COMBINATORIAL KNOT FLOER HOMOLOGY AND CYCLIC DOUBLE BRANCHED COVERS

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Abstract. Using a Heegaard diagram for the pullback of a knot $K \subset S^3$ in its cyclic double branched cover $\Sigma_2(K)$, we give a combinatorial proof for the invariance of knot Floer homology over $\mathbb{Z}$.

1. Introduction

Heegaard Floer homology, introduced by Ozsváth and Szabó [5] and independently by Rasmussen [8] is a collection of invariants for closed oriented three-manifolds. $HF(Y)$ is defined by counting some holomorphic discs in the symmetric product of a Riemann surface. There is a relative version of the theory for (Y, K), where K is a nullhomologous knot in Y. In [9], Sarkar proposed an algorithm to compute Heeagaard Floer homology combinatorially.

Given a knot $K \in S^3$, a *grid diagram* associated to K is an $n \times n$ planar grid, with $\mathbb{X} = \{X_i\}_{i=1}^{i=n}$ and $\mathbb{O} = \{O_i\}_{i=1}^{i=n}$ basepoints. Each column and each row contains exactly one X and one O inside. We view this grid diagram as a torus $T^2 \in S^3$ by standard edge identifications. Here each horizontal line is an α circle and each vertical line is a β circle. Manolescu, Ozsváth and Sarkar [1] showed that such diagrams can be used to compute $\widehat{HFK}(S^3, K)$ combinatorially. In [3], Levine gave a construction of a Heegaard diagram for $(\Sigma_m(K), \widetilde{K})$ to compute $\widehat{HFK}(\Sigma_m(K), \widetilde{K})$ combinatorially. In this paper we use that construction to establish knot Floer homology of the pullback of a knot $K \in S^3$ in its m-fold cyclic branched cover $\Sigma_m(K)$ over $\mathbb{Z}_2$ in combinatorial terms. Later we use a recent work of Ozsváth, Stipsicz and Szabó [4] where they assigned signs to the rectangles and bigons for an arbitrary Heegaard diagram and consider the homology with $\mathbb{Z}$ coefficients.

Let $\widetilde{T}$ be the surface obtained by gluing together m copies of the torus T^2 (where T^2 is the torus which describes the grid diagram $\mathcal{D}$ of K) along branch cuts connecting X and O in each column. Denoted the copies by $T_0, \dots, T_{m-1}$. Whenever X is above O, glue the left side of the branch cut in T_k to the right side of the same cut in T_{k+1}; if the O is above X, then glue the left side of the branch cut in T_k to the right side of the same cut in T_{k-1} (indices modulo m). The projection map $\pi : \widetilde{T} \to T$ is an m-fold cyclic branched cover, branched around the basepoints. Each α and β circle in T^2 bounds a disk in $S^3 - K$ so each of them has m distinct lifts to $\Sigma_m(K)$. Each lift of each α circle intersects exactly one lift of each β circle. This can be shown as in Figure 1 with m disjoint grids. Denote by $\widetilde{\beta}^i_j$ for $i = 0, 1$ and $j = 0, \ldots, n-1$ the vertical arcs, call the arcs which has intersection with $\widetilde{\beta}^i_0$, $\widetilde{\alpha}^i_j$. We call each horizontal line in each grid an α *line*, which consists of distinct segments of different lifts of an α circle. Let us denote this Heegaard diagram by $\widetilde{\mathcal{D}}$.

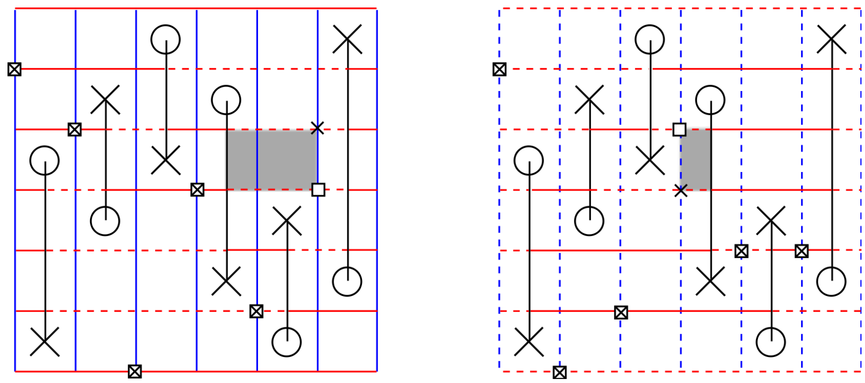

FIGURE 1. A Heegaard diagram $\widetilde{D} = (\widetilde{T}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \mathbb{O}, \mathbb{X})$ for $\widetilde{K} \in \Sigma_m(K)$, where K is the figure eight knot. Here the horizontal (resp. vertical) lines represent $\widetilde{\alpha}$ (resp.$\widetilde{\beta}$) circles. Here a rectangle in $\mathcal{R}$ is illustrated which contributes in the differential of $C(\widetilde{\mathcal{D}})$ and connects a generator $\mathbf{x}$ which is shown with crosses to a generator $\mathbf{y}$ which is shown with hollow rectangles.

Denote by $\mathcal{R}_{\mathcal{D}}$ the set of embedded rectangles in T which do not contain any basepoints and whose left and right edges are arcs of β circles and whose upper and lower edges are arcs of α circles. Each rectangle in $\mathcal{R}_{\mathcal{D}}$ has m disjoint lifts to $\widetilde{T}$ (possibly passing through the branch cuts); denote the set of such lifts by $\mathcal{R}$ for convenience.

Let the set of generators $\mathcal{S}$ be the set of unordered $2n$-tuples $\mathbf{x}$ of intersection points between $\widetilde{\alpha}^i_j$ and $\widetilde{\beta}^i_j$ for $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$, such that each of $\widetilde{\alpha}^i_j$ and $\widetilde{\beta}^i_j$ has exactly one component of $\mathbf{x}$. also denote by $\mathcal{S}_{\mathcal{D}}$ the set of generators for $\mathcal{D}$. In [3], Levine showed that any generator $\mathbf{x} \in \mathcal{S}$ can be decomposed (non-uniquely) as $\mathbf{x} = \widetilde{\mathbf{x}}_1 \cup \cdots \cup \widetilde{\mathbf{x}}_m$, where $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are generators in $\mathcal{S}_{\mathcal{D}}$, and $\widetilde{\mathbf{x}}_i$ is a lift of $\mathbf{x}_i$ to $\widetilde{\mathcal{D}}$.

The Alexander grading of a generator $\mathbf{x} \in \mathcal{S}$ is defined as follows. Given two finite set of points A, B in the plane, let $\mathcal{I}(A, B)$ be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ with $a_1 < b_1$ and $a_2 < b_2$. Let $\mathcal{J}(A, B) = \dfrac{1}{2}(\mathcal{I}(A, B) + \mathcal{I}(B, A))$. Given $\mathbf{x}_i \in \mathcal{S}_{\mathcal{D}}$, define

$$A(\mathbf{x}_i) = \mathcal{J}(\mathbf{x}_i - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X} - \mathbb{O}) - (\frac{n-1}{2}).$$

For any generator $\mathbf{x} \in \mathcal{S}$, consider one of the decompositions $\mathbf{x} = \widetilde{\mathbf{x}}_1 \cup \cdots \cup \widetilde{\mathbf{x}}_m$ and define

$$A(\mathbf{x}) = \frac{1}{m} \textstyle\sum_i A(\mathbf{x}_i).$$

A simple calculation shows that the above definition is well-defined.

Let C be the $\mathbb{Z}_2$ vector space generated by $\mathcal{S}$. Define a differential ∂ on C by considering the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ nonzero if and only if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ agree along all but two vertical circles and there is a rectangle $R \in \mathcal{R}$ whose lower-left and upper-right corners are in $\mathbf{x}$ and whose lower-right and upper-left corners are in $\mathbf{y}$, and which does not contain any X, O, or components of $\mathbf{x}$ in its interior. We denote by $Rect(\mathbf{x}, \mathbf{y})$ the set of such rectangles.

$$\partial\mathbf{x} = \sum_{\mathbf{y} \in \mathcal{S}} \sum_{R \in Rect(\mathbf{x},\mathbf{y})} \mathbf{y}$$

Later we consider C as a vector space over $\mathbb{Z}$.

Theorem 1. *Let K be an oriented knot and $\widetilde{K}$ be its pullback in the cyclic double branched cover $\Sigma_2(K)$, the filtered quasi-isomorphism type of the complex (C, ∂) over $\mathbb{Z}$ is an invariant of the knot type and the grid number.*

In Section 2, we have an overview of the required definitions and basic properties of multi-point Heegaard diagrams and the construction of cyclic branched cover from grid diagrams. In Section 3, we give a combinatorial prove of the statement of Theorem 1 over $\mathbb{Z}_2$. In Section 4, using sign assignments for Heegaard diagrams from [4], we stablish the knot Floer homology of the pullback of a knot $K \subset S^3$ in its cyclic double branched cover $\Sigma_2(K)$ over $\mathbb{Z}$.

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2. Preliminaries

Let us begin by reminding the reader the construction of knot Floer homology from multi-pointed Heegaard diagrams. See [5, 1] for details.

Let S be a surface of genus g, $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_{g+k-1}\}$ (resp. $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_{g+k-1}\}$) be a collection of pairwise disjoint, embedded closed curves in S which span a half-dimensional subspace of $H_1(S, \mathbb{Z})$ (so they give a handlebody H_α (resp. H_β) with boundary equal to S (resp. $-S$)), and let $\mathbf{w} = \{w_1, ..., w_k\}$ and $\mathbf{z} = \{z_1, ..., z_k\}$ be set of distinct points, such that each component of $S \backslash \boldsymbol{\alpha}$ and each component of $S \backslash \boldsymbol{\beta}$ contains exactly one point of $\mathbf{w}$ and one point of $\mathbf{z}$. We call $(S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ a Heegaard diagram. Note that this set of information specifies a Heegaard decomposition $H_\alpha \bigcup_S H_\beta$ for an oriented closed 3-manifold Y and a knot $K \in Y$ which is obtained by connecting the w (resp. z) basepoints to z (resp. w) basepoints with arcs in the complement of α (resp. β) curves and pushing the arcs into H_α (resp. H_β). The orientation of the knot K is such that it intersects S positively at the z basepoints. Another way to obtain the Heegaard diagram is to consider a compatible self-indexing Morse function f on Y with n critical points of index 0 and 3, and $g + n - 1$ of index 1 and 2. In this terms, the induced Heegaard surface is $S = f^{-1}(\frac{3}{2})$, taking gradient vector field $\bigtriangledown f$, α_i (resp. β_i) corresponds to the set of points that flow down (resp. up) to a critical point of index 1 (resp. 2), and we have $H_\alpha = f^{-1}([0, \frac{3}{2}])$ and $H_\beta = f^{-1}([\frac{3}{2}, 3])$.

In order to construct m-fold cyclic branched cover of a knot $K \in S^3$, let Y be the link exterior cut along a Seifert surface F. Take m copies of Y and denote them by $Y_1, ..., Y_m$. The boundary of Y_i has two components $F_{i,+}$ and $F_{i,-}$ which are copies of F. The m-fold cyclic cover $\widehat{\Sigma}_m(K)$ of the knot exterior is formed by the disjoint union $Y_1 \sqcup ... \sqcup Y_m$ where $F_{i,-}$ is identified with $F_{i+1,+}$ for each i (indices are considered mod m). Then $\Sigma_m(K)$ is created by glueing back a solid torus to the boundary of $\widehat{\Sigma}_m(K)$. Denote by $\pi : \Sigma_m(K) \to S^3$ the projection map whose downstairs branch locus is K.

