On extensions of Lie algebras

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Abstract

In the note some construction of Lie algebras is introduced. It is proved that the construction has the same property as a well known wreath product of groups [1]: Any extension of groups can be embedded into their wreath product [2].

Let M and L be Lie algebras over an arbitrary field K, $U = U(L)$ - a universal enveloping algebra of Lie algebra L, $\{e_i, i \in I\}$ - a well-ordered basis in L. We can convert the linear space $Hom_K(U, M)$ into a Lie algebra if we define a Lie product by Leibniz formula

$$[f, h](E) = \sum_{I*J=E} [f(I), h(J)],$$

where E, I, J are standard monomials [3]: for example $E = e_j e_k \cdots e_m e_n$, $e_j \leq e_k \cdots \leq e_m \leq e_n$, $I * J$ is a product in a symmetrical algebra of L and f, h are elements of $Hom_K(U, L)$.

Define an action of L on $Hom_K(U, M)$ by a rule

$$(fu)(E) = f(uE),$$

where $f \in Hom_K(U, M)$, $u \in L$ and the product of u and E is taken in algebra U.

It can be immediately checked that $Hom_K(U, M)$ is indeed a Lie algebra with respect to the above defined product and L acts on $Hom_K(U, M)$ as a Lie algebra of derivations of the Lie algebra $Hom_K(U, L)$.

We denote a semidirect product of $Hom_K(U, M)$ and L by $M\ Wr\ L$ and call it a wreath product of Lie algebras M and L.

The notation and the name are justified by the following Therem that we prove here:

Any extension N of Lie algebra M by Lie algebra L can be embedded into their wreath product $M\ Wr\ L$

The theorem is similar to the well known theorem of Kaloujnine and Krasner [2].

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To prove it, we need the following way of constructing the extension N via a factor set $g(u,v)$.

Let M and L be Lie algebras and suppose that elements of L act on M as derivations of algebra M, that is

$$[x,y]u = [xu,y] + [x,yu].$$

Let $g : L \times L \to M$ be a bilinear mapping, such that

$$\begin{aligned}
&\text{(a) } g(u,v) = -g(v,u)\\
&\text{(b) } g(u,v)w + g([u,v],w) + g(v,w)u + g([v,w],u)+\\
&\qquad g(w,u)v + g([w,u],v) = 0\\
&\text{(c) } (xu)v - (xv)u = x[u,v] + [x, g(u,v)]
\end{aligned}$$

where $x, y \in M$, $u, v \in L$. Then the direct product $N = M \times L$ of linear spaces M and L can be converted into a Lie algebra by the formula

$$[(x,u),(y,v)] = ([x,y] + xv - yu + g(u,v), [u,v]).$$

It can be verified that N is the extension of M by L with a given factor set $g(u,v)$ and given an action of elements of L on M.

We will henceforth assume that N, as the extension of M by L, is given as just described.

We are now coming to the proof of the Theorem. We will construct an embedding $\varphi : N \to M\ Wr\ L$.

If $(x,u) \in N$, then $(x,u) = (x,0) + (0,u)$. Therefore it is enough to determine $\varphi((x,0))$ and $\varphi((0,u))$. In turn, if $\{z_q, q \in Q\}$ is a basis in M and

$$x = \sum_q \beta_q z_q,$$

then $\varphi((x,0))$ must equal $\sum_q \beta_q \varphi((z_q,0))$. Therefore it suffices to determine $\varphi((z_q,0))$. Equally, to determine $\varphi((0,u))$ we need to know $\varphi((0,e_i))$.

Next, set $\varphi((x,u)) = (f_{(x,u)}, u)$, where $f_{(x,u)} \in Hom_K(U,M)$. In the same sense we will use notations $f_{(x,0)}$, $f_{(0,u)}$, $f_{(z_q,0)}$, $f_{(0,e_i)}$. For example, $\varphi((x,0)) = (f_{(x,0)}, 0)$ and $\varphi((0,e_i)) = (f_{(0,e_i)}, e_i)$.

