Developments in perfect simulation for Gibbs measures

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Abstract

This paper deals with the problem of perfect sampling from a Gibbs measure with infinite range interactions. We present some sufficient conditions for the extinction of processes which are like supermartingales when large values are taken. This result has profound consequences on perfect simulation, showing that local modifications on the interactions of a model do not affect the simulability. We also pose the question to optimize over a sequence of sets and we completely solve the question in the case of finite range interactions.

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1 Introduction

In this paper we deal with the problem of perfect simulation of Gibbs measures. The first algorithm of this kind was realized in [PW96]. Their paper opened a new field of research which is evolving in different directions. In [MG98] they extended the results of [PW96] to a continuous state space. In [HS00] the study of perfect sampling from a Gibbs measure started and in [DSP08] the authors showed the importance of percolation in perfect simulation algorithms for Gibbs measures with finite range interactions. In [CFF02] they dealt with long memory processes which means that the state of the process at time zero depends on all its past history. In [GLO10] the authors consider the problem of perfect sampling from a Gibbs measure with infinite range interactions.

We start from [GLO10] and we pose new questions. Their algorithm is based on a probability mass function that we improve in several ways. Firstly we make the probability mass function stochastically smaller and, as we shall see in Section 2, this will always be an advantage. Furthermore the probability mass function depends on the choice of a sequence of sets which was fixed in their paper. In Section 3, we pose the question to optimize over some sets with appropriate properties. We provide a number of improvements and we completely solve the problem in the case of finite range interactions. In Section 4, we present some sufficient conditions such that a discrete process extinguishes almost surely. Theorem 5 presents this result and it has applications in various areas. The assumptions of Theorem 5 are weaker than the ones for the extinction of Galton-Watson process which is solved as a particular case. This result has implications for the perfect simulation algorithm, see Corollary 3, because it supplies a weaker sufficient condition, for the applicability of their algorithm, than the one given in [GLO10]. Moreover, we establish an equivalence relation among interactions in the sense that two interactions are equivalent if they only differ locally. By Theorem 3, we prove that given two equivalent interactions, if one respects the sufficient condition for the perfect sampling, then the other one satisfies it too.

2 Synopsis

We present some results of the work [GLO10] with few changes of notation that will be useful in our paper.

Let $S = \{-1, 1\}^{\mathbb{Z}^d}$ be the set of spin configurations. We endow S with $\mathcal{S}$, the σ-algebra generated by cylinders. A point $v \in \mathbb{Z}^d$ is called a *vertex*. Let $\sigma(v) \in \{-1, 1\}$ be the value of the configuration σ at vertex $v \in \mathbb{Z}^d$.

We write $A \Subset \mathbb{Z}^d$ to denote that A is a finite subset of $\mathbb{Z}^d$. The cardinality of a set A is denoted by $|A|$. An *interaction* $\mathbf{J} = \{J_B \in \mathbb{R} : B \Subset \mathbb{Z}^d\}$ is a collection of real numbers indexed by $B \Subset \mathbb{Z}^d$, with $|B| \geq 2$ such that

$$\sup_{v \in \mathbb{Z}^d} \sum_{B: v \in B} |J_B| < \infty. \tag{1}$$

We denote by $\mathcal{J}$ the collection of all the interactions.

A probability measure π on $(S, \mathcal{S})$ is said to be a Gibbs measure relative to the interaction $\mathbf{J}$ if for all $v \in \mathbb{Z}^d$ and for any $\zeta \in S$

$$\pi(\sigma(v) = \zeta(v) | \sigma(u) = \zeta(u) \ \forall u \neq v) = \frac{1}{1 + \exp[-2 \sum_{B: v \in B} (J_B \prod_{w \in B} \sigma(w))]} \ \ a.s.$$

Let us define the set $\mathcal{A}_v = \{B \Subset \mathbb{Z}^d : v \in B, J_B \neq 0\}$, for $v \in \mathbb{Z}^d$; the set $\mathcal{A}_v$ is finite or countable, therefore we write $\mathcal{A}_v = \{A_{i,v} : i < N_v + 1\}$ where $N_v = |\mathcal{A}_v|$. We now introduce a sequence of sets with appropriate properties that will replace the balls used in [GLO10].

Let $\mathbf{B}_v = \{B_v(k), k \in \mathbb{N}\}$, for $v \in \mathbb{Z}^d$, be a sequence of finite subsets in $\mathbb{Z}^d$ such that

1) $B_v(0) = \{v\}$;

2) $B_v(k) \subset B_v(k+1)$ and $B_v(k+1) \setminus B_v(k) \neq \emptyset$, for $k \in \mathbb{N}$;

3) $\bigcup_{k \in \mathbb{N}} B_v(k) \supset \bigcup_{i < N_v + 1} A_{i,v}$.

We denote by $\mathcal{B}_v$ the space of the sequences verifying 1), 2) and 3).

In this section we will use a fixed choice of $\mathbf{B}_v \in \mathcal{B}_v$ that will not be mentioned later.

In [GLO10] they present a perfect simulation algorithm for a Gibbs measure π with long range interaction. It can be divided into two steps: *the backward sketch procedure* and *the forward spin procedure*. For the applicability of the algorithm they only have a condition on its first part, i.e. on the backward sketch procedure. The algorithm is defined through a Glauber dynamics having π as reversible measure. For any $v \in \mathbb{Z}^d$ and $\sigma \in S$, let $c_v(\sigma)$ be the rate at which the spin in v flips when the system is in the configuration σ,

$$c_v(\sigma) = \exp\left(- \sum_{B: v \in B} \left(J_B \prod\nolimits_{v \in B} \sigma(v)\right)\right).$$

The difficulty of dealing with a measure with long range interaction is overcome through a decomposition of the rates $c_v(\sigma)$ as in (10). For $l \geq 1$

$$c_v^{[l]}(\sigma) = \exp\left(- \sum_{B, v \in B, B \not\subset B_v(l)} |J_B| \right) \exp\left(- \sum_{B: v \in B, B \subset B_v(l)} \left(J_B \prod\nolimits_{v \in B} \sigma(v)\right)\right), \tag{2}$$

which are the rates relative to a process having a reversible measure with the same interaction truncated at range l. Notice that by condition (1),

$$\lim_{l\to\infty} c_v^{[l]}(\sigma) = c_v(\sigma). \tag{3}$$

To present the decomposition we define two probability mass functions of which the first one selects a random region of dependence and the second one updates the value of the spin. For $v \in \mathbb{Z}^d$, $\mathbf{J} \in \mathcal{J}$, let

$$\lambda_{v,\mathbf{J},\mathbf{B}_v}(k) = \begin{cases} \exp(-\sum_{B:v\in B} |J_B|) & \text{if } k = 0, \\ \exp(-\sum_{B:v\in B, B\not\subset B_v(k)} |J_B|) - \exp(-\sum_{B:v\in B, B\not\subset B_v(k-1)} |J_B|) & \text{if } k \geq 1. \end{cases} \tag{4}$$

Note that, for $v \in \mathbb{Z}^d$, $(\lambda_{v,\mathbf{J},\mathbf{B}_v}(k), k \in \mathbb{N})$ is a probability mass function on $\mathbb{N}$ because of properties 1), 2) and 3) of $\mathcal{B}_v$. For brevity of notation we will omit the indices $\mathbf{J}$, $\mathbf{B}_v$, putting $\lambda_v = \lambda_{v,\mathbf{J},\mathbf{B}_v}$ when there is no ambiguity.

