# An alternative unified formulation for integer and fractional quantum Hall effects

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(Dated: July 28, 2011)

In this paper, a new unified formulation of integer and fractional quantum Hall effect are presented. Firstly, under the condition of strong magnetic field and symmetry gauge, the Pauli equation is solved in which the wave function and energy levels are given explicitly, then after the calculation of the degeneracy density for 2-dimensional idea electron gas system, the Hall resistance of the system is obtained, where the quantum Hall number $\nu$ is introduced. The new defined $\nu$ , called filling factor in the literature, is related to radial quantum number $n_\rho$ and angular quantum number $|m|$, different $n_\rho$ and $|m|$ corresponding to different $\nu$, which gives unification explaination for integer and fractional quantum Hall effects, and we also predicate more new cases of fractional quantum Hall effects to be confirmed by experiment.

PACS numbers: 72.20.My, 71.45.-d, 73.40.Lq
Keywords: quantum Hall effect, fractional quantum effect, degeneracy density, Hall quantum number.



## INTRODUCTION

In 1879, E. H. Hall discovered that when a conductor carrying an electric current perpendicular to an applied magnetic field develops a voltage gradient which is transverse to both the current and the magnetic field. This phenomenon is called Hall effect. About 100 years after the discovery of Hall effect, Klaus von Klitzing, in 1980, made the unexpected discovery that, under low temperature and strong magnetic field, the Hall conductivity was exactly quantized[1], in which the Hall conductivity $\sigma$ takes on the quantized values, i.e. $\sigma = \nu\frac{e^2}{h}$ and $\nu$ takes integer values, we call it quantum Hall effect (QHF). For this finding, von Klitzing was awarded the 1985 Nobel Prize in Physics[2]. Very soon after that, under much more low temperature, the fractional quantum Hall effect(FQHE) was experimentally discovered in 1982 by Daniel Tsui and Horst Stormer[3], in which $\nu$, called filling factor, takes fractional values. Each particular value of the magnetic field corresponds to $\nu = \frac{p}{q}$, where p and q are integers with no common factors. Here q turns out to be an odd number with the exception of two $\nu$'s 5/2 and 7/2. The principal series of such fractions are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, etc.,and $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, etc., all fractions have an odd denominator. The effect was explained by Laughlin in 1983, using a novel quantum liquid phase that accounts for the effects of interactions between electrons. Tsui, Stormer, and Laughlin were awarded the 1998 Nobel Prize in physics for their work.

The theoretical study of integer and fractional quantum Hall effect has about thirty years' history[5]. For examples, the theory of Laughlin which introduced several novel concepts in correlated quantum fluids, inspired analogous effects in other subfields of physics; the quantum Hall effect was generalized to four dimensions [6] in order to study the interplay between quantum correlations and dimensionality in strongly correlated systems; two-dimensional electron systems were modeled by strings interacting with D-branes [7], where the fractionally-charged quasi-particles and composite fermions were described in the language of string theory; an interesting analogy between the quantum Hall effect and black hole has been reported, and in particular, the edge properties of a quantum Hall effect system have been used to model black hole physics from the point of view of an external observer [8]. Important developments of the quantum Hall effect have also taken place from the field theoretical point of view[9].

In this paper, we give an alternative unification description for integer and fractional quantum Hall effect, in which we use purely quantum mechanics theory and do not need any other suppose conditions or hypothesis.

## THE SOLUTION OF PAULI EQUATION

In this section, we study the wave function and energy levels for the electron moving in strong magnetic field, and give the expectation value of the moving range of single electron, then after a new definition of quantum Hall number $\nu$, the Hall resistance formula is given explicitly.

