Monte Carlo Methods Appl. **16** (2010), 361–377
DOI 10.1515/MCMA.2010.013 

# Increasing the number of inner replications of multifactor portfolio credit risk simulation in the $t$-copula model

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**Abstract.** We consider the problem of simulating tail loss probabilities and expected losses conditioned on exceeding a large threshold (expected shortfall) for credit portfolios. Instead of the commonly used normal copula framework for the dependence structure between obligors, we use the $t$-copula model. We increase the number of inner replications using the so-called geometric shortcut idea to increase the efficiency of the simulations. The paper contains all details for simulating the risk of the $t$-copula credit risk model by combining outer importance sampling (IS) with the geometric shortcut. Numerical results show that the applied method is efficient in assessing tail loss probabilities and expected shortfalls for credit risk portfolios. We also compare the tail loss probabilities and expected shortfalls under the normal and $t$-copula model.

**Keywords.** Monte Carlo simulation, credit risk, geometric shortcut, VaR, expected shortfall, variance reduction, extremal dependence.

**2010 Mathematics Subject Classification.** 91G60.

## 1 Introduction

The most widely used model for credit risk is the normal copula model of CreditMetrics [13], which is a Merton [21] type model; if a latent variable of an obligor crosses a level (determined by the marginal default probability of the obligor), the obligor is assumed to default. Latent variables for obligors have some common risk factors to introduce dependence across obligors and one idiosyncratic risk factor.

In recent papers the dependence structure of multivariate variables had been shown to be better represented by the $t$-copula. Among others [6] and [7] stress the necessity of using copulas to model the dependence structure of multivariate variables in finance. Furthermore, [20] conclude that the $t$-copula fits empirically better than the normal copula without assuming specific marginal distributions for asset and equity returns. In a more recent work, [17] apply goodness-of-fit tests to the $t$, normal and Gumbel copula for the risk management of linear asset portfolios. The $t$-copula is preferred to the normal and Gumbel copulas because of capturing the dependence better in the non-extremes and extremes (tails).

Although there are some approximations (see e.g., [18,19]) for the tail behaviors of loss distributions for credit portfolios under the normal and $t$-copula framework, there are no closed-form solutions. Monte Carlo simulation is a better alternative than these approximations, since it gives a confidence interval for the computed point estimate for credit risk. To reach sufficient precision in acceptable computation time variance reduction techniques for simulating tail loss probabilities and expected shortfalls in the normal or in the $t$-copula framework are of greatest practical importance and attracted attention in recent years. Although, there are lots of papers in the normal copula framework (see e.g., [4, 11, 15, 23]), to our knowledge there are only three $t$-copula papers [2, 3, 16].

The validity of routinely applied quantitative methods like the normal copula model in risk management are questioned by the current credit crisis (see [5]). This also underlies the practical importance of efficient simulations for the $t$-copula model. If we look at the proposed variance reduction techniques closer in the $t$-copula papers; [3] consider conditional Monte Carlo approach [1] in contrast to [16] and [2] who use importance sampling (IS). The main computational burden of conditional Monte Carlo approach is the computation of threshold values of each common risk factor for each obligor and sorting these threshold values for each replication. We follow [16] and [2] by proposing an IS strategy as part of our simulation method in this paper. [16] and [2] use the classical generation for the multivariate $t$ distribution; a multivariate normal vector is divided by the square root of a univariate chi-square random variable. These papers primarily use asymptotics for portfolio credit risk to propose an IS on the chi-square random variable. These two ingenious methods have problems in implementation because of being computationally intensive for multidimensional risk factors. Searching for a $q$-minimal index set and finding the mean shift and exponential twisting parameter for conditional default probabilities for every generated chi-square random variable is computationally intensive as pointed out in [16]. Getting a random variate from exponentially twisted probability distribution function of the chi-square distribution which changes for every generated chi-square random variable and applying numerical integration afterwards to compute the likelihood ratio is computationally exhausting in [2]. This motivated us to look for an easily applicable simulation method to reduce the variance of the simulations for tail loss probability and expected shortfall computation. [23] consider increasing the number of inner replications by using a geometric-shortcut in the normal copula framework. The proposed idea allows a very efficient simulation for the case of independent obligors. The asymptotic efficiency rate of the new idea is higher than that of the naive algorithm when the average default probability tends to zero. They use this method to replace inner IS (see e.g. [11]) in the simulation of tail loss probability and expected shortfall. In this paper we apply the same idea in the $t$-copula framework. One additional contribution of our paper is to compare the tail loss probabilities and expected shortfalls under the normal and $t$-copula model for a variety of multidimensional credit portfolio examples so that one can have a rough estimate of the difference between the computed risks under the normal and $t$-copula model.