If

$$\varphi([(x,u),(y,v)] = [\varphi((x,u)), \varphi((y,v))],$$

then

$$\begin{aligned}
&(1)\ f_{(g(u,v),0)} + f_{(0,[u,v])} = [f_{(0,u)}, f_{(0,v)}] + f_{(0,u)}v - f_{(0,v)}u\\
&(2)\ f_{(xu,0)} = [f_{(x,0)}, f_{(0,u)}] + f_{(x,0)}u\\
&(3)\ f_{([x,y],0)} = [f_{(x,0)}, f_{(y,0)}]
\end{aligned}$$

and vice versa.

Now, determine $f_{(z_q,0)}$ and $f_{(0,e_i)}$ on standard monomials E by induction in such a way that assures (1), (2), (3).

Put $f_{(x,0)}(1) = x$ and $f_{(0,u)}(1) = 0$.

If $E = e_j$ then for $u = e_i$ and $v = e_j$, (1) gives us:

$$f_{(g(e_i,e_j),0)}(1) + f_{(0,[e_i,e_j])}(1) = [f_{(0,e_i)}, f_{(0,e_j)}](1) + (f_{(0,e_i)}e_j)(1) - (f_{(0,e_j)}e_i)(1)$$

or

$$g(e_i, e_j) = f_{(0,e_i)}(e_j) - f_{(0,e_j)}(e_i).$$

Define $f_{(0,e_i)}(e_j)$ in the form $\alpha g(e_i, e_j)$ where $\alpha \in K$ is to be determined. Then

$$f_{(0,e_j)}(e_i) = \alpha g(e_j, e_i) = -\alpha g(e_i, e_j) = -f_{(0,e_i)}(e_j)$$

and $g(e_i, e_j) = 2f_{(0,e_i)}(e_j)$. Hence

$$f_{(0,e_i)}(e_j) = \frac{1}{2} g(e_i, e_j).$$

To determine $f_{(z_q,0)}(e_j)$ we use (2) and put $u = e_j$ and $x = z_q$:

$$f_{(z_q e_j,0)}(1) = [f_{(z_q,0)}, f_{(0,e_j)}](1) + (f_{(z_q,0)}e_j)(1).$$

So $f_{(z_q,0)}(e_j) = z_q e_j$. It is immediate that

$$f_{(x,0)}(e_j) = xe_j.$$

If $E = e_j e_k$ then we use

$$f_{(g(e_i,e_j),0)}(e_k) + f_{(0,[e_i,e_j])}(e_k) = [f_{(0,e_i)}, f_{(0,e_j)}](e_k) + (f_{(0,e_i)}e_j)(e_k) - (f_{(0,e_j)}e_i)(e_k).$$

By the previous $f_{(g(e_i,e_j),0)}(e_k) = g(e_i, e_j)e_k$. Next if

$$[e_i, e_j] = \sum_r \alpha_r e_r$$

then

$$f_{(0,[e_i,e_j])}(e_k) = \sum_r \alpha_r f_{(0,e_r)}(e_k)$$

and the values $f_{(0,e_r)}(e_k)$ are already known. We have also

$$[f_{(0,e_i)}, f_{(0,e_j)}](e_k) = [f_{(0,e_i)}(e_k), f_{(0,e_j)}(1)] + [f_{(0,e_i)}(1), f_{(0,e_j)}(e_k)] = 0,$$

$$(f_{(0,e_i)}e_j)(e_k) = f_{(0,e_i)}(e_j e_k),$$

$$(f_{(0,e_j)}e_i)(e_k) = f_{(0,e_j)}(e_ie_k).$$

If $e_i \leq e_k$ then we set $f_{(0,e_j)}(e_ie_k) = -f_{(0,e_i)}(e_je_k)$. Then

$$f_{(0,e_i)}(e_je_k) = \frac{1}{2}(f_{(g(e_i,e_j),0)}(e_k) + f_{(0,[e_i,e_j])}(e_k)).$$

In the case of $e_i > e_k$ we have

$$f_{(0,e_j)}(e_ie_k) = f_{(0,e_j)}(e_ke_i) + f_{(0,e_j)}([e_i,e_k])$$

and $e_j \leq e_k < e_i$. We set as before $f_{(0,e_k)}(e_je_i) = -f_{(0,e_j)}(e_ke_i)$ in

$$f_{(g(e_j,e_k),0)}(e_i) + f_{(0,[e_j,e_k])}(e_i) =$$

$$[f_{(0,e_j)}, f_{(0,e_k)}](e_i) + (f_{(0,e_j)}e_k)(e_i) - (f_{(0,e_k)}e_j)(e_i).$$