The Dirac equation for an electron moving in magnetic filed is usually expressed as,

$$i\hbar\frac{\partial}{\partial t}\psi = [c\vec{\alpha}\cdot(\vec{p}+\frac{e}{c}\vec{A}) - e\phi + \mu c^2\beta]\psi. \qquad (1)$$

Let's consider the situation of the electron moving in $x-y$ plane, and uniform magnetic field applied in $z$-direction, then under the symmetric gauge, i.e. $\vec{A} = (-\frac{B}{2}y, \frac{B}{2}x)$, without electric field, the non-relativistic approximation of Dirac equation (1) is cast to Pauli equation as follows,

$$i\hbar\frac{\partial}{\partial t}\psi=(\frac{1}{2\mu}(p_x-\frac{eB}{2c}y)^2+\frac{1}{2\mu}(p_y+\frac{eB}{2c}x)^2+\frac{1}{2\mu}p_z^2+\frac{e\hbar B}{2\mu c}\sigma_z)\psi, \quad (2)$$

last term is the Stern-Gerlach term.

In cylinder coordinate description, we find, after tedious calculation, the stationary wave function is obtained,

$$\psi_{n_\rho l\lambda}(\rho,\varphi,z,s,t)=[\frac{(n_\rho+|l|)!}{\pi n_\rho!}]^{1/2}\frac{1}{a}\xi^{|l|}e^{-\xi^2/2}\cdot \Sigma_{k=0}^{n_\rho}\frac{(-1)^k\mathrm{Binomial}(n_\rho,k)}{(k+|l|)!}\xi^{2k+|l|}\chi_\lambda(s)e^{il\varphi}e^{-ip_z z/\hbar}e^{-iEt/\hbar}, \quad (3)$$

where $\xi=\frac{\rho}{a}$ and $a=(2\hbar c/eB)^{1/2}$, which defines the moving scale of the electron. When magnetic strength $B=17T$, the valve of $a$ is $93.68\mathring{A}$, which is 177 times as large as Bohr radius. So under strong magnetic field electron gas system can be considered as an idea gas, the interaction among electrons can be neglected. The energy levels for the stationary state can be expressed as,

$$E_{n_\rho m\lambda}=\frac{1}{2\mu}p_z^2+(2n_\rho+|m|+m+\lambda+1)\frac{e\hbar B}{2\mu c}, \quad n_\rho=0,1,2,\ldots;\ m=0,\pm1,\pm2,\ldots;\lambda=-1,1 \quad (4)$$

where, $n_\rho$'s are radial quantum number, $m$'s are angular quantum number ($z$-direction) and $\lambda$'s are spin quantum number in the Pauli representation. When the angular momentum quantum number $m\le 0$, we have $m+|m|=0$, then the energy formula (4) can be replaced as

$$E_n=\frac{1}{2\mu}p_z^2+(n_\rho+\frac{\lambda+1}{2})\frac{e\hbar B}{\mu c}\ ,\ n_\rho=0,1,2,\ldots;\ \lambda=-1,1 \quad (5)$$

This energy level formula shows that when $m\le 0$, the energy levels do not depend on angular quantum number, namely, the energy degeneracy is infinite. On other words, because of the low temperature and strong magnetic field, electron is forced to be in the states of infinite energy degeneracy. This is just the very condition for quantum Hall effect. Theoretically, because of the infinite energy degeneracy, there should be infinite number of electrons occupying one energy level, but it is not true for reality. Because every electron has certain moving space, so in a finite plane, there should be finite electron occupying one energy level. For two dimensional idea electron gas, the expectation value of moving area for single electron is depended on quantum number $n_\rho$ and $m$,

$$<S>_{n_\rho,m}=<\pi\rho^2>_{n_\rho,m}=\pi\int_0^{2\pi}d\varphi\int_0^\infty\rho^3|\phi(\rho,\varphi,z,s)|^2d\rho =\pi(2n_\rho+|m|+1)\frac{2\hbar c}{eB}, \quad (6)$$

the detail calculation can be found in [10]. The physics conditions of quantum Hall effect are: 1) electrons are fully polarized, namely, $\lambda=-1$; 2), $m\le 0$. The energy level does not depend on $m$, so for one energy level the angular quantum number can take $(|m|+1)$ values as $0,-1,-2,-3,\cdots,-m$. Therefore, for the electron gas with total electron number $N$, in strong magnetic field, on the plane range of $N<S>_{n_\rho,m}$, the number of electron state is

$$N_B=N(|m|+1). \quad (7)$$

So the energy degeneracy density (energy degeneracy on per unit area) for electron gas in strong magnetic field is

$$n_B=\frac{N_B}{N<S>_{n_\rho,m}}=\frac{|m|+1}{2n_\rho+|m|+1}\frac{eB}{hc}. \quad (8)$$

this shows that the energy degeneracy on unit area depends on radius quantum number $n_\rho$ and angular quantum number $|m|$, and is also proportional to magnetic strength $B$.

