MOMENT INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS WITH SPECTRUM IN CURVED HYPERSURFACES

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(0). Summary

In this note we develop further the technique from [B-G], based on the multi-linear restriction theory from [B-C-T], to establish some new inequalities on the distribution of trigonometric polynomials on the n-dimensional torus $\mathbb{T}^n$, $n \geq 2$, of the form

$$f(x) = \sum_{z \in \mathcal{E}} a_z e^{2\pi i x.z} \tag{0.1}$$

where $\mathcal{E}$ stands for the set of $\mathbb{Z}^n$-points on some dilate D.S of a fixed compact, smooth hypersurface S in $\mathbb{R}^n$ with positive definite second fundamental form. More precisely, we prove that for $p \leq \frac{2n}{n-1}$ and any fixed $\varepsilon > 0$, the bound

$$\|f\|_{L^p(\mathbb{T}^n)} \leq C_\varepsilon D^\varepsilon \|f\|_{L^2(\mathbb{T}^n)} \tag{0.2}$$

holds.

In particular, if Δ stands for the Laplacian on $\mathbb{T}^n$ and

$$-\Delta f = Ef \tag{0.3}$$

we have that for $p \leq \frac{2n}{n-1}$, $n \geq 2$

$$\|f\|_{L^p(\mathbb{T}^n)} \ll_\varepsilon E^\varepsilon \|f\|_{L^2(\mathbb{T}^n)}. \tag{0.4}$$

Recall that if $n = 2$, one has the inequality, for f satisfying (0.3),

$$\|f\|_{L^4(\mathbb{T}^2)} \leq C \|f\|_{L^2(\mathbb{T}^2)} \tag{0.5}$$

due to Zygmund and Cook. For $n = 3$, arithmetical considerations permit to obtain a bound

$$\|f\|_{L^4(\mathbb{T}^3)} \ll_\varepsilon E^\varepsilon \|f\|_{L^2(\mathbb{T}^3)} \tag{0.6}$$

For $n \geq 4$, no estimate of the type (0.4) for some $p > 2$ seemed to be known. Recall also that it is *conjectured* that one has uniform bounds

$$\|f\|_{L^q(\mathbb{T}^n)} \leq C_q \|f\|_{L^2(\mathbb{T}^n)} \text{ if } q < \frac{2n}{n-2} \tag{0.7}$$

and

$$\|f\|_{L^q(\mathbb{T}^n)} \leq C_q E^{\frac{1}{2}(\frac{n-2}{2} - \frac{n}{q})} \|f\|_{L^2(\mathbb{T}^n)} \text{ if } q > \frac{2n}{n-2} \tag{0.8}$$

if f satisfies (0.3). The inequality (0.8) was proven in [B1] (using the Hardy-Littlewood circle method) under the assumption

$$q > \frac{2(n+1)}{n-3} \tag{0.9}$$

(up to an E^ε-factor).

Another application of (0.2) relates to the periodic Schrödinger group $e^{it\Delta}$. For $n \geq 1$, one has the Strichartz' type inequality

$$\|(e^{it\Delta} f)(x)\|_{L^q(\mathbb{T}^{n+1})} \ll R^\varepsilon \|f\|_{L^2(\mathbb{T}^n)} \tag{0.10}$$

for $q \leq \frac{2(n+1)}{n}$ and f satisfying $\operatorname{supp} \hat{f} \subset \mathbb{Z}^n \cap B(0, R)$.

Combined with results from [B3], (0.10) implies that for $q > \frac{2(n+3)}{n}$

$$\|(e^{it\Delta} f)(x)\|_{L^q(\mathbb{T}^{n+1})} \leq C_q R^{\frac{n}{2} - \frac{n+2}{q}} \|f\|_{L^2(\mathbb{T}^n)} \tag{0.11}$$

for f as above. Note that inequality (0.11) is optimal. This result is new (and of interest to the theory of the nonlinear Schrödinger equations with periodic boundary conditions) for $n \geq 4$. (See [B3] for more details).