Now we turn to Levine [3] construction of a Heegaard diagram for $(\Sigma_m(K), \widetilde{K})$ from a grid diagram G for $K \in S^3$. First we may isotope K to lie entirely within H_α, take a Seifert surface contained in a ball in H_α then isotope K and F so that K returns to its original place. In this way F intersects the Heegaard surface exactly in n arcs which connect the two basepoints in each column and $F \cap H_\beta$ consists of n strips. We cut the grid diagram along these n arcs and consider m copies of it, then glue these copies as we explained in Section 1. If $f : S^3 \to \mathbb{R}$ is a self-indexing Morse function compatible with the grid diagram G for K, then $\widetilde{f} = f \circ \pi : \Sigma_m(K) \to \mathbb{R}$ is a self-indexing Morse function for $(\Sigma_m(K), \widetilde{K})$ and describes a Heegaard decomposition $\Sigma_m(K) = \widetilde{H}_\alpha \cup_{\widetilde{T}} \widetilde{H}_\beta$ where $\widetilde{T}$ is the surface obtained from T as described. So we obtain a Heegaard diagram $\widetilde{D} = (\widetilde{T}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$, where $\widetilde{\boldsymbol{\alpha}}$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\mathbf{w}}$, $\widetilde{\mathbf{z}}$ are the lifts of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{w}$, $\mathbf{z}$ to $\widetilde{T}$.

3. Invariance of Knot Floer Homology

In this section we want to show that the combinatorial knot Floer homology of a cyclic double branched cover is independent of the grid diagram. As a result of the work of Cromwell [10], any two grid diagram of the same knot can be connected by a sequence of following elementary moves:

(1) **Cyclic Permutation** This move corresponds to cyclically permuting the rows or the columns of the grid diagram for the knot K and consequently obtaining a Heegaard diagram for $\widetilde{K}$.
(2) **Commutation** Consider two consecutive columns in a grid diagram G of knot K, the X and O decorations of one of the columns separate the vertical circle into two arcs. If both of the X and O decorations of the other column is in one of the arcs, switching these two columns is a commutation move for G and lead to another grid diagram H for K. For the knot $\widetilde{K}$, we consider the Heegaard diagram $\widetilde{H}$ to be obtained from $\widetilde{G}$ by a commutation move. Commutation can be alternatively done by reversing the roles of rows and columns.
(3) **Stabilization** Consider a row in a grid diagram G of knot K, in order to get stabilized grid diagram H, we split this row in two and introduce a new column. For convenience, label the original diagram so it has decorations $\{O_i\}_{i=2}^{i=n+1}$, $\{X_i\}_{i=2}^{i=n+1}$. We Copy O_i onto one of the new rows and the X_i in the other row. Then we add O_1 and X_1 in the two squares which are the intersections of these two rows with the new column. We denote by β_1 (resp. β_2) the left (resp. right) vertical circle. We let α denote the new horizontal circle in H which separates O_1 from X_1. In this way we actually added two consecutive breaks in the knot. We call this a stabilization move of the knot K. For the knot $\widetilde{K}$, we consider the Heegaard diagram $\widetilde{H}$ to be obtained from $\widetilde{G}$ by a stabilization. Note that we can consider only certain stabilization moves after combining the previous stabilizations with commutation moves. In these reduced stabilization moves the new column is introduced next to O_i or X_i. There are different type s of stabilization corresponding the different ways of placing the new X and O in the new rows Also there is a similar move where the roles of rows and columns will be interchanged.

3.1. **Cyclic Permutation.** Let $\widetilde{G}$ be a Heegaard diagram for a knot $\widetilde{K}$, and let $\widetilde{H}$ be a different Heegaard diagram obtained by permuting the rows of the grid diagram corresponding to K, such that the $(i+1)^{th}$ row becomes the i^{th} row, mod n. In [3], Levine showed that for $i = 0, \ldots, n-1$ and $j = 0, \ldots, n-1$ and $k = 0, 1$, the generators of C can be described as $\mathbf{x} = \{x_{ij}^k = \widetilde{\beta}_i^k \cap \widetilde{\alpha}_j^{k-w(i,j)}\}$, where $w(i,j)$ is the winding number and $k - w(i,j)$ is considered mod 2.

We define a chain map $\Phi : C(\widetilde{G}) \to C(\widetilde{H})$ by defining it on the components of each generator, $\Phi(x_{ij}^k) = x_{i,j-1}^l = \widetilde{\beta}_i^l \cap \widetilde{\alpha}_{j-1}^{k-w(i,j)}$, where $l = k - w(i,j) + w'(i,j-1)$.

Lemma 1. *The map $\Phi : C(\widetilde{G}) \to C(\widetilde{H})$ is a filtered chain map.*

Proof. A simple calculation shows that $w(i,j+b)-w(i,j) = w'(i,j+b-1)-w'(i,j-1)$, where w is the winding number in diagram G and w' is the winding number for the new diagram H, and the coordinates of each point are understood mod n. Using this equality, the fact that the map Φ is a chain complex is straightforward. To prove that Φ preserves the Alexander filtration, we use Lemma 3.1 and Proposition 3.5 in [3], which proves that any generator $\mathbf{x}$ of $C(\widetilde{\mathcal{D}})$ can be decomposed (non-uniquely) as $\mathbf{x} = \tilde{\mathbf{x}}_1 \cup \cdots \cup \tilde{\mathbf{x}}_m$, where $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are generators of $C(\mathcal{D})$, and $\tilde{\mathbf{x}}_i$ is a lift of $\mathbf{x}_i$ to $\tilde{\mathcal{D}}$, also note that the Alexander grading of $\mathbf{x}$ is equal to the average of the Alexander gradings of $\mathbf{x}_1, \ldots, \mathbf{x}_m$. But we know that the Alexander grading of a generator of $C(\mathcal{D})$ is preserved under the cyclic permutation of the grid diagram corresponding to K. □

A similar reasoning proves the result for cyclic permutations that change columns.

3.2. **Commutation.** Let $\widetilde{G}$ be a Heegaard diagram for $\widetilde{K}$, and let $\widetilde{H}$ be the Heegaard diagram obtained by commuting two columns.

We define a Heegaard diagram E associated to the specific commutation, consisting of two $n \times n$ grids where the opposite sides of each grid are identified. The X and O decorations in the right grid are the same as for the knot K itself, in the left grid we just interchange decorations of the two columns where we want to make the commutation move. Let $\widetilde{\beta}^i_j$ denote the vertical arcs in the Heegaard diagram for $i = 0, 1$ and $j = 0, \ldots, n-1$. For the horizontal arcs, we denote by $\widetilde{\alpha}^i_j$ the arc which has intersection with $\widetilde{\beta}^i_0$.

Let the set of generators $\mathcal{S}_E$ be the set of unordered $2n$-tuples $\mathbf{x}$ of intersection points between $\widetilde{\alpha}^i_j$ and $\widetilde{\beta}^i_j$ for $i = 0, 1$ and $j = 0, \ldots, n-1$, such that each of $\widetilde{\alpha}^i_j$ and $\widetilde{\beta}^i_j$ has exactly one component of $\mathbf{x}$.

Define the set of allowed regions $\mathcal{R}_E$ to be the set of topological rectangles whose upper and lower edges are parts of $\widetilde{\alpha}^i_j$ and whose left and right edges are parts of $\widetilde{\beta}^i_j$.

Let $C(E)$ be the $\mathbb{Z}_2$ vector space generated by $\mathbf{S}(E)$. Define a differential ∂_E on E by considering the coefficient of $\mathbf{y}$ in $\partial_E \mathbf{x}$ nonzero if and only if $\mathbf{x}, \mathbf{y} \in \mathcal{S}_E$ agree along all but two vertical circles and there is a rectangle $R \in \mathcal{R}_E$ whose lower-left and upper-right corners are in $\mathbf{x}$ and whose lower-right and upper-left corners are in $\mathbf{y}$, and which does not contain any X, O, or components of $\mathbf{x}$ in its interior. See Figure 2.

It is convenient to draw the new Heegaard diagram E in the same diagram as of the Heegaard diagram of $\widetilde{G}$, replacing a distinguished vertical circle $\widetilde{\beta}$ in $\widetilde{G}$ with a different one γ in E. The circles $\widetilde{\beta}$ and γ intersect each other in two points a and b, which are not on any horizontal circle. See Figure 3.

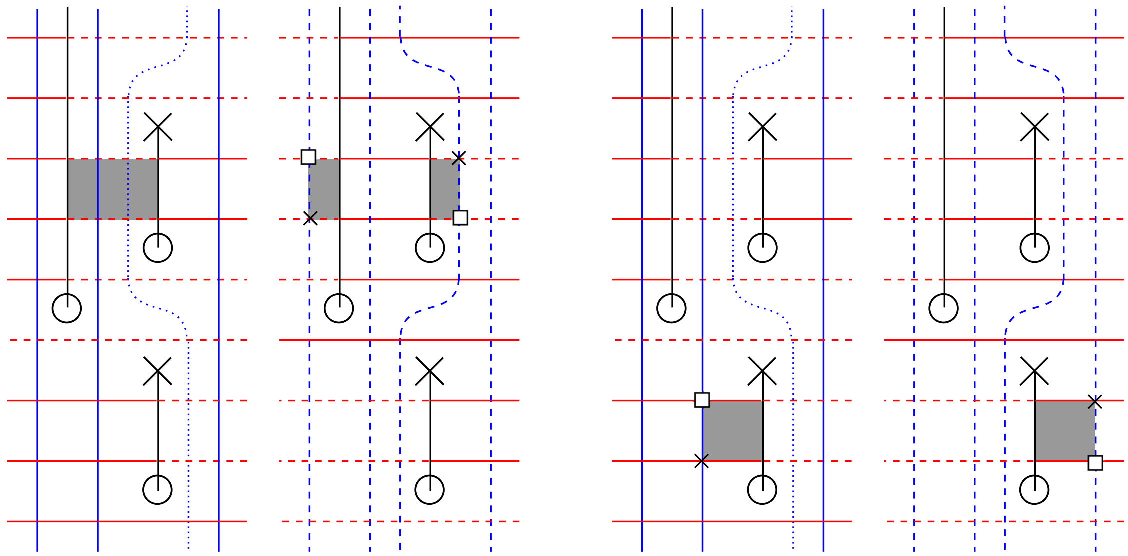

FIGURE 2. Here we illustrated two topological rectangles in $\mathcal{R}_E$. In each Heegaard diagram, the region changes the generator which is shown by crosses to the one which is shown by hollow squares.

