

ECHELON MODIFICATIONS OF VECTOR BUNDLES

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ABSTRACT. We study a filtered generalization of the operation of elementary modification of vector bundles. The generalization is motivated by applications to the degeneration theory of linear systems.

The notion of *elementary modification* of a vector bundle along a divisor on a scheme is well known and is a standard method for constructing vector bundles. The purpose of this note is to define a generalization of this to a certain *filtered* setting, on both the bundle and divisor sides. The generalization, whose main properties are given in Theorem 2.2 below, is motivated by situations which occur in the study of linear systems on a family of curves with reducible fibres. The construction made here will be applied in [4] to yield boundary modifications of the Hodge bundle on the moduli space of curves. These modifications will play a key role in our work on the closure in $\overline{\mathcal{M}}_g$ of g_d^r loci in \mathcal{M}_g . See also [5].

We will work in a general setting of vector bundles on a scheme. However, we wish to draw attention to a couple of relatively subtle points arising in the construction, which may seem surprising from such a general vantage point, and which are both motivated by the applications. One is the need for an appropriate ‘persistence condition’, which is needed to ensure that the divisor filtration and the bundle filtration interact well. The other is the fact that the divisors involved in the echelon modifications are, in a sense, smaller than the ‘obvious’ divisors one could work with. In the situation of curve families with reducible fibres, this implies working on the *total* space of the family (or something like it), and constructing modifications that are not a pullback from the base. This feature is moreover critical for the universal property of echelon modifications (Theorem 2.2, (iii)).

This note was mostly contained in [4] originally, but will be published separately.

1. ECHELON DATA AND THEIR NORMAL FORMS

Our purpose here is to define the notion of *echelon datum* on a scheme. Roughly speaking, such a datum consists of a vector bundle E together with a descending chain of full-rank locally free subsheaves E^i , such that the degeneracy loci of the inclusion maps

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$E^i \rightarrow E$ form a (usually non-reduced) divisorial chain, more specifically that E^i has minimal generators that have zeros of size $\delta_j, j \leq i$ as sections of E , where $\delta_1 \leq \delta_2 \dots$ is a suitable ascending chain of divisors. The definition is as follows.

Definition 1.1. • A pre-echelon datum of length m , (E, δ) on a scheme X consists of the following items

(i) a descending chain of locally free sheaves of the same rank

$$E = E^0 \supset E^1 \dots \supset E^m;$$

(ii) a collection of Cartier divisors $\delta_0 = 0, \delta_1, \dots, \delta_m$, such that $d\delta_i := \delta_i - \delta_{i-1}$ is effective;

which satisfy the following

'persistence condition': for all i , we have $E^i \supset E^{i-1}(-d\delta_i)$ and the quotient $E^i/E^{i-1}(-d\delta_i)$ maps isomorphically to a locally free, locally split $\mathcal{O}_{d\delta_i}$ -submodule of $E^j \otimes \mathcal{O}_{d\delta_i}$, for all $j < i$.

• An echelon datum $\chi = (E, \delta, D)$ consists of a pre-echelon datum (E, δ) plus a collection of Cartier sub-divisors $(D, \leq \delta)$ such that $dD_i := D_i - D_{i-1}$ is effective and D_i has no components in common with $D_i^\dagger := \delta_i - D_i$.

Remark 1.2. Note that for all $0 \leq j < i$, E^i contains $E^j(\delta_j - \delta_i)$, so that

$$E^i/E^j(\delta_j - \delta_i) \subset E^j \otimes \mathcal{O}_{\delta_i - \delta_j}$$

is a well-defined subsheaf, and we have commutative

$$\begin{array}{ccc} E^i/E^j(\delta_j - \delta_i) & \rightarrow & E^i/E^{i-1}(-d\delta_i) \\ \downarrow & & \downarrow \\ E^j \otimes \mathcal{O}_{d\delta_i} & \leftarrow & E^{i-1} \otimes \mathcal{O}_{\delta_i} \end{array}$$

The persistence condition means that the right arrow is an isomorphism to a locally free and cofree (i.e. split) subsheaf of its target, and likewise for the composite of the right and bottom arrows. See example 1.5 for motivation for the persistence condition.

Remark 1.3. An echelon datum is said to be *scalar* if $\delta_i = n_i\delta, D_i = n_iD$ for all i and fixed effective divisors δ, D . This cases considered in this paper have this property.