More generally, fix a smooth function $\psi : U \to \mathbb{R}$ on a neighborhood U of $0 \in \mathbb{R}^n$ such that $D^2\psi$ is positive definite. For $q \leq \frac{2(n+1)}{n}$ and $R \to \infty$,

$$\Big[\int_{[0,1]^{n+1}} \Big| \sum_{z \in \mathbb{Z}^n, |z| < R} a_z e^{2\pi i (x.z + R^2 t \psi(\frac{z}{R}))} \Big|^q \, dx dt \Big]^{1/q} \ll R^\varepsilon \Big(\sum |a_z|^2 \Big)^{\frac{1}{2}}. \tag{0.12}$$

Taking $\psi(x) = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2, \alpha_1, \ldots, \alpha_n > 0$, generalizes (0.10) to irrational tori (cf. [B]).

(1). Multilinear Estimates

Fix a smooth, compact hyper-surface S in $\mathbb{R}^n$ with positive definite second fundamental form. For $x \in S$, denote $x' \in S^{(n-1)} = [|x| = 1]$ the normal vector at the point x and let $\sim: S^{(n-1)} \to S$ be the Gauss map. Thus $\tilde{x'} = x$ for $x \in S$. Let σ be the surface measure of S.

The estimates below depend on the multi-linear theory developed in [BCT] to bound oscillatory integral operators. We recall the following version for later use. Let

$$\phi(x, y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \big(\langle Ay, y\rangle + O(|y|^3)\big) \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$ is restricted to a small neighborhood of 0 and A is symmetric and definite (in particular, A is non-degenerate).

Denote

$$Z(x, y) = \partial_{y_1}(\nabla_x \phi) \wedge \cdots \wedge \partial_{y_{n-1}}(\nabla_x \phi). \tag{1.2}$$

Fix $2 \leq k \leq n$ and disjoint balls $U_1, \ldots, U_k \subset \mathbb{R}^{n-1}$ such that the transversality condition holds

$$|Z(x, y^{(1)}) \wedge \cdots \wedge Z(x, y^{(k)})| > c \text{ for all } x \text{ and } y^{(i)} \in U_i. \tag{1.3}$$

Then

$$\Big\| \Big(\prod_{i=1}^k |Tf_i|\Big)^{\frac{1}{k}} \Big\|_{L^q(B_R)} \ll R^\varepsilon \Big(\prod_{i=1}^k \|f_i\|_2\Big)^{\frac{1}{k}} \tag{1.4}$$

with $q = \frac{2k}{k-1}$, provided supp $f_i \subset U_i$.

(2). Preliminary Lemmas

We recall a few estimates from [B-G], §3.

Lemma 1.

Let $U_1, \ldots, U_n \subset S$ be small caps such that $|x_1' \wedge \cdots \wedge x_n'| > c$ for $x_i \in U_i$.

Let M be large and $\mathcal{D}_i \subset U_i (1 \leq i \leq n)$ discrete sets of $\frac{1}{M}$-separated points.

Let $B_M \subset \mathbb{R}^n$ be a ball of radius M. Then, for $q = \frac{2n}{n-1}$

$$\fint_{B_M} \prod_{i=1}^n \Big| \sum_{\xi \in \mathcal{D}_i} a(\xi) e^{ix.\xi} \Big|^{q/n} \ll M^\varepsilon \prod_{i=1}^n \Big[\sum_{\xi \in \mathcal{D}_i} |a(\xi)|^2 \Big]^{\frac{q}{2n}} \tag{2.1}$$

where $\fint$ denotes the average.

Proof.

This is just a discretized version of (2.4) with $k = n$; our assumption ensures the required transversality condition (1.3)

We can assume B_M centered at 0. Introduce functions g_i on U_i defined by

$$\begin{cases} g_i(\zeta) = a(\xi) \text{ if } |\zeta - \xi| < \dfrac{c}{M}, \xi \in \mathcal{D}_i \\ g_i(\zeta) = 0 \text{ otherwise.} \end{cases} \tag{2.2}$$

($c > 0$ a small constant). One may then replace $\sum_{\xi \in \mathcal{D}_i} a(\xi) e^{ix.\xi}$ by $c' M^{n-1} \int_S g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta)$ if $x \in B_M$. Hence

$$\begin{aligned} &\int_{B_M} \prod_{i=1}^n \Bigg| \sum_{\zeta \in \mathcal{D}_i} a(\xi) e^{ix.\xi} \Bigg|^{q/n} dx \lesssim \\ &M^{(n-1)q} \int_{B_M} \prod_{i=1}^n \Big| \int_S g_i(\zeta) e^{ix\zeta} \sigma(d\zeta) \Big|^{q/n} dx \overset{(1.4)}{\ll} \\ &M^{(n-1)q+\varepsilon} \prod_{i=1}^n \|g_i\|_{L^2(U_i)}^{q/n} \sim M^{\frac{n-1}{2}q+\varepsilon} \prod_{i=1}^n \Bigg[\sum_{\xi \in \mathcal{D}_i} |a(\xi)|^2 \Bigg]^{\frac{q}{2n}}. \end{aligned} \tag{2.3}$$

Since $\fint_{B_M}$ refers to the average, (2.1) follows, since $q = \frac{2n}{n-1}$.