We define a chain map $\Phi_{\widetilde{\beta}\gamma} : C(\widetilde{G}) \longrightarrow C(E)$ by counting pentagons. Let $\mathbf{x} \in \mathbf{S}(\widetilde{G})$ and $\mathbf{y} \in \mathbf{S}(E)$, define $Pent_{\widetilde{\beta}\gamma}(\mathbf{x}, \mathbf{y})$ to be the space of pentagons where each pentagon is an embedded disk in $\widetilde{T}$ whose boundary consists of five arcs, each is contained in a horizontal or a vertical circle. We start at the $\widetilde{\beta}$ component of $\mathbf{x}$, traverse the arc of a horizontal circle, meet its corresponding $\mathbf{y}$ component, continue through an arc of a vertical circle, meet a component of $\mathbf{x}$, proceed to another horizontal circle, meet the component of $\mathbf{y}$ which is on the distinguished circle γ, continue to an arc in γ, meet an intersection point of $\widetilde{\beta}$ with γ, and finally, traverse an arc in $\widetilde{\beta}$ until we come back at the initial component of $\mathbf{x}$. All the angles here are required to be acute and all the pentagons are empty from O or X decorations and $\mathbf{x}$ components. By using such pentagons, we change $\mathbf{x}$ to $\mathbf{y}$ which differ from each other in exactly two components. See Figure 3.

Given $\mathbf{x} \in \mathbf{S}(\widetilde{G})$, define

$$\Phi_{\widetilde{\beta}\gamma}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(E)} \sum_{p\in Pent_{\widetilde{\beta}\gamma}(\mathbf{x},\mathbf{y})} \mathbf{y} \in C(E) \tag{3.1}$$

Lemma 2. *The map $\Phi_{\widetilde{\beta}\gamma} : C(\widetilde{G}) \longrightarrow C(E)$ preserves the Alexander filtration, and is a chain map.*

Proof. The fact that the Alexander filtration will not change is straightforward. For proving that $\Phi_{\widetilde{\beta}\gamma}$ is a chain map we should consider regions which consist either of a

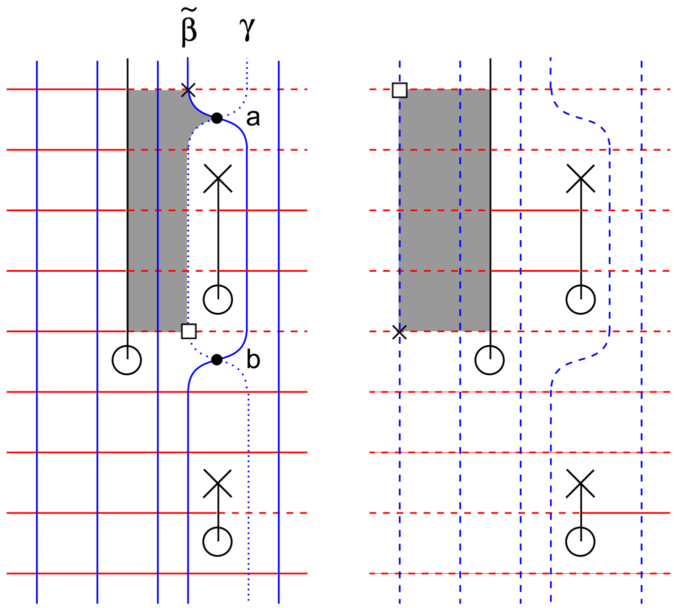

FIGURE 3. Here we showed two Heegaard diagrams G and E and a pentagon in $Pent_{\widetilde{\beta}\gamma}(\mathbf{x}, \mathbf{y})$, where components of $\mathbf{x}$ are indicated by crosses and those of $\mathbf{y}$ are indicated by hollow squares.

disjoint rectangle and pentagon, a rectangle and pentagon with overlapping interiors, or a rectangle and a pentagon which meet along an edge. For the first two cases just changing the order in which we use the rectangle or pentagon, will cancel each other out. For juxtaposing rectangles and pentagons, generally the composite region has two type of decomposition to a rectangle and a pentagon. However, there is a special case when the width of the pentagon and the rectangle which have an edge in common, is one. In this case we will consider another pentagon which is disjoint from the previous domain and an appropriate rectangle as shown in Figure 4.

□

In order to define chain homotopy operators, we count hexagons. Given $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\widetilde{G})$, we let $Hex_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x}, \mathbf{y})$ denote the set of embedded hexagons in $\widetilde{T}$. The boundary of a hexagon consists of six arcs each is contained in a horizontal or a vertical circle. More specifically, under the orientation induced on the boundary of h, we start at the $\widetilde{\beta}$-component of $\mathbf{x}$, traverse the arc of a horizontal circle, meet its corresponding component of $\mathbf{y}$, continue through an arc of a vertical circle, meet its corresponding component of $\mathbf{x}$, proceed to another horizontal circle, meet its component of $\mathbf{y}$, which is contained in the distinguished circle $\widetilde{\beta}$, continue along $\widetilde{\beta}$ until the intersection point b of $\widetilde{\beta}$ and γ, proceed on γ to the intersection point a of $\widetilde{\beta}$ with γ, and finally, continue on $\widetilde{\beta}$ to the $\widetilde{\beta}$-component of $\mathbf{x}$, which was also our initial point. All the angles of our

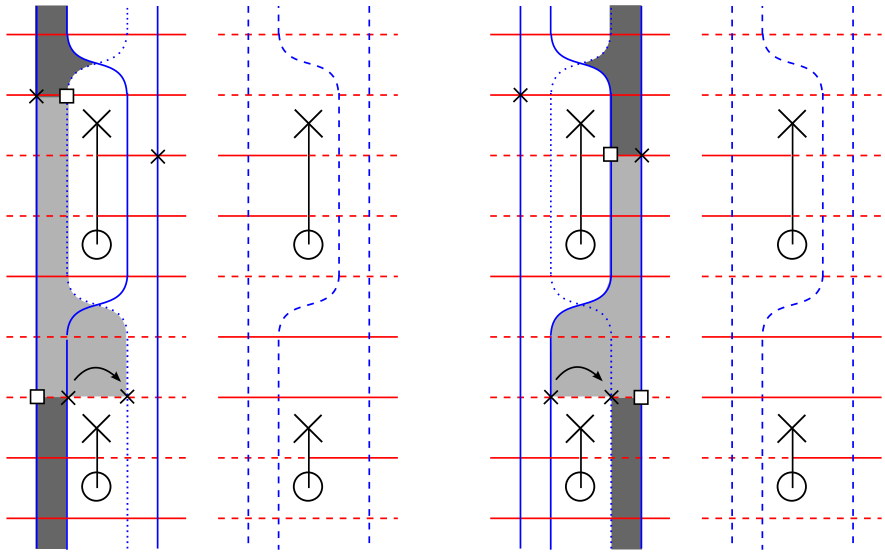

FIGURE 4. Here we showed a special case where two different composite regions have the same contribution so two terms cancel each other out. In the left diagram we first used the pentagon and then the rectangle. In the right diagram we first used a rectangle and then a pentagon.

hexagon required to be acute and all the hexagons are empty from O or X decorations and $\mathbf{x}$ components. See Figure 5. We now define the function $H_{\widetilde{\beta}\gamma\widetilde{\beta}} : C(\widetilde{G}) \longrightarrow C(\widetilde{G})$ by

$$H_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\widetilde{G})} \sum_{h\in Hex_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x},\mathbf{y})} \mathbf{y}. \tag{3.2}$$

Proposition 1. *The map $\Phi_{\widetilde{\beta}\gamma} : C(\widetilde{G}) \longrightarrow C(E)$ is a chain homotopy equivalence. In other words*

$$\mathbb{I} + \Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma} + \partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}} + H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial = 0$$
$$\mathbb{I} + \Phi_{\widetilde{\beta}\gamma} \circ \Phi_{\gamma\widetilde{\beta}} + \partial \circ H_{\gamma\widetilde{\beta}\gamma} + H_{\gamma\widetilde{\beta}\gamma} \circ \partial = 0.$$

Proof. Generally considering disjoint hexagon and rectangle or hexagon and rectangle with overlapping interiors, terms in $\partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}}$ will cancel out with the terms in $H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$. The composite regions which are made up of juxtaposition of a hexagon and a rectangle will cancel out with terms that represent juxtaposing of two pentagons or juxtaposing of a hexagon and a rectangle in another way. However, there is one type of composite regions which are counted once in $\Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma} + \partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}} + H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$ depending on the place of the generator components on each such composite region, this term is also counted once in the identity map. See Figure 6.

□

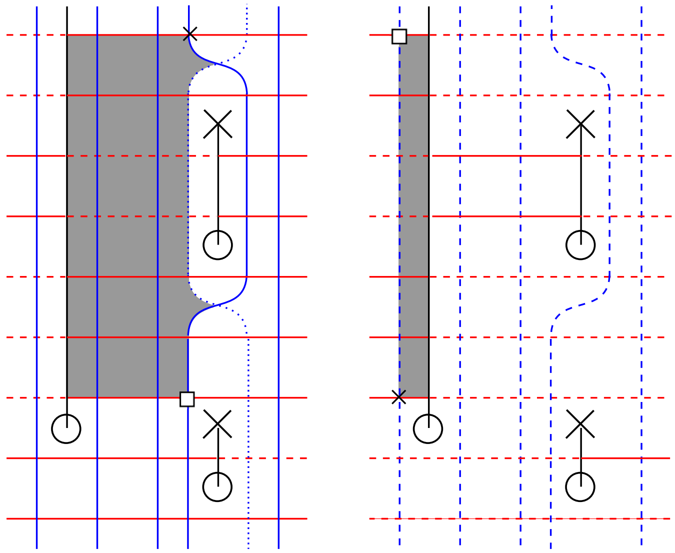

FIGURE 5. Here we showed a hexagon in $Hex_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x},\mathbf{y})$, where components of $\mathbf{x}$ are indicated by crosses and those of $\mathbf{y}$ are indicated by hollow squares.

A similar argument between the Heegaard diagram $\widetilde{H}$ and E proves the desired result.

3.3. **Stabilization.** Let $\widetilde{G}$ be a Heegaard diagram and $\widetilde{H}$ denote a stabilization. Let $B = C(\widetilde{G})$ and $C = C(\widetilde{H})$ and C' be the mapping cone of zero map between B and B i.e., $C' = B \oplus B$, endowed with the differential $\partial' : C' \longrightarrow C'$ given by

$$\partial'(a, b) = (\partial a, -\partial b)$$

where ∂ denotes the differential within B. Let $\mathcal{L}$ and $\mathbb{R} \cong B$ be the subgroups of C' of elements of the form $(c, 0)$ and $(0, c)$ for $c \in B$, respectively. The module $\mathbb{R}$ inherits Alexander grading from its identification with B and $\mathcal{L}$ is given Alexander grading which is one less than that it inherits from its identification with B.