Moreover, for each $\sigma \in S$ and $v \in \mathbb{Z}^d$, let

$$p_v^{[0]}(1) = p_v^{[0]}(-1) = \frac{1}{2},$$

and for $k \geq 1$

$$p_v^{[k]}(-\sigma(v)|\sigma) = \frac{\exp(-\sum_{B,v\in B, B\subset B_v(k-1)} J_B\chi_B(\sigma))}{2} \cdot$$

$$\cdot \frac{\exp(-\sum_{B,v\in B,B\subset B_v(k),B\not\subset B_v(k-1)} J_B\chi_B(\sigma)) - \exp(-\sum_{B,v\in B,B\subset B_v(k),B\not\subset B_v(k-1)} |J_B|)}{1 - \exp(-\sum_{B,v\in B,B\subset B_v(k),B\not\subset B_v(k-1)} |J_B|)}, \tag{5}$$

$$p_v^{[k]}(\sigma(v)|\sigma) = 1 - p_v^{[k]}(-\sigma(v)|\sigma).$$

Notice that for each $a \in \{-1, 1\}$, $p_v^{[0]}(a)$ does not depend on v and that, by construction, for any $k \geq 1$, $p_v^{[k]}(a|\sigma)$ depends only on the restriction of the configuration σ to the set $B_v(k)$; it is an important property that links the backward sketch procedure to the forward spin procedure.

Note that for each $v \in \mathbb{Z}^d$, $k \geq 1$ and $\sigma \in S$, the probability mass functions $\lambda_v(\cdot)$ on $\mathbb{N}$ and $p_v^{[k]}(\cdot|\sigma)$ on $\{-1, 1\}$ have been defined differently than in [GLO10]. These changes will be clarified in Remark 1 and in Remark 2.

The first part of their algorithm constructs a process that we are going to define. Let M_v be the mass to be associated to each vertex v. Given $v \in \mathbb{Z}^d$, let $\{C_n\}_{n\in\mathbb{N}} = \{C_n^{v,\mathbf{B}_v}\}_{n\in\mathbb{N}}$ be a process with homogeneous Markovian dynamics and which takes values in $\mathcal{C} = \{A \Subset \mathbb{Z}^d\}$ for $n \in \mathbb{N}$. Let $C_0 = C_0^{v,\mathbf{B}_v} = \{v\}$.

If $C_n = \emptyset$ then $C_{n+1} = \emptyset$.

If $C_n \neq \emptyset$, then the set C_{n+1} is constructed as follows. A random vertex W_n is selected, proportionally to its mass, with

$$\mathbb{P}(W_n = w|C_n) = \frac{M_w}{\sum_{z\in C_n} M_z} \quad \text{for } w \in C_n. \tag{6}$$

We will choose all the M_v's equal to a positive constant, therefore the probability in (6) will be equal to $1/|C_n|$. Formula (6) will be used to define more general models in Section 5. Then a random value $K_{w,n}$ is drawn by using the probability mass function λ_w, so $\mathbb{P}(K_{w,n} = k) = \lambda_w(k)$, for $k \in \mathbb{N}$. If $K_{w,n} = 0$ then $C_{n+1} = C_n \setminus \{w\}$; if $K_{w,n} = k$, for $k \in \mathbb{N}_+$, then $C_{n+1} = C_n \cup B_w(K_{w,n}) = C_n \cup B_w(k)$. The following proposition is a summary of some results in [GLO10] that we write in a more general form about $\mathbf{B}_v$.

Proposition 1. *The perfect simulation algorithm in [GLO10] generates a random field with distribution π if and only if for any $v \in \mathbb{Z}^d$*

$$\limsup_{n\to\infty} C_n^{v,\mathbf{B}_v} = \emptyset \ \ a.s. \tag{7}$$

A sufficient condition, given in [GLO10], for (7) is

$$\sup_{v\in\mathbb{Z}^d} \sum_{k=1}^{\infty} |B_v^*(k)|\lambda_v(k) < 1, \tag{8}$$

where $B_v^*(k)$ is the ball, in norma L^1, centered in v and radius k.

In Section 5 we will show that (8) can be replaced by the following weaker assumption, for some choice of $\mathbf{B}_v \in \mathcal{B}_v$,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \sum_{k=1}^{\infty} |B_v(k)|\lambda_v(k) < 1. \tag{9}$$

Remark 1. *According to the definition of λ_v, we have also set, differently than [GLO10], $M_v = 2$ instead of $M_v = 2\exp(\sum_{B:v\in B} |J_B|)$ and, for $k \in \mathbb{N}$, $p_v^{[k]}$ as in (5). In Theorem 2 of [GLO10] they prove the decomposition*

$$c_v(\sigma) = M_v\left[\frac{\lambda_v(0)}{2} + \sum_{k=1}^{\infty} \lambda_v(k) p_v^{[k]}(-\sigma(v)|\sigma)\right], \tag{10}$$

by combining (3) with the decomposition

$$c_v^{[l]}(\sigma) = M_v \left[\frac{\lambda_v(0)}{2} + \sum_{k=1}^{l} \lambda_v(k) p_v^{[k]}(-\sigma(v)|\sigma) \right] \text{ for all } l \geq 1. \tag{11}$$

Now we have to show that our choices still satisfy the decomposition (11). To this end it suffices to prove that for any $k \geq 1$ and $\sigma \in S$,

$$c_v^{[k]}(\sigma) - c_v^{[k-1]}(\sigma) = M_v \lambda_v(k) p_v^{[k]}(-\sigma(v)|\sigma), \tag{12}$$

where the rates $c_v^{[k]}(\sigma)$ for $k \geq 1$ and $c_v^{[0]}(\sigma) = \frac{1}{2} M_v \lambda_v(0)$ are defined as in [GLO10]. Since for $k \geq 2$ the right hand side of (12) does not depend on M_v and moreover there is no difference between our measures λ_v, $p_v^{[k]}$ and theirs, (12) is still valid. For $k = 1$ we have

$$c_v^{[1]}(\sigma) - c_v^{[0]}(\sigma) =$$

$$= \exp\left(-\sum\nolimits_{B: v\in B, B\not\subset B_v(1)} |J_B|\right) \exp\left(-\sum\nolimits_{B: v\in B, B\subset B_v(1)} J_B \chi_B(\sigma)\right) - \frac{M_v}{2} \exp\left(-\sum_{B: v\in B} |J_B|\right) =$$

$$= \exp\left(-\sum\nolimits_{B: v\in B, B\not\subset B_v(1)} |J_B|\right) \exp\left(-\sum\nolimits_{B: v\in B, B\subset B_v(1)} J_B \chi_B(\sigma)\right)$$

$$- \exp\left(-\sum\nolimits_{B: v\in B, B\not\subset B_v(0)} |J_B|\right) \exp\left(-\sum\nolimits_{B: v\in B, B\subset B_v(0)} J_B \chi_B(\sigma)\right) =$$

$$= M_v \lambda_v(1) p_v^{[1]}(-\sigma(v)|\sigma),$$

where the second equality is valid since it is never verified that $B \Subset \mathbb{Z}^d$ with $|B| \geq 2$ and $B \subset B_v(0) = \{v\}$.

3 An optimization problem for perfect simulation

In this section we deal with the optimal choice of $\mathbf{B}_v \in \mathcal{B}_v$, reaching concrete results. We start with some definitions.

Definition 1. *For $v \in \mathbb{Z}^d$, the sequence $\mathbf{B}_v \in \mathcal{B}_v$ is* less refined *than $\mathbf{B'}_v \in \mathcal{B}_v$, in symbols $\mathbf{B}_v \preceq \mathbf{B'}_v$, if $\mathbf{B}_v$ is a subsequence of $\mathbf{B'}_v$.*

This relation between two sequences of $\mathcal{B}_v$ is a partial order. The set $\mathcal{B}_v$ has no minimum, nor maximum, nor even minimal elements; nevertheless it has an uncountable infinite number of maximal elements, corresponding to the sequences of sets which increase by only one vertex at a time.

