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COUNTERPARTY RISK FOR CREDIT DEFAULT SWAPS: IMPACT OF SPREAD VOLATILITY AND DEFAULT CORRELATION

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We consider counterparty risk for Credit Default Swaps (CDS) in presence of correlation between default of the counterparty and default of the CDS reference credit. Our approach is innovative in that, besides default correlation, which was taken into account in earlier approaches, we also model credit spread volatility. Stochastic intensity models are adopted for the default events, and defaults are connected through a copula function. We find that both default correlation and credit spread volatility have a relevant impact on the positive counterparty-risk credit valuation adjustment to be subtracted from the counterparty-risk free price. We analyze the pattern of such impacts as correlation and volatility change through some fundamental numerical examples, analyzing wrong-way risk in particular. Given the theoretical equivalence of the credit valuation adjustment with a contingent CDS, we are also proposing a methodology for valuation of contingent CDS on CDS.

Keywords: Counterparty risk; credit valuation adjustment; Credit Default Swaps; contingent credit default swaps; credit spread volatility; default correlation; stochastic intensity; copula functions; wrong way risk.

1. Introduction

We consider counterparty risk for Credit Default Swaps (CDS) in presence of correlation between default of the counterparty and default of the CDS reference credit. We assume the party that is computing the counterparty risk adjustment to be default free, as a possible approximation to situations where this party has a much

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higher credit quality than the counterparty. Our approach is innovative in that, besides default correlation, which was taken into account in earlier approaches, we also model explicitly credit spread volatility. This is particularly important when the underlying reference contract itself is a CDS, as the counterparty credit valuation adjustment involves CDS options, and modeling options without volatility in the underlying asset is quite undesirable. We investigate the impact of the reference volatility on the counterparty adjustment as a fundamental feature that is ignored or not studied explicitly in other approaches.

Hull and White [13] address the counterparty risk problem for CDS by resorting to default barrier correlated models, without considering explicitly credit spread volatility in the reference CDS. Leung and Kwok [14], building on Collin-Dufresne *et al.* [12], model default intensities as deterministic constants with default indicators of other names as feeds. The exponential triggers of the default times are taken to be independent and default correlation results from the cross feeds, although again there is no explicit modeling of credit spread volatility. Furthermore, most models in the industry, especially when applied to Collateralized Debt Obligations or k-th to default baskets, model default correlation but ignore credit spread volatility. Credit spreads are typically assumed to be deterministic and a copula is postulated on the exponential triggers of the default times to model default correlation. This is the opposite of what used to happen with counterparty risk for interest rate underlyings, for example in Sorensen and Bollier [15] or Brigo and Masetti [8], where correlation was ignored and volatility was modeled instead. Here we rectify this, with a model that takes into account credit spread volatility besides the still very important correlation. Ignoring correlation among underlying and counterparty can be dangerous, especially when the underlying instrument is a CDS. Indeed, this credit underlying case involves default correlation, that is perceived in the market as more relevant than the dubious interest-rate/credit-spread correlation of the interest rate or commodity underlying case. It is not so much that the latter is less relevant because it would have no impact in counterparty risk credit valuation adjustments. We have seen in Brigo and Pallavicini [10, 11] and Brigo and Bakkar [4] that changing this correlation parameter has a relevant impact for interest rate and commodities underlyings, respectively. Still, the value of said correlation is difficult to estimate historically or imply from market quotes, and the historical estimation often produces a very low or even slightly negative correlation parameter in the interest rate case. So even if this parameter has an impact, it is difficult to assign a value to it and often this value would be practically null for interest rate payouts. On the contrary, default correlation is more clearly perceived, as measured also by implied correlation in the quoted indices tranches markets (i-Traxx and CDX).

To investigate the impact of both default correlation and credit spread volatility, tractable stochastic intensity diffusive models with possible jumps are adopted for the default events and defaults are connected through a copula function on the exponential triggers of the default times. We find that both default correlation and credit spread volatility have a relevant impact on the positive credit valuation adjustment one needs to subtract from the default free price to take into account counterparty risk. We analyze the pattern of such impacts as volatility and correlation parameters vary through some fundamental numerical examples, and find that results under extreme default correlation (wrong way risk) are very sensitive to credit spread volatility. This points out that credit spread volatility should not be ignored in these cases. Given the theoretical equivalence of the credit valuation adjustment with a contingent CDS, we are also proposing a methodology for valuation of contingent CDS on CDS. This can be particularly relevant for a financial institution that has bought protection or insurance on CDS from other institutions whose credit quality is deteriorating. The case of mono-line insurers after the sub-prime crisis is just a possible example.

We finally describe the structure of the paper, and how to benefit most of it from the point of view of readers with different backgrounds.

The essential results are described in the case study in Sec. 6, so the reader aiming at getting the main message of the paper with minimal technical implications can go directly to this section, that has been written to be as self-contained as possible. Otherwise, Sec. 2 describes the counterparty risk valuation problem in quite general terms and, apart a few technicalities on filtrations that can be overlooked at first reading, is quite intuitive. Sec. 3 describes the reduced form model setup of the paper with stochastic intensities and a copula on the exponential triggers. A detailed presentation of the shifted squared root (jump) diffusion (SSRJD) model and of its calibration to CDS, previously analyzed in Brigo and Alfonsi [3], Brigo and Cousot [6], and Brigo and El-Bachir [7], is given. Section 4 details how the general formula for the counterparty credit valuation adjustment given in Sec. 2 can be written under the specific CDS payoff and modeling assumptions of the paper, although formulas derived here will not be used, as we will proceed through a more direct numerical approach. These calculations can however give a feeling for the complexity of the problem and for the kind of issues one has to face in these situations, and for this reason are presented. Section 5 details the numerical techniques that are used to compute the credit valuation adjustment in the case study. Finally, Sec. 6 briefly recaps the modeling assumptions and illustrates the paper conclusions with the case study itself.

2. General Valuation of Counterparty Risk

We denote by τ_1 the default time of the credit underlying the CDS, and by τ_2 the default time of the counterparty. We assume the investor who is considering a transaction with the counterparty to be default-free. This assumption is removed in Brigo and Capponi [5], that deals with bilateral counterparty risk with focus on CDS, where we allow the investor to default as well and the valuation to become symmetric.

We place ourselves in a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$. The filtration $(\mathcal{G}_t)_t$ models the flow of information of the whole market, including credit and defaults. $\mathbb{Q}$ is the risk neutral measure. This space is endowed also with a right-continuous and complete sub-filtration $\mathcal{F}_t$ representing all the observable market quantities but the default events (hence $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\{\tau_1 \leq u\}, \{\tau_2 \leq u\} : u \leq t)$ is the right-continuous filtration generated by the default events).

We set $\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot \,|\, \mathcal{G}_t)$, the risk neutral expectation leading to prices.

Let us call T the final maturity of the payoff we need to evaluate. If $\tau_2 > T$ there is no default of the counterparty during the life of the product and the counterparty has no problems in repaying the investors. On the contrary, if $\tau_2 \leq T$ the counterparty cannot fulfill its obligations and the following happens. At τ_2 the Net Present Value (NPV) of the residual payoff until maturity is computed: If this NPV is negative (respectively positive) for the investor (defaulted counterparty), it is completely paid (received) by the investor (counterparty) itself. If the NPV is positive (negative) for the investor (counterparty), only a recovery fraction R_{EC} of the NPV is exchanged.