Let $(I, I) \subset \mathbf{S}(\widetilde{H})$ be the set of those generators which have both of their α components on the lifts of β_1. There is a natural (point-wise) identification between $\mathbf{S}(\widetilde{G})$ and (I, I), which drops the Alexander filtration by one. For $\mathbf{x} \in \mathbf{S}(\widetilde{G})$, let $\phi(\mathbf{x}) \in \mathbf{S}(\widetilde{H})$ be the induced generator in (I, I). We have

$$A_{C(\widetilde{G})}(\mathbf{x}) = A_{C(\widetilde{H})}(\phi(\mathbf{x})) + 1 = A_{C'}(0, \mathbf{x}) = A_{C'}(\mathbf{x}, 0) + 1 \qquad (3.3)$$

Definition 1. A domain is said to be of type R if it is rectangular and placed as each of the regions which are shown in Figure 7. We say a domain is of type L if either it is trivial or it has the shape in Figure 8. For $\mathbf{x} \in \mathbf{S}(\widetilde{H})$ and $\mathbf{y} \in (I, I) \subset \mathbf{S}(\widetilde{H})$, we use a

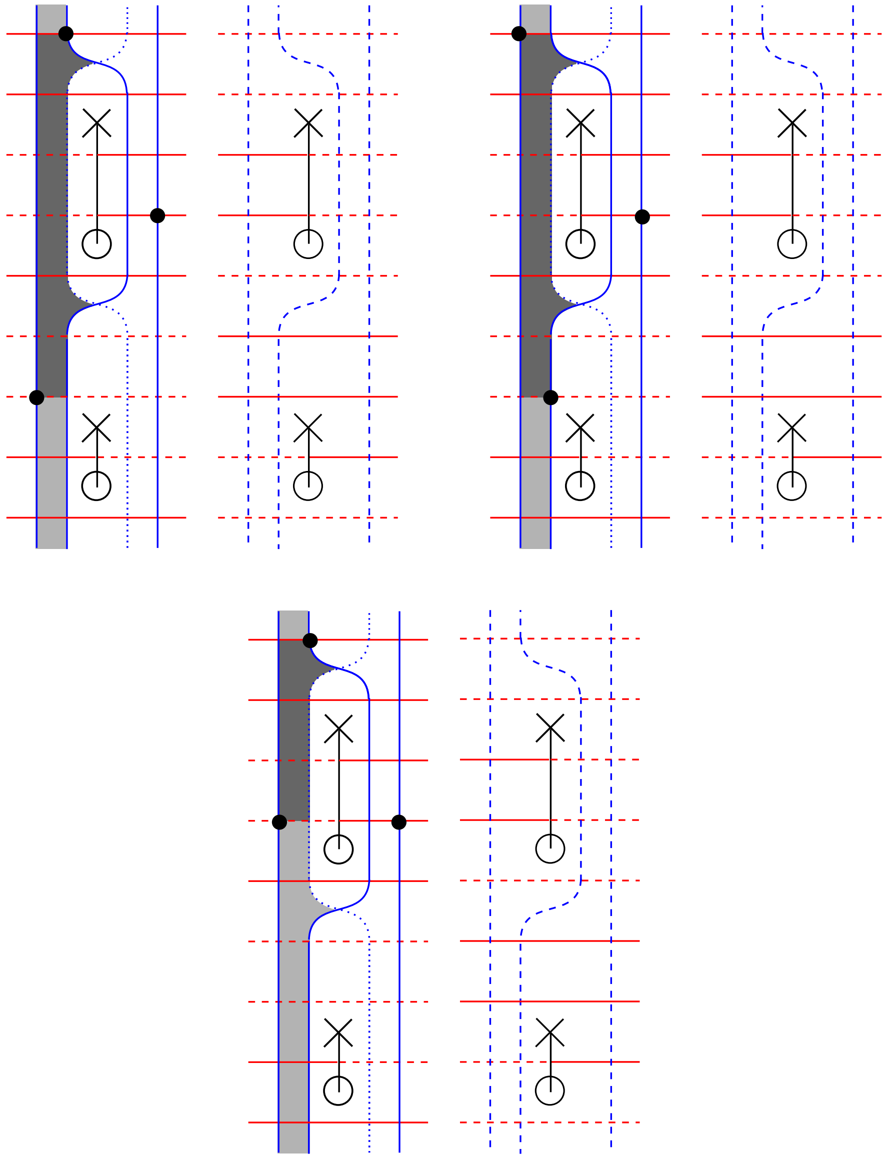

Figure 6. In this figure we illustrated a special case of composite regions which do not change the location of generator components. In the first composite region first we used a hexagon and then a rectangle, in the second one first we used a rectangle and then a hexagon, and in the last one we used two juxtaposing pentagons. There are three more such regions where the annulus is at the right of the decorations. Each of these regions counted once in $\Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma} + \partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}} + H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$ and once in the identity map.

double-region, which is either two regions of type R or two regions of type L, in each sheet.

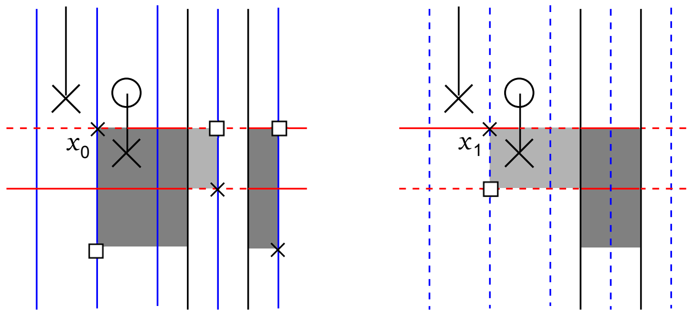

FIGURE 7. We have illustrated two type R domains, which are considered as a double-region of type R.

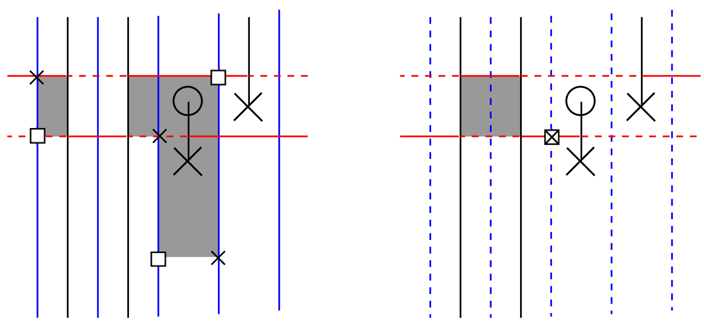

FIGURE 8. We have illustrated two domains of type L, where one of them is trivial. We consider them as a double-region of type L.

We now define maps

$$F^L : C(\widetilde{H}) \to \mathcal{L}$$
$$F^R : C(\widetilde{H}) \to \mathcal{R}$$

where F^L (resp. F^R) counts double-regions of type L (resp. R), more precisely define

$$F^L(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}} \sum_{p \in \pi^L(\mathbf{x},\mathbf{y})} \mathbf{y}$$
$$F^R(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}} \sum_{p \in \pi^R(\mathbf{x},\mathbf{y})} \mathbf{y}$$

where $\pi^L(\mathbf{x},\mathbf{y})$ (resp. $\pi^R(\mathbf{x},\mathbf{y})$) denotes the set of double-regions of type L (resp. R). We set $\pi^F(\mathbf{x},\mathbf{y}) = \pi^L(\mathbf{x},\mathbf{y}) \cup \pi^R(\mathbf{x},\mathbf{y})$ and define

$$F = \begin{pmatrix} F^L \\ F^R \end{pmatrix} : C(\widetilde{H}) \to C'$$

Lemma 3. *The map $F : C(\widetilde{H}) \to C'$ preserves the Alexander filtration, and is a chain map.*

Proof. To idea to prove that F is a chain map is similar to Lemma 3.5 in [2]. We group together different possibilities of terms in $\partial \circ F$ and $F \circ \partial$. Denote by x_0 (resp. x_1) the intersection of the lifts of α with $\widetilde{\beta}_1^0$ (resp. $\widetilde{\beta}_1^0$). Note that in addition to empty rectangles that we count in ∂, there are rectangles which contain $x_0 \in \mathbf{x}$ (or $x_1 \in \mathbf{x}$), O_1 and X_1 in their interior, but we count them as an empty rectangle in $\widetilde{G}$. We call these rectangles Type 2, and the ordinary ones Type 1. Also note that a rectangle can not intersect both regions of a double-region, otherwise the image of the composite region would wrap the grid diagram horizontally and it contradicts the assumption of emptiness of rectangles ,L and R regions.

If r is of Type 1, we have the following cases:

I(0) A composition of a rectangle r of Type 1 and p, a double-region of type L or R, where they do not have any corner in common. This composition can be counted in either $\partial \circ F$ or $F \circ \partial$.

I(1) A composition in either way of r, a rectangle of Type 1 and $p \in \pi^F$, where they have one corner in common and r does not contain x_0 (or x_1) in its boundary. This composition has a unique concave corner, cutting in either order gives two different decompositions. See Figure 9.

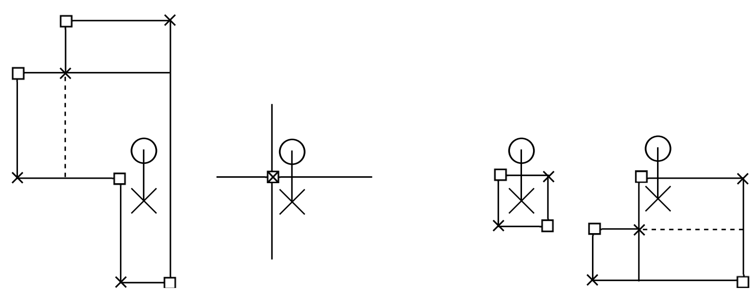

FIGURE 9. Here we illustrated terms from I(1). The first Figure shows a type L region and juxtaposing rectangle in one sheet and trivial type L region in the other sheet. The second Figure shows double-region of type R and a juxtaposing rectangle.

I(1′) A composition of a rectangle r of Type 1 and $p \in \pi^F$, where they share one corner and x_0 (or x_1) is in the boundary of r. There are two cases here. First x_0 (or x_1) is in the right edge of r. Second, the bottom right corner of r is x_0 (or x_1) and p is trivial in the same sheet as r. See Figure 10.

I(2) A composition in either way of a rectangle r of Type 1 and $p \in \pi^F$, where they share two corners and x_0 (or x_1). In this case the only possibility for p is to have a nontrivial type L region in at least one sheet. See Figure 11.

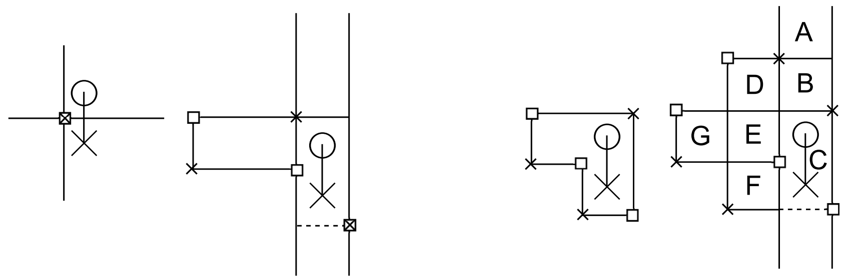

FIGURE 10. In this Figure we have pairs which cancel out with contributions from I′(1). The left picture is a contribution from I(3). The right picture is a contribution from II(1).

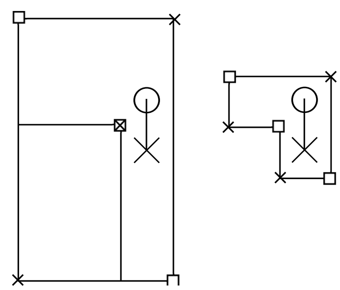

FIGURE 11. This picture shows a term from I(2), which can be alternatively decomposed as a trivial type L region and a Type 2 rectangle.

I(3) A domain which contains a vertical column of width one in one of the two sheets. In this case the only possibility for p is to have a complexity 3 region in at least one sheet and shares three corners with r. See Figure 10.

If r is of Type 2, it can only appear in terms of the form $F \circ \partial$ and we have the following cases:

II(0) The rectangle r of Type 2 is disjoint from p. See Figure 11.