We follow [16] and [2] in the construction of multivariate $t$ distribution. In this paper we use an IS density that shifts the mean of the multivariate normal distribution and a gamma distribution with shape parameter $\nu/2$ and scale parameter smaller than two. A similar IS strategy is applied for market risk in [24].

We shortly define the normal and $t$-copula model in Section 2. In Section 3, we summarize the geometric shortcut idea for dependent obligors in the $t$-copula model. In Section 4, we add outer-IS to the inner replications of the geometric shortcut for a portfolio having dependent obligors. While Sections 3–4 concentrate on the efficient simulation of tail loss probabilities, Section 5 consider the simulation of expected shortfall. We report our numerical results in Section 6.

## 2 The normal and $t$-copula model

We first describe the details of the normal copula model of CreditMetrics [13] for the dependence structure across obligors. The notation used throughout the paper follows [11]:

| | |
|---|---|
| $m$: | number of obligors in the portfolio |
| $Y_j$: | default indicator for the $j$th obligor (equal to 1 if default occurs, 0 otherwise) |
| $c_j$: | loss resulting from the default of the $j$th obligor |
| $p_j$: | marginal default probability of the $j$th obligor |
| $L = \sum_{j=1}^{m} c_j Y_j$: | total loss of the portfolio |
| $n$: | number of replications in a simulation |

We only consider a fixed horizon, over which we are interested in the distribution of tail loss probability and ES. The exposure values $c_j$ and the marginal default probabilities $p_j$ are assumed to be constant and known.

The normal copula model introduces a multivariate normal vector $(X_1, \ldots, X_m)$ of latent variables to obtain dependence across obligors. The relationship between the default indicators and the latent variables is described by

$$Y_j = \mathbf{1}\{X_j > x_j\}, \quad j = 1, \ldots, m,$$

where $X_j$ has standard normal distribution and $x_j = \Phi^{-1}(1 - p_j)$, with $\Phi^{-1}$ inverse of the cumulative normal distribution. Obviously, the threshold value $x_j$ is chosen such that $P(Y_j = 1) = p_j$.

The correlations among the $X_j$ are modeled by defining

$$X_j = b_j \epsilon_j + a_{j1} Z_1 + \cdots + a_{jd} Z_d, \quad j = 1, \ldots, m, \tag{2.1}$$

where $\epsilon_j$ and $Z_1, \ldots, Z_d$ are independent standard normal random variables with $b_j^2 + a_{j1}^2 + \cdots + a_{jd}^2 = 1$. While, $(Z_1, \ldots, Z_d)$ are systematic risk factors affecting all of the obligors, $\epsilon_j$ is the idiosyncratic risk factor affecting only obligor $j$. Furthermore, $a_j = (a_{j1}, \ldots, a_{jd})$ are constant and nonnegative factor loadings, assumed to be known. Thus, given the vector $Z = (Z_1, \ldots, Z_d)^T$, we have the conditionally independent default probabilities

$$p_j(Z) = P(Y_j = 1|Z) = \Phi\left(\frac{a_j Z + \Phi^{-1}(p_j)}{b_j}\right), \quad j = 1, \ldots, m. \tag{2.2}$$

In the $t$-copula model latent variables have multivariate $t$-distribution instead of multivariate normal distribution. The model that has been widely used (see, e.g., [2, 16]) is

$$X_j' = \frac{(b_j \epsilon_j + a_{j1} Z_1 + \cdots + a_{jd} Z_d)}{\sqrt{V/\nu}}, \quad j = 1, \ldots, m, \tag{2.3}$$

where the definitions of $Z, \epsilon_j, a_j$ and $b_j$ are the same as in (2.1), and $V$ denotes a chi-square random variable with $\nu$ degrees of freedom that is independent of $Z$ and $\epsilon_j$. Since $X_j'$ is $t$-distributed random variable, the threshold value for the indicator function $Y_j = \mathbf{1}\{X_j' > x_j'\}$ should be $x_j' = F_\nu^{-1}(1-p_j)$ ($F_\nu$ is the cdf of the $t$ distribution with $\nu$ degrees of freedom) to preserve the marginal default probabilities. Finally, given the vector $Z$ and $V$, we have the conditionally independent default probabilities

$$p_j(Z, V) = P(Y_j = 1|Z, V) = \Phi\left(\frac{a_j Z - \sqrt{V/\nu} F_\nu^{-1}(1 - p_j)}{b_j}\right), \quad j = 1, \ldots, m. \tag{2.4}$$

## 3 Inner replications using geometric shortcut: dependent obligors

The geometric shortcut idea was introduced in Section 3 of [23] to simulate tail loss probability ($P(L > x)$) and expected shortfall for independent obligors. The idea is simply to generate instead of many Bernoulli random variates with small probabilities a geometric random variate that is used as index of the next default. It is no problem to apply the same idea to dependent obligors under the normal copula framework, as, conditional on $Z = z$, obligors default independently. Changing the model to the $t$-copula does not change the structure of the problem. This time conditional on $Z = z$ and $V = v$, obligors default independently. We start with Algorithm 1 that describes the naive simulation algorithm for the full model.