Example 1.4. One (scalar) example to keep in mind is the following: $\pi : X \rightarrow B$ is a proper morphism (e.g. a family of curves), \mathcal{L} is a line bundle on X , $E = \pi^*(\pi_*(\mathcal{L}))$, $\delta_1 = \pi^*(\delta)$, $D = D_1$ is a component of δ_1 , and

$$E^i = \pi^*(\pi_*(\mathcal{L}(-iD))). \quad \square$$

In analyzing echelon data locally, a useful tool is a normal form called *echelon decomposition*, constructed as follows. Let t_i be an equation for $\delta_i - \delta_{i-1}$. Using the persistence

condition, we can find a free submodule A_0 of E^m which maps isomorphically to its image in E^{m-1} , and a free complement to the latter, $A'_1 \subset E^{m-1}$, so that

$$E^m = A_0 \oplus t_m A'_1 \subset E^{m-1} = A_0 \oplus A'_1$$

By assumption, A_0 persists in E^{m-2} ; therefore we can find free submodules $A_1, A'_2 \subset E^{m-2}$ so that

$$E^{m-1} = A_0 \oplus A_1 \oplus t_{m-2} A'_2 \subset E^{m-2} = A_0 \oplus A_1 \oplus A'_2,$$

therefore as submodule of E^{m-2} , we have

$$E^m = A_0 \oplus t_m A_1 \oplus t_m t_{m-1} A'_{m-2}$$

Continuing in this way and setting finally $A'_m = A_m$ we obtain what we will call an *echelon decomposition*

$$E = A_0 \oplus \dots \oplus A_m$$

such that

$$(1) \quad E^i = A_0 \oplus \dots \oplus A_{m-i} \oplus t_i A_{m-i+1} \oplus \dots \oplus t_i \dots t_1 A_m, i = 0, \dots, m.$$

In particular,

$$E^m = A_0 \oplus t_m A_1 \oplus t_m t_{m-1} A_2 \oplus \dots \oplus t_m \dots t_1 A_m.$$

Example 1.5 (Example 1.4, cont). In the situation of the last example, where $t_i = t = xy$ is an equation for δ and y is an equation for D , the summand $A_0 \oplus \dots \oplus A_{m-i}$ corresponds to the image of general sections of $\mathcal{L}(-iD)$ (divisible by y^i as section of \mathcal{L}); $t A_{m-i+1}$ comes from sections of $\mathcal{L}(-iD)$ divisible by x , etc.; $t^i A_m$ comes from sections of $\mathcal{L}(-iD)$ divisible by x^i . Because multiplication by y does not affect linear independence modulo x , it is easy to see that the persistence condition is satisfied.

2. ECHELON MODIFICATION

Our purpose is to associate to an echelon datum $\chi = (E, \delta, D)$ a type of birational modification of the bundle $E = E^0$ which generalizes the familiar notion of elementary modification. This echelon modification will be an *ascending* chain of vector bundles

$$E = E_0 \subset E_1 \dots \subset E_m = \text{Mod}(\chi, E),$$

all equal to E off D_m . $\begin{matrix} \text{Z} \\ \text{I} \end{matrix}$: $E_i \neq E^i$. The process works iteratively, first producing E_1 carrying an echelon datum of length $m - 1$, etc.

Let (E, δ, D) denote an echelon datum on X . For $i \geq 1$, set

$$dD_i = D_i - D_{i-1}, L_i = \mathcal{O}_{dD_i}(dD_i), i \geq 1, D_0 := 0.$$

L_i is an invertible $\mathcal{O}(dD_i)$ -module. Also let

$$G^i = \text{coker}(E^i \rightarrow E^{i-1}), H^i = E^i / \mathcal{O}(-d\delta_i) E^{i-1}$$

which are by assumption locally free \mathcal{O}_{δ_i} -modules. We have a locally split exact sequence of locally free \mathcal{O}_{δ_i} -modules

$$(2) \quad 0 \rightarrow H^i \rightarrow E^{i-1} \otimes \mathcal{O}_{d\delta_i} \rightarrow G^i \rightarrow 0$$

This yields a similar exact sequence upon restriction on dD_i . In particular, the kernel of the natural map $E^{i-1} \rightarrow G^i \otimes \mathcal{O}_{dD_i}$ is generated by E^i and $E(-D_i)$. For H^i , there is another exact sequence

$$(3) \quad 0 \rightarrow E^{i-1}(-\delta_i) \rightarrow E^i \rightarrow H^i \rightarrow 0$$

In term of a local echelon decomposition (1), we have

$$H^i = (A_0 \oplus \dots \oplus A_{m-i}) \otimes \mathcal{O}_{d\delta_i}$$

To start our ascending chain, define a subsheaf $E_1 \subset E(D_1)$ by the exact diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \rightarrow & E^1(D_1) & \rightarrow & E(D_1) & \rightarrow & G^1(D_1) \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \rightarrow & E_1 & \rightarrow & E(D_1) & \rightarrow & G^1 \otimes L_1 \rightarrow 0. \end{array}$$