Let us define, for $v \in \mathbb{Z}^d$, a new probability mass function obtained from λ_v as follows

$$\begin{array}{ll} \hat{\lambda}_{v,\mathbf{J},\mathbf{B}_v}(|B_v(l)|-1) = \lambda_{v,\mathbf{J},\mathbf{B}_v}(l), & \text{for } l \in \mathbb{N}, \\ \hat{\lambda}_{v,\mathbf{J},\mathbf{B}_v}(i-1) = 0, & \text{for } i \notin \{|B_v(l)|, l \in \mathbb{N}\}. \end{array} \tag{13}$$

Theorem 1. *Let $v \in \mathbb{Z}^d$, $\mathbf{J} \in \mathcal{J}$, and $\mathbf{B}_v$, $\mathbf{B'}_v \in \mathcal{B}_v$ such that $\mathbf{B}_v \preceq \mathbf{B'}_v$. Then $\hat{\lambda}_{v,\mathbf{J},\mathbf{B'}_v} \preceq_{st} \hat{\lambda}_{v,\mathbf{J},\mathbf{B}_v}$.*

Proof. For brevity of notation we write $\hat{\lambda}_v = \hat{\lambda}_{v,\mathbf{J},\mathbf{B}_v}$ and $\hat{\lambda}'_v = \hat{\lambda}_{v,\mathbf{J},\mathbf{B'}_v}$. To show the stochastic ordering $\hat{\lambda}'_v \preceq_{st} \hat{\lambda}_v$ we equivalently prove that for each $n \in \mathbb{N}$,

$$F'(n) = \sum_{l=0}^{n} \hat{\lambda}'_v(l) \geq \sum_{l=0}^{n} \hat{\lambda}_v(l) = F(n). \tag{14}$$

The functions $F(n)$ and $F'(n)$ are the probability distributions relative to $\hat{\lambda}_v$ and $\hat{\lambda}'_v$ respectively. They are piecewise constant functions whose jumps occur only in the points of the set $\{|B_v(l)|-1, l \in \mathbb{N}\}$ and $\{|B'_v(l)|-1, l \in \mathbb{N}\}$ respectively, i.e.

$$F(n) = \hat{\lambda}_v(0) + ... + \hat{\lambda}_v(n) = \lambda_v(0) + ... + \lambda_v(j), \text{ where } j = \max\{l \in \mathbb{N} : |B_v(l)| - 1 \leq n\},$$

$$F'(n) = \hat{\lambda}'_v(0) + ... + \hat{\lambda}'_v(n) = \lambda'_v(0) + ... + \lambda'_v(j'), \text{ where } j' = \max\{l \in \mathbb{N} : |B'_v(l)| - 1 \leq n\}.$$

Now we show that for each $m \in \{|B_v(l)|-1, l \in \mathbb{N}\}$,

$$F(m) = F'(m). \tag{15}$$

Let $m \in \{|B_v(l)|-1, l \in \mathbb{N}\}$, then

$$F(m) = \lambda_v(0) + ... + \lambda_v(j), \text{ where } j \text{ is the unique index such that } |B_v(j)| - 1 = m,$$

$$F'(m) = \lambda'_v(0) + ... + \lambda'_v(j'), \text{ where } j' \text{ is the unique index such that } |B'_v(j')| - 1 = m,$$

from which, by the hypothesis of the theorem,

$$B_v(j) = B'_v(j'). \tag{16}$$

Note that the following sums are telescopic, hence

$$\sum_{l=0}^{n} \lambda_v(l) = \exp\left(-\sum\nolimits_{B:v\in B, B\not\subset B_v(n)} |J_B|\right) \text{ and } \sum_{l=0}^{n} \lambda'_v(l) = \exp\left(-\sum\nolimits_{B:v\in B, B\not\subset B'_v(n)} |J_B|\right), \tag{17}$$

for $n \in \mathbb{N}_+$.

From (16) and (17),

$$\sum_{l=0}^{j} \lambda_v(l) = \sum_{l=0}^{j'} \lambda'_v(l)$$

immediately follows and it implies (15). Since F and F' are nondecreasing, from (15) and

$$\{|B_v(l)|, l \in \mathbb{N}\} \subset \{|B'_v(l)|, l \in \mathbb{N}\}$$

we obtain (14). □

Analogously to [GLO10], see (8), we introduce the following quantity that it will be used later; we call it the *birth-death expectation*,

$$\mu_{v,\mathbf{J}}(\mathbf{B}_v) = \sum_{l=1}^{\infty} |B_v(l)| \lambda_{v,\mathbf{J},\mathbf{B}_v}(l) - 1, \tag{18}$$

for $\mathbf{J} \in \mathcal{J}$, $v \in \mathbb{Z}^d$, $\mathbf{B}_v \in \mathcal{B}_v$.

We are now in the position to present our result concerning the birth-death expectation, it will be involved in their and our sufficient condition for the perfect sampling.

Corollary 1. *Let* $\mathbf{J} \in \mathcal{J}$, $v \in \mathbb{Z}^d$, $\mathbf{B}_v$, $\mathbf{B'}_v \in \mathcal{B}_v$ *such that* $\mathbf{B}_v \preceq \mathbf{B'}_v$. *Then* $\mu_{v,\mathbf{J}}(\mathbf{B'}_v) \leq \mu_{v,\mathbf{J}}(\mathbf{B}_v)$.

Proof. Let $\mathbf{J} \in \mathcal{J}$, $v \in \mathbb{Z}^d$, $\mathbf{B}_v$, $\mathbf{B'}_v \in \mathcal{B}_v$ such that $\mathbf{B}_v \preceq \mathbf{B'}_v$ and let $\hat{\lambda}_v = \hat{\lambda}_{v,\mathbf{J},\mathbf{B}_v}$, $\hat{\lambda'}_v = \hat{\lambda}_{v,\mathbf{J},\mathbf{B'}_v}$ be the corresponding measures. Consider the random variables $X_v \sim^{\mathcal{L}} \hat{\lambda}_v$ and $X'_v \sim^{\mathcal{L}} \hat{\lambda'}_v$. From Theorem 1, it follows that $\mathbb{E}(f(X_v)) \geq \mathbb{E}(f(X'_v))$ for each nondecreasing function $f : \mathbb{N} \to \mathbb{R}$ which has finite expected value $\mathbb{E}(f(X_v))$. Note that

$$\mu_{v,\mathbf{J}}(\mathbf{B}_v) = \sum_{l=1}^{\infty} |B_v(l)| \lambda_v(l) - 1 = \sum_{l=1}^{\infty} (|B_v(l)| - 1) \lambda_v(l) - \lambda_v(0) \tag{19}$$

$$= \sum_{l=1}^{\infty}(|B_v(l)|-1)\hat{\lambda}_v(|B_v(l)|-1) - \hat{\lambda}_v(0) = \sum_{i=1}^{\infty}(i-1)\hat{\lambda}_v(i-1) - \hat{\lambda}_v(0) = \sum_{i=1}^{\infty} i\ \hat{\lambda}_v(i) - \hat{\lambda}_v(0),$$

therefore (19) is the expected value of the random variable $g(X_v)$ where,

$$g(i) = \begin{cases} -1 & \text{if } i = 0; \\ i & \text{if } i \geq 1. \end{cases} \tag{20}$$

The function in (20) is nondecreasing. Thus, by the stochastic ordering, $\mu_{v,\mathbf{J}}(\mathbf{B'}_v) = \mathbb{E}(g(X'_v)) \leq \mathbb{E}(g(X_v)) = \mu_{v,\mathbf{J}}(\mathbf{B}_v)$. ☐

Finally we show the utility of our choice of λ_v which is simpler than theirs.