**Algorithm 1** Tail loss probability computation using naive simulation for dependent obligors.

**for** replications $k = 1, \ldots, n$ **do**
  generate $z_l \sim N(0,1), l = 1, \ldots, d$ and $v$ from $\chi^2_\nu$ distribution independently
  calculate $p_j(z, v)$, $j = 1, \ldots, m$, as in (2.4) where $z = (z_1, \ldots, z_d)$
  **for** obligors $j = 1, \ldots, m$ **do**
    generate a $U(0,1)$ variate $U$
    if $(U < p_j(z, v))$ set $L^{(k)} = L^{(k)} + c_j$
  **end for**
**end for**
return $\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{L^{(k)} > x\}$

The naive algorithm consists of an outer part with $n$ independent generations of the $z$-vector and $V$ and an inner part where, depending on the values of the $z$-vector and $V$, the default probabilities are calculated and the defaults are simulated. [23] consider increasing the number of repetitions of the simulation of the defaults from 1 to $n_{\text{in}} > 1$. They argue that the optimum number of $n_{\text{in}}$ depends on the contribution of the inner and the outer repetitions on the variance. Although, they give approximate analytical results on $n_{\text{in}}$, their conclusion is to use $n_{\text{in}} = \min(\lfloor 1/\bar{p}_z \rfloor, m)$ where $\bar{p}_z$ denotes the average value of the default probabilities $p_j(z)$ for the current $z$ vector under the normal copula model. The intuition behind this selection is not to increase the computation time of the new algorithm much more than the naive algorithm ($n_{\text{in}} = 1$). Thus, in this paper we use $n_{\text{in}} = \min(\lfloor 1/\bar{p}_{z,v} \rfloor, m)$ where $\bar{p}_{z,v}$ denotes the average value of the default probabilities $p_j(z, v)$ for the current $z$ and $v$ vector. Algorithm 2 shows how to implement the geometric shortcut for the dependent obligor case under the $t$-copula model.

**Algorithm 2** Tail loss probability simulation using inner replications and the geometric shortcut for dependent obligors.

**for** replications $k = 1, \dots, n$ **do**
  generate $z_l \sim N(0, 1), l = 1, \dots, d$ and $v$ from $\chi^2_v$ distribution independently
  calculate $p_j(z, v)$, $j = 1, \dots, m$, as in (2.4) where $z = (z_1, \dots, z_d)$
  construct a loss vector $L_{(\mathrm{in})}$ of size $n_{\mathrm{in}} = \min(\lfloor 1/\bar{p}_{z,v} \rfloor, m)$
  **for** obligors $j = 1, \dots, m$ **do**
    initialize $i$ to zero and cont fo false
    **repeat**
      generate the $U(0, 1)$ random variate $U$
      set $i = i + \text{ceiling}(\log(1 - U)/\log(1 - p_j))$
      **if** $(i > n_{\mathrm{in}})$ **then** set cont to true
      **else** set $L^{(i)}_{(\mathrm{in})} = L^{(i)}_{(\mathrm{in})} + c_j$
    **until** cont = true
  **end for**
  compute $\bar{p}^{(k)}_{\mathrm{in}} = \frac{1}{n_{\mathrm{in}}} \sum_{i=1}^{n_{\mathrm{in}}} \mathbf{1}\{L^{(i)}_{(\mathrm{in})} > x\}$ where $\bar{p}^{(k)}_{\mathrm{in}}$ denotes the average loss probability of the $k$th outer replication
**end for**
return $\frac{1}{n} \sum_{k=1}^{n} \bar{p}^{(k)}_{\mathrm{in}}$

## 4 Integrating IS with inner replications using the geometric shortcut: dependent obligors

Implementing inner replications using the geometric shortcut alone is not sufficient to decrease the variance for highly dependent obligors. Thus, we make use of IS for further decreasing the variance of the simulations. A vector $Z$ of i.i.d. normal variates and an independent chi square random variate $V$ are generated in the beginning of the loop of Algorithm 1. An importance sampling strategy is best applied directly to these variates. We change their distribution in order to increase the frequency of very high loss values $L$. To obtain easy IS-densities and simple likelihood ratios we only change the mean values of the normal variates by adding a mean shift vector with positive entries to the normal vector $Z$. As the chi-square distribution is a special case of the gamma distribution with shape parameter $\frac{v}{2}$ and scale parameter 2, a natural choice for the IS density for $V$ is the gamma distribution. As $X'_j$ is inversely proportional to $V$ in (2.3), a decrease of $V$ will result in an increase in the dependence of default of obligors. Thus as IS density we use a gamma distribution with the same shape parameter but a smaller scale parameter as this increases the probability of very high losses.