Because the right vertical map is surjective with kernel $E \otimes \mathcal{O}_{D_1^\dagger}, D_1^\dagger := \delta_1 - D_1$, E_1 is generated by E and $E^1(D_1)$, i.e.

$$(5) \quad E_1 = E^1(D_1) + E \subset E(D_1).$$

Because G^1 is locally free over \mathcal{O}_{δ_1} , it follows that $G^1 \otimes L_1$ is a locally free \mathcal{O}_{D_1} -module, hence E_1 is an elementary modification of E and in particular is locally free over X . Also, the snake lemma yields exact

$$(6) \quad 0 \rightarrow E^1(D_1) \rightarrow E_1 \rightarrow G^1 \otimes \mathcal{O}_{D_1^\dagger} \rightarrow 0.$$

In particular, E_1 is just $E^1(D_1)$ if $D_1 = \delta_1$. We have another the exact diagram

$$(7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & E & = & E & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & E_1 & \rightarrow & E(D_1) & \rightarrow & G^1 \otimes L_1 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & H^1 \otimes L_1 & \rightarrow & E \otimes L_1 & \rightarrow & G^1 \otimes L_1 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

In terms of a local decomposition as in (1), we can write, where generally $t_i = x_i y_i$ with y_i an equation for dD_i ,

$$E_1 = \frac{1}{y_1} (A_0 \oplus \dots \oplus A_{m-1}) \oplus A_m.$$

Next, define subsheaves $E_1^i \subset E_1$ by

$$E_1^i = E^{i+1}(\delta_1) \cap E_1 \subset E(\delta_1).$$

In particular,

$$E_1^1 = E^2(\delta_1) \cap E_1.$$

In terms of a local echelon decomposition, we have

$$E_1^i = \frac{1}{y_1} (A_0 \oplus \dots \oplus A_{m-i-1} \oplus t_{i+1} A_{m-i} \oplus \dots \oplus t_{i+1} \dots t_2 A_{m-1}) \oplus t_{i+1} \dots t_2 A_m$$

Then set, analogously to the above,

$$E_2 = E_1^1(dD_2) + E_1 \subset E_1 \otimes dD_2$$

In terms of an echelon decomposition, this is

$$E_2 = \frac{1}{y_1 y_2} (A_0 \oplus \dots \oplus A_{m-2}) \oplus \frac{1}{y_1} A_{m-1} \oplus A_m$$

Then we have exact

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow H^2 \otimes \mathcal{O}(D_2) \rightarrow 0$$

Note the natural inclusion

$$(8) \quad E^2(D_2) \rightarrow E_2$$

In general, we define inductively

$$(9) \quad \begin{aligned} E_j^i &= E_{j-1}^{i+1}(d\delta_j) \cap E_j, i \geq 1 \\ E_{j+1} &= E_j^1(dD_{j+1}) + E_j \subset E_j(dD_{j+1}). \end{aligned}$$

Again we have an inclusion, $\forall i$,

$$E^i(D_i) \rightarrow E_i.$$

In terms of a local echelon decomposition as in (1), we can describe the E_i as follows.

$$(10) \quad \begin{aligned} E_i &= \frac{1}{y_1 \dots y_i} (A_0 \oplus \dots \oplus A_{m-i}) \oplus \frac{1}{y_1 \dots y_{i-1}} A_{m-i+1} \oplus \dots \oplus A_m, \\ E_m &= \frac{1}{y_1 \dots y_m} A_0 \oplus \frac{1}{y_1 \dots y_{m-1}} A_1 \oplus \dots \oplus \frac{1}{y_1} A_{m-1} \oplus A_m \end{aligned}$$

Remark 2.1. Note that via the various exact sequences (e.g. (6), (7)) above, $K(X)$ - group elements, Chern classes and similar attributes of the modifications E_i are computable in terms of similar attributes of the echelon data.

We summarize some of the main properties of elementary modifications as follows. All of them follow directly from the explicit construction and local forms given above. Property (iii), the universal property of echelon modifications, is from our perspective the main *raison d'être* for the construction.

Theorem 2.2. Let (E, χ) be an echelon datum on an integral scheme X .

- (i) $\text{Mod}(\chi, E)$ is a locally free sheaf containing and generically equal to E , and depends functorially on (χ, E) .
- (ii) If $f : Y \rightarrow X$ is a dominant morphism from another integral scheme, then

$$f^* \text{Mod}_X(\chi, E) = \text{Mod}_Y(f^*(\chi), f^*(E)).$$

- (iii) If $\phi : E \rightarrow L$ is a map to a line bundle, such that for each i , $\phi(E^i) \subset L(-D_i)$, then ϕ extends to a map

$$\text{Mod}(\chi, \phi) : \text{Mod}(\chi, E) \rightarrow L.$$

If E' is any locally free sheaf containing and generically equal to E such that ϕ extends to E' , then E' is contained in $\text{Mod}(\chi, E)$.