Let us define a quantum Hall number $\nu$ as:

$$\nu=\frac{|m|+1}{2n_\rho+|m|+1}, \quad (9)$$

then the equation (8) can be rewrite as,

$$n_B=\frac{N_B}{N<S>_{n_\rho,m}}=\nu\frac{eB}{hc}. \quad (10)$$

From the definition of Hall resistance and equation (10), we get the Hall resistance as

$$\rho_{xy}=\frac{B}{n_Bec}=\frac{1}{\nu}\frac{h}{e^2} \quad (11)$$

Obviously, the Hall resistance is only depended on the quantum Hall number $\nu$, and the later depends on the ratio of $|m|+1$ and $2n_\rho+|m|+1$. This is the key result of this paper, from this result, the quantum Hall effect and fractional quantum Hall effect can be unified formulated (see section below).

## UNIFICATION FORMULATION OF INTEGER AND FRACTIONAL QUANTUM HALL EFFECTS

In this section, we will use the results above to give a unification description for both quantum Hall effect and fractional Hall effect. From the formula of quantum Hall number (9), we can see that different quantum number $n_\rho$ and $m$ corresponding different quantum states, and $\nu$ can take value 1 and also fraction, they relate respectively to integer quantum Hall effect and fractional quantum Hall effect.

1, When $n_\rho=0$, from equation(9) we get $\nu=1$. And now the angular quantum number can still take one of $|m|+1$ values: $0,-1,-2\cdots-|m|$. On other words, on one energy level, the electrons of the electron gas system can fill different state with different angular quantum number $0,-1,-2\cdots-|m|$. the less absolute value of angular

TABLE I: fractional quantum Hall number

| $m\backslash n_\rho$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1/3 | 1/5 | 1/7 | 1/9 | 1/11 | 1/13 | 1/15 | 1/17 | 1/19 |
| -1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 | 1/10 |
| -2 | 3/5 | 3/7 | 1/3 | 3/11 | 3/13 | 1/5 | 3/17 | 3/19 | 1/7 |
| -3 | 2/3 | 1/2 | 2/5 | 1/3 | 2/7 | 1/4 | 2/9 | 1/5 | 2/11 |
| -4 | 5/7 | 5/9 | 5/11 | 5/13 | 1/3 | 5/17 | 5/19 | 5/21 | 5/23 |
| -5 | 3/4 | 3/5 | 1/2 | 3/7 | 3/8 | 1/3 | 3/10 | 3/11 | 1/4 |
| -6 | 7/9 | 7/11 | 7/13 | 7/15 | 7/17 | 7/19 | 1/3 | 7/23 | 7/25 |
| -7 | 4/5 | 2/3 | 4/7 | 1/2 | 4/9 | 2/5 | 4/11 | 1/3 | 4/13 |
| -8 | 9/11 | 9/13 | 3/5 | 9/17 | 9/19 | 3/7 | 9/23 | 9/25 | 1/3 |
| -9 | 5/6 | 5/7 | 5/8 | 5/9 | 1/2 | 5/11 | 5/12 | 5/13 | 5/14 |

quantum number correspond to the more stable state and less moving range. Firstly, when all electrons occupy the state of angular quantum number equals 0, and thus Hall plateau appears. When the magnetic field becomes less, electron density of electron gas also gets less, then the extra electrons of state $m = 0$ will fill the states of $m = -1$, when these states are full filled, the Hall plateau appears again. When the extern magnetic field gets less and less, the electrons will fill the states of $m = -2, -3, \cdots$, other Hall plateaus appear one by one, this is the integer Hall effect.

2, When $n_\rho \neq 0$, the electrons of the electron gas system stay in excited sates, different $n_\rho$ and different $|m|$ corresponding to different fractional quantum Hall number (see the table I), and thus correspond to different Hall effects, which is just the fractional Hall effect.