It is well known that a main practical problem in the application of importance sampling is the choice of the parameters of the IS-distribution. We utilize the general idea (see e.g., [10]) to select the parameters such that the mode of the resulting IS density is equal to the mode of the zero-variance IS function which is for our problem defined by

$$f_0(z, v) = P(L > x | Z = z, V = v)\, f_N(z)\, f_V(v), \tag{4.1}$$

where $f_N(z)$ denotes the density of i.i.d. standard normal variates and $f_V(v)$ is the density of the chi-square distribution with $\nu$ degrees of freedom.

Finding the mode $\mu$ of $f_0(z, v)$ requires the solution of the multidimensional optimization problem;

$$\max_{z,v}(P(L > x | Z = z, V = v) e^{-z^T z/2} f_V(v)) \,. \tag{4.2}$$

Finding the exact solution for (4.2) is usually difficult. Thus, we need an approximate method. [11] lists some of the possible approximations for computing $P(L > x | Z = z)$ for the normal copula framework. These could be directly applied to compute $P(L > x | Z = z, V = v)$, since obligors default independently conditional on the random variables in both models. For information on the advantages and disadvantages of the different strategies to select the mean shift for IS see Chapter 5 of [22]. There it is concluded from an empirical comparison that the reached variance reductions are very similar for all methods. We therefore describe and use the normal approximation of $P(L > x | Z, V)$. Since, $E[L|Z = z, V = v] = \sum_{j=1}^{m} c_j\, p_j(z, v)$ and $V[L|Z = z, V = v] = \sum_{j=1}^{m} c_j^2 \left[p_j(z, v) - p_j(z, v)^2\right]$ we have:

$$P\,(L > x | Z = z, V = v) \approx 1 - \Phi\left(\frac{x - E[L|Z = z, V = v]}{\sqrt{\mathrm{Var}[L|Z = z, V = v]}}\right).$$

Thus, to obtain the mode $\mu$ for the IS distribution we solve the optimization problem

$$\max_{\mu}\left[1 - \Phi\left(\frac{x - E[L|(Z, V) = \mu]}{\sqrt{\mathrm{Var}[L|(Z, V) = \mu]}}\right)\right] e^{-\mu_z^T \mu_z/2} f_V(\mu_v) \tag{4.3}$$

where $\mu_z$ and $\mu_v$ are $z$ and $v$ component of the mode vector $\mu$.

We use the multidimensional optimization function, `nmsimplex2`, of GSL [8] to solve (4.3). This function is a very efficient implementation of Nead–Melder method which does not require gradients in contrast to quasi-Newton methods. In our numerical experiments, we could find the optimal mode at most in 5 seconds even for a 21-factor model by just using the same starting value of 1.0 for all dimensions.

After solving $\mu$ using (4.3), we can directly use the $\mu_z$ as the optimal mean shift for $Z$ since it has a multinormal distribution. To calculate the optimal scale parameter $\theta$ of the gamma IS-density for $V$, we use

$$\theta = \frac{\mu_v}{\nu/2-1}, \tag{4.4}$$

as this scale parameter implies that the mode is equal to $\mu_v$. Then the likelihood ratio is

$$w_{\mu_z,\theta}(Z,V) = \exp(-\mu_z^T Z + \mu_z^T \mu_z/2 - V/2 + V/\theta + \log(\theta/2)\nu/2), \tag{4.5}$$

where $\exp(-\mu_z^T Z + \mu_z^T \mu_z/2)$ accounts for the mean shift we have added to the normal vector and the term $\exp(-V/2 + V/\theta + \log(\theta/2)\nu/2)$ relates the density of the $\chi^2_\nu$ distribution to that of the gamma distribution with shape parameter $\nu/2$ and scale parameter $\theta$. The full algorithm is given in Algorithm 3 for the sake of completeness.

**Algorithm 3** Tail loss probability simulation using integration of IS with inner replications using the geometric shortcut for dependent obligors.