II(1) A domain which contains a vertical column of width one in one of the two sheets. In this case the only possibility for p is to have a nontrivial type L region in that sheet and shares one corner with r. See Figure 10.

Contributions from case I(0) cancel each other out and as we mentioned contributions from I(1) cancel each other out. We claim that contributions from I(1′) will cancel out with contributions from I(3) and II(1). In both cases for I(1′), we add the vertical column which contains O_1 and X_1. In the first case the alternative decomposition is in II(2) and in the second case it is in I(3). Contributions from I(2) cancel out with contributions from II(0), See Figure 11.

□

We decompose the set of generators as:

$$\mathbf{S}(\widetilde{H}) = (I,I) \cup (I,J) \cup (I,N) \cup (J,I) \cup (N,I) \cup (J,J) \cup (J,N) \cup (N,J) \cup (N,N)$$

Having I in the first component always shows that $\widetilde{\alpha}^0$ component is in $\widetilde{\beta}_1^0$ or $\widetilde{\beta}_1^1$ and having I in the second component shows that $\widetilde{\alpha}^1$ component is in the other lift of $\widetilde{\beta}_1$. In case of having J in the first (resp. second) coordinate, we deal with the $\widetilde{\alpha}^0$ (resp. $\widetilde{\alpha}^1$) component on the lifts of β_2. The N shows that the configuration has its component neither on the lifts of β_1 nor on the lifts of β_2 in the corresponding α lift.

The corresponding decomposition of modules is

$$C = C^{I,I} \oplus C^{I,J} \oplus C^{I,N} \oplus C^{J,I} \oplus C^{N,I} \oplus C^{J,J} \oplus C^{J,N} \oplus C^{N,J} \oplus C^{N,N}$$

In order to see that F is a quasi-isomorphism, we will introduce an appropriate filtration. Let Q be the union of two $(n-1) \times (n-1)$ squares of dots placed in all the squares which do not appear in the rows or columns through O_1. Given $h \in \mathbb{Z}$, let $C(\widetilde{H}, h)$ denote the summand generated by generators $\mathbf{x}$ with Alexander gradings equal to h. For fixed $\mathbf{x}, \mathbf{y} \in \mathbf{S}_h$, there is at most one domain $p \in \pi(\mathbf{x}, \mathbf{y})$ with $O_i(p) = X_i(p) = 0$ for all i, so we can find a function $\mathcal{F}$ such that

$$\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y}) = \#(Q \cap p)$$

The function $\mathcal{F}$ determines a filtration on $C(\widetilde{H}, h)$, whose associated graded object counts only those rectangles which contain no O_i, X_i, or points in Q. Thus these rectangles must be supported in the row or column through O_1. We let C_Q denote this associated graded object, and typically drop h from the notation.

In order to study the boundary map between different submodules and calculate the homology groups, we need a new definition.

Definition 2. In order to get the boundary map we count rectangles, we call an empty rectangle *undone* if the components of the generator are in its upper-right and lower-left corners. We call it *done* if the generator components occupy upper-left and lower-right corners. Also a *done* or *undone* empty vertical (resp. horizontal) rectangle refers to empty rectangles which are placed in the column (resp. row) through O_1. See Figure 12.

Lemma 4. *$H_*(C_Q)$ is isomorphic to free $\mathbb{F}_2$-module generated by elements of (I,I) and (J,J) .*

Proof. There are two cases, according to whether the X_2 marks the square to the left or the right of O_1. Suppose X_2 is in the square just to the left of the square marked O_1 . Then we have a direct splitting $C_Q = C_Q^{I,I} \oplus B_1 \oplus B_2 \oplus B_3$, where the differentials in $C_Q^{I,I}$ are trivial, hence its homology is the free $\mathbb{F}_2$-module generated by elements of (I, I); also B_1, B_2 and B_3 are chain complexes fitting into these exact sequences

$$
\begin{array}{c}
0 \longrightarrow C_Q^{I,N} \longrightarrow B_1 \longrightarrow C_Q^{I,J} \longrightarrow 0 \\
0 \longrightarrow C_Q^{N,I} \longrightarrow B_2 \longrightarrow C_Q^{J,I} \longrightarrow 0 \\
0 \longrightarrow B_4 \longrightarrow B_3 \longrightarrow C_Q^{J,J} \longrightarrow 0
\end{array}
$$

Here B_4 fits into the below short exact sequence

$$0 \longrightarrow C_Q^{N,N} \longrightarrow B_4 \longrightarrow C_Q^{J,N} \oplus C_Q^{N,J} \longrightarrow 0$$

First we show that $H_*(C_Q^{N,N})$ is zero. In the first Heegaard diagram in Figure 12, there are four disjoint *done* or *undone* rectangles, we can think of the generator which is shown there to be $e_1 \wedge e_1' \wedge e_2$. This notion shows the undone rectangles that we count in boundary map. Then we have

$$\partial(e_1 \wedge e_1' \wedge e_2) = e_1 \wedge e_1' + e_1' \wedge e_2 + e_1 \wedge e_2$$

Here $e_1 \wedge e_1'$ is the term that comes from counting e_2, in other words it represent a generator with undone rectangles e_1, e_1' and done rectangles e_2, e_2'. This shows that we can partition our complex as a union of subcomplexes each isomorphic to the exterior algebra over a vector space with coefficients in $\mathbb{Z}_2$. So the homology of $C_Q^{N,N}$ being equal to the direct sum of homology of these subcomplexes is trivial. In The above case the vector space is four dimensional. As another example see the second Heegaard diagram in Figure 12, there are three disjoint *undone* rectangles, and the generator would be represented also by $e_1 \wedge e_1' \wedge e_2$, and we will again have

$$\partial(e_1 \wedge e_1' \wedge e_2) = e_1 \wedge e_1' + e_1' \wedge e_2 + e_1 \wedge e_2$$

Note that here the exterior algebra is over a three dimensional vector space. The same idea shows that $H_*(C_Q^{I,N}) = 0$ and $H_*(C_Q^{N,I}) = 0$.

In order to compute the homology of B_1 it is enough to compute the homology of $C_Q^{I,J}$ as a quotient. Each generator in $C_Q^{I,J}$ can have at most two terms in its boundary, one of which is in $C_Q^{I,N}$, so $H_*(B_1) = H_*(C_Q^{I,J}) = 0$, See Figure 13. In the same way we can prove that $H_*(B_2) = H_*(C_Q^{J,I}) = 0$ and $H_*(B_4) = H_*(C_Q^{J,N} \oplus C_Q^{N,J}) = 0$, See Figure 14. Finally the differentials in $C_Q^{J,J}$ are trivial, so its homology is the free $\mathbb{F}_2$-module generated by elements of (J, J).

Suppose on the other hand that X_2 is just to the right of O_1. Then there is a direct sum splitting $C_Q = C_Q^{J,J} \oplus B_1' \oplus B_2' \oplus B_3'$, where B_1', B_2' and B_3' are chain complexes fitting into these exact sequences

$$
\begin{array}{c}
0 \longrightarrow C_Q^{J,I} \longrightarrow B_1' \longrightarrow C_Q^{J,N} \longrightarrow 0 \\
0 \longrightarrow C_Q^{I,J} \longrightarrow B_2' \longrightarrow C_Q^{N,J} \longrightarrow 0
\end{array}
$$

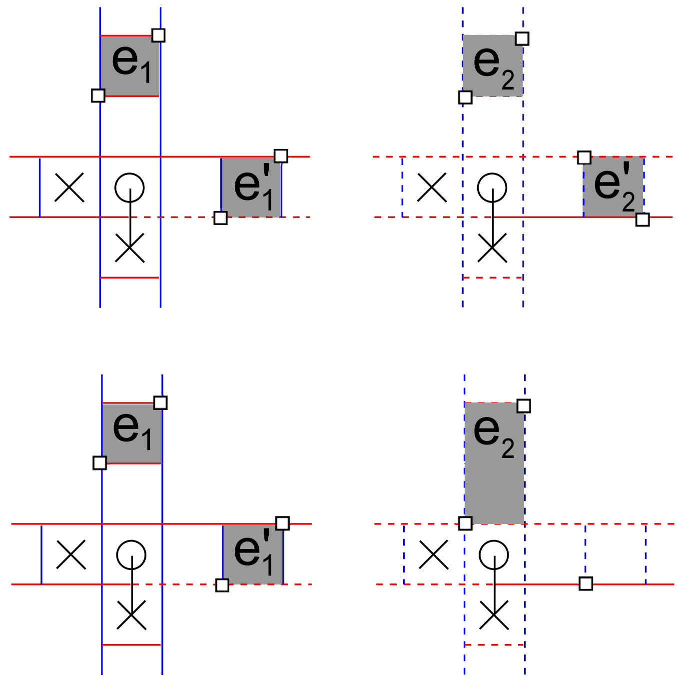

FIGURE 12. Hollow squares show a generator in $C_Q^{N,N}$. Note that in the first Heegaard diagram, the rectangle e_2' is done and the rectangles e_1, e_1', e_2 are undone. In the second Heegaard diagram all the shaded rectangles are undone.

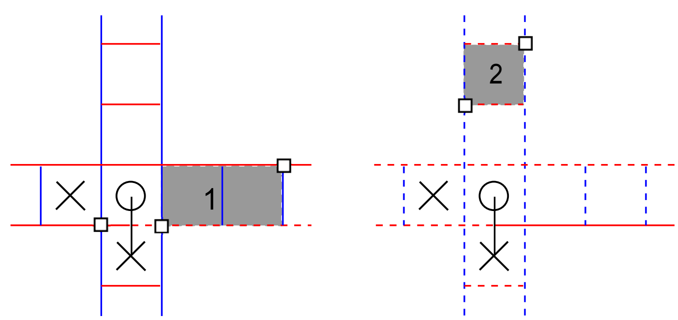

FIGURE 13. Hollow squares show a generator in $C_Q^{I,J}$. We might use either one of the regions (number 1 or 2), using region 1 leads to a generator in $C_Q^{I,N}$. If we use region 2 the result is a generator in $C_Q^{I,J}$.

$$0 \longrightarrow B_4' \longrightarrow B_3' \longrightarrow C_Q^{N,N} \longrightarrow 0$$

Here B_4' fits in the below exact sequence

$$0 \longrightarrow C_Q^{I,I} \longrightarrow B_4' \longrightarrow C_Q^{I,N} \oplus C_Q^{N,I} \longrightarrow 0$$

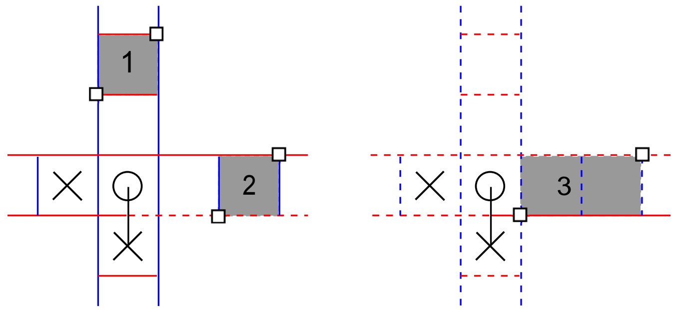

FIGURE 14. Hollow squares show a generator in $C_Q^{J,N}$. We might use one of the regions shown above in the boundary map, using region 1 or 2 lead to a generator in $C_Q^{J,N}$. If we use region 3 the result is a generator in $C_Q^{N,N}$.