Remark 2. *For each $v \in \mathbb{Z}^d$ the sequence $(\lambda_v(k))_{k\geq 0}$, defined in (4), is preferable than the one given in [GLO10] which we indicate with $(\lambda^*_v(k))_{k\geq 0}$. Indeed, given an interaction $\mathbf{J} = \{J_B \in \mathbb{R}, B \Subset \mathbb{Z}^d\}$, since $\lambda_v(0) > \lambda^*_v(0)$, $\lambda_v(1) < \lambda^*_v(1)$ and $\lambda_v(k) = \lambda^*_v(k)$ for $k \geq 2$, then the two measures respect the stochastic ordering $\lambda_v \preceq_{st} \lambda^*_v$ for each $v \in \mathbb{Z}^d$. Since $\hat{\lambda}_v(0) = \lambda_v(0) > \lambda^*_v(0) = \hat{\lambda}^*_v(0)$ and $\hat{\lambda}_v(0) + \hat{\lambda}_v(k) = \hat{\lambda}^*_v(0) + \hat{\lambda}^*_v(k)$ for $k \geq |B_v(1)| - 1$, then $\hat{\lambda}_v \preceq_{st} \hat{\lambda}^*_v$. From the latter stochastic ordering, the equalities in (19), and the fact that the function in (20) is nondecreasing, (18) calculated by λ_v is smaller than (18) calculated by λ^*_v. Hence our choice of λ_v facilitates their sufficient condition (8) and our sufficient condition (9) for the applicability of the algorithm.*

By the next two theorems, we see that if an interaction $\mathbf{J}$ verifies (9), then all the interactions obtained from $\mathbf{J}$ by changing them on a finite region and by lowing them in absolute value elsewhere, still verify (9). By Corollary 3 all the associated Gibbs measures are perfectly simulable.

Theorem 2. *Let $v \in \mathbb{Z}^d$, $\mathbf{B}_v \in \mathcal{B}_v$, $\mathbf{J}$, $\widetilde{\mathbf{J}} \in \mathcal{J}$ such that $|\widetilde{J}_B| \leq |J_B|$ for each $B \Subset \mathbb{Z}^d$. Then $\lambda_{v,\widetilde{\mathbf{J}},\mathbf{B}_v} \preceq_{st} \lambda_{v,\mathbf{J},\mathbf{B}_v}$. Hence $\mu_{v,\widetilde{\mathbf{J}}}(\mathbf{B}_v) \leq \mu_{v,\mathbf{J}}(\mathbf{B}_v)$.*

Proof. For brevity of notation we write $\lambda_v = \lambda_{v,\mathbf{J},\mathbf{B}_v}$ and $\widetilde{\lambda}_v = \lambda_{v,\widetilde{\mathbf{J}},\mathbf{B}_v}$. To show the stochastic ordering, we equivalently prove that for each $v \in \mathbb{Z}^d$, $n \in \mathbb{N}$

$$\sum_{l=0}^{n} \widetilde{\lambda}_v(l) \geq \sum_{l=0}^{n} \lambda_v(l). \tag{21}$$

Since $|\widetilde{J}_B| \leq |J_B|$ for each $B \Subset \mathbb{Z}^d$, then

$$\widetilde{\lambda}_v(0) = \exp\left(-\sum\nolimits_{B:v\in B} |\widetilde{J}_B|\right) \geq \exp\left(-\sum\nolimits_{B:v\in B} |J_B|\right) = \lambda_v(0),$$

and for $n \geq 1$

$$\sum_{l=0}^{n} \widetilde{\lambda}_v(l) = \exp\left(-\sum\nolimits_{B:v\in B, B\not\subset B_v(n)} |\widetilde{J}_B|\right) \geq \exp\left(-\sum\nolimits_{B:v\in B, B\not\subset B_v(n)} |J_B|\right) = \sum_{l=0}^{n} \lambda_v(l).$$

□

Theorem 3. *Given the interactions* $\mathbf{J}$, $\widetilde{\mathbf{J}} \in \mathcal{J}$, *if the cardinality of* $\mathcal{C} = \{B \Subset \mathbb{Z}^d : |J_B| \neq |\widetilde{J}_B|\}$ *is finite, then for* $v \in \mathbb{Z}^d$ *and* $\mathbf{B}_v \in \mathcal{B}_v$,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \mu_{v,\mathbf{J}}(\mathbf{B}_v) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \mu_{v,\widetilde{\mathbf{J}}}(\mathbf{B}_v). \tag{22}$$

Proof. Note that the measures $\lambda_{v,\mathbf{J},\mathbf{B}_v}$, $\lambda_{v,\widetilde{\mathbf{J}},\mathbf{B}_v}$ are equal for each v such that all the finite subsets B containing v do not belong to $\mathcal{C}$. In fact if $\{B \Subset \mathbb{Z}^d : v \in B, B \in \mathcal{C}\} = \emptyset$, then for each B including v we have $|J_B| = |\widetilde{J}_B|$, hence $\lambda_{v,\mathbf{J},\mathbf{B}_v} = \lambda_{v,\widetilde{\mathbf{J}},\mathbf{B}_v}$ for each $k \geq 0$. Therefore for $\Lambda \supset \bigcup_{B\in\mathcal{C}} B$

$$\sup_{v\notin\Lambda} \mu_{v,\mathbf{J}}(\mathbf{B}_v) + 1 = \sup_{v\notin\Lambda} \sum_{k=1}^{\infty} |B_v(k)| \lambda_{v,\mathbf{J},\mathbf{B}_v}(k) = \sup_{v\notin\Lambda} \sum_{k=1}^{\infty} |B_v(k)| \lambda_{v,\widetilde{\mathbf{J}},\mathbf{B}_v}(k) = \sup_{v\notin\Lambda} \mu_{v,\widetilde{\mathbf{J}}}(\mathbf{B}_v) + 1. \tag{23}$$

Since the cardinality of $\mathcal{C}$ is finite, then $\bigcup_{B\in\mathcal{C}} B$ is finite. Therefore passing to the limit in (23) for $\Lambda \uparrow \mathbb{Z}^d$, we obtain (22). □

Condition (9) is equivalent to $\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \mu_{v,\mathbf{J}}(\mathbf{B}_v) < 0$, therefore we are interested in finding the infimum value

$$\inf_{\mathbf{x}\in\mathcal{B}_v} \mu_{v,\mathbf{J}}(\mathbf{x}). \tag{24}$$

We define $\mathcal{E}_v$ by distinguishing two cases $N_v = \infty$, $N_v < \infty$. In the first case let $\mathcal{E}_v$ be a subset of $\mathcal{B}_v$ such that each element $(B_v(l))_{l\in\mathbb{N}} \in \mathcal{E}_v$ has the property that there exists a sequence $\{i_k\}_{k\in\mathbb{N}}$ where $B_v(l) = \bigcup_{k=1}^{l} A_{i_k,v}$ for $l \in \mathbb{N}_+$. When $N_v < \infty$, let $\mathcal{E}_v$ be a subset of $\mathcal{B}_v$ such that each element $(B_v(l))_{l\in\mathbb{N}} \in \mathcal{E}_v$ has the property that

$$\exists \bar{l} : B_v(\bar{l}) = \bigcup_{k=1}^{N_v} A_{k,v},\ \exists (i_1, \ldots, i_{\bar{l}}) : B_v(l) = \bigcup_{k=1}^{l} A_{i_k,v}\ \forall l \leq \bar{l}. \tag{25}$$

We notice that, for each $l > \bar{l}$, $\lambda_v(l) = 0$ for any choice of $B_v(l)$ verifying 2).