3. POLYECHELON DATA

In practice, one needs to work with multiple echelon data on a given bundle. This is feasible provided the data satisfy a reasonable condition of *transversality*, a strong version of which follows.

Definition 3.1. Let $(\chi_j = (E_j, \delta_{j,\cdot}, D_{j,\cdot}))$ be a collection of echelon data of respective lengths m_j on a given bundle E . This collection is said to be *mutually transverse* if for any choice of subset $S \subset \{1, \dots, m\}$ and assignment $S \ni j \mapsto i(j)$,

- the sequence of divisors $(D_{j,i(j)} - D_{j,i(j)-1} : j \in S)$ is regular, i.e. its intersection has codimension $|S|$;
- for all $i \notin S$, $(E_i^k \cap \bigcap_{j \in S} E_j^{i(j)}, \delta_{i,k}, D_{i,k} : k = 0, \dots, m_i)$ is an echelon datum on $\bigcap_{j \in S} E_j^{i(j)}$.

A polyechelon datum is a mutually transverse collection of echelon data as above.

Now a key, albeit elementary, observation is the following. If χ, χ' are transverse echelon data, with corresponding filtrations $F \cdot (E), (F') \cdot (E)$, then for each k , there is an echelon datum $(F \cdot (E) \cap (F')^k(E), \delta_{\cdot, k}, D_{\cdot, k})$ and performing the corresponding echelon modifications leads to a new bundle $\text{Mod}(\chi, (F')^k(E))$. Together these bundles form echelon datum

$$\text{Mod}(\chi, \chi') = (\text{Mod}(\chi, (F')^k(E), \delta'_k, D'_k : k = 0, \dots, m').$$

This is an echelon datum on $\text{Mod}(\chi, E)$.

This operation can be iterated: given a transverse collection of echelon data (χ_1, \dots, χ_k) , echelon modifications yield a new transverse collection of echelon data

$$(\chi_{1,2} = \text{Mod}(\chi_1, \chi_2), \dots, \chi_{1,k} = \text{Mod}(\chi_1, \chi_k))$$

on $M_1 = \text{Mod}(\chi_1, E)$, etc. Iterating, we get an increasing chain

$$E = M_0 \subset M_1 \subset \dots \subset M_k$$

and we will call this sequence (or sometimes just its last member) the *poly-echelon modification* of E with respect to the poly-echelon data $(\chi.)$, denoted respectively

$$(M.) = (\text{Mod} .(\chi., E)), M_k = \text{Mod}(\chi., E).$$

The following is elementary

Proposition 3.2. *Notations as above,*

- (i) $(\text{Mod} .(\chi., E))$ is a sequence of locally free sheaves and generic isomorphisms;
- (ii) $\text{Mod}(\chi., E)$ is independent of the ordering or the sequence $(\chi.)$.

Example 3.3. This is the sort of situation we have in mind for echelon data. Let

$$\pi : X \rightarrow B$$

be a family of nodal curves and $\delta \subset B$ a boundary component corresponding to a relative separating node θ . We assume the boundary family X_δ splits globally as

$${}_L X \cup {}_R X,$$

with the two components having local equations x, y , respectively with $xy = t$ a local equation of δ . Let L be a line bundle on X and

$$(n.) = (n_0 = 0 < n_1 < \dots < n_m)$$

be an increasing sequence of integers with the property that for each i ,

$$E^i := \pi_* L(-n_i {}_R X)$$

is locally free and its formation commutes with base-change, as will be the case whenever $R^1 \pi_* L(-n_i {}_R X)$ is locally free or equivalently, $h^1(X_t, L(-n_i {}_R X) \otimes \mathcal{O}_{X_t})$ is independent of t . Let ${}_R D_i = n_i {}_R X$. Then

$${}_R \chi = (E, n, \delta, n, {}_R D)$$

is an echelon datum on X . Likewise for the analogous ${}_L \chi$. Clearly ${}_R \chi$ and ${}_L \chi$ are transverse. We may construct the two echelon modifications of $(E.)$

$${}_R M = \text{Mod}({}_R \chi, E), {}_L M = \text{Mod}({}_L \chi, E)$$

and

$$M_\theta = \text{Mod}(\text{Mod}({}_R \chi, {}_L \chi), {}_L M) = \text{Mod}(\text{Mod}({}_L \chi, {}_R \chi), {}_R M).$$

A fundamental point here, which follows from Theorem 2.2, is that the natural map $E \rightarrow L$ factors through M_θ . Heuristically that is because a section s of E_i yields a section of L vanishing to order n_i on ${}_R X$, hence s/y^{n_i} still yields a regular section of L .

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