Lemma 2.

Let $S \subset \mathbb{R}^n$ be as above and $2 \le m \le n$. Let V be an m-dimensional subspace of $\mathbb{R}^n$, $P_1, \dots, P_m \in S$ such that

$$P_1', \dots, P_m' \in V \text{ and } |P_1 \wedge \cdots \wedge P_m| > c \tag{2.4}$$

and $U_1, \dots, U_m \subset S$ sufficiently small neighborhoods of $P_1, \dots, P_m$.

Let M be large and $\mathcal{D}_i \subset U_i$ $(1 \le i \le m)$ discrete sets of $\frac{1}{M}$-separated points $\xi \in S$ such that $\operatorname{dist}(\xi', V) < \frac{c}{M}$. *Let $g_i \in L^\infty(U_i) (1 \le i \le m)$. Then* *letting* $q = \frac{2m}{m-1}$

$$\int_{B_M} \prod_{i=1}^m \Big| \sum_{\xi \in \mathcal{D}_i} \Big(\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) \Big) \Big|^{q/m} dx \ll \tag{2.5}$$
$$M^\varepsilon \Big\{ \int_{B_M} \prod_{i=1}^m \Big[\sum_{\xi \in \mathcal{D}_i} \Big| \int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) \Big|^2 \Big]^{1/2m} \Big\}^q.$$

Proof.

Performing a rotation, we may assume $V = [e_1, \dots, e_m]$ and denote $\tilde{V} \subset S$ the image of $V \cap S^{(n-1)}$ under the Gauss map. Let again B_M be centered at 0. For each $\xi \in \bigcup_{i=1}^m \mathcal{D}_i$ there is by assumption some $\hat{\xi} \in \tilde{V}$. $|\xi - \hat{\xi}| < \frac{c}{M}$. Write

$$\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta) e^{ix.\zeta} \sigma(d\zeta) = e^{ix\hat{\xi}} \int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta) e^{ix.(\zeta-\hat{\xi})} \sigma(d\zeta). \tag{2.6}$$

Since in the second factor of (2.6), $|\zeta - \hat{\xi}| = o(\frac{1}{M})$, we may view it as constant $a(\xi)$ on $B_M \subset \mathbb{R}^n$.

Thus we need to estimate

$$\int_{B_M} \Big\{ \prod_{i=1}^m \Big| \sum_{\xi \in \mathcal{D}_i} e^{ix.\hat{\xi}} a(\xi) \Big|^{q/m} \Big\} dx. \tag{2.7}$$

Writing $x = (u, v) \in B_M^{(m)} \times B_M^{(n-m)}$, (2.7) may be bounded by

$$\max_{v \in B_M^{(n-m)}} \int_{B_M^{(m)}} \Big\{ \prod_{i=1}^m \Big\} \Big| \sum_{\xi \in \mathcal{D}_i} e^{iu.\pi_m(\hat{\xi})} a_v(\xi) \Big|^{q/m} \Big\} du \tag{2.8}$$

with $a_v(\xi) = e^{iv.\hat{\xi}} a(\xi)$.

Since S has positive definite second fundamental form, $\pi_m(\tilde{V}) \subset V = [e_1, \dots, e_m]$ is a hypersurface in V with same property and the normal vector at $\pi_m(\hat{\xi}) = (\hat{\xi})' \in V$. Since (2.4), application of (2.1) with n replaced by m and $\mathcal{D}_i$ by $\{\pi_m \hat{\xi}; \xi \in \mathcal{D}_i\}$ gives the estimate on (2.7)

$$\ll M^\varepsilon \prod_{i=1}^m \Big[\sum_{\xi \in \mathcal{D}_i} |a(\xi)|^2 \Big]^{q/2m}$$

and (2.5) follows .