In the next theorem we restrict the research of the infimum from $\mathcal{B}_v$ to $\mathcal{E}_v$. This produces a sensitive improvement in the case that N_v is finite for each vertex $v \in \mathbb{Z}^d$, in this case the infimum is a minimum because there is a finite number of choices in (25).

Theorem 4. *Let* $\mathbf{J} \in \mathcal{J}$*,* $v \in \mathbb{Z}^d$*, then*

$$\inf_{\mathbf{x}\in\mathcal{B}_v} \mu_{v,\mathbf{J}}(\mathbf{x}) = \inf_{\mathbf{x}\in\mathcal{E}_v} \mu_{v,\mathbf{J}}(\mathbf{x}).$$

Proof. First we consider the case $N_v = \infty$. We endow $\mathcal{B}_v$ with the discrete topology. To prove the theorem we will show that for each $\mathbf{x} \in \mathcal{B}_v$ there exists $\mathbf{y} \in \mathcal{E}_v$ such that $\mu_{v,\mathbf{J}}(\mathbf{y}) \leq \mu_{v,\mathbf{J}}(\mathbf{x})$. By starting from $\mathbf{x} = (x(l))_{l\in\mathbb{N}} \in \mathcal{B}_v$, we will construct a sequence of points $\{\mathbf{x}^{(n)} \in \mathcal{B}_v\}_{n\in\mathbb{N}}$ such that $\mathbf{x}^{(0)} = \mathbf{x}$ and $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{y} \in \mathcal{E}_v$; we will prove that, for $n \in \mathbb{N}$, $\mu_{v,\mathbf{J}}(\mathbf{x}^{(n+1)}) \leq \mu_{v,\mathbf{J}}(\mathbf{x}^{(n)})$ and then, by Fatou's lemma, $\mu_{v,\mathbf{J}}(\mathbf{y}) \leq \liminf_{n\to\infty} \mu_{v,\mathbf{J}}(\mathbf{x}^{(n)})$, from which $\mu_{v,\mathbf{J}}(\mathbf{y}) \leq \mu_{v,\mathbf{J}}(\mathbf{x})$.

Let $\mathbf{x}^{(0)} = \mathbf{x} = \{x(l)\}_{l\in\mathbb{N}} \in \mathcal{B}_v$, we now give the rules to construct $\mathbf{x}^{(1)}$. Define

$$k_0 = 1 + \sup\{l \in \mathbb{N}_+ : \exists (i_1, \ldots, i_l) \text{ s.t. } x(j) = \bigcup_{k=1}^{j} A_{i_k,v} \text{ for any } j = 1, \ldots, l\},$$

if $k_0 = \infty$, then $\mathbf{x} \in \mathcal{E}_v$ and there is nothing to prove. If $k_0 < \infty$ then define

$$I = \{i : A_{i,v} \subset x(k_0)\},$$

$$I^- = \{i : A_{i,v} \subset x(k_0 - 1)\}.$$

If $I = I^-$ then eliminate $x(k_0)$ from the sequence obtaining $x^{(1)}(l) = x(l)$, for $l \leq k_0 - 1$, $x^{(1)}(l) = x(l + 1)$, for $l \geq k_0$. In this case $\mu_{v,\mathbf{J}}(\mathbf{x}^{(0)}) = \mu_{v,\mathbf{J}}(\mathbf{x}^{(1)})$.

If $I \neq I^-$, consider $j = \min\{i : i \in I \setminus I^-\}$, define $x^{(1)}(l) = x(l)$, for $l \leq k_0 - 1$, $x^{(1)}(k_0) = x(k_0 - 1) \cup A_{j,v}$, $x^{(1)}(l) = x(l - 1)$, for $l \geq k_0 + 1$. It is easy to check that the sequence $\mathbf{x}^{(1)}$ verify the conditions 1), 2) and 3). In this case the sequence $\mathbf{x}^{(0)}$ is less refined than $\mathbf{x}^{(1)}$, therefore $\mu_{v,\mathbf{J}}(\mathbf{x}^{(0)}) \geq \mu_{v,\mathbf{J}}(\mathbf{x}^{(1)})$, by Corollary 1.

We repeat the procedure to construct $\mathbf{x}^{(n+1)}$ from $\mathbf{x}^{(n)}$, for any $n \in \mathbb{N}_+$. Obviously there exists $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{y} \in \mathcal{E}_v$. Since $\hat{\lambda}_{v,\mathbf{J},\mathbf{z}}(0)$ does not depend on $\mathbf{z} \in \mathcal{B}_v$ we set $\hat{\lambda}_{v,\mathbf{J}}(0) = \hat{\lambda}_{v,\mathbf{J},\mathbf{z}}(0)$, therefore we can write

$$\mu_{v,\mathbf{J}}(\mathbf{y}) = -\hat{\lambda}_{v,\mathbf{J},\mathbf{y}}(0) + \sum_{i=1}^{\infty} i\hat{\lambda}_{v,\mathbf{J},\mathbf{y}}(i) = -\hat{\lambda}_{v,\mathbf{J}}(0) + \sum_{i=1}^{\infty} \liminf_{n\to\infty} i\hat{\lambda}_{v,\mathbf{J},\mathbf{x}^{(n)}}(i)$$

$$\leq -\hat{\lambda}_{v,\mathbf{J}}(0) + \liminf_{n\to\infty} \sum_{i=1}^{\infty} i\hat{\lambda}_{v,\mathbf{J},\mathbf{x}^{(n)}}(i) = \liminf_{n\to\infty} \mu_{v,\mathbf{J}}(\mathbf{x}^{(n)}) \leq \mu_{v,\mathbf{J}}(\mathbf{x}),$$

where the first inequality follows by Fatou's lemma. The case $N_v < \infty$ is simpler and in a finite number n_0 of steps one obtains that $\mathbf{x}^{(n_0)}$ is in $\mathcal{E}_v$. $\square$

Remark 3. *If for any $v \in \mathbb{Z}^d$ the number N_v is* small *and if it can be proved that for some $(\mathbf{x}_v \in \mathcal{B}_v)_{v\in\mathbb{Z}^d}$*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \mu_{v,\mathbf{J}}(\mathbf{x}_v) < 0, \tag{26}$$

then one can write the algorithm. Proving (26) is a little easier than proving their sufficient condition, and in both cases it should be done a priori. *In the backward sketch procedure a random vertex w is selected, now the algorithm calculates $\widehat{\mathbf{x}}_w = \arg\min_{\mathbf{x}\in\mathcal{E}_w} \mu_{w,\mathbf{J}}(\mathbf{x})$ with a finite number of elementary operations because, for any $\mathbf{x} \in \mathcal{E}_w$, $\lambda_{w,\mathbf{J},\mathbf{x}}(l)$ must be calculated for $l = 1, \ldots, N_w$ and also all the sums involved in the definition of $\lambda_{w,\mathbf{J},\mathbf{x}}$ and of $\mu_{w,\mathbf{J}}(\mathbf{x})$ are finite. Moreover the $\mathbf{x}$'s in $\mathcal{E}_w$ to be considered are at most $(N_w!)$. By comparing the finite list (having at most $N_w!$ elements) of $\mu_{w,\mathbf{J}}(\mathbf{x})$ with $\mathbf{x} \in \mathcal{E}_w$, the algorithm finds $\widehat{\mathbf{x}}_w \in \mathcal{E}_w$ such that $\mu_{w,\mathbf{J}}(\widehat{\mathbf{x}}_w) = \min_{\mathbf{x}\in\mathcal{E}_w} \mu_{w,\mathbf{J}}(\mathbf{x})$. This procedure is repeated for all the selected vertices, which are almost surely finite. Hence the problem is computable and the previous procedure is really an algorithm. The computability is guaranteed by the fact that N_v is finite, further the algorithm runs in reasonable time if N_v is small.*

If N_v is large *or equal to* infinity, *if one succeed to calculate a $(\mathbf{x}_v \in \mathcal{B}_v)_{v\in\mathbb{Z}^d}$ such that condition (26) is satisfied, then the algorithm can use this particular choice.*

4 A general result on the extinction of a population

The following theorem gives a generalization of the results on Galton-Watson's process and it applies to processes that behave like a supermartingale when they assume large values.