Let us denote by $\Pi^D(t,T)$ the sum of all payoff terms between t and T, all terms discounted back at t, and subject to counterparty default risk. We denote by $\Pi(t,T)$ the analogous quantity when counterparty risk is not considered. All these payoffs are seen from the point of view of the safe "investor" (i.e. the company facing counterparty risk). Then we have the net present value at time τ_2 as $\text{NPV}(\tau_2, T) = \mathbb{E}_{\tau_2}\{\Pi(\tau_2, T)\}$ and

$$\begin{aligned}\Pi^D(t,T) = \mathbf{1}_{\{\tau_2 > T\}}\Pi(t,T) + \mathbf{1}_{\{t < \tau_2 \leq T\}}[\Pi(t,\tau_2) \\ + D(t,\tau_2)(\text{R}_{\text{EC}}(\text{NPV}(\tau_2,T))^+ - (-\text{NPV}(\tau_2,T))^+)]\end{aligned} \tag{2.1}$$

being $D(u,v)$ the stochastic discount factor at time u for maturity v. This last expression is the general price of the payoff under counterparty risk. Indeed, if there is no early counterparty default this expression reduces to risk neutral valuation of the payoff (first term in the right hand side); in case of early default, the payments due before default occurs are received (second term), and then if the residual net present value is positive only a recovery of it is received (third term), whereas if it is negative it is paid in full (fourth term).

We notice incidentally that our definition involves an expectation $\mathbb{E}_{\tau_2}$, i.e. conditional on $\mathcal{G}_{\tau_2}$ where

$$\mathcal{G}_{\tau_2} := \sigma(\mathcal{G}_t \cap \{t \leq \tau_2\}, t \geq 0), \quad \mathcal{F}_{\tau_2} := \sigma(\mathcal{F}_t \cap \{t \leq \tau_2\}, t \geq 0).$$

It is possible to prove the following

Proposition 2.1. (General counterparty risk pricing formula) *At valuation time t, and on $\{\tau_2 > t\}$, the price of our payoff under counterparty risk is*

$$\mathbb{E}_t\{\Pi^D(t,T)\} = \mathbb{E}_t\{\Pi(t,T)\} - \underbrace{L_{GD}\,\mathbb{E}_t\{\mathbf{1}_{\{t<\tau_2\leq T\}}D(t,\tau_2)\,(NPV(\tau_2))^+\}}_{Positive\ counterparty\text{-}risk\ adj.\,(CR\text{-}CVA)} \tag{2.2}$$

where $L_{GD} = 1 - R_{EC}$ is the Loss Given Default *and the recovery fraction R_{EC} is assumed to be deterministic. It is clear that the value of a defaultable claim is the value of the corresponding default-free claim minus an option part, in the specific a call option (with zero strike) on the residual NPV giving nonzero contribution* *only in scenarios where $\tau_2 \leq T$. This adjustment, including the* Lgd *factor, is called counterparty-risk credit valuation adjustment (CR-CVA). Counterparty risk adds an optionality level to the original payoff.*

For a proof see for example Brigo and Masetti [8].

Notice finally that the previous formula can be approximated as follows. Take $t = 0$ for simplicity and write, on a discretization time grid $T_0, T_1, \ldots, T_b = T$,

$$\begin{aligned}\mathbb{E}[\Pi^D(0,T_b)] &= \mathbb{E}[\Pi(0,T_b)] - \mathrm{L}_{\mathrm{GD}} \sum_{j=1}^{b} \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \leq T_j\}} D(0,\tau_2)(\mathbb{E}_{\tau_2}\Pi(\tau_2,T_b))^+] \\ &\approx \mathbb{E}[\Pi(0,T_b)] - \underbrace{\mathrm{L}_{\mathrm{GD}} \sum_{j=1}^{b} \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \leq T_j\}} D(0,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T_b))^+]}_{\text{approximated (positive) adjustment}}\end{aligned} \tag{2.3}$$

where the approximation consists in postponing the default time to the first T_i following τ_2. From this last expression, under independence between Π and τ_2, one can factor the outer expectation inside the summation in products of default probabilities times option prices. This way we would not need a default model for the counterparty but only survival probabilities and an option model for the underling market of Π. This is only possible, in our case of a CDS as underlying contract, if the default correlation between the CDS reference credit and the counterparty is zero. This is what led to earlier results on swaps with counterparty risk in interest rate payoffs in Brigo and Masetti [8]. In this paper we do not assume zero correlation, so that in general we need to compute the counterparty risk without factoring the expectations. To do so we need a default model for the counterparty, to be correlated with the default model for the underlying CDS.

2.1. *Contingent CDS*

A Contingent Credit Default Swap (CCDS) is a CDS that, upon the default of the reference credit, pays the loss given default on the residual net present value of a given portfolio if this is positive.

It is immediate then that the default leg CCDS valuation, when the CCDS underlying portfolio constituting the protection notional is Π, is simply the counterparty credit valuation adjustment in Formula (2.2). When Π is an underlying CDS, our adjustments calculations above can then be interpreted also as examples of pricing contingent CDS on CDS.

3. Modeling Assumptions

In this section we consider a reduced form model that is stochastic in the default intensity both for the counterparty and for the CDS reference credit. We will not correlate the spreads with each other, as typically spread correlation has a much lower impact on dependence of default times than default correlation. The latter is rigorously defined as a dependence structure on the exponential random variables characterizing the default times of the two names. This dependence structure is typically modeled with a copula function.

More in detail, we assume that the counterparty default intensity λ_2, and the cumulated intensity $\Lambda_2(t) = \int_0^t \lambda_2(s)ds$, are independent of the default intensity for the reference CDS λ_1, whose cumulated intensity we denote by Λ_1. We assume intensities to be strictly positive, so that $t \mapsto \Lambda(t)$ are invertible functions.

We assume deterministic default-free instantaneous interest rate r (and hence deterministic discount factors $D(s,t),\ldots$), but all our conclusions hold also under stochastic rates that are independent of default times.

We are in a Cox process setting, where

$$\tau_1 = \Lambda_1^{-1}(\xi_1), \quad \tau_2 = \Lambda_2^{-1}(\xi_2),$$

with ξ_1 and ξ_2 standard (unit-mean) exponential random variables whose associated uniforms $U_j = 1 - \exp(-\xi_j)$, $j = 1, 2$, are correlated through a copula function. We assume

$$U_j = 1 - \exp(-\xi_j), \;\; j = 1, 2, \quad \mathbb{Q}(U_1 < u_1, U_2 < u_2) =: C(u_1, u_2).$$

In the case study below we assume the copula C to be Gaussian and with correlation parameter ρ, although the choice can be easily changed, as the framework is general.