Lemma 3. *Let*

$$p = \frac{2n}{n-1}.$$

Take $K_n \gg K_{n-1} \gg \cdots \gg K_1 \gg 1$. For $1 \le j \le n$, denote by $\{U_\alpha^{(j)}\}$ a partition of S in cells of size $\frac{1}{K_j}$. Then, for $R > K_n$ and $g \in L^2(S)$,

$$\Big\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_R)} \ll_\varepsilon$$

$$C(K_n) R^\varepsilon \Big[\int_S |g(\xi)|^2 \sigma(d\xi) \Big]^{\frac{1}{2}} + \sum_{2 \le j \le n} C(K_{j-1}) K_j^\varepsilon \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(j)} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p(B_R)} \Big\}^{\frac{1}{2}}$$

$$+ \Big\{ \sum_\alpha \Big\| \int_{U_\alpha^{(1)}} g(\xi) e^{ix\xi} \sigma(d\xi) \Big\|^2_{L^p(B_R)} \Big\}^{\frac{1}{2}} \tag{2.9}$$

where $C(K)$ denotes some polynomial function of K.

Proof. We follow the analysis from §3 in [B-G].

For $x \in B_R$, let

$$(2.10) = \int_S g(\xi) e^{ix.\xi} \sigma(d\xi)$$

Start decomposing $S = \bigcup_\alpha U_\alpha(\frac{1}{K_n})$ in caps of size $\frac{1}{K_n}$ and write

$$(2.10) = \sum_\alpha \int_{U_\alpha(\frac{1}{K_n})} g(\xi) e^{ix.\xi} \sigma(d\xi) = \sum_\alpha c_\alpha(x).$$

Fixing x, there are 2 possibilities

(2.11) There are $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$|c_{\alpha_1}(x)|, \dots, |c_{\alpha_n}(x)| > K_n^{-(n-1)} \max_\alpha |c_\alpha(x)| \tag{2.12}$$

and

$$|\xi_1 \wedge \cdots \wedge \xi_n| \gtrsim K_n^{-n} \text{ for } \xi_i \in U_{\alpha_i}. \tag{2.13}$$

(2.14) The negation of (2.11), which implies that there is an $(n-1)$-dim subspace V_{n-1} such that

$$|c_\alpha(x)| \le K_n^{-(n-1)} \max_\alpha |c_\alpha(x)| \ \text{ if } \ \operatorname{dist}(U_\alpha, \tilde{V}_{n-1}) \gtrsim \frac{1}{K_n}.$$

If (2.11), it follows from (2.12) that

$$\Big|\int_S g(\xi)e^{ix.\xi}\sigma(d\xi)\Big| \leq K_n^{n-1}\max|c_\alpha(x)| \leq K_n^{2n-2}\Big[\prod_{i=1}^n |c_{\alpha_i}(x)|\Big]^{\frac{1}{n}}$$

and the corresponding contribution to the $L^p_{B_R}$-norm of (4.1) is bounded by

$$\int_{B_R}^{(2.11)}\Big|\int_S g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p$$

$$\lesssim K_n^{2p(n-1)}\sum_{\substack{\alpha_1,\dots,\alpha_n\\(2.13)}}\int_{B_R}\prod_{i=1}^n\Big|\int_{U_{\alpha_i(\frac{1}{K_n})}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{\frac{p}{n}}. \tag{2.15}$$

In view of (2.13), the [BCT]-estimate (1.4) with $k=n$ applies to each (2.15) term. Thus

$$\int_{B_R}\prod_{i=1}^n\Big|\int_{U_{\alpha_i(\frac{1}{K_n})}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{\frac{2}{n-1}}dx \ll C(K_n)R^\varepsilon\Big[\int_S|g(\xi)|^2\sigma(d\xi)\Big]^{\frac{n}{n-1}}. \tag{2.16}$$

Next consider the case (2.14).Thus

$$|(2.10)| \leq \Big|\int_{\text{dist}\,(\xi,\tilde V_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big| + \max_\alpha\Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|$$
$$= (2.17) + (2.18)$$

where V_{n-1} depends on x.

Note however that, from its definition, we may view $|c_\alpha(x)|$ as 'essentially' constant on balls of size K_n. Making this claim rigorous requires some extra work and one replaces $|c_\alpha(x)|$ by a majorant $|c_\alpha| * \eta_{K_n}$, $\eta_K(x) = \frac{1}{K^n}\eta\Big(\frac{x}{K}\Big)$ and η a suitable bump-function. We may then ensure that $|c_\alpha| * \eta_{K_n}$ is approximately constant at scale K_n. But we will not sidetrack the reader with these technicalities that may be found in [B-G], §2.