compute $\mu$ using (4.3), compute $\theta$ using (4.4).
**for** replications $k = 1, \ldots, n$ **do**
  generate $z_l \sim N(\mu_{z,l}, 1), l = 1, \ldots, d$, independently
  generate $v$ from the gamma distribution with shape parameter $\nu/2$ and scale parameter $\theta$;
  calculate $w^{(k)}_{\mu_z,\theta}$ as in (4.5)
  calculate $p_j(z,v)$, $j = 1, \ldots, m$, as in (2.4) where $z = (z_1, \ldots, z_d)$
  initialize a loss vector $L_{(\text{in})}$ of length $n_{\text{in}} = \min(\lfloor 1/\bar{p}_{z,v} \rfloor, m)$ to zero.
  **for** obligors $j = 1, \ldots, m$ **do**
    initialize $i$ to zero and cont fo false
    **repeat**
      generate the $U(0,1)$ random variate $U$
      set $i = i + \text{ceiling}(\log(1-U)/\log(1-p_j))$
      **if** $(i > n_{\text{in}})$ **then** set cont to true
      **else** set $L^{(i)}_{(\text{in})} = L^{(i)}_{(\text{in})} + c_j$
    **until** cont $=$ true
  **end for**
  compute $\bar{p}^{(k)}_{\text{in}} = \frac{1}{n_{\text{in}}} \sum_{i=1}^{n_{\text{in}}} \mathbf{1}\{L^{(i)}_{(\text{in})} > x\}$ where $\bar{p}^{(k)}_{\text{in}}$ stands for the average loss probability of the $k$th outer replication
**end for**
return $\frac{1}{n} \sum_{k=1}^{n} w^{(k)}_{\mu_z,\theta} \bar{p}^{(k)}_{\text{in}}$

## 5 ES simulation for dependent obligors

We described our new methodology for tail loss probability computation. It is time to explain how the same methodology can be used for the computation of ES. We define ES as the expected losses conditioned on exceeding a large threshold $x$:

$$\mathrm{ES}(x) = E[L|L > x]\,.$$

If we assume that $P(L \geq x) > 0$, ES can be written as

$$r = E[L|L \geq x] = \frac{E\left[L\,\mathbf{1}\{L \geq x\}\right]}{P(L \geq x)}\,.$$

The naive simulation estimate for this ratio is

$$\hat{r}^{\text{naive}} = \frac{\sum_{k=1}^{n} L^{(k)}\mathbf{1}\{L^{(k)} \geq x\}}{\sum_{k=1}^{n} \mathbf{1}\{L^{(k)} \geq x\}}\,. \tag{5.1}$$

Algorithm 4 gives all the details of how to use that estimate.

**Algorithm 4** Naive simulation for computing ES for dependent obligors.

1: **for** replications $k = 1, \ldots, n$ **do**
2: generate $z_l \sim N(0,1), l = 1, \ldots, d$ and $v$ from $\chi^2_\nu$ distribution independently
3: calculate $p_j(z, v), j = 1, \ldots, m$, as in (2.4) where $z = (z_1, \ldots, z_d)$
4: **for** obligors $j = 1, \ldots, m$ **do**
5: generate a $U(0,1)$ variate $U$
6: if $(U < p_j(z, v))$ set $L^{(k)} = L^{(k)} + c_j$
7: **end for**
8: **end for**
9: return $\hat{r}^{\text{naive}}$ using (5.1)

If we use outer IS and the geometric shortcut our new estimate of expected shortfall is

$$\hat{r}^{\text{new}} = \frac{\sum_{k=1}^{n} \frac{w^{(k)}}{n_k} \sum_{i=1}^{n_k} L^{(k,i)} 1\{L^{(k,i)} \geq x\}}{\sum_{k=1}^{n} \frac{w^{(k)}}{n_k} \sum_{i=1}^{n_k} 1\{L^{(k,i)} \geq x\}} = \frac{\sum_{k=1}^{n} w^{(k)} \bar{L}_{\text{in}}^{(k)}}{\sum_{k=1}^{n} w^{(k)} \bar{p}_{\text{in}}^{(k)}} \tag{5.2}$$

where $\bar{L}_{\text{in}}^{(k)}$ stands for the average of the loss values that are greater than $x$ and $\bar{p}_{\text{in}}^{(k)}$ stands for the average loss probability of the $k$th outer replication.

To estimate the accuracy of (5.2) we use a general result for ratio estimates given on page 234 of [9]. It is applicable as the values $(\bar{L}_{\text{in}}^{(k)}, \bar{p}_{\text{in}}^{(k)})$ for $k = 1, \ldots, n$ are i.i.d. Thus we use the confidence interval

$$\hat{r}^{\text{new}} \pm z_{\delta/2} \frac{\hat{\sigma}^{\text{new}}}{\sqrt{n}} \tag{5.3}$$

where

$$\hat{\sigma}^{\text{new}} = \left( \frac{n \sum_{k=1}^{n} (w^{(k)} \bar{L}_{\text{in}}^{(k)} - \hat{r}^{\text{new}} w^{(k)} \bar{p}_{\text{in}}^{(k)})^2}{(\sum_{k=1}^{n} w^{(k)} \bar{p}_{\text{in}}^{(k)})^2} \right)^{1/2}. \tag{5.4}$$

The full algorithm is given in Algorithm 5.