3.1. *CIR++ stochastic intensity models*

For the stochastic intensity model we set

$$\lambda_j(t) = y_j(t) + \psi_j(t;\beta_j), \quad t \geq 0, \;\; j = 1, 2 \tag{3.1}$$

where ψ is a deterministic function, depending on the parameter vector β (which includes y_0), that is integrable on closed intervals. The initial condition y_0 is one more parameter at our disposal: We are free to select its value as long as

$$\psi(0;\beta) = \lambda_0 - y_0.$$

We take each y to be a Cox Ingersoll Ross (CIR) process (see for example Brigo and Mercurio [9]):

$$\mathrm{d}y_j(t) = \kappa(\mu - y_j(t))\mathrm{d}t + \nu\sqrt{y_j(t)}\,\mathrm{d}Z_j(t), \quad j = 1, 2$$

where the parameter vectors are $\beta_j = (\kappa_j, \mu_j, \nu_j, y_j(0))$, with κ, μ, ν, y_0 positive deterministic constants. As usual, the Z are standard Brownian motion processes under the risk neutral measure, representing the stochastic shock in our dynamics.

Usually, for the CIR model one assumes a condition ensuring the origin to be inaccessible, the condition being $2\kappa\mu > \nu^2$. However, this limits the CDS implied volatility generated by the model when imposing also positivity of the shift ψ, which is a condition we will always impose in the following to avoid negative intensities.

This is why we do not enforce the condition $2\kappa\mu > \nu^2$ and in our case study below it will be violated.

Correlation in the spreads is a minor driver with respect to default correlation, so we assume that the two Brownian motions Z are independent. We will often use the integrated quantities

$$\Lambda(t) = \int_0^t \lambda_s ds, \quad Y(t) = \int_0^t y_s ds, \quad \text{and} \quad \Psi(t,\beta) = \int_0^t \psi(s,\beta)ds.$$

This kind of models and the related calibration to CDS has been investigated in detail in Brigo and Alfonsi [3], while Brigo and Cousot [6] examine the CDS implied volatility patterns associated with the model.

Notice that we can easily introduce jumps in the diffusion process. Brigo and El-Bachir [7] consider a formulation where

$$\mathrm{d}y_j(t) = \kappa(\mu - y_j(t))\mathrm{d}t + \nu\sqrt{y_j(t)}\mathrm{d}Z_j(t) + \mathrm{d}J_j(t), \quad j = 1, 2,$$

with

$$J_j(t) = \sum_{i=1}^{N_j(t)} Y_j^i$$

and N standard Poisson process with intensity α counting the jumps, and the Y's i.i.d. exponential random variables with mean γ representing the jump sizes. Besides deriving log-affine survival probability formulas re-shaped exactly in the same form as in the CIR model without jumps, Brigo and El-Bachir [7] derive a closed form solution for CDS options as well.

In the sequel we take $\alpha = 0$ and assume no jumps. However, all calculations and also the fractional Fourier transform method are exactly applicable to the extended model with jumps.

3.2. *CIR++ model: CDS calibration*

We focus on the calibration of the default model for the counterparty, the one for the reference credit being completely analogous. Since we are assuming deterministic rates, the default time τ_2 and interest rate quantities $r, D(s,t), \ldots$ are trivially independent. It follows that the (receiver) CDS valuation, for a CDS selling protection at time 0 for defaults between times T_a and T_b in exchange of a periodic premium rate S becomes

$$\begin{aligned}\mathrm{CDS}_{a,b}(0, S, \mathrm{L_{GD}}; \mathbb{Q}(\tau_2 > \cdot)) = S\Bigg[&-\int_{T_a}^{T_b} P(0,t)(t - T_{\gamma(t)-1})\mathrm{d}_t \boxed{\mathbb{Q}(\tau_2 \geq t)} \\ &+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau_2 \geq T_i)}\Bigg] \\ &+ \mathrm{L_{GD}}\left[\int_{T_a}^{T_b} P(0,t)\ \mathrm{d}_t \boxed{\mathbb{Q}(\tau_2 \geq t)}\right], \end{aligned} \qquad (3.2)$$

where in general $T_{\gamma(t)}$ is is the first T_j following t. This formula is model independent. This means that if we strip survival probabilities from CDS in a model independent way at time 0, to calibrate the market CDS quotes we just need to make sure that the survival probabilities we strip from CDS are correctly reproduced by the CIR++ model. Since the survival probabilities in the CIR++ model are given by

$$\mathbb{Q}(\tau_2 > t)_{\text{model}} = \mathbb{E}(e^{-\Lambda_2(t)}) = \mathbb{E}\exp\left(-\Psi_2(t,\beta) - Y_2(t)\right) \tag{3.3}$$

we just need to make sure

$$\mathbb{E}\exp\left(-\Psi_2(t,\beta_2) - Y_2(t)\right) = \mathbb{Q}(\tau_2 > t)_{\text{market}}$$

from which

$$\Psi_2(t,\beta_2) = \ln\left(\frac{\mathbb{E}(e^{-Y_2(t)})}{\mathbb{Q}(\tau_2 > t)_{\text{market}}}\right) = \ln\left(\frac{P^{\text{CIR}}(0,t,y_2(0);\beta_2)}{\mathbb{Q}(\tau_2 > t)_{\text{market}}}\right) \tag{3.4}$$

where we choose the parameters β_2 in order to have a positive function ψ_2 (i.e. an increasing Ψ_2) and P^{CIR} is the closed form expression for bond prices in the time homogeneous CIR model with initial condition $y_2(0)$ and parameters β_2 (see for example Brigo and Mercurio [9]). Thus, if ψ_2 is selected according to this last formula, as we will assume from now on, the model is easily and automatically calibrated to the market survival probabilities for the counterparty (possibly stripped from CDS data).

A similar procedure goes for the reference credit default time τ_1.

Once we have done this and calibrated CDS data through $\psi(\cdot,\beta)$, we are left with the parameters β, which can be used to calibrate further products. However, this will be interesting when single name option data on the credit derivatives market will become more liquid. Currently the bid-ask spreads for single name CDS options are large and suggest to either consider these quotes with caution, or to try and deduce volatility parameters from more liquid index options. At the moment we content ourselves of calibrating only CDS's. To help specifying β without further data we set some values of the parameters implying possibly reasonable values for the implied volatility of hypothetical CDS options on the counterparty and reference credit.

4. CDS Options Embedded in the Counterparty Risk Adjustment

We now move to computing the counterparty risk adjustment, as in Eq. (2.3).