Then

$$(f_{(0,e_j)}(e_ke_i) = \frac{1}{2}(f_{(g(e_j,e_k),0)}(e_i) + f_{(0,[e_j,e_k])}(e_i)).$$

This imlies

$$f_{(0,e_i)}(e_je_k) = f_{(g(e_i,e_j),0)}(e_k) +$$

$$f_{(0,[e_i,e_j])}(e_k) + \frac{1}{2}(f_{(g(e_j,e_k),0)}(e_i) + f_{(0,[e_j,e_k])}(e_i)) + f_{(0,e_j)}([e_i,e_k]).$$

Next we determine $f_{(z_q,0)}(e_je_k)$. We set $u = e_j$ and $x = z_q$ in (2). We have

$$f_{(z_qe_j,0)}(e_k) = [f_{(z_q,0)}, f_{(0,e_j)}](e_k) + (f_{(z_q,0)}e_j)(e_k)$$

or

$$f_{(z_qe_j,0)}(e_k) = [f_{(z_q,0)}(1), f_{(0,e_j)}(e_k)] + f_{(z_q,0)}(e_je_k)$$

or

$$f_{(z_q,0)}(e_je_k) = z_qe_je_k - \frac{1}{2}[z_q, g(e_j,e_k)].$$

It follows immediately, that

$$f_{(x,0)}(e_je_k) = xe_je_k - \frac{1}{2}[x, g(e_j,e_k)].$$

Suppose now that $f_{(x,0)}$ and $f_{(0,e_i)}$ are already defined for any standard monomial of degree less than n and let $E = e_je_kF$ be a standard monomial of degree n.

It has to be by (1)

$$f_{(g(e_i,e_j),0)}(e_kF) + f_{(0,[e_i,e_j])}(e_kF) = [f_{(0,e_i)}, f_{(0,e_j)}](e_kF) +$$

$$(f_{(0,e_i)}e_j)(e_kF) - (f_{(0,e_j)}e_i)(e_kF)$$

or

$$(f_{(0,e_i)}(e_je_kF) - (f_{(0,e_j)}(e_ie_kF) = f_{(g(e_i,e_j),0)}(e_kF) + f_{(0,[e_i,e_j])}(e_kF) -$$

$$[f_{(0,e_i)}, f_{(0,e_j)}](e_kF).$$

If $e_i \leq e_k$ we put $f_{(0,e_j)}(e_ie_kF) = -f_{(0,e_i)}(e_je_kF)$ and obtain

$$f_{(0,e_i)}(E) = \frac{1}{2}(f_{(g(e_i,e_j),0)}(e_kF) + f_{(0,[e_i,e_j])}(e_kF) - [f_{(0,e_i)}, f_{(0,e_j)}](e_kF)).$$

In the case of $e_i > e_k$ we have

$$e_ie_kF = e_kG + \sum_s \alpha_s H_s,\ \alpha_s \in K.$$

Here G is a standard monomial which is equal to the product of e_i and all factors of F and H_s are standard monomials of degree less than n.The first factor of G can be e_i or the first factor e_m of F. We have $e_j \leq e_k < e_i$ for the first case and $e_j \leq e_k \leq e_m$ for the second one.

So we set as before $f_{(0,e_k)}(e_jG) = -f_{(0,e_j)}(e_kG)$ in

$$f_{(g(e_j,e_k),0)}(G)+f_{(0,[e_j,e_k])}(G) = [f_{(0,e_j)}, f_{(0,e_k)}](G)+f_{(0,e_j)}(e_kG)-f_{(0,e_k)}(e_jG).$$

Then

$$f_{(0,e_j)}(e_kG) = \frac{1}{2}(f_{(g(e_j,e_k),0)}(G) + f_{(0,[e_j,e_k])}(G) - [f_{(0,e_j)}, f_{(0,e_k)}](G))$$

Ultimately we have

$$f_{(0,e_i)}(E) = f_{(g(e_i,e_j),0)}(e_kF) + f_{(0,[e_i,e_j])}(e_kF) - [f_{(0,e_i)}, f_{(0,e_j)}](e_kF) +$$

$$\frac{1}{2}(f_{(g(e_j,e_k),0)}(G) + f_{(0,[e_j,e_k])}(G) - [f_{(0,e_j)}, f_{(0,e_k)}](G)) + \sum_s \alpha_s f_{(0,e_j)}(H_s),$$

and values of functions in the right hand side are known.