**Algorithm 5** ES simulation using integration of IS with inner replications using the geometric shortcut for dependent obligors.

compute $\mu$ using (4.3). Then, compute $\theta$ using (4.4).
**for** replications $k = 1, \ldots, n$ **do**
    generate $z_l \sim N(\mu_{z,l}, 1), l = 1, \ldots, d$, independently
    generate $v$ from the gamma distribution with shape parameter $\nu/2$ and scale parameter $\theta$;
    calculate $w_{\mu_z,\theta}^{(k)}$ as in (4.5)
    calculate $p_j(z, v)$, $j = 1, \ldots, m$, as in (2.4) where $z = (z_1, \ldots, z_d)$
    construct a loss vector $L_{(\text{in})}$ of size $n_{\text{in}} = \min(\lfloor 1/\bar{p}_{z,v} \rfloor, m)$
    **for** obligors $j = 1, \ldots, m$ **do**
        initialize $i$ to zero and cont fo false
        **repeat**
            generate the $U(0, 1)$ random variate $U$
            set $i = i + \text{ceiling}(\log(1 - U)/\log(1 - p_j))$
            **if** $(i > n_{\text{in}})$ **then** set cont to true
            **else** set $L_{(\text{in})}^{(i)} = L_{(\text{in})}^{(i)} + c_j$
        **until** cont = true
    **end for**
    compute $\bar{p}_{\text{in}}^{(k)} = \frac{1}{n_{\text{in}}} \sum_{i=1}^{n_{\text{in}}} \mathbf{1}\{L_{(\text{in})}^{(i)} > x\}$ where $\bar{p}_{\text{in}}^{(k)}$ stands for the average loss probability of the $k$th outer replication
    compute $\bar{L}_{\text{in}}^{(k)} = \frac{1}{n_{\text{in}}} \sum_{i=1}^{n_{\text{in}}} L_{(\text{in})}^{(i)} \mathbf{1}\{L_{(\text{in})}^{(i)} > x\}$ where $\bar{L}_{\text{in}}^{(k)}$ stands for the average of the loss values that are greater than $x$ for the $k$th outer replication
**end for**
return $\hat{r}^{\text{new}}$ using (5.2)

## 6 Numerical results

In this section we compare the performance of our new approach for the $t$-copula model with naive Monte Carlo simulation. The efficiency of a simulation method is inversely proportional to the product of the sampling variance and the required simulation time (see e.g., [14] and [12]). This classical definition of simulation efficiency is valid for our problem as for fixed credit portfolios the sampling variance is $O(1/n)$ and the simulation time is $O(n)$ (the canonical case of [12]). We therefore report as a main result of our comparison the efficiency ratio (E.R.), simply the ratio of the product of the sampling variance and the execution time of the naive algorithm and our new algorithm.

To assess the efficiency of the proposed method, [23] make use of two credit portfolio examples of [11] and add a third example that has somewhat smaller dependence between obligors. We use the same numerical examples to measure the efficiency of our method.

The first numerical example of [11] is a portfolio of $m = 1000$ obligors in a 10-factor model. The marginal default probabilities $p_j = 0.01(1+\sin(16\Pi j/m))$ thus vary between 0 and 2%; the exposures $c_j = (\lceil 5j/m \rceil)^2$ take the values $1, 4, 9, 16$, and 25, with 200 obligors at each level. These parameters represent a significant departure from a homogeneous model. The factor loadings $a_{jl}$ are generated independently and uniformly from the interval $(0, 1/\sqrt{10})$; the upper limit of this interval ensures that the sum of squared entries for each obligor does not exceed 1. Note that this upper limit also implies that for some of the obligors the sum of the squares of the $a_{ji}$ values are close to 1 indicating that this credit portfolio contains strongly correlated obligors. We report the point estimates and the half length of the 95% confidence intervals in percent (95% C.I.) for tail loss probability and ES for three different $x$-values for the $t$-copula ($\nu = 10$) model in Table 1. We also report the observed execution times and the efficiency ratio (E.R.) of the new algorithm and the naive method; E.R. values larger than one indicate that the new algorithm has a higher efficiency than naive method. Table 2 shows how the degrees of freedom for the $t$-copula affects the computed results given in Table 1. Note that we use the simulation strategy proposed in [23] for $\nu = \infty$, since this is the normal copula case.

The second numerical example of [11] is a 21-factor model with 1000 obligors. The marginal default probabilities fluctuate as in the first example, and the exposures $c_j$ increases from 1 to 100 linearly as $j$ increases from 1 to 1000. The matrix of the factor loadings, $A = (a_{jl}, j = 1, \ldots, 1000, l = 1, \ldots, 21)$, has the following block structure:

$$A = \left( R \left| \begin{matrix} F & & \\ & \ddots & \\ & & F \end{matrix} \right| \begin{matrix} G \\ \vdots \\ G \end{matrix} \right), \quad G = \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix},$$

with $R$ a column vector of 1000 entries all equal to 0.8; F, a column vector of 100 entries all equal to 0.4; $G$ a $100 \times 10$ matrix, and $g$, a column vector of 10 entries, all equal to 0.4. Note that now each obligor only has three non zero $a_{ji}$ values, 0.8, 0.4 and 0.4. Considering that the sum of the squares of these values is 0.96 and also that the entry of 0.8 is constant for the first column, we can see that in this example the dependence between obligors is strong. The performance results for this example are summarized in Tables 3 and 4.