The only non-trivial term to compute is

$$\mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}}(\mathbb{E}\{\Pi(T_j,T_b) \,|\, \mathcal{G}_{T_j}\})^+] \tag{4.1}$$

Now let us assume we are dealing with a counterparty "2" from which we are buying protection at a given spread S through a CDS on the relevant reference credit "1". This is the position where we would be in the most critical situation in case of counterparty default. We are thus holding a payer CDS on the reference credit "1". Therefore $\Pi(T_j,T_b)$ is the residual NPV of a payer CDS between T_a and

T_b at time T_j, with $T_a < T_j \le T_b$. The NPV of a payer CDS at time T_j can be written similarly to (3.2), except that now valuation occurs at T_j and has to be conditional on the information available in the market at T_j, i.e. $\mathcal{G}_{T_j}$. We can write:

$$\begin{aligned}
&\mathrm{CDS}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1)\\
&\quad = \mathbf{1}_{\{\tau_1 > T_j\}} \overline{\mathrm{CDS}}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1)\\
&\quad = \mathbf{1}_{\{\tau_1 > T_j\}} \Bigg\{ S \Bigg[-\int_{\max(T_a,T_j)}^{T_b} P(T_j,t)(t - T_{\gamma(t)-1}) d_t \boxed{\mathbb{Q}(\tau_1 \ge t \,|\, \mathcal{G}_{T_j})}\\
&\qquad + \sum_{i=\max(a,j)+1}^{b} P(T_j, T_i)\alpha_i \boxed{\mathbb{Q}(\tau_1 \ge T_i \,|\, \mathcal{G}_{T_j})} \Bigg]\\
&\qquad + \mathrm{L{\scriptstyle GD}}_1 \left[\int_{\max(T_a,T_j)}^{T_b} P(T_j,t)\, d_t \boxed{\mathbb{Q}(\tau_1 \ge t \,|\, \mathcal{G}_{T_j})} \right] \Bigg\}
\end{aligned} \tag{4.2}$$

The T_j-credit valuation adjustment for counterparty risk would read

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}} (\mathbb{E}\{\Pi(T_j, T_b) \,|\, \mathcal{G}_{T_j}\})^+]\\
&\quad = \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}} (\mathrm{CDS}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1))^+]\\
&\quad = \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}} 1_{\{\tau_1 > T_j\}} (\overline{\mathrm{CDS}}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1))^+]\\
&\quad = \mathbb{E}[\mathbb{E}\{\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}} 1_{\{\tau_1 > T_j\}} (\overline{\mathrm{CDS}}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1))^+ \,|\, \mathcal{F}_{T_j}\}]\\
&\quad = \mathbb{E}[(\overline{\mathrm{CDS}}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1))^+ \mathbb{E}\{\mathbf{1}_{\{T_{j-1} < \tau_2 \le T_j\}} 1_{\{\tau_1 > T_j\}} \,|\, \mathcal{F}_{T_j}\}]\\
&\quad = \mathbb{E}\{(\overline{\mathrm{CDS}}_{a,b}(T_j, S, \mathrm{L{\scriptstyle GD}}_1))^+ \; [\exp(-\Lambda_2(T_{j-1})) - \exp(-\Lambda_2(T_j))\\
&\qquad - C(1 - \exp(-\Lambda_1(T_j)), 1 - \exp(-\Lambda_2(T_j)))\\
&\qquad + C(1 - \exp(-\Lambda_1(T_j)), 1 - \exp(-\Lambda_2(T_{j-1})))]\}
\end{aligned} \tag{4.3}$$

This can be easily computed through simulation of the processes λ up to T_j if we know the formula for $\mathbb{Q}(\tau_1 \ge u \,|\, \mathcal{G}_{T_j})$ for all $u \ge T_j$ in terms of $\lambda_1(T_j)$.

This valuation, leading to an easy formula for $\overline{\mathrm{CDS}}_{a,b}(T_j)$, would be simple if we were to compute the above probabilities under the filtration $\mathcal{G}^1_{T_j}$ of the default time τ_1 alone, rather than $\mathcal{G}_{T_j}$ incorporating information on τ_2 as well. Indeed, in such a case we could write

$$\begin{aligned}
\mathbb{Q}(\tau_1 \ge u \,|\, \mathcal{G}^1_{T_j}) \;&=\; \mathbf{1}_{\{\tau_1 > T_j\}} \mathbb{E}\left[\exp\left(-\int_{T_j}^{u} \lambda_1(s) ds \right) \Bigg| \mathcal{F}^1_{T_j} \right]\\
&=\; \mathbf{1}_{\{\tau_1 > T_j\}} P^{CIR++}(T_j, u; y_1(T_j))\\
&:=\; \mathbf{1}_{\{\tau_1 > T_j\}} \exp(-(\Psi(u) - \Psi(T_j))) P^{CIR}(T_j, u; y_1(T_j))
\end{aligned} \tag{4.4}$$

i.e. the bond price in the CIR++ model for λ_1, $P^{CIR}(T_j, u; y_1(T_j))$ being the non-shifted time homogeneous CIR bond price formula for y_1. Substitution in (4.2) would give us the NPV at time T_j, since $\overline{\mathrm{CDS}}(T_j)$ would be computed using indeed (4.4) in (4.2). So finally, we would have all the needed components to compute our counterparty risk adjustment (2.3) through mere simulation of the λ's up to T_j.

However, there is a fatal drawback in this approach. Indeed, the survival probabilities contributing to the valuation of $\overline{\mathrm{CDS}}(T_j)$ have to be calculated conditional also on the information on the counterparty default τ_2 available at time T_j.

We can write the correct formula for this survival probability as follows.

$$\begin{aligned}
&\mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\mathbb{Q}(\tau_1 \ge u \,|\, \mathcal{G}_{T_j}) \\
&\quad = \mathbb{E}\left[\mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\mathbf{1}_{\{\tau_1>u\}} \,|\, \mathcal{G}_{T_j}\right] \\
&\quad = \mathbb{E}\left[\mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\mathbf{1}_{\{\tau_1>T_j\}}\mathbf{1}_{\{\tau_1>u\}} \,|\, \mathcal{G}_{T_j}\right] \\
&\quad = \mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\ \mathbb{E}\left[\mathbf{1}_{\{\tau_1>u\}} \,|\, \mathcal{G}_{T_j}, \tau_1 > T_j, T_{j-1} < \tau_2 \le T_j\right] \\
&\quad = \mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\mathbb{E}\left[\mathbf{1}_{\{\tau_1>u\}} \,|\, \mathcal{F}_{T_j}, \tau_1 > T_j, T_{j-1} < \tau_2 \le T_j\right] \\
&\quad = \mathbf{1}_{\{T_{j-1}<\tau_2\le T_j\}}\ \frac{\mathbb{Q}(\tau_1 > u, T_{j-1} < \tau_2 \le T_j \,|\, \mathcal{F}_{T_j})}{\mathbb{Q}(\tau_1 > T_j, T_{j-1} < \tau_2 \le T_j \,|\, \mathcal{F}_{T_j})} \\
&\quad = \mathbf{1}_{\{\cdot\}}\frac{A}{B}, \\
&A = e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda_2(T_j)} + \mathbb{E}[C(1-e^{-\Lambda_1(u)}, 1-e^{-\Lambda_2(T_{j-1})}) \\
&\qquad - C(1-e^{-\Lambda_1(u)}, 1-e^{-\Lambda_2(T_j)})|\mathcal{F}_{T_j}], \\
&B = e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda_2(T_j)} + C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_{j-1})}) \\
&\qquad - C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_j)})
\end{aligned}$$

The residual expectation in the numerator accounts for randomness of $\Lambda_1(u) - \Lambda_1(T_j)$, that is not accounted for in $\mathcal{F}_{T_j}$, and is thus incorporated by taking an expectation with respect to the density of $\Lambda_1(u) - \Lambda_1(T_j)$ (that, in case of the CIR model, can be obtained through Fourier methods).