Thus, upon viewing the $|c_\alpha|$ approximatively constant at scale K_n, the bound (2.17) + (2.18) may clearly be considered valid on $B(\bar x, K_n)$ with the same linear space V_{n-1}.

Obviously

$$(2.18) \leq \Big(\sum_\alpha\Big|\int_{U_\alpha(\frac{1}{k_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)|^p\Big)^{\frac{1}{p}}$$

and the corresponding $L^p_{B_R}$-contribution is bounded by

$$\Big\{\sum_\alpha \Big\|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big\|^2_{L^p_{B_R}}\Big\}^{1/2}. \tag{2.19}$$

Consider the term (2.17). Proceeding similarly, write for $x\in B(\bar x, K_n)$

$$\begin{gathered}\int_{\text{dist}\,(\xi, V_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi)=\\ \sum_\alpha\int_{U_\alpha(\frac{1}{K_{n-1}})\cap[\text{dist}\,(\xi,\tilde V_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi)=\sum_\alpha c^{(n-1)}_\alpha(x).\end{gathered} \tag{2.20}$$

We distinguish the cases

(2.20) There are $\alpha_1,\dots,\alpha_{n-1}$ such that

$$|c^{(n-1)}_{\alpha_1}(x)|,\dots,|c^{(n-1)}_{\alpha_{n-1}}(x)|>K_{n-1}^{-(n-2)}\max_\alpha|c^{(n-1)}_\alpha(x)| \tag{2.21}$$

and

$$|\xi'_1\wedge\dots\wedge\xi'_{n-1}|\gtrsim K_{n-1}^{-(n-1)}\quad\text{for}\quad \xi_i\in U_{\alpha_i}\Big(\frac{1}{K_{n-1}}\Big). \tag{2.22}$$

(2.23) Negation of (2.20), implying that there is an $(n-2)$-dim subspace $V_{n-2}\subset V_{n-1}$ (depending on x) such that

$$|c^{(n-1)}_\alpha(x)|<K_{n-1}^{-(n-2)}\max_\alpha|c^{(n-1)}_\alpha(x)|\text{ for dist}\,(U_\alpha,\tilde V_{n-2})\gtrsim\frac{1}{K_{n-1}}.$$

This space V_{n-2} can then again be taken the same on a K_{n-1}-neighborhood of x.

We analyze the contribution of (2.20). By (2.21)

$$|(2.19)|<K_{n-1}^{2n-4}\Big[\prod_{i=1}^{n-1}|c^{(n-1)}_{\alpha_i}(x)|\Big]^{\frac{1}{n-1}} \tag{2.24}$$

and hence

$$\begin{gathered}\fint_{\substack{B(\bar x,K_n)\\ x\text{ satisfies (2.20)}}}\Big|\int_{\text{dist}\,(\xi,\tilde V_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p\le\\ K_{n-1}^{p(2n-4)}\sum_{\substack{\alpha_1,\dots,\alpha_{n-1}\\ (2.22)}}\fint_{B(\bar x,K_n)}\Big\{\prod_{i=1}^{n-1}\Big|\int_{U_{\alpha_i}(\frac{1}{K_{n-1}})\cap[\text{dist}\,(\xi,\tilde V_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{p/n-1}\Big\}.\end{gathered} \tag{2.25}$$

We use the bound (2.5) to estimate the individual integrals

$$(2.26)\quad \int\!\!\!\!\!\!-_{B(\bar{x},K_n)} \Big\{\prod_{i=1}^{n-1}\Big|\int_{U_{\alpha_i}(\frac{1}{K_{n-1}})\cap[\text{dist}\,(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|\Big\}^{\frac{q}{n-1}} \text{ with } q=\frac{2(n-1)}{n-2}.$$

Thus $m=n-1, V=V_{n-1}$ and P_i is the center of $U_{\alpha_i}(\frac{1}{K_{n-1}})$. Let $M=K_n$ and $\mathcal{D}_i$ the centers of a cover of $U_{\alpha_i}(\frac{1}{K_{n-1})}\cap[\text{dist}\,(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_N}]$ by caps $U_\alpha(\frac{1}{K_n})$.