| $x$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 500 | 5.02e−02(30) | ±2.7% | 5.09e−02(52) | ±0.72% | 7.98 |
| 2000 | 4.15e−03(30) | ±9.6% | 4.20e−03(51) | ±0.94% | 60.5 |
| 5000 | 1.30e−04(30) | ±54% | 1.02e−04(47) | ±1.2% | 2210 |
| 500 | 1048.0(30) | ±1.7% | 1050.9(52) | ±0.39% | 11.2 |
| 2000 | 2803.8(30) | ±2.5% | 2790.5(51) | ±0.22% | 77.6 |
| 5000 | 5686.6(31) | ±7.3% | 5732.0(47) | ±0.12% | 2586 |

Table 1. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula ($\nu = 10$) model for the 10-factor model. $n = 100{,}000$. Execution times (in seconds) are in parentheses.

| $\nu$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 5 | 8.22e−03(31) | ±6.8% | 7.47e−03(50) | ±0.91% | 42.3 |
| 15 | 2.81e−03(30) | ±11.7% | 2.99e−03(50) | ±0.95% | 79.8 |
| $\infty$ | 7.70e−04(26) | ±22% | 8.48e−04(45) | ±1.0% | 236 |
| 5 | 2972.8(31) | ±2.2% | 3011.3(50) | ±0.27% | 42.3 |
| 15 | 2663.0(31) | ±2.7% | 2704.1(50) | ±0.20% | 109 |
| $\infty$ | 2404.0(26) | ±4.0% | 2474.4(44) | ±0.15% | 421 |

Table 2. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula model with $\nu$ degrees of freedom for the 10-factor model. $x = 2000$ and $n = 100{,}000$. Execution times (in seconds) are in parentheses.

| $x$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 2500 | 4.65e−02(38) | ±2.8% | 4.67e−02(57) | ±0.93% | 6.04 |
| 20,000 | 3.96e−03(37) | ±9.9% | 3.94e−03(59) | ±1.1% | 52.4 |
| 40,000 | 1.60e−04(37) | ±49% | 2.02e−04(59) | ±1.5% | 446 |
| 2500 | 8684.5(38) | ±2.4% | 8626.6(56) | ±0.63% | 9.96 |
| 20,000 | 27,712.4(37) | ±2.2% | 27,486.2(60) | ±0.23% | 56.9 |
| 40,000 | 42,432.4(38) | ±2.3% | 43,117.4(59) | ±0.073% | 612 |

Table 3. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula ($\nu = 10$) model for the 21-factor model. $n = 100{,}000$. Execution times (in seconds) are in parentheses.

| $\nu$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 5 | 5.03e−03(36) | ±8.7% | 4.92e−03(58) | ±1.1% | 40.8 |
| 15 | 3.68e−03(38) | ±10.2% | 3.52e−03(59) | ±1.2% | 49.8 |
| $\infty$ | 2.49e−03(32) | ±12% | 2.71e−03(50) | ±1.1% | 65.5 |
| 5 | 27,818.4(36) | ±1.9% | 28,215.1(57) | ±0.25% | 33.0 |
| 15 | 27,598.2(37) | ±2.6% | 27,138.3(58) | ±0.25% | 71.7 |
| $\infty$ | 26,485.9(33) | ±2.6% | 26,405.1(49) | ±0.21% | 103 |

Table 4. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula model with $\nu$ degrees of freedom for the 10-factor model. $x = 20{,}000$ and $n = 100{,}000$. Execution times (in seconds) are in parentheses.

The third numerical example is a 5-factor model with 1200 obligors. Default probabilities are generated independently and uniformly from the interval $[0, 0.02]$ and exposure levels are defined by $c_j = (\lceil 20j/m \rceil)^2$. To define the factor loadings the obligors are separated into 6 segments of size 200. For each segment the factors are generated uniformly from the interval $(0, \max)$. For the structure of the matrix and the max values see Table 5. Note that the maximal sum of the squares of the $a_{ji}$ are 0.51, 0.26 and 0.5 for segments 1, 2 and 3 respectively. Thus the dependence of the obligors is clearly smaller than for examples 1 and 2. The results of the numerical experiments are reported in Tables 6 and 7.