It is clear that this last expression we obtained is much more complex than (4.4). One can check that if the chosen copula is the independence copula, $C(u_1, u_2) = u_1 u_2$, then our last expression reduces indeed to (4.4).

The difference, in correctly taking into account the dependence of default time τ_1 conditional on the information on default time τ_2, manifests itself in the copula terms. Indeed, with respect to the earlier and incorrect formula taking into account only information of name 1, we made the transition

$$\mathbb{E}[e^{-\int_{T_j}^{u}\lambda_1(u)du}] \to \mathbb{E}[C(1-e^{-\int_{T_j}^{u}\lambda_1(u)du}e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_{j \text{ or } j-1})}) \,|\, \Lambda_1(T_j), \Lambda_2(T_j)]$$

that clearly involves directly the copula.

By substituting our last formula for $\mathbb{Q}(\tau_1 \ge u \,|\, \mathcal{G}_{T_j})$ in (4.2) and then the resulting expression in (4.3), we conclude.

This procedure is however quite demanding, and the idea of partitioning the default interval in periods $[T_{j-1}, T_j]$ is not as effective here as in other situations (such as Brigo and Masetti [8]) and we approach the problem in a more direct numerical way in the next section.

5. Direct Numerical Methodology: Monte Carlo and Fourier Transform

In this section we abandon the choice of bucketing the counterparty default time τ_2 in intervals and move to implementing directly the original formula (2.2), whose relevant term in our case reads

$$\begin{aligned}&\mathbb{E}_t\{\mathbf{1}_{\{t<\tau_2\le T_b\}}D(t,\tau_2)(\mathrm{CDS}_{a,b}(\tau_2,S,T_b))^+\}\\&\quad=\mathbb{E}_t\{\mathbf{1}_{\{t<\tau_2\le T_b\}}D(t,\tau_2)(1_{\{\tau_1>\tau_2\}}\overline{\mathrm{CDS}}_{a,b}(\tau_2,S,T_b))^+\}\end{aligned}$$

Recall the formula in (4.2) for $\overline{\mathrm{CDS}}$ and keep in mind that this is to be computed at the random time τ_2. In the $\overline{\mathrm{CDS}}$ formula, all we need to know is the survival probability $1_{\{\tau_1\ge\tau_2\}}\mathbb{Q}(\tau_1>u\,|\,\mathcal{G}_{\tau_2})=1_{\{\tau_1\ge\tau_2\}}\mathbb{Q}(\tau_1>u\,|\,\mathcal{G}_{\tau_2},\tau_1\ge\tau_2)$.

Summarizing: To effectively compute counterparty risk, we aim at determining the value of the CDS contract on the reference credit "1" at the point in time τ_2 where the counterparty "2" defaults. The reference name "1" has survived this point, and there is a copula C that connects the two default times. The stochastic intensities λ_1 and λ_2 of names "1" and "2" are independent and the default times are connected uniquely through the copula, that is however the most important source of default dependence, correlation among the λ being in general only a secondary source of dependence.

We need to compute the probability

$$\mathbb{Q}(\tau_1>T\,|\,\mathcal{G}_{\tau_2},\tau_1>\tau_2)=\mathbb{Q}(U_1>1-\exp\{-Y_1(T)-\Psi_1(T;\beta_1)\}\,|\,\mathcal{G}_{\tau_2},\tau_1>\tau_2)$$

for any $T>\tau_2$, where U_1 is a uniform random variable, $\lambda_1=y_1+\psi_1$ is the intensity process, Ψ_1 is the integrated deterministic shift $\Psi_1(T)=\int_0^T\psi_1(t)dt$ and analogously Y_1 is the integrated y_1 process.

The information $\mathcal{G}_{\tau_2}$ will determine uniquely τ_2 and hence the value U_2, since the intensity λ_2 is also measurable w.r.t. $\mathcal{G}$. In addition, it includes the quantity $\Lambda_1(\tau_2)$, which is measurable as well.

Now, by conditioning on the value U_1, the above probability can be written as

$$\mathbb{E}[P(U_1)\,|\,\mathcal{G}_{\tau_2},\tau_1>\tau_2]$$

for

$$P(u_1)=\mathbb{Q}(u_1>1-\exp\{-Y_1(T)-\Psi_1(T;\beta_1)\}\,|\,\mathcal{G}_{\tau_2})$$

The conditional probability can be expressed as the cumulative probability of the integrated CIR process

$$P(u_1)=\mathbb{Q}(Y_1(T)-Y_1(\tau_2)<-\log(1-u_1)-Y_1(\tau_2)-\Psi_1(T;\beta_1)\,|\,\mathcal{G}_{\tau_2})$$

The characteristic function of the integrated CIR process $Y_1(T) - Y_1(\tau_2)$ is known in closed form at time τ_2, with a calculation much resembling the CIR bond price formula. The probabilities $P(u_1)$ can therefore be retrieved for an array of u_1 using fractional FFT methods.

Moving to the conditional expectation with respect to U_2, we first need to ascertain the conditional distribution

$$C_{1|2}(u_1; U_2) := \mathbb{Q}(U_1 < u_1 \,|\, \mathcal{G}_{\tau_2}, \tau_1 > \tau_2)$$

Essentially the conditions give us the following information on U_1:

- The default time τ_2 provides $U_2 = 1 - \exp\{-Y_2(\tau_2) - \Psi_2(\tau_2; \beta_1)\}$
- The inequality $\tau_1 > \tau_2$ yields $U_1 > 1 - \exp\{-Y_1(\tau_2) - \Psi_1(\tau_2; \beta_1)\} =: \bar{U}_1$

Thus, we can write for $u_1 > \bar{U}_1$

$$\begin{aligned} C_{1|2}(u_1; U_2) &= \mathbb{Q}(U_1 < u_1 \,|\, U_2, U_1 > \bar{U}_1) \\ &= \frac{\mathbb{Q}(U_1 < u_1, U_1 > \bar{U}_1 \,|\, U_2)}{\mathbb{Q}(U_1 > \bar{U}_1 \,|\, U_2)} \\ &= \frac{\mathbb{Q}(U_1 < u_1 \,|\, U_2) - \mathbb{Q}(U_1 < \bar{U}_1 \,|\, U_2)}{1 - \mathbb{Q}(U_1 < \bar{U}_1 \,|\, U_2)} \end{aligned}$$

Recall that there is a copula $C(u_1, u_2) = \mathbb{Q}(U_1 < u_1, U_2 < u_2)$ that connects the realizations of U_1 and U_2. Then the above probability is readily computable. In particular, if the copula is differentiable one can write

$$C_{1|2}(u_1; U_2) = \frac{\frac{\partial}{\partial u_2} C(u_1, U_2) - \frac{\partial}{\partial u_2} C(\bar{U}_1, U_2)}{1 - \frac{\partial}{\partial u_2} C(\bar{U}_1, U_2)}$$

For several copulas the above expression is known in closed form. Note that $C_{1|2}(u_1; U_2) = 0$ for $u_1 < \bar{U}_1$.