To determine $f_{(z_q,0)}(E)$ we use (2) and put $u = e_j$ and $x = z_q$. We have

$$f_{(z_qe_j,0)}(e_kF) = [f_{(z_q,0)}, f_{(0,e_j)}](e_kF) + (f_{(z_q,0)}e_j)(e_kF).$$

This implies

$$f_{(z_q,0)}(E) = f_{(z_qe_j,0)}(e_kF) - [f_{(z_q,0)}, f_{(0,e_j)}](e_kF)$$

and

$$f_{(x,0)}(E) = f_{(xe_j,0)}(e_kF) - [f_{(x,0)}, f_{(0,e_j)}](e_kF).$$

To guarantee that φ preserves Lie multiplication, it remains to prove (3) for $f_{(x,u)}$ defined above.

We apply induction on degree n of the standard monomial E. If $n = 0$ (3) is evident. Suppose we have proved (3) for any standard monomial E of degree less than n and let eE be a standard monomial of degree n.

We have

$$f_{([x,y],0)}(eE) = f_{([x,y]e,0)}(E) - \sum_{I*J=E} [f_{([x,y],0)}(I), f_{(0,e)}(J)] =$$

$$f_{([xe,y],0)}(E) + f_{([x,ye],0)}(E) - \sum_{I*J=E} [\sum_{F*H=I} [f_{(x,0)}(F), f_{(y,0)}(H)], f_{(0,e)}(J)] =$$

$$f_{([xe,y],0)}(E) - \sum_{I*J=E} \sum_{F*H=I} [[f_{(x,0)}(F), f_{(0,e)}(J)], f_{(y,0)}(H)]+$$

$$f_{([x,ye],0)}(E) - \sum_{I*J=E} \sum_{F*H=I} [f_{(x,0)}(F), [f_{(y,0)}(H), f_{(0,e)}(J)]] =$$

$$\sum_{S*H=E} [f_{(xe,0)}(S) - \sum_{F*J=S} [f_{(x,0)}(F), f_{(0,e)}(J)], f_{(y,0)}(H)]+$$

$$\sum_{F*R=E} [f_{(x,0)}(F), f_{(ye,0)}(R) - \sum_{H*J=R} [f_{(y,0)}(H), f_{(0,e)}(J)]] =$$

$$\sum_{S*H=E} [(f_{(xe,0)} - [f_{(x,0)}, f_{(0,e)}])(S), f_{(y,0)}(H)]+$$

$$\sum_{F*R=E} [f_{(x,0)}(F), (f_{(ye,0)} - [f_{(y,0)}, f_{(0,e)}])(R)] =$$

$$\sum_{S*H=E} [f_{(x,0)}(eS), f_{(y,0)}(H)] + \sum_{F*R=E} [f_{(x,0)}(F), f_{(y,0)}(eR)] =$$

$$\sum_{P*Q=eE} [f_{(x,0)}(P), f_{(y,0)}(Q)] = [f_{(x,0)}, f_{(y,0)}](eE).$$

The mapping φ is one-to-one. Indeed, $\varphi((x,u)) = \varphi((y,v))$ implies $u = v$ and therefore $f_{(x,u)} = f_{(y,u)}$. But $f_{(x,u)} = f_{(x,0)} + f_{(0,u)}$ and $f_{(y,u)} = f_{(y,0)} + f_{(0,u)}$. Therefore $f_{(x,0)} = f_{(y,0)}$. In particular, $f_{(x,0)}(1) = f_{(y,0)}(1)$ and $x = y$.

Thus we have built the mapping which embeds an extension N of Lie algebra M by Lie algebra L into the wreath product $MWrL$.

References

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