By (2.5) we get an estimate

$$(2.26)\ll K_n^\varepsilon C(K_{n-1})\Big\{\int\!\!\!\!\!\!-_{B(\bar{x},K_n)}\prod_{i=1}^{n-1}\Big[\sum_\alpha^{(i)}\Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2(n-1)}}\Big\}^q \tag{2.27}$$

where in $\sum^{(i)}$ the sum is over those α such that $U_\alpha(\frac{1}{K_n})\subset U_{\alpha_i}(\frac{1}{K_{n-1}})$ and $U_\alpha(\frac{1}{K_n})\cap\tilde{V}_{n-1}\neq\phi$. Hence, we certainly have

$$(2.26)\ll K_n^\varepsilon\ C(K_{n-1})\Big\{\int\!\!\!\!\!\!-_{B(\bar{x},K_n)}\Big[\sum_\alpha\Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2}}\Big\}^q$$

and therefore, since $p<q$,

$$(2.25)\ll K_n^\varepsilon C(K_{n-1})\Big\{\int\!\!\!\!\!\!-_{B(\bar{x},K_n)}\Big[\sum_\alpha\Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{p/2}\Big\}. \tag{2.28}$$

Hence the collected contribution over B_R of (2.28) is bounded by

$$K_n^\varepsilon C(K_{n-1})\Big\{\sum_\alpha\Big\|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big\|^2_{L^p(B_R)}\Big\}^{1/2}. \tag{2.29}$$

Next, we analyze the contribution of (2.23) which is similar to that of (2.14) with $n-1$ replaced by $n-2$ and K_n by K_{n-1}. The local estimate (2.27) becomes

$$K_{n-1}^\varepsilon C(K_{n-2})\Big\{\int\!\!\!\!\!\!-_{B(\bar{x},K_{n-1})}\prod_{i=1}^{n-2}\Big[\sum_\alpha^{(i)}\Big|\int_{U_\alpha(\frac{1}{K_{n-1}})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2(n-2)}}\Big\}^q \tag{3.30}$$

with $q=\frac{2(n-2)}{n-3}$ and where in $\sum^{(i)}$ the sum is over those α such that

$$U_\alpha\Big(\frac{1}{K_{n-1}}\Big)\subset U_{\alpha_i}\Big(\frac{1}{K_{n-2}}\Big) \text{ and } U_\alpha\Big(\frac{1}{K_{n-1}}\Big)\cap\tilde{V}_{n-2}\neq\phi.$$

The collected contribution of (2.30) to the $L^p_{B_R}$-norm of (2.10) is bounded by

$$K^\varepsilon_{n-1} C(K_{n-2}) \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{1}{K_{n-1}})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{(B_R)}} \Big\}^{\frac{1}{2}}. \tag{3.31}$$

The continuation of the process is now clear and leads to the bound (2.9). This proves Lemma 3.

Taking $K_j > K_{j-1}^{C/\varepsilon}$ in Lemma 3, we obtain

Lemma 4. *Fix $\varepsilon > 0$. Let $K_1 \gg 1$ be large enough and assume $R > K_1^{C(\varepsilon)}$. Then, with $p = \frac{2n}{n-1}$*

$$\Big\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{B_R}} \le R^\varepsilon \Big[\int_S |g(\xi)|^2 \sigma(d\xi) \Big]^{\frac{1}{2}} + \max_{K_1 < K < K_1^{C(\varepsilon)}} \Big\{ K^\varepsilon \sum_\alpha \Big\| \int_{U_\alpha(\frac{1}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{B_R}} \Big\}^{1/2} \tag{2.32}$$

with $\{U_\alpha(\frac{1}{K})\}$ a cover of S by $\frac{1}{K}$-size caps.

The first term on the right side of (2.32) may be eliminated.

Observe first that since $|x| < R$, the left side may be replaced by

$$\Big\| \int G(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{B_R}} \tag{2.33}$$

where G is a smoothing of g at scale $\frac{1}{R}$.

Applying (2.32) with g replaced by G, the first term on the right

$$\Big[\int_S |G(\xi)|^2 \sigma(d\xi) \Big]^{\frac{1}{2}} \lesssim \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{c}{R})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{B_R}} \Big\}^{\frac{1}{2}} \tag{2.34}$$

and the other terms may be majorized by

$$\Big\| \int_{U_\alpha(\frac{1}{K})} G(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{B_R}} \lesssim \Big\| \int_{U_\alpha(\frac{1}{K})} g_1(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{B_R}} \tag{2.35}$$

for some $g_1 = \eta g$ with η a smooth function.