all the differential in $C_Q^{J,J}$ are trivial so its homology is the $\mathbb{F}_2$-module generated by elements of (J,J). It is easy to see that $H_*(C_Q^{J,I})=0$. If we consider $C_Q^{J,N}$ as a quotient chain complex it is also easy to show that $H_*(C_Q^{J,N})=0$ so we have $H_*(B_1')=0$. In the same way we can prove that $H_*(B_2')=0$. The differentials in $C_Q^{I,I}$ are trivial so its homology is the $\mathbb{F}_2$-module generated by elements of (I,I). Also $H_*(C_Q^{I,N}\oplus C_Q^{N,I})=0$ as a quotient chain complex. A simple calculation shows that $H_*(C_Q^{N,N})=0$. Considering the above exact sequences, we get the desired result. $\square$

Lemma 5. *Suppose that $F : C \longrightarrow C'$ is a filtered chain map which induces an isomorphism on the homology of the associated graded object. Then F is a quasi-isomorphism.*

Proposition 2. *The map F is a filtered quasi-isomorphism.*

Proof. We consider the map induced by F:

$$F_Q : C_Q \longrightarrow C'_Q$$

C'_Q splits as a direct sum of chain complexes $\mathcal{L}_Q\oplus\mathbb{R}_Q$, both of which are freely generated by elements in (I,I). There are two cases. First take the case where X_2 is in the square just to the left of the square marked O_1. Consider the subcomplex $C_Q^{(I,I)}\oplus C_Q^{(J,J)}\subset C_Q$. By Lemma 4, this subcomplex carries the homology, and hence it suffices to show that the restriction of F_Q to this subcomplex induces an isomorphism in homology. We know that F_Q^L restricted to $C_Q^{(I,I)}$ is an isomorphism. In addition, F_Q^R restricted to $C_Q^{(J,J)}$ counts double-regions of type R made up of rectangles supported in the rows and columns through O_1, which contain X_1 in their interiors and the double-region end up in (I,I). But for each element in (J,J) there is a unique way of assigning such a double-region. Thus F_Q is a quasi-isomorphism when X_2 is just to the left of O_1. In the second case, When X_2 marks the square just to the right of O_1, a similar proof works. We now use Lemma 5 to conclude that F is a quasi-isomorphism. $\square$

4. Sign Assignment

In [4], it is shown that there is a sign assignment for (stable) Heegaard Floer diagrams, using a special case of that we want to assign signs to our chain maps and chain homotopies, and show that they are indeed chain homotopies with right signs.

Denote by $\mathcal{A}$ the union of the set of all empty rectangles and the set of all empty bigons. We show an element of $\mathcal{A}$ by $\phi : \mathbf{x} \to \mathbf{y}$ that means ϕ is a rectangle or bigon by initial generator $\mathbf{x}$ and terminal generator $\mathbf{y}$. If $\phi_1 : \mathbf{x} \to \mathbf{y}$ and $\phi_2 : \mathbf{y} \to \mathbf{z}$ then $\phi_1 * \phi_2$ shows the composite region that has $\mathbf{x}$ as its initial generator and $\mathbf{z}$ as the terminal generator. If a Heegaard diagram has n α-curves, we say that the Heegaard diagram is of power n.

Definition 3. A sign assignment $\mathcal{S}$ is a map from $\mathcal{A}$ into $\{\pm 1\}$ with the following properties:

(S-1) if the composition of two rectangles r_1 and r_2 make a column, then
$$\mathcal{S}(r_1)\mathcal{S}(r_2) = 1$$
(S-2) if the two pairs (ϕ_1, ϕ_2) and (ϕ_3, ϕ_4) satisfy the equality $\phi_1 * \phi_2 = \phi_3 * \phi_4$ then,
$$\mathcal{S}(\phi_1)\mathcal{S}(\phi_2) + \mathcal{S}(\phi_3)\mathcal{S}(\phi_4) = 0.$$

We can construct new sign assignments from an old one.

Definition 4. Let $\mathcal{S}$ be a sign assignment and u be any map from $\mathbf{S}$ the set of generators of power n to $\{\pm 1\}$, then we can define a new sign assignment $\mathcal{S}'$ such that $\mathcal{S}'(\phi) = u(\mathbf{x})\mathcal{S}(\phi)u(\mathbf{y})$, for any $\phi : \mathbf{x} \to \mathbf{y}$ in $\mathcal{A}$. If $\mathcal{S}$ and $\mathcal{S}'$ are related as above, we say that $\mathcal{S}$ and $\mathcal{S}'$ are gauge equivalent sign assignments.

Theorem 2. *For any power n there is, up to gauge equivalence a unique sign assignment. The map $\partial^{\mathbb{Z}}$ over $\mathbb{Z}$ satisfies $(\partial^{\mathbb{Z}})^2 = 0$ and the resulting Floer homology $\widetilde{HF}^{Z}$ is independent of the choice of $\mathcal{S}$, and order of the α- and β-curves, and invariant under nice moves.*

Now we turn to the proof of the main theorem with signs. We use the same method from section 3. The case of cyclic permutation is simple so we start by proving the invariance over $\mathbb{Z}$ for commutation.

4.1. **Commutation.**

In order to obtain a chain map, we need to assign signs to each of the pentagons. There are two types of pentagons: either the pentagon is on the left of the curve $\widetilde{\beta}$ which we call it a left pentagon, or on the right side of the curve $\widetilde{\beta}$

which is a right pentagon. We use isotopy move of the curve $\widetilde{\beta}$ (resp. γ) for left (resp. right) pentagons to create new intersections with α circles and induce new rectangles, each rectangle corresponds to one of the pentagons. In case of left pentagons, the corresponding rectangle has the same initial points as the pentagon, also the upper-right terminal point is the same. The lower-left terminal point for the pentagon is on the intersection of γ and an α circle and the lower-left terminal point of the corresponding rectangle is on the intersection of $\widetilde{\beta}$ and the same α circle. See Figure 15. In case of right pentagons, the terminal points and the upper-left initial point of the corresponding rectangle are the same as pentagon, and the lower-left initial point is on the intersection of the same α circle as before and γ circle. We call the corresponding rectangle $r(p)$ and define the sign for pentagons as follows:

$$\varepsilon(p) = \begin{cases} \mathcal{S}(r(p)) & \text{if p is a left pentagon} \\ -\mathcal{S}(r(p)) & \text{if p is a right pentagon} \end{cases}$$

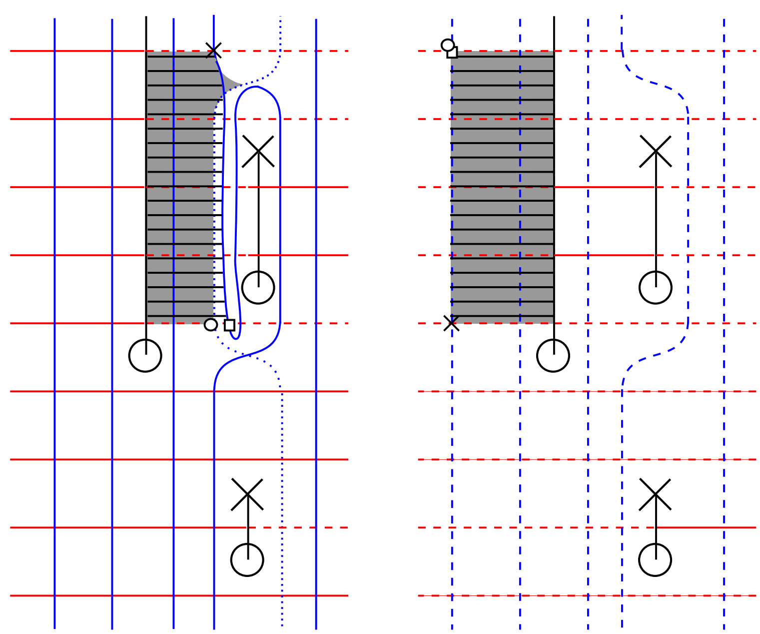

FIGURE 15. **Signs for pentagons.** Here we have shown a left pentagon p (shaded in gray) and the corresponding rectangle $r(p)$ (shaded horizontally). The initial generator $\mathbf{x}$ is marked by crosses. The terminal generator is marked by hollow rectangles for the pentagon and by hollow circles in case of the corresponding rectangle.

Given $\mathbf{x} \in \mathbf{S}(\widetilde{G})$, define

$$\Phi_{\widetilde{\beta}\gamma}(\mathbf{x}) = \sum_{\mathbf{y} \in \ \mathbf{S}(E)} \sum_{p \in Pent_{\widetilde{\beta}\gamma}(\mathbf{x},\mathbf{y})} \varepsilon(p)\mathbf{y} \in C(E) \tag{4.1}$$

Lemma 6. *The map $\Phi_{\widetilde{\beta}\gamma} : C(\widetilde{G}) \longrightarrow C(E)$ preserves the Alexander filtration, and is anti-chain map.*

Proof. The fact that the Alexander filtration will not change follows from the definition. In order to show that $\Phi_{\widetilde{\beta}\gamma}$ is anti-chain map we consider different compositions of a pentagon and a rectangle, whether they are disjoint or have overlapping interiors or have an edge in common. In most of the cases the composed region has two different decomposition and the consistency of signs follows from property S-2 in Definition **??** and the universal property of sign assignments. However there is one special case of regions that have a unique decomposition. They can be paired and each pair have the same initial and terminal points, one of the decompositions represents a term in $\partial \circ \Phi_{\widetilde{\beta}\gamma}$ and the other one represents a term in $\Phi_{\widetilde{\beta}\gamma} \circ \partial$; See Figure 16. From property S-1 and the universal property of sign assignments, we know that in case of left pentagons we have $\epsilon(p)\mathcal{S}(r) = -1$ and in case of right pentagons we have $\epsilon(p)\mathcal{S}(r) = 1$. So the paired regions have different signs and cancel each other out.

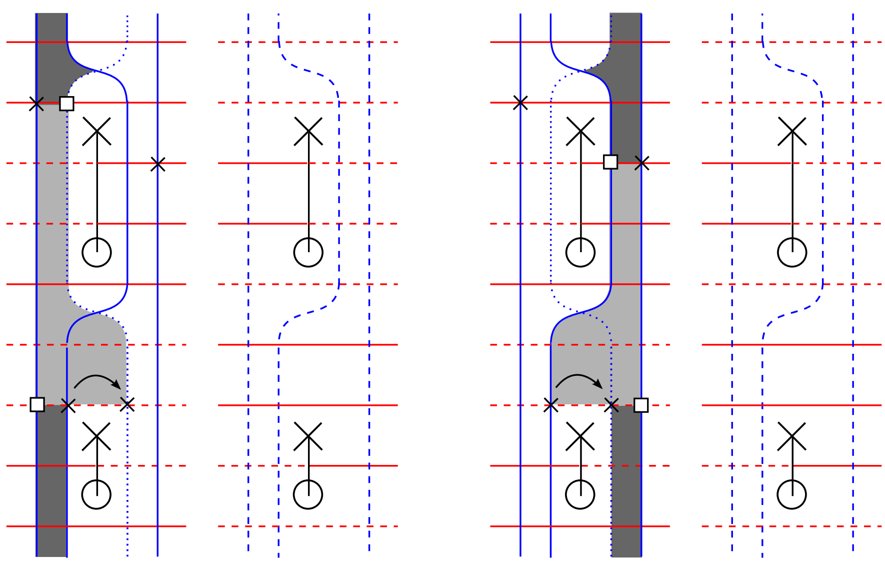

FIGURE 16. Here we showed a special case where two different composite regions have the same contribution. In the left diagram we first used the pentagon and then the rectangle. In the right diagram we first used a rectangle and then a pentagon. This two terms cancel out each other because the product of the signs are opposite.