Putting the two together, we compute the survival probability as the numerical integral

$$\mathbb{Q}(\tau_1 > T \,|\, \mathcal{G}_{\tau_2}, \tau_2 > \tau_1) = \int_{u=\bar{U}_R}^{1} P(u) \mathrm{d}C_{1|2}(u; U_2)$$

which is easily computed given the grid output of the fractional FFT procedure.

The numerical procedure we implement is the following:

(1) Produce the default times τ_2 and τ_1 using the copula and the intensities.
(2) If $\tau_1 > \tau_2$, then assume that we sit at the counterparty default time
(3) We bucket, assuming that default actually happens at the next payment date (we could use finer bucketing but for practical purposes this is enough)
(4) Compute U_2 and $\bar{U}_1$.
(5) We aim at building the survival curve, and to do that we loop over the payment times T_k, from τ_2 to the CDS maturity T_b.

(a) Given the model parameters and the spot intensity $y_1(\tau_2)$, we use the fractional FFT to produce the cumulative probability density of the random variable $X = Y_1(T_k) - Y_1(\tau_2)$, which follows the integrated CIR process for maturity T_k

$$p_j = \mathbb{Q}(X < x_j), \quad \text{for a grid } x_j$$

(b) From the abscissas x_j we can compute the corresponding values of the support for the uniform U_1, as

$$u_j = 1 - \exp\{-x_j - Y_1(\tau_2) - \Psi_1(T_k)\}$$

(c) Based on the conditional distribution for U_1 we compute the quantities $f_j = C_{1|2}(u_j; U_2)$.

(d) The survival probability is given by the trapezoidal integration

$$\mathbb{Q}(\tau_1 > T_k \,|\, \mathcal{G}(\tau_2), \tau_1 > \tau_2) \approx \sum_j \frac{p_{j+1} + p_j}{2} \Delta f_j$$

(6) Given the survival curve for the reference entity over the points T_k we can compute the value of the CDS.

(7) By taking the positive part, discounting and averaging, we produce the counterparty adjustment.

6. A Case Study

We consider a default-free institution trading a CDS on a reference name "1" with counterparty "2", where the counterparty "2" is subject to default risk. The default free assumption can also be an approximation for situations where the credit quality of the first institution is much higher than the credit quality of the counterparty. The CDS on the reference credit "1", on which we compute counterparty risk, is a five-years maturity CDS with recovery rate 0.3. The CDS spreads both for the underlying name "1" and the counterparty name "2" for the basic set of parameters we will consider are given in Table 2.

We aim at checking the separated and combined impact of two important quantities on the counterparty-risk credit valuation adjustment (CR-CVA): Default correlation and credit spread volatility. In order to do this, we devise a modeling apparatus accounting for both features. What is novel in our analysis is especially the second feature, as earlier attempts focused mostly on the first one.

To account for credit spread volatilities, we assume default intensities (or instantaneous credit spreads) of both names to follow CIR dynamics, and intensities to be uncorrelated, as explained more in detail in Sec. 3.1:

$$\lambda_j(t) = y_j(t) + \psi_j(t),$$

$$\mathrm{d}y_j(t) = \kappa_j(\mu_j - y_j(t))\mathrm{d}t + \nu_j\sqrt{y_j(t)}\mathrm{d}Z_j(t) + \mathrm{d}J_j(t), \quad \text{for } j = 1, 2$$

As before, we take $\alpha_j = 0$ in the Poissons driving the intensities jumps J and hence assume pre-default intensities λ to have no jumps, as we are interested in valuing the overall impact of credit spread volatility rather than the impact of a fine-tuned realistic intensity dynamics. However, all the above calculations and also the fractional Fourier transform method are exactly applicable to the extended model with intensity jumps, for which the characteristic function of the integrated intensity is still known (see Brigo and El-Bachir [7] for several calculations on the jump-extended model).

The base-case intensities parameter values that we use are given in the Table 1. We work with a counterparty that is of higher credit quality than the reference credit on which the traded CDS is issued, with default intensities which are three times smaller ($y(0)$ and μ are smaller) and significantly less volatile (higher κ and lower ν). To benchmark our results we use the case with no counterparty risk. The spread for a 5 year CDS, assuming a flat risk-free interest rate curve at 3% and recovery rates of 30%, is equal to 252bp (where $1bp = 10^{-4}$). The curve of spot CDS spreads across maturities corresponding to the two parameters sets is in Table 2.

In order to model "default correlation", or more precisely the dependence of the two names defaults we postulate a Gaussian copula on the exponential triggers of the default times, although we could use any other tractable copula. By "default correlation" parameter we mean the Gaussian copula parameter ρ.

In this context, if we define the cumulated intensities $\Lambda_j(t) := \int_0^t \lambda_j(u)du$, $j = 1, 2$, the default times τ_1 and τ_2 of the reference credit and the counterparty, respectively, are given by $\tau_j = \Lambda_j^{-1}(\xi_j)$, with ξ_1 and ξ_2 unit-mean exponential random variables connected through the Gaussian copula with correlation parameter ρ.

Table 1. Intensity parameters for the reference credit "1" and the counterparty "2".

	$y(0)$	κ	μ	ν
Reference 1	0.03	0.50	0.05	0.50
Counterparty 2	0.01	0.80	0.02	0.20

Table 2. CDS spreads for different maturities corresponding to the intensity parameters given in Table 1 with shifts ψ to zero. LGD for both CDS is 0.7.

	Spread (in bp)	
Maturity	Reference "1"	Counterparty "2"
1y	234	92
2y	244	104
3y	248	112
4y	251	117
5y	252	120
6y	253	123
7y	253	125
8y	254	126
9y	254	127
10y	254	128

When we say "credit spread volatility" parameters we mean ν_1 for the reference credit and ν_2 for the counterparty. As the focus is mostly on credit spread volatility for the reference credit, we also check what implied CDS volatilities are produced by our choice of the ν_1 and other parameters for hypothetical reference credit's CDS options, maturing in one year and in case of exercise entering a CDS that is four years long at option maturity. This way we have a more direct market quantity linked to our parameter for credit spread volatility. For examples of implied volatility in the CDS market with specific lognormal dynamics, see for example Brigo [1] and [2].

We begin with a case where the credit spread for the counterparty, as driven by λ_2, is almost deterministic. We assume here that $\nu_2 = 0.01$.

Table 3 reports our results. We notice a number of interesting patterns. First, one can examine the table columns. Let us start from the first five columns. We see that as the correlation increases, the CR-CVA for the payer CDS increases, except on the very end of the correlation spectrum. Indeed, when increasing correlation in the final step from 0.9 to 0.99, the CR-CVA goes down.

This is somehow reasonable given the way default times are modeled, and we may explain it as follows. Let us take the case of the first column. Here the volatility parameter of the reference credit ν_1 is also very small. So essentially the intensities λ_1 and λ_2 are almost deterministic. Suppose they are also constant in time, for

Table 3. CR-CVA in basis points for the case $\nu_2 = 0.01$ including the LGD = 0.7 factor; numbers within round brackets represent the monte-carlo standard error; the reference credit CDS also has LGD = 0.7 and a five year maturity.

