\det \Delta_A^{(0)}}. \tag{A.3}$$

The analytic torsion

$$T_X(A) := \prod_k (\det \Delta_A^{(k)})^{(-1)^{k+1}k/2}$$

is an invariant of Riemannian manifolds defined by Ray and Singer, which proved to be equal to the Reidemeister torsion $\tau_X(A)$ by work of Cheeger and Müller after choosing the volume form on cohomology induced by the metric on the manifold. Since $\det \Delta_A^{(k)} = \det \Delta_A^{(3-k)}$ by Poincaré duality, we deduce from Equations (A.2) and (A.3)

$$\sqrt{\tau_X(A)} = (\det \Delta_A^{(0)})^{3/4} (\det \Delta_A^{(1)})^{-1/4} = \frac{\det \Delta_A^{(0)}}{\sqrt{|\det D_A|}} = \frac{|\det J(A)|}{\sqrt{|\det \mathrm{Hess}_A \mathrm{CS}\,|}}. \tag{A.4}$$

Let us turn to the analogue of the signature. In finite dimensions we have for a path x_t between two nondegenerate critical points x_0 and x_1 we get

$$\operatorname{sign}(\mathrm{Hess}_{x_1}(f)) - \operatorname{sign}(\mathrm{Hess}_{x_0}(f)) = 2\,\mathrm{SF}(\nabla \operatorname{grad}_{x_t} f),$$

where the spectral flow SF is defined in Section 8. Therefore, instead of the signature of the Hessian, we can use twice the spectral flow of H_{A_t} for a path of connections A_t from the trivial connection θ to some flat connection $A = A_1$. Since S_{A_t} has symmetric spectrum and $D_{A_t} - H_{A_t}$ is a compact operator for all t, we can use $2\,\mathrm{SF}(D_{A_t}) = 2\,\mathrm{SF}(H_{A_t})$. Keep in mind, that this procedure neglects the signature at the trivial connection. Note, that this is the idea behind the gauge-theoretic version of Casson's invariant for homology 3–spheres by Taubes [43] and its generalizations. This turned out to be the perfect approach for the Casson invariant, because we needed an integer-valued analogue to the signature. The case of the Witten-Reshetikhin-Turaev invariants allows for an alternative approach.

We can consider the η–invariant defined in (6.1) as a generalized signature. This has the immediate merit of being defined for every connection, but it is metric-dependent and not necessarily an integer. Since the ρ–invariant defined in (6.3) is independent of the metric, we will choose it as a generalized signature, keeping in mind that we introduced $\eta(D_\theta)$. By following the arguments in [48], $\eta(D_\theta)$ can be altered into a prefactor, which is a topological invariant of a framed, oriented manifold. It was observed in [24] that this prefactor vanishes for the (canonical) Atiyah 2–framing. For further details we refer to [48, Section 2] and [24, Section 1].

It has been mentioned by Jeffrey [29, Section 5.2.2] that Reidemeister torsion can be used as a density, thereby extending the above use of Reidemeister torsion in the formal application of the stationary phase method to degenerate critical points. We need this idea to allow for critical components of positive dimension. In order to do this, we need to identify $T_A\mathcal{M}(X) \cong H^1(X, d_A)$ and identify $H^2(X, d_A)$ with $(H^1(X, d_A))^*$ using Poincaré duality. Note that Poincaré duality depends on a choice of inner product on $\mathfrak{g}$, which is possibly a multiple of our original choice of inner product on $\mathfrak{g}$. Furthermore, we need to choose a suitable volume form or density on $H^0(X, d_A)$ and $(H^3(X, d_A))^*$, for example we might take the one induced by the inner product on $\mathfrak{g}$.

Since we allow higher-dimensional components in the moduli space of flat connections and the stationary phase approximation in finite dimensions (A.1) includes the factor $k^{-\frac{n-\dim G}{2}}$, this suggests, that we should shift our result by the factor k^{d_c}, where d_c is half the real dimension or equivalently the complex dimension d_c of the critical component $\mathcal{M}(X)_c$, which is expected to be

$$d_c = \frac{1}{2} \max_{A \in \mathcal{M}(X)_c} \Big(\dim(H^1(X, d_A)) - \dim(H^0(X, d_A))\Big)$$

by the growth rate conjecture (see [6, Lemma 7.2] for evidence). By the same argument we may like to introduce the factor $\frac{1}{(2\pi)^{d_c}}$. For similar reasons Rozansky [41, Equation (2.33)] includes such a factor. We will simply set every factor of the form K^{d_c} for a constant $K > 0$, to 1 by remarking, that a change of normalization for Poincaré duality (used to treat Reidemeister torsion as a density) by a factor K results in the factor K^{-d_c} in the stationary phase approximation. The other factors, which only depend on n and the dimension of G, we need to omit, because both $\mathcal{A}$ and $\mathcal{G}$ are infinite-dimensional.

If we therefore replace the Signature of the Hessian by the ρ–invariant (6.3), replace the rest via Equation (A.4) and normalize Poinacré duality appropriately (independently of X and G), the following conjecture is justified.

Conjecture A.1. *Let G be a simple, simply-connected, compact Lie group. Let X be a closed 3–manifold and C the set of all connected components of $\mathcal{M}(X)$. Then, in the Atiyah 2–framing, the leading order asymptotic behavior of $Z_G^{(k)}(X)$ in the limit $k \to \infty$ is given by*

$$Z_G^{(k)}(G) \dot{\sim} \sum_{c \in C} \frac{1}{|Z(G)|} \int_{A \in \mathcal{M}(X)_c} \sqrt{\tau_X(A)} e^{2\pi i \operatorname{CS}_X(A) k} e^{\frac{\pi i}{4}\rho_A(X)} k^{d_c}. \tag{A.5}$$

Theorem 8.1 immediately yields the more familiar version (1.2) of (A.5) stated in the introduction. Observe that the Chern-Simons invariant is constant on connected components of flat connections, we could therefore put it in front of the integral. On reducible subsets it will be necessary to interpret these conjectures in a suitable way, however we do not consider the reducible case in this paper.

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