Sometimes we will write for brevity of notation $\mathbf{i}_h^k$ in place of the vector $(i_h, ..., i_k)$, for $h < k$.

Theorem 5. *Let $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ be a stochastic process over $\mathbb{N}$. Suppose that there exists $N \in \mathbb{N}$ such that the following relations hold:*

1) $\mathbb{P}(X_{n+1} = 0 | X_n = 0) = 1$, *for* $n \in \mathbb{N}$;

2) for $i \leq N$ there exists $n_i \in \mathbb{N}_+$ such that

$$q_i = \operatorname{ess\,inf}_{m\in\mathbb{N}, i_0,\ldots,i_{m-1}\in\mathbb{N}_+} \mathbb{P}(X_{m+n_i} = 0 | X_0 = i_0, \ldots, X_m = i_m) > 0, \ i_m = i;$$

3) $\mathbb{E}(X_{n+1} | X_0 = i_0, \ldots, X_n = i_n) \leq i_n$ *a.s. for* $n \in \mathbb{N}$, $i_0, \ldots, i_{n-1} \in \mathbb{N}$, $i_n > N$*;*

4) $p_i = \operatorname{ess\,inf}_{m\in\mathbb{N}, i_0,\ldots,i_{m-1}\in\mathbb{N}_+} \mathbb{P}(X_{m+1} \neq i | X_0 = i_0, \ldots, X_m = i_m) > 0$, $i_m = i > N$.

Then

$$\lim_{n\to\infty} X_n = 0 \ a.s.$$

Proof. Let $A = \{0, 1, ..., N\}$, $B = \{N+1, N+2, ...\}$ where N is given in the theorem. Let us define

$$T^{(1)}_{A\to B} = \inf\{n \geq 0 : X_n \in B\}, \quad T^{(1)}_{B\to A} = \inf\{n > T^{(1)}_{A\to B} : X_n \in A\}, \tag{27}$$

$$T^{(h)}_{A\to B} = \inf\{n > T^{(h-1)}_{B\to A} : X_n \in B\}, \quad T^{(h)}_{B\to A} = \inf\{n > T^{(h)}_{A\to B} : X_n \in A\}, \tag{28}$$

for $h \geq 2$.

The random variables $\{T^{(h)}_{A\to B}, T^{(h)}_{B\to A} : h \in \mathbb{N}\}$ are stopping time. We put $T^{(h)}_{A\to B} = \infty$ if the set, on which the infimum is defined, is empty or if $T^{(h-1)}_{B\to A} = \infty$. Similarly we write $T^{(h)}_{B\to A} = \infty$ if the set, on which the infimum is defined, is empty or if $T^{(h)}_{A\to B} = \infty$. The following inequalities are obtained directly by definitions (27) and (28)

$$T^{(1)}_{A\to B} \leq T^{(1)}_{B\to A} \leq T^{(2)}_{A\to B} \leq \ldots \leq T^{(h)}_{A\to B} \leq T^{(h)}_{B\to A} \leq \ldots$$

The previous inequalities are strict until one of these stopping times becomes infinite.

Let us define the stopped process $\{Y^{(m)}_n = X_{n\wedge T^{(m)}_{B\to A}} : n \in \mathbb{N}\}$ on $\{T^{(m)}_{A\to B} < \infty\}$, for $m \in \mathbb{N}_+$. We do a partition of $\{T^{(m)}_{A\to B} < \infty\}$ in the sets $\{\{T^{(m)}_{A\to B} = k\} : k \in \mathbb{N}_+\}$. On every set $\{T^{(m)}_{A\to B} = k\}$, the elements of A are absorbing states for $Y^{(m)}_n$ when $n \geq k$, therefore $\{Y^{(m)}_n\}_{n\geq k}$ is a non-negative supermartingale on $\{T^{(m)}_{A\to B} = k\}$, by hypothesis 3). Thus, see [Wil91], there exists

$$\lim_{n\to+\infty} Y^{(m)}_n < \infty \quad \text{on } \{T^{(m)}_{A\to B} < \infty\} \ a.s. \tag{29}$$

We will prove that the limit in (29) belongs to A almost surely.

Given $k \in \mathbb{N}_+$, we prove (29) on the set $\{T^{(m)}_{A\to B} = k\}$. In fact if, by contradiction, $i \in B$

$$\mathbb{P}(\lim_{n\to+\infty} Y^{(m)}_n = i | T^{(m)}_{A\to B} = k) = \mathbb{P}\bigg(\bigcup_{h=k+1}^{\infty} \bigcap_{n=h}^{\infty} \{Y^{(m)}_n = i\} \bigg| T^{(m)}_{A\to B} = k \bigg) \leq$$

$$\leq \sum_{h=k+1}^{\infty} \mathbb{P}\bigg(\bigcap_{n=h}^{\infty} \{Y^{(m)}_n = i\} \bigg| T^{(m)}_{A\to B} = k \bigg) \leq \sum_{h=k+1}^{\infty} \prod_{r=h+1}^{\infty} \mathbb{P}(Y^{(m)}_r = i | Y^{(m)}_h = \ldots = Y^{(m)}_{r-1} = i, T^{(m)}_{A\to B} = k) \ =$$

$$= \sum_{h=k+1}^{\infty} \prod_{r=h+1}^{\infty} \mathbb{P}(X_r = i | X_h = \ldots = X_{r-1} = i, T^{(m)}_{A\to B} = k) \ , \tag{30}$$

the latter equality is a consequence of the fact that if the limit belongs to B then the process $\{X_n\}_{n\geq k}$ never visits A and so, in this case, the processes $\{Y^{(m)}_n\}_{n\geq k}$ and $\{X_n\}_{n\geq k}$ coincide. Now, by using hypothesis 4) and a standard argument on the partition of the trajectories, we obtain the following upper bound for (30)

$$\sum_{h=k+1}^{\infty} \prod_{r=h}^{\infty} (1-p_i) = 0. \tag{31}$$

Hence we get that

$$\lim_{n\to+\infty} Y^{(m)}_n \in A \ a.s.$$

or equivalently that

$$\mathbb{P}\left(\{T^{(m)}_{A\to B} < \infty\} \setminus \{T^{(m)}_{B\to A} < \infty\} \right) = 0.$$

From which it follows that

$$\mathbb{P}(\cdot | T^{(m-1)}_{A\to B} < \infty) = \mathbb{P}(\cdot | T^{(m-1)}_{B\to A} < \infty). \tag{32}$$

Notice that, if the numbers n_i, for $i = 0, \ldots, N$, verify hypothesis 2) of the theorem, then, by taking a $n \geq \max\{n_i : i \leq N\}$, condition 2) is still verified. In fact if the process visits the state zero, then it indefinitely remains in zero, which directly follows by hypothesis 1). Therefore let us define $\widetilde{n} = \max\{n_i : i \leq N\} \in \mathbb{N}_+$, then hypothesis 2) is satisfied by using $\widetilde{n}$ instead of n_i where the values of the q_i's can only increase by replacing all the n_i's with $\widetilde{n}$. Hence all the q_i's calculated setting $n_i = \widetilde{n}$ are greater than some positive constant q which can be chosen equal to $\inf\{q_i : i = 1, \ldots, N\}$.