□

Similarly we define $\Phi_{\gamma\widetilde{\beta}}$, and a similar argument shows that it is anti-chain map.

We now want to define signs for hexagons, in order to show that homotopy operators which are defined by counting hexagons are chain homotopies with regard to sign assignment. We start with defining signs for width one pentagons. Note that each width one hexagon corresponds to a rectangle in grid diagram $\widetilde{G}$ with a cut along the connecting line of O and X that are inside it. We define the sign to be minus the sign of the empty width one rectangle which made a complete column together with the hexagon.

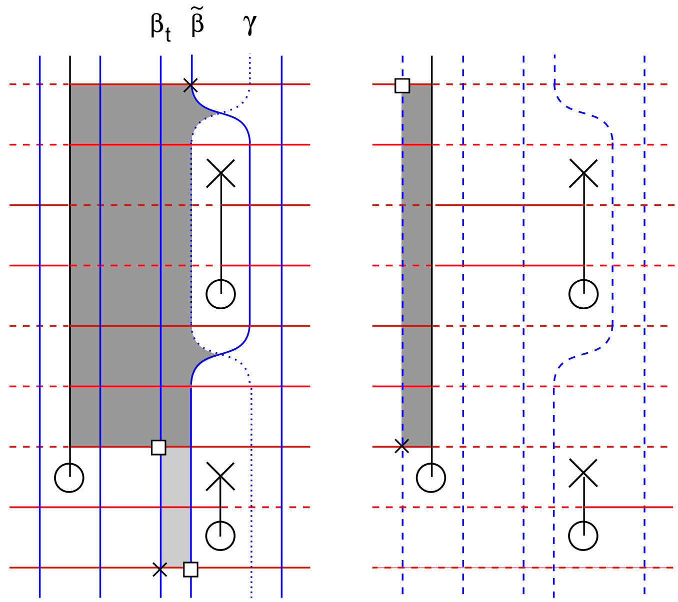

FIGURE 17. **Sign assignment for hexagons.** We define the sign for a hexagon of width more than one by considering two different decompositions of the shaded region.

We proceed to define signs for all hexagons. For each hexagon h of width $w > 1$, consider the component of the generator on the β circle just before $\widetilde{\beta}$, let us call that β_t. Form the composite region made up of hexagon itself and a rectangle r of width one (Figure 17), this composite region has another decomposition made up of a width one hexagon h' and a rectangle r' of width $w-1$. We have already defined signs for width one hexagons, so let us define $\varepsilon(h) := -\mathcal{S}(r)\varepsilon(h')\mathcal{S}(r')$. Consider the following homotopy operator $H_{\widetilde{\beta}\gamma\widetilde{\beta}} : C(\widetilde{G}) \longrightarrow C(\widetilde{G})$

$$H_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\widetilde{G})} \sum_{h\in Hex_{\widetilde{\beta}\gamma\widetilde{\beta}}(\mathbf{x},\mathbf{y})} \varepsilon(h)\mathbf{y}. \tag{4.2}$$

where $\varepsilon(h)$ is as defined above. However there is another way to define signs for hexagons with width more than one, using another composed region which is made up of hexagon itself and a thin rectangle above the hexagon. Note that this new method results in the same sign assignment for hexagons. This can be easily shown using property S-1 of sign assignments for Heeagaard diagrams and the definition of sign for thin hexagons. Similarly define $H_{\widetilde{\beta}\gamma\widetilde{\beta}} : C(H) \longrightarrow C(H)$.

Proposition 3. *The map $\Phi_{\widetilde{\beta}\gamma} : C(\widetilde{G}) \longrightarrow C(E)$ is a chain homotopy equivalence with respect to sign assignments. In other words*

$$\mathbb{I} + \Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma} + \partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}} + H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial = 0$$
$$\mathbb{I} + \Phi_{\widetilde{\beta}\gamma} \circ \Phi_{\gamma\widetilde{\beta}} + \partial \circ H_{\gamma\widetilde{\beta}\gamma} + H_{\gamma\widetilde{\beta}\gamma} \circ \partial = 0.$$

Proof. We want to show that the same pairing as in proposition 1 works here with regard to sign assignments.

Consider a term of the form $H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$ which is paired by a term in $\Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma}$, in order to show that these two decompositions give opposite signs, we use a cube of generators as in Figure 18. To each face of the cube we assign a sign which is the product of the signs of its four edges. The front face demonstrates the decomposition at hand, and we want to show that its sign is -1. The upper face and left face have sign -1 by definition of the sign of thin hexagons (note that $\Phi_{\gamma\widetilde{\beta}}$ is anti-chain map and the sign of each thin hexagon is defined based on the sign of its complementary rectangle). The right face has sign -1 by definition of sign for hexagons with width more than one. According to property S-2 of sign assignments for grid diagrams, the back face has sign -1 (it shows two different ways for decomposing a region). The map $\Phi_{\widetilde{\beta}\gamma}$ is anti-chain map so the lower face has sign -1 as well. This argument shows that the sign of the front face is also -1 as desired.

For other type of pairings of $\Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma}$, $\partial \circ H_{\widetilde{\beta}\gamma\widetilde{\beta}}$, $H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$ a similar argument using cubes will give the desired result.

However there are special types of composite regions that have exactly one decomposition (Figure 6). These regions made up of either a hexagon and a rectangle or two pentagons and they can be paired with identity, so we need to show that the product of sign of the decomposition in each case is -1. For a hexagon and a pentagon the result came from the definition of thin hexagon sign. The desired property for composite of two pentagons follows from property S-1 and universality of sign assignment for Heeagaard diagrams.

□

4.2. **Stabilization.** Now that we proved commutation invariance we want to show the Stabilization invariance. Let $\mathcal{S}$ be a sign assignment for $\widetilde{H}$, we want to define a

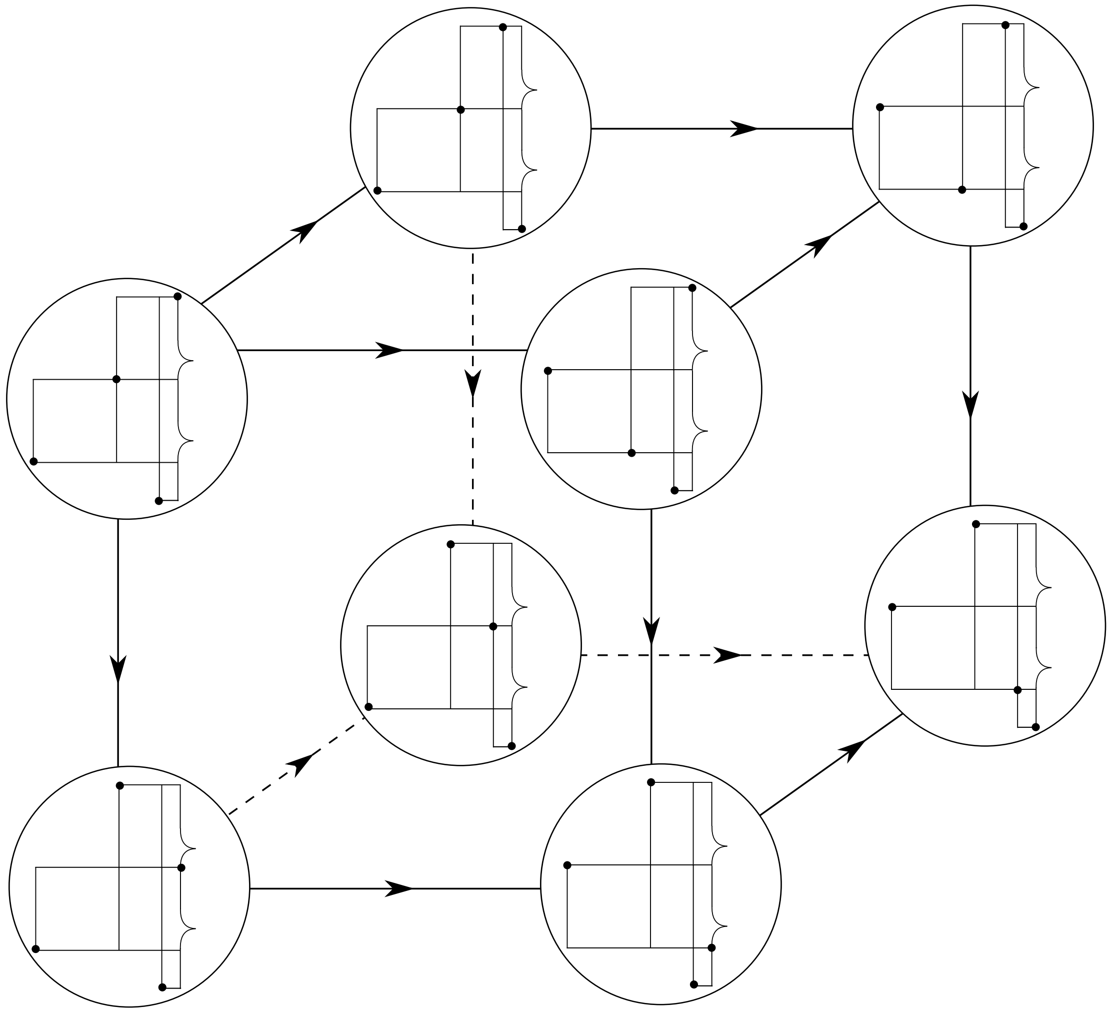

FIGURE 18. Here we have illustrated a cube to show that a paring of two terms in $H_{\widetilde{\beta}\gamma\widetilde{\beta}} \circ \partial$ and $\Phi_{\gamma\widetilde{\beta}} \circ \Phi_{\widetilde{\beta}\gamma}$ have opposite product of signs (front face). In order to do that we show that all other five faces have -1 sign.

sign assignment for $\widetilde{G}$. Note that we identified $\mathbf{S}(\widetilde{G})$ with $(I, I) \subset \mathbf{S}(\widetilde{H})$. For fixed $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\widetilde{G})$ and $r \in Rect(\mathbf{x}, \mathbf{y})$, there is a corresponding rectangle r' connecting $\mathbf{x}', \mathbf{y}' \in \mathbf{S}(\widetilde{H})$ which is either of Type 1 or Type 2. We decompose a rectangle of Type 2 as a product of smaller rectangles. See Figure 19.

To assign a sign to a region r of width one that has a cut that connects two branched points inside, with initial generator $\mathbf{x}$ and terminal generator $\mathbf{y}$, we use the empty rectangle $\overline{r} \in Rect(\mathbf{y}, \mathbf{x})$ (Figure 19) and define $\mathcal{S}(r) = -\mathcal{S}(\overline{r})$.