When we look at Tables 1, 3 and 6, we first observe that the new method is always more efficient than naive (efficiency ratios are greater than one). At the extremes the efficiency ratios increase which is attributed to IS. The performance of the new method can also be assessed by solely looking at the 95% confidence intervals.

| Segment | Obligor $j$ | $a_{j,1}$ | $a_{j,2}$ | $a_{j,3}$ | $a_{j,4}$ | $a_{j,5}$ |
|---|---|---|---|---|---|---|
| $1A$ | 1–200 | $U(0, 0.5)$ | $U(0, 0.5)$ | $U(0, 0.1)$ | | |
| $1B$ | 201–400 | $U(0, 0.5)$ | $U(0, 0.1)$ | $U(0, 0.5)$ | | |
| $2A$ | 401–600 | $U(0, 0.4)$ | | $U(0, 0.3)$ | $U(0, 0.1)$ | |
| $2B$ | 601–800 | $U(0, 0.4)$ | | $U(0, 0.1)$ | $U(0, 0.3)$ | |
| $3A$ | 801–1000 | $U(0, 0.5)$ | | | $U(0, 0.4)$ | $U(0, 0.3)$ |
| $3B$ | 1001–1200 | $U(0, 0.5)$ | | | $U(0, 0.3)$ | $U(0, 0.4)$ |

Table 5. Distributions used to generate the factor loadings for the 5-factor model.

| $x$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 5000 | 8.70e−02(33) | ±2.0% | 8.72e−02(61) | ±0.61% | 5.93 |
| 15,000 | 1.27e−02(33) | ±5.5% | 1.28e−02(61) | ±0.76% | 27.3 |
| 30,000 | 1.87e−03(33) | ±14% | 1.64e−03(60) | ±0.89% | 184 |
| 5000 | 10,256.7(33) | ±1.3% | 10,207.6(61) | ±0.36% | 6.79 |
| 15,000 | 22,120.1(33) | ±1.8% | 22,181.8(62) | ±0.22% | 35.3 |
| 30,000 | 38,904.0(33) | ±3.3% | 38,181.2(60) | ±0.15% | 263 |

Table 6. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula ($\nu = 10$) model for the 5-factor model. $n = 100{,}000$. Execution times (in seconds) are in parentheses.

| $\nu$ | prob./ES (naive) | 95% C.I. | prob./ES (new) | 95% C.I. | E.R. |
|---|---|---|---|---|---|
| 5 | 2.37e−02(33) | ±4.0% | 2.30e−02(61) | ±0.76% | 15.7 |
| 15 | 7.77e−03(33) | ±7.0% | 8.40e−03(61) | ±0.79% | 36.8 |
| $\infty$ | 9.80e−04(28) | ±20% | 8.72e−04(53) | ±0.86% | 358 |
| 5 | 25,308.5(34) | ±1.8% | 25,576.2(61) | ±0.32% | 16.8 |
| 15 | 20,844.0(33) | ±2.0% | 20,966.5(61) | ±0.19% | 55.5 |
| $\infty$ | 17,941.7(28) | ±4.1% | 18,119.1(54) | ±0.10% | 788 |

Table 7. Tail loss probabilities or expected shortfalls and 95% confidence intervals as percentage of the point estimates for the $t$-copula model with $\nu$ degrees of freedom for the 10-factor model. $x = 15{,}000$ and $n = 100{,}000$. Execution times (in seconds) are in parentheses.

In Tables 2, 4 and 7, we increase the degrees of freedom of the $t$-copula. Computed tail loss probabilities and expected shortfalls decrease for an increase in the degrees of freedom of the $t$-copula. This is a quite expected result since the $t$-copula approaches to the normal copula ($\nu = \infty$) as we increase the degrees of freedom. This results in an increase in the efficiency ratios as smaller tail loss probabilities imply higher variance reduction. However, the increase of the efficiency ratios is different for each of the examples. While, we have a very large increase for the 5-factor model, the increase is moderate for the 21-factor model. Here we see the effect of the magnitude of correlation of defaults between obligors. Note that the correlation is weakest in the 5-factor model and strongest in the 21-factor model. Thus, if we have highly correlated obligors in our model then we expect less difference between the normal and $t$-copula model.

We also tried the approach of using only outer importance sampling and a single naive inner repetition. Due to the higher extremal dependence in the $t$-copula model and faster execution time, the efficiency of that approach was not much worse than the combination of IS with inner replications of the geometric shortcut.

## 7 Conclusions

We presented an efficient method for simulating tail loss probabilities and expected shortfalls under the $t$-copula model. We use the geometric shortcut idea presented for the normal copula framework in [23]. We also combined this idea with an efficient and easy-to-apply IS. We tested the performance of the proposed method on various numerical examples. Our numerical results showed that the proposed method is much more efficient than the naive method for small tail loss probabilities. Its efficiency decreases for increasing tail loss probabilities, but it is still greater than 7 for a tail loss probability of 0.05. Our numerical results also show that the differences between tail loss probabilities and expected shortfalls under the normal and $t$-copula model increase when the correlation between obligors becomes weaker.

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Received September 25, 2009; revised September 21, 2010.

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