Then we get, for $k \in \mathbb{N}_+$, almost surely

$$\mathbb{P}(T_{A\to B}^{((k+1)\widetilde{n})} = \infty | T_{A\to B}^{(k\widetilde{n})} < \infty) = \mathbb{P}(T_{A\to B}^{((k+1)\widetilde{n})} = \infty | T_{B\to A}^{(k\widetilde{n})} < \infty), \tag{33}$$

by (32). By denoting the set of trajectories $M_{n,k} = \{\mathbf{i}_0^n \in \mathbb{N}^n : \{\mathbf{X}_0^n = \mathbf{i}_0^n\} \subset \{T_{B\to A}^{(k\widetilde{n})} = n\}\}$, from the previous relation we obtain almost surely

$$\mathbb{P}(T_{A\to B}^{((k+1)\widetilde{n})} = \infty | T_{B\to A}^{(k\widetilde{n})} < \infty) =$$

$$= \sum_{n=1}^{\infty} \sum_{\mathbf{i}_0^n \in M_{n,k}} \mathbb{P}(T_{A\to B}^{((k+1)\widetilde{n})} = \infty | T_{B\to A}^{(k\widetilde{n})} = n, \mathbf{X}_0^n = \mathbf{i}_0^n) \mathbb{P}(T_{B\to A}^{(k\widetilde{n})} = n, \mathbf{X}_0^n = \mathbf{i}_0^n | T_{B\to A}^{(k\widetilde{n})} < \infty) \geq$$

$$\geq \sum_{n=1}^{\infty} \sum_{\mathbf{i}_0^n \in M_{n,k}} \mathbb{P}(X_{n+\widetilde{n}} = 0 | \mathbf{X}_0^n = \mathbf{i}_0^n) \mathbb{P}(T_{B\to A}^{(k\widetilde{n})} = n, \mathbf{X}_0^n = \mathbf{i}_0^n | T_{B\to A}^{(k\widetilde{n})} < \infty) \geq q > 0.$$

Thus indicating $m = \lfloor n/\widetilde{n} \rfloor$ for a generic $n \in \mathbb{N}_+$, we obtain the following relation

$$\mathbb{P}(T_{A\to B}^{(n)} < \infty) \leq \prod_{k=2}^{m} \mathbb{P}(T_{A\to B}^{(k\widetilde{n})} < \infty | T_{A\to B}^{((k-1)\widetilde{n})} < \infty) \leq (1-q)^{m-1}.$$

Since, for each $n \in \mathbb{N}_+$, $\{T_{A\to B}^{(n)} < \infty\} \supset \{T_{A\to B}^{(n+1)} < \infty\}$, by the monotone convergence theorem

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \{T_{A\to B}^{(n)} < \infty\} \right) = \lim_{n\to+\infty} \mathbb{P}(T_{A\to B}^{(n)} < \infty) \leq \lim_{n\to+\infty} (1-q)^{\lfloor n/\widetilde{n} \rfloor - 1} = 0.$$

Hence almost surely there exists a finite random index $S = 2, 3, \ldots$ such that $T_{A\to B}^{(S-1)} < \infty$, $T_{B\to A}^{(S-1)} < \infty$ and $T_{A\to B}^{(S)} = \infty$, then $X_n \in A$ for any $n \geq T_{B\to A}^{(S-1)}$. It remains to show that the process can not stay indefinitely in $\{1, 2, \ldots, N\}$.

Let us define

$$\widetilde{X}_k = X_{k\widetilde{n}}, \text{ for } k \in \mathbb{N}.$$

Note that for the process $\widetilde{\mathbf{X}} = \{\widetilde{X}_n : n \in \mathbb{N}\}$ there exists a random time almost surely finite

$$\widetilde{T}_A = \inf\{n : \widetilde{X}_k \in A, \text{ for } k \geq n\},$$

such that the process remains indefinitely in A after $\widetilde{T}_A$. Moreover observe that $\widetilde{T}_A$ is not a stopping time and it shall be taken into account the information provided by the value of $\widetilde{T}_A$. Directly from hypothesis 2) it follows that

$$\tilde{q} = \operatorname{ess\,inf}_{m\in\mathbb{N}, i_0, i_1, \ldots, i_{m-1} \in \mathbb{N}, i_m \in A} \mathbb{P}(\widetilde{X}_{m+1} = 0 | \widetilde{\mathbf{X}}_0^m = \mathbf{i}_0^m)$$

is positive.

Now we will show that for each $n \in \mathbb{N}_+$,

$$\operatorname{ess\,inf}_{m\geq n,\mathbf{i}_0^{n-2}\in\mathbb{N}^{n-1},i_{n-1}\in B,i_n,\dots,i_m\in A}\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)\geq\tilde{q}>0. \qquad (34)$$

We notice that for $\mathbf{i}_0^{n-2}\in\mathbb{N}^{n-1}$, $i_{n-1}\in B$, $i_n,\dots,i_m\in A$,

$$\{\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m,\widetilde{X}_{m+1}=0\}\subset\{\widetilde{T}_A=n\},$$

from which

$$\mathbb{P}(\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m,\widetilde{X}_{m+1}=0)\leq\mathbb{P}(\widetilde{T}_A=n).$$

Hence almost surely

$$\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)=\frac{\mathbb{P}(\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m,\widetilde{X}_{m+1}=0)}{\mathbb{P}(\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)}=$$

$$=\frac{\mathbb{P}(\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m,\widetilde{X}_{m+1}=0)}{\mathbb{P}(\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)}\geq\frac{\mathbb{P}(\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m,\widetilde{X}_{m+1}=0)}{\mathbb{P}(\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)}=\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m).$$

From which by taking the essential infimum,

$$\operatorname{ess\,inf}_{m\geq n,\mathbf{i}_0^{n-2}\in\mathbb{N}^{n-1},i_{n-1}\in B,i_n,\dots,i_m\in A}\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{T}_A=n,\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)\geq$$

$$\geq\operatorname{ess\,inf}_{m\geq n,\mathbf{i}_0^{n-2}\in\mathbb{N}^{n-1},i_{n-1}\in B,i_n,\dots,i_m\in A}\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)\geq$$

$$\geq\operatorname{ess\,inf}_{m\in\mathbb{N},i_0,i_1,\dots,i_{m-1}\in\mathbb{N},i_m\in A}\mathbb{P}(\widetilde{X}_{m+1}=0|\widetilde{\mathbf{X}}_0^m=\mathbf{i}_0^m)=\tilde{q}>0.$$

Analogously to (31), by the latter inequalities and standard arguments on the partition of trajectories, one obtain that the process $\widetilde{\mathbf{X}}$ will be eventually equals to zero. Obviously the same property is obtained for the original process $\mathbf{X}$, i.e. $\lim_{n\to+\infty}X_n=0$ *a.s.* □

Remark 4. *We note that, in the previous theorem, the process* $\{X_n\}_{n\in\mathbb{N}}$ *could be a non-homogeneous Markov chain. In particular one can consider a culture of bacteria in which the number of its population affects the ability of reproduction of the bacteria by changing the probability that the cell dies before its mitosis. In some way we can think that a process* $\{X_n\}_{n\in\mathbb{N}}$*, verifying the assumptions of Theorem 5, can be chosen as a model for these biological cultures. Therefore the bacteria cultures will die in a finite time.*

5 Applications of Theorem 5 to perfect simulation

Let us consider a probability mass function ψ_v indexed by $v \in \mathbb{Z}^d$ and let $\sum_{l=0}^{\infty} \psi_v(l) = 1$. Moreover, for each $v \in \mathbb{Z}^d$, let $\psi_v(0) > 0$.