ρ	Vol parameter ν_1 CDS Implied vol	0.01 1.5%	0.10 15%	0.20 28%	0.30 37%	0.40 42%	0.50 42%
−99	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	39(2)	38(2)	42(2)	38(2)	40(2)	41(2)
−90	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	39(2)	38(2)	41(2)	39(2)	40(2)	41(2)
−60	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	1(0)
	Receiver adj	37(2)	36(1)	38(1)	35(1)	38(1)	37(1)
−20	Payer adj	0(0)	0(0)	1(0)	3(0)	3(0)	4(1)
	Receiver adj	18(1)	16(1)	18(1)	18(1)	20(1)	21(1)
0	Payer adj	3(0)	4(0)	6(0)	7(1)	6(1)	6(1)
	Receiver adj	0(0)	2(0)	5(0)	7(0)	10(0)	12(1)
+20	Payer adj	28(1)	27(1)	23(1)	21(1)	17(2)	15(1)
	Receiver adj	0(0)	0(0)	1(0)	1(0)	2(0)	3(0)
+60	Payer adj	87(4)	78(4)	73(4)	66(4)	55(3)	52(3)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
+90	Payer adj	80(6)	81(6)	77(5)	82(5)	78(5)	73(5)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
+99	Payer adj	2(1)	7(2)	30(3)	66(5)	61(5)	84(5)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)

simplicity. Then under default correlation 0.99 also the exponential triggers ξ_1 and ξ_2 are almost perfectly correlated, say $\xi_1 \approx \xi_2 =: \xi$. Then we have $\tau_1 = \xi/\lambda_1$, $\tau_2 = \xi/\lambda_2$. As $\lambda_1 > \lambda_2$, we get that $\xi/\lambda_1 < \xi/\lambda_2$ in all scenarios, so that $\tau_1 < \tau_2$ in all scenarios. But if this happens, then the residual NPV of the CDS on the reference credit "1" at the default time τ_2 of the counterparty is zero, since the reference credit always defaults before the counterparty does. This explains why we find almost zero CR-CVA when λ_1's volatility is very small.

If we increase λ_1's volatility,[1] then $\xi/\lambda_1 < \xi/\lambda_2$ is no longer going to happen in all scenarios, since randomness in λ_1 can produce some paths where actually λ_1 is now smaller than λ_2, and hence $\tau_1 > \tau_2$. Indeed, as we increase the volatility, following the last row of the table we see that the payer adjustment gets away from zero and increases in value, as the increased randomness in λ_1 produces more and more paths where λ_1 is smaller than λ_2. We reach an extreme case for correlation equal to 0.99: in this case the CR-CVA for correlation 0.99 does not even go back and keeps on increasing with respect to the case with correlation 0.9. In this sense the last column of the table is qualitatively different from all others, in that it is the only one where CR-CVA keeps on increasing until the end of the considered correlation spectrum.

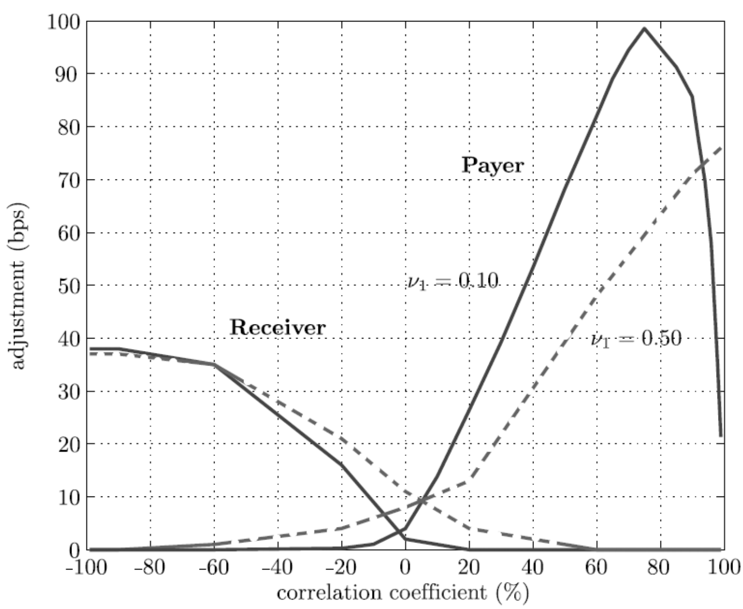

Fig. 1. CR-CVA patterns in correlations for payer and receiver CDS and for low (0.1) and high (0.5) reference credit volatility ν_1, when counterparty volatility ν_2 is 0.1

[1]In our idealized example we still keep λ_1 constant in time but increase its variance as a static random variable.

We zoom on these patterns for the later case with $\nu_2 = 0.1$ in Fig. 1, as exemplified by the continuous "payer" graph for the case with low volatility $\nu_1 = 0.1$ and the dashed "payer" one for the case with high volatility $\nu_1 = 0.5$. The continuous graph reverts towards zero in the end, whereas the dashed one keeps increasing.

Notice also that typically the payer CDS CR-CVA vanishes for very negative correlations. This happens because, in that region, when the counterparty defaults the underlying CDS does not. In such a case, we have a CDS option at the counterparty default time where the underlying CDS spread had a negative large jump due to the copula contagion coming from default of the counterparty. This negative jump causes the option to become worthless as the underlying goes below the strike in almost all scenarios.

We may also analyze the receiver adjustment, which evolves in a more stylized pattern. The adjustment remains substantially decreasing as default correlation increases, and goes to zero for high correlations. This happens because in this case, in the few scenarios where $\tau_1 > \tau_2$ and the reference CDS has still value at the counterparty default, the positive correlation induces a contagion copula-related term on the intensity of the survived reference name "1". This causes in turn the option to go far out of the money and hence to be negligible, leading to a null CR-CVA.

As the counterparty volatility ν_2 increases first to 0.1 and then to 0.2 all qualitative features we described above are maintained, although somehow smoothed by the larger counterparty volatility. Detailed results are given in Tables 4 and 5.

Table 4. CR-CVA for the case $\nu_2 = 0.1$ including the LGD = 0.7 factor; numbers within round brackets represent the monte-carlo standard error; the reference credit CDS also has LGD = 0.7 and a five year maturity.

ρ	Vol parameter ν_1 CDS Implied vol	0.01 1.5%	0.10 15%	0.20 28%	0.30 37%	0.40 42%	0.50 42%
−99	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	40(2)	38(2)	39(2)	38(2)	36(1)	37(1)
−90	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	39(2)	38(2)	38(2)	38(2)	35(1)	37(2)
−60	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	1(0)
	Receiver adj	36(1)	35(1)	36(1)	36(1)	32(1)	35(1)
−20	Payer adj	0(0)	0(0)	1(0)	2(0)	3(0)	4(1)
	Receiver adj	16(1)	16(1)	17(1)	19(1)	18(1)	21(1)
0	Payer adj	3(0)	4(0)	5(0)	7(1)	7(1)	8(1)
	Receiver adj	0(0)	2(0)	5(0)	8(0)	10(0)	11(1)
+20	Payer adj	27(1)	25(1)	23(1)	20(1)	16(2)	13(1)
	Receiver adj	0(0)	0(0)	1(0)	2(0)	2(0)	4(0)
+60	Payer adj	80(4)	82(4)	67(4)	64(4)	55(3)	48(3)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
+90	Payer adj	87(6)	86(6)	88(6)	78(5)	80(5)	71(4)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
+99	Payer adj	10(2)	21(3)	52(5)	68(5)	73(5)	76(5)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)

Table 5. CR-CVA for the case $\nu_2 = 0.2$ including the LGD $= 0.7$ factor; numbers within round brackets represent the monte-carlo standard error; the reference credit CDS also has LGD $= 0.7$ and a five year maturity.