The fractional quantum Hall effect corresponding to the filling of electron to exciting states. The principal two series of $n_\rho$, mentioned in the introduction part, i.e. $\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \cdots$ .,and $\nu = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \cdots$ are completely included in the table I. Besides, there are also some other fractional $\nu$ in tableI which ,may correspond to new fractional Hall effects, they should be experimentally checked.

## REMARKS AND CONCLUSIONS

From the calculations and analysis above, we give some remarks as follows.

1, From the table I, we can see that the $\nu$ of all known fractional quantum Hall effects are concluded in the table, for examples, $\nu = \frac{1}{3}, \nu = \frac{2}{3}, \nu = \frac{2}{5}, \nu = \frac{3}{5}, \cdots$ The $\nu = \frac{1}{3}$ appears firstly in the table and also most frequently. From this point of view, its first discovering by experiment is easy understood, and also it corresponds to the excited state with most low energy. 2, The $\nu$ which second frequently appears in the table I is $\nu = \frac{1}{2}$, the first $\nu = \frac{1}{2}$ in the table corresponds to the quantum number $n_\rho = 1, m = -1, \lambda = -1$, the energy level, for single electron, the wave function and the probability density of this state are:

$$\phi_{1,-1}(\xi, \varphi, z, s) = \frac{2-\xi^2}{a\sqrt{2\pi}} \xi e^{-\xi^2/2} \chi_\lambda(s) e^{-i\varphi} e^{-ip_z z/\hbar} e^{-iE_{1\lambda}t/\hbar}, \tag{12}$$

$$E_{1\lambda} = \frac{1}{2\mu} p_z^2 + \frac{(\lambda+3)e\hbar B}{2\mu c} \ , \tag{13}$$

$$f_{1,-1} = (2-\xi^2)\xi^3 e^{-\xi^2}. \tag{14}$$

The second $\nu = \frac{1}{2}$ corresponds to the quantum number $n_\rho = 2, m = -3, \lambda = -1$, the single particle energy, wave function and probability density of this state are:

$$\phi_{2,-3}(\xi, \varphi, z, s) = \frac{20-10\xi^2+\xi^4}{a\sqrt{240\pi}} \xi^3 e^{-\xi^2/2} \chi_\lambda(s) e^{-3i\varphi} e^{-ip_z z/\hbar} e^{-iE_{2\lambda}t/\hbar}, \tag{15}$$

$$E_{2\lambda} = \frac{1}{2\mu} p_z^2 + \frac{(\lambda+5)e\hbar B}{2\mu c} \ , \tag{16}$$

$$f_{2,-3} = \frac{20-10\xi^2+\xi^4}{120} \xi^7 e^{-\xi^2}. \tag{17}$$

Other $\nu = \frac{1}{2}$ situations can also be discussed similarly.

3, For $m = 0$, the Hall quantum number is reduced to $\nu = \frac{1}{2n_\rho+1}$, the electron gas system in these states are so called incompressible quantum liquid, in this case, electrons are in s-wave states, namely in circular Bohr orbit.

4, Besides the some quantum Hall numbers defined in (9) fully description the known fraction Hall effects, the fractional Hall effect for other quantum Hall numbers defined in (9) may also exist, we expect they can be experimentally checked very soon.

In conclusion, using the method of quantum mechanics, we firstly got the expectation value of the move range of single electron, and then the degeneracy density for electron gas system is obtained, at same time the new defined quantum Hall number $\nu$ was introduced, which is called filling factor in the literature. The most importance of our formulation is that we gave out a unified explaination for integer and fractional quantum Hall effects, and some possible fractional quantum Hall effects was also predicted, and what is more, we do not need any suppose conditions, such as the fractionally-charged quasi-particles, composite fermions and extension states etc. which are needed to be introduced to explain the fractional Hall effects in the literature.

This work was supported by the National Natural Science Foundation of China (10447005, 10875035), an open topic of the State Key Laboratory for Superlattices and

Microstructures (CHJG200902), the scientific research project in Shaanxi Province (2009K01-54) and Natural Science Foundation of Zhejiang Province (Y6110470),

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