Lemma 7. *For any sign assignment $\mathcal{S}$ for $\widetilde{H}$, define*

$$\mathcal{S}_0(r) = \begin{cases} \mathcal{S}(r') & \textit{if } r \textit{ is of Type 1} \\ \mathcal{S}(r_1)\mathcal{S}(r_2)\mathcal{S}(r_3) & \textit{if } r \textit{ is of Type 2} \end{cases}$$

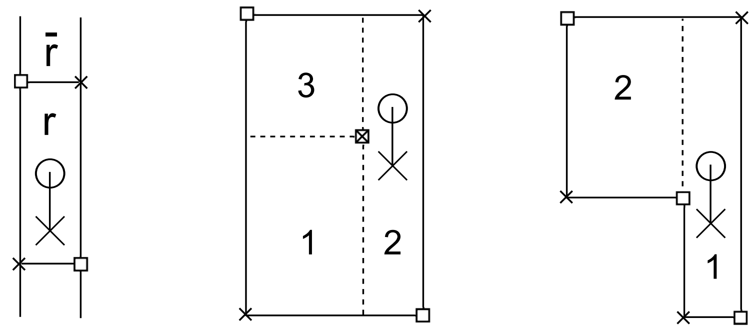

FIGURE 19. First picture shows two complementary regions r and $\overline{r}$, their union make a column, and we define $\mathcal{S}(r) = -\mathcal{S}(\overline{r})$. The second picture shows a decomposition of a Type 2 rectangle and the order in which we use the regions to define the sign for it. The third picture shows the decomposition for a Type L region that we use to define its sign.

$\mathcal{S}_0$ is a sign assignment for $\widetilde{G}$. See Figure 19.

Proof. The proof is similar to Lemma 4.26 [2]. □

Now we want to define signs μ from the regions that we count in the map which we used in the proof of Stabilization invariance into $\{\pm 1\}$,

$$F = \begin{pmatrix} F^L \\ F^R \end{pmatrix} : C(\widetilde{H}) \to C',$$

where

$$F^L(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}} \sum_{p\in\pi^L(\mathbf{x},\mathbf{y})} \mu(p)\mathbf{y}$$

$$F^R(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}} \sum_{p\in\pi^R(\mathbf{x},\mathbf{y})} \mu(p)\mathbf{y}$$

For a type L region define the sign as follows (Figure 19):

$$\mu(p) = \begin{cases} 1 & \text{if p is trivial} \\ \mu(r_1)\mu(r_2) & \text{if p is nontrivial} \end{cases}$$

For a double-region just multiply the signs of individual regions.

Lemma 8. *F^L is a chain map.*

Proof. We prove this lemma by studying cases in lemma 3. Obviously, Contributions in case I(0) and case I(1) cancel each other out. For terms in I(2) and II(0) cancellations come from the definition of sign for Type 2 rectangles. Cancellations of pairs in I(3) and I′(1) also come from the definition of sign for type L regions. The last case is a pairing of II(1) and I′(1) terms, let p and p' be nontrivial type L regions, r be a

Type 2 rectangle and r' be a Type 1 rectangle which satisfy the following equality, (See Figure 10)

$$p * r = r' * p'$$

Here for simplicity we call the regions with alphabets, note that $\mathcal{S}$ depends on the initial and terminal points of regions and the terms cannot be freely commuted. So the above equality can be rewritten as:

$$ACEG * BCDEF = DEF * CEG$$

The following computation proves the desired result.

$$\begin{aligned}
S(p)S(r) &= S(AC)[S(EG)S(F)]S(BC)S(DE) \\
&= -[S(AC)S(EF)][S(G)S(BC)]S(DE) \\
&= -S(CEF)[S(A)S(BC)][S(G)S(DE)] \\
&= -S(CEF)[-1][-S(D)S(EG)] \\
&= [-S(CEF)S(D)]S(EG) \\
&= S(r')S(C)S(EG) \\
&= S(r')S(p').
\end{aligned}$$

□

We now turn to type R regions, note that the two seemingly separate regions have actually been glued together along the branch cut so the shape of a double-region of type R is an octagon (Figure 20). Performing an isotopy move on the α curve shown in Figure 20 result in new rectangles and bigons that allows us to define the sign for octagons as follows

$$\mu(\theta) = \mathcal{S}(R_1)\mathcal{S}(B_1)\mathcal{S}(R_2)\mathcal{S}(B_2)\mathcal{S}(R_3),$$

where θ is an octagon, R_1, R_2 and R_3 are rectangles and B_1 and B_2 are bigons.

Lemma 9. *F^L is anti-chain map.*

Proof. We prove this lemma in two steps, first consider a rectangle r and an octagon θ that are disjoint and the composites $r * \theta$ have an alternative decomposition as $\theta' * r'$, where r' is a rectangle with same domain as r and θ' is an octagon with the same domain as θ, then we have

$$\begin{aligned}
\mathcal{S}(r)\mu(\theta) &= \mathcal{S}(r)\mathcal{S}(R_1)\mathcal{S}(B_1)\mathcal{S}(R_2)\mathcal{S}(B_2)\mathcal{S}(R_3) \\
&= -\mathcal{S}(R_1)\mathcal{S}(B_1)\mathcal{S}(R_2)\mathcal{S}(B_2)\mathcal{S}(R_3)\mathcal{S}(r) \\
&= -\mu(\theta)\mathcal{S}(r),
\end{aligned}$$

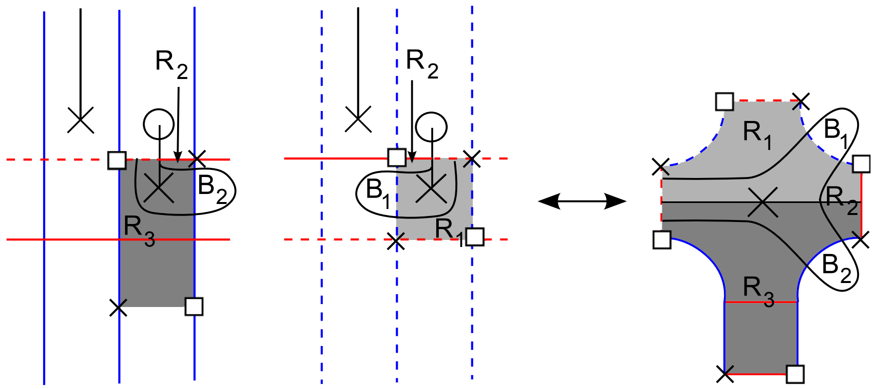

FIGURE 20. Here a thin octagon and regions which are created after an isotopy move of an α circle is illustrated.

the minus sign in the second equality comes from commuting disjoint rectangles and bigons, an odd number of times.

In the second case when r and θ have an edge in common, note that r can not contain the bigon B_1, because there are no Type 2 rectangles in Heegaard diagram $\widetilde{H}$ (there is a X just to the left of O_1), also a rectangle in $\widetilde{G}$ do not have β as one of its edges.

Let us study different cases when a rectangle r has an edge in common with an octagon θ_1. In Figure 21, left edge of r is in common with the octagon and we will have the following equations

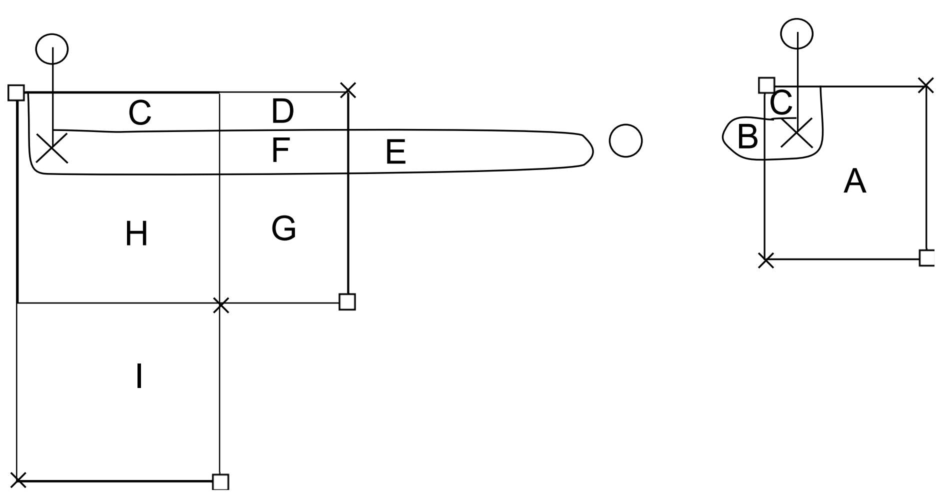

FIGURE 21. Here the left edge of $r = DFG$ is in common with the octagon $\theta_1 = ACHI$, the compisite region has an alternate decomposition I*θ_2.

$$\begin{aligned} r * \theta_1 &= (DFG) * A * B * C * (EF) * (HI) \\ &= A * B * (DFG) * C * (EF) * (HI) \\ &= A * B * (CD) * (FG) * (EF) * (HI) \\ &= A * B * (CD) * E * G * (HI) \\ &= A * B * (CD) * E * I * (GH) \\ &= I * A * B * (CD) * E * (GH) = I * \theta_2. \end{aligned}$$

In above equations, rectangles and bigons are commuted an odd number of times so

$$\mathcal{S}(r)\mu(\theta_1) = -\mathcal{S}(I)\mu(\theta_2).$$

Next consider a case when lower edge of r is in common with octagon θ_1 (Figure 22), then

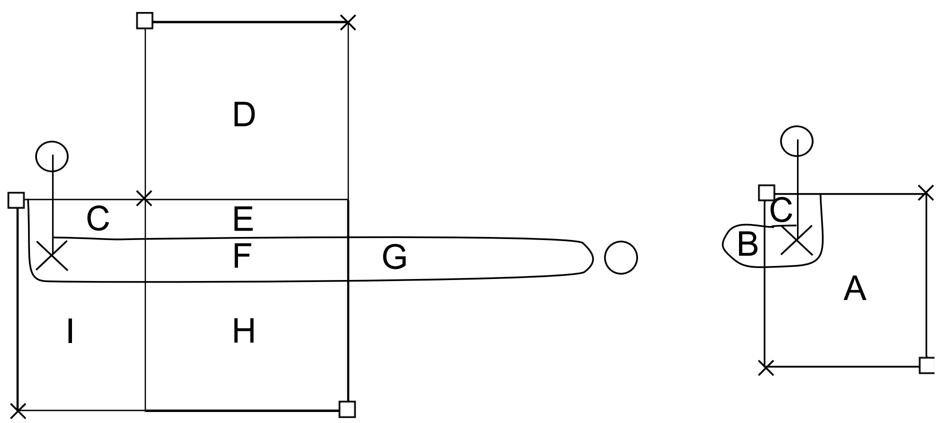

FIGURE 22. Here the lower edge of $r = D$ is in common with octagon $\theta_1 = ACEFHI$, this composite region has an alternate decomposition $\theta_2 * t$, where $\theta_2 = ACI$ is an octagon and $t = DEFH$ is a rectangle.

$$\begin{aligned} r * \theta_1 = D * \theta_1 &= D * A * B * (CD) * G * (HI) \\ &= A * B * D * (CE) * G * (HI) \\ &= A * B * C * (DE) * G * (HI) \\ &= A * B * C * (FG) * (DEF) * (HI) \\ &= A * B * C * (FG) * I * (DEFH) \\ &= \theta_2 * (DEFH) = \theta_2 * t \end{aligned}$$

again here terms are commuted an odd number of times so

$$\mathcal{S}(r)\mu(\theta_1) = -\mu(\theta_2)\mathcal{S}(t)$$

the remaining cases are similar. □

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