Let us associate to each vertex $v \in \mathbb{Z}^d$ a sequence $\mathbf{S}_v = \{S_v(l) \Subset \mathbb{Z}^d : l \in \mathbb{N}_+\}$ and a mass M_v such that $\inf_{v\in\mathbb{Z}^d} M_v \geq 1$.

Let $v \in \mathbb{Z}^d$ and $\{D_n\}_{n\in\mathbb{N}}$ be a homogeneous Markov chain with countable state space $\mathcal{C} = \{A \Subset \mathbb{Z}^d\}$.

At time zero the Markov chain has a initial measure $\nu^{(0)}$. The rules of the dynamics are given in Section 2, it only needs to replace C_n, $\mathbf{B}_v$, λ_v with D_n, $\mathbf{S}_v$, ψ_v respectively.

Let us define, for each $v \in \mathbb{Z}^d$,

$$\eta_v = -\psi_v(0) + \sum_{l=1}^{\infty} |S_v(l)|\psi_v(l), \tag{35}$$

which is similar a 18 and it plays the same role.

We are now in the position to present our result on the almost surely extinction of processes above defined.

Corollary 2. *Let η_v as in (35), if $\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{v\notin\Lambda} \eta_v < 0$, then $\limsup_{n\to\infty} D_n = \emptyset$ almost surely.*

Proof. Let $X_n = |D_n|$, we want to show that the process $\{X_n\}_{n\in\mathbb{N}}$ verifies all the hypotheses of Theorem 5. Hypothesis 1) is trivially verified since that if $D_n = \emptyset$, then $D_{n+1} = \emptyset$. We verify now hypothesis 3). First of all note that from the assumption of the corollary it follows the existence of a $\delta > 0$ such that the set

$$R_\delta = \{v : \eta_v > -\delta\} \tag{36}$$

has finite cardinality.

Let us fix $\delta > 0$ such that $|R_\delta| < \infty$, we define $a = \max\{0, M_v\eta_v : v \in R_\delta\}$.

Let us consider $D_n \neq \emptyset$, we easily see that

$$\mathbb{E}(X_{n+1}|D_n) = \mathbb{E}(|D_{n+1}|\ |D_n) \leq |D_n| + \sum_{v\in D_n} \frac{M_v}{\sum_{u\in D_n} M_u}\eta_v.$$

Under the assumption of the corollary and since $M_v \geq 1$ for each $v \in \mathbb{Z}^d$, we obtain

$$\mathbb{E}(X_{n+1}|D_n) \leq |D_n| + \frac{1}{\sum_{u\in D_n} M_u}\left[a|R_\delta| - \delta(|D_n| - |R_\delta|)\right].$$

We get that if

$$|D_n| \geq \left\lceil \frac{a|R_\delta|}{\delta} + |R_\delta| \right\rceil \equiv N, \tag{37}$$

then $\mathbb{E}(X_{n+1}|D_n) \leq X_n$. Since

$$\mathbb{E}(X_{n+1}|\mathbf{X}_0^n = \mathbf{i}_0^n) = \sum_{A\Subset\mathbb{Z}^d:|A|=i_n} \mathbb{E}(X_{n+1}|D_n = A)\mathbb{P}(D_n = A|\mathbf{X}_0^n = \mathbf{i}_0^n), \tag{38}$$

we have that (38) is lesser or equal to $X_n = i_n$ when $i_n \geq N$. Hence the property 3) is obtained by choosing N as in (37), because all the summands in (38) are non-positive.

Now we show that

$$\xi = \inf_{v\in\mathbb{Z}^d} \psi_v(0) > 0.$$

Note that

$$\rho = \inf\{\psi_v(0) : v \in R_\delta\} > 0$$

because it is an infimum on a finite set of positive numbers. Moreover, from (35), it follows

$$\rho' = \inf\{\psi_v(0) : v \in R_\delta^c\} \geq \delta > 0.$$

Hence

$$\xi = \min\{\rho, \rho'\} > 0.$$

Therefore the hypothesis 2) is verified for $n_i = N$ and the q_i's are larger or equal than $\xi^N > 0$, for $i \leq N$.

We also obtain 4) by observing that $p_i \geq \xi > 0$ for each $i \in \mathbb{N}_+$.

Thus, from Theorem 5,

$$\lim_{n\to+\infty} X_n = 0 \ a.s.$$

There exists an almost surely finite random time Y such that $C_Y = \emptyset$. □

Given $\mathbf{J} \in \mathcal{J}$, $\mathbf{B}_v \in \mathcal{B}_v$, we set

$$S_v(l) = B_v(l) \setminus \{v\} \text{ for } l \in \mathbb{N}_+, v \in \mathbb{Z}^d,$$

and $\psi_v = \lambda_{v,\mathbf{J},\mathbf{B}_v}$, then, by a simple calculation, $\eta_v = \mu_{v,\mathbf{J}}(\mathbf{B}_v)$. Putting $M_u = const.$, for each $u \in \mathbb{Z}^d$, and $\nu^{(0)} = \delta_{\{v\}}$ the process $\{D_n\}_n$ coincides with $\{C_n\}_n$ defined in Section 2.

Corollary 3. *Let* $\mathbf{J} \in \mathcal{J}$*,* $v \in \mathbb{Z}^d$*,* $\mathbf{B}_v \in \mathcal{B}_v$*. If* $\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \mu_{v,\mathbf{J}}(\mathbf{B}_v) < 0$*, then* $\limsup_{n\to\infty} C_n = \emptyset$ *almost surely. Moreover a sufficient condition for the perfect sampling from a Gibbs measure is* $\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \mu_{v,\mathbf{J}}(\mathbf{B}_v) < 0$*.*

The second part of the corollary is a direct consequence of the first part of the corollary and Proposition 1.

References

[CFF02] Francis Comets, Roberto Fernández, and Pablo A. Ferrari. Processes with long memory: regenerative construction and perfect simulation. *Ann. Appl. Probab.*, 12(3):921–943, 2002.

[DSP08] Emilio De Santis and Mauro Piccioni. Exact simulation for discrete time spin systems and unilateral fields. *Methodol. Comput. Appl. Probab.*, 10(1):105–120, 2008.

[GLO10] A. Galves, E. Löcherbach, and E. Orlandi. Perfect simulation of infinite range Gibbs measures and coupling with their finite range approximations. *J. Stat. Phys.*, 138(1-3):476–495, 2010.

[HS00] Olle Häggström and Jeffrey E. Steif. Propp-Wilson algorithms and finitary codings for high noise Markov random fields. *Combin. Probab. Comput.*, 9(5):425–439, 2000.

[MG98] D. J. Murdoch and P. J. Green. Exact sampling from a continuous state space. *Scand. J. Statist.*, 25(3):483–502, 1998.

[PW96] James Gary Propp and David Bruce Wilson. Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures Algorithms*, 9(1-2):223–252, 1996.

[Wil91] David Williams. *Probability with martingales*. Cambridge University Press, Cambridge, 1991.