ρ	Vol parameter ν_1 CDS Implied vol	0.01 1.5%	0.10 15%	0.20 28%	0.30 37%	0.40 42%	0.50 42%
−99	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	41(2)	40(2)	39(2)	40(2)	40(2)	40(2)
−90	Payer adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
	Receiver adj	41(2)	39(2)	39(2)	41(2)	40(2)	40(2)
−60	Payer adj	0(0)	0(0)	0(0)	0(0)	1(0)	1(0)
	Receiver adj	39(1)	37(1)	37(1)	37(1)	36(1)	35(1)
−20	Payer adj	0(0)	0(0)	2(0)	3(0)	3(0)	4(1)
	Receiver adj	17(1)	17(1)	17(1)	19(1)	21(1)	20(1)
0	Payer adj	3(0)	5(0)	6(0)	7(1)	6(1)	6(1)
	Receiver adj	0(0)	2(0)	4(0)	7(0)	10(0)	12(1)
+20	Payer adj	25(1)	24(1)	23(1)	20(1)	17(1)	15(1)
	Receiver adj	0(0)	0(0)	1(0)	2(0)	2(0)	4(0)
+60	Payer adj	74(4)	74(4)	69(4)	59(3)	54(3)	52(3)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	1(0)
+90	Payer adj	91(6)	90(6)	88(5)	80(5)	81(5)	81(5)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)
+99	Payer adj	43(4)	56(5)	57(5)	72(5)	74(5)	78(5)
	Receiver adj	0(0)	0(0)	0(0)	0(0)	0(0)	0(0)

Table 6. CR-CVA for three cases: the first column tabulates the example given in Fig. 1 for the Payer case with $\nu_1 = 0.1$ (and $\nu_2 = 0.1$). The second column shows the same adjustments in case we swap the parameters in Table 1, so that now the counterparty "2" is riskier than the reference credit of the CDS "1". The third case shows what happens if, under the original parameters again, we increase the reference credit initial level and long term mean to $\lambda_1(0) = 0.05$ and $\mu_1 = 0.07$.

ρ	Base	Risky counterparty	High intensity
10	14	12	15
20	25	29	28
30	39	46	40
40	53	66	53
50	68	88	65
60	82	115	75
65	89	131	79
70	94	148	81
75	99	168	81
80	95	191	74
85	91	220	65
90	86	254	48
99	21	359	2

We also check what happens if we swap the reference credit and the counterparty CIR parameters, now having the counterparty to be riskier. Results are in Table 6. We see that λ_2 now tends to be larger than λ_1. As a consequence, in the case with correlation 0.99 and almost deterministic intensities, we would have this time that $\tau_1 = \xi/\lambda_1 > \xi/\lambda_2 = \tau_2$ in most scenarios, so that we do not expect any more the CR-CVA to be killed or reduced for extreme correlations. And indeed we see that in the "risky counterparty" column of Table 6 the adjustment keeps on increasing even for very high correlation.

Finally, we check what happens if we increase the levels (rather than volatilities) of intensities for the reference credit. If we do this, the inversion of the CR-CVA pattern (for the payer case) as correlation increases towards extreme values arrives earlier, as expected.

6.1. *Conclusions*

We see from the above analysis that both credit spread volatility and default correlation matter considerably in valuing counterparty risk. And we see that the patterns of the adjustments in credit spread volatility depend qualitatively on correlation, in that they can be either flat, decreasing or increasing according to the particular default correlation value one fixes. As concerns the pattern in correlation, this too depends qualitatively on the credit spread volatility that is chosen. For payer CDS, extreme correlation (sometimes referred to as "wrong way risk") may result in counterparty risk getting smaller with respect to more moderated correlation values, *unless the credit spread volatility is large enough*. Indeed, to have a relevant impact of wrong way risk for counterparty risk on Payer CDS we need also credit spread volatility to go up. This is a feature of the copula model of which we need to be aware. In a copula model with deterministic credit spreads (a standard assumption in the industry), by ignoring credit spread volatility we would have that wrong-way risk causes counterparty risk almost to vanish with respect to cases with lower correlation. To get a relevant impact of wrong way risk we need to put back credit spread volatility into the picture, if we are willing to use a reduced form copula-based model.

References

[1] D. Brigo, Market models for CDS options and callable floaters, *Risk* **January** (2005).
[2] D. Brigo, Constant maturity credit default swap valuation with market models, *Risk* **June** (2006).
[3] D. Brigo and A. Alfonsi, Credit default swaps calibration and derivatives pricing with the SSRD stochastic intensity model, *Finance and Stochastic* **9**(1) (2005) 29–42.
[4] D. Brigo and I. Bakkar, Accurate counterparty risk valuation for energy-commodities swaps, *Energy Risk* **March** (2009) 106–111.
[5] D. Brigo and A. Capponi, Bilateral counterparty risk with stochastic dynamical models (2008), available at SSRN.com and arXiv.org.

[6] D. Brigo and L. Cousot, A comparison between the SSRD model and the market model for CDS options pricing, *International Journal of Theoretical and Applied Finance* **9**(3) (2006) 315–399.
[7] D. Brigo and N. El-Bachir, An exact formula for default swaptions pricing in the SSRJD stochastic intensity model, to appear in *Mathematical Finance* (2008).
[8] D. Brigo and M. Masetti, Risk neutral pricing of counterparty risk, in *Counterparty Credit Risk Modeling: Risk Management, Pricing and Regulation*, ed. M. Pykhtin (Risk Books, London, 2006).
[9] D. Brigo and F. Mercurio, *Interest Rate Models: Theory and Practice — With Smile, Inflation and Credit*, 2nd edn. (Springer Verlag, 2006).
[10] D. Brigo and A. Pallavicini, Counterparty risk under correlation between default and interest rates, in *Numerical Methods for Finance*, eds. J. Miller, D. Edelman and J. Appleby (Chapman Hall, 2007).
[11] D. Brigo and A. Pallavicini, Counterparty risk and Contingent CDS with stochastic intensity hybrid models, *Risk Magazine* **February** (2008).
[12] P. Collin-Dufresne, R. Goldstein and J. Hugonnier, A general formula for pricing defaultable securities, *Econometrica* **72** (2002) 1377–1407.
[13] J. Hull and A. White, Valuing credit default swaps II: Modeling default correlations, Working paper, University of Toronto (2000).
[14] S. Y. Leung and Y. K. Kwok, Credit default swap valuation with counterparty risk, *The Kyoto Economic Review* **74**(1) (2005) 25–45.
[15] E. H. Sorensen and T. F. Bollier, Pricing swap default risk, *Financial Analysts Journal* **50** (1994) 23–33.