Hence we obtain

Lemma 5. *Fix $\varepsilon > 0$. Let $K_1 \gg 1$ be large enough and assume $R > K_1^{C(\varepsilon)}$. Then, with $p = \frac{2n}{n-1}$, we have*

$$\Big\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{B_R}} < R^\varepsilon \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{c}{R})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{(R)}} \Big\}^{\frac12} + \max_{K_1 < K < K_1^{C(\varepsilon)}} \Big\{ K^\varepsilon \sum_\alpha \Big\| \int_{U_\alpha(\frac{1}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{(R)}} \Big\}^{\frac12} \tag{2.36}$$

where $L^p_{(R)} = L^p\big(\omega(\frac{x}{R})dx\big)$ with $0 < \omega < 1$ some rapidly decaying function on $\mathbb{R}^n$.

In order to iterate (2.36), we rely on rescaling.

Parametrize S (locally, after affine coordinate change) as

$$\begin{cases} \xi_i = y_i (1 \le i \le n-1) \\ \xi_n = y_1^2 + \cdots + y_{n-1}^2 + O(|y|^3) \end{cases} \tag{2.37}$$

with y taken in a small neighborhood of 0.

Let $U(\rho)$ be a ρ-cap on S and evaluate

$$\Big\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_R)}. \tag{2.38}$$

Thus in view of (2.37), (2.38) amounts to

$$\Big\| \int_{B(a,\rho)} g(y) e^{i\varphi(x,y)} dy \Big\|_{L^p(B_R)} \tag{2.39}$$

with

$$\varphi(x,y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \big(|y|^2 + O(|y|^3)\big) \tag{2.40}$$

and $B(a,\rho) \subset \mathbb{R}^{n-1}$.

A shift $y \mapsto y - a$ and change of variables $x_i' = x_i + x_n(2a_i + \cdots)$ $(1 \le i < n)$ permits to set $a = 0$. By parabolic rescaling

$$y = \rho y' \text{ and } \rho x_i = x_i' (1 \le i < n), \rho^2 x_n = x_n' \tag{2.41}$$

we obtain a new phase function $\psi(x', y')$ and (2.39) becomes

$$\rho^{n-1-\frac{n+1}{p}} \Big\| \int_{B(0,1)} g(a+\rho y') e^{i\psi(x',y')} dy' \Big\|_{L^p(\Omega)} \tag{2.42}$$

where $\Omega = [|x_i'| < \rho R (1 \leq i < n), |x_n'| < \rho^2 R]$.

Partition $\Omega = \bigcup \Omega_s$ in size-$\rho^2 R$ balls Ω_s and apply Lemma 5 on each Ω_s with R replaced by $\rho^2 R$. Assuming

$$R > \rho^{-2} K_1^{C(\varepsilon)} \tag{2.43}$$

(2.36) implies that

$$\Big\| \int_{B(0,1)} g(a+\rho y') e^{i\psi(x',y')} dy' \Big\|_{L^p(\Omega_s)} <$$

$$(\rho^2 R)^\varepsilon \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{c}{\rho^2 R})} g(a+\rho y') e^{i\psi(x',y')} dy' \Big\|^2_{L^p(\omega(\frac{x'-b_s}{\rho^2 R}) dx')} \Big\}^{\frac12} +$$

$$\max_{K_1 < K < K_1^{C(\varepsilon)}} K^\varepsilon \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{1}{K})} g(a+\rho y') e^{i\psi(x',y')} dy' \Big\|^2_{L^p(\omega(\frac{x'-b_s}{\rho^2 R}) dx')} \Big\}^{\frac12} \tag{2.44}$$

with b_s the center of Ω_s.

Note that certainly

$$\sum_s \omega\Big(\frac{x'-b_s}{\rho^2 R}\Big) < \omega_1\Big(\frac{x}{R}\Big).$$

Summing $(2.44)^p$ over s and reversing the coordinate changes clearly implies that

$$(2.39), (2.42) <$$

$$(\rho^2 R)^\varepsilon \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\frac{c}{\rho R})} g(y) e^{i\varphi(x,y)} dy \Big\|^2_{L^p_{(R)}} \Big\}^{\frac12} +$$

$$\max_{K_1 < K < K_1^{C(\varepsilon)}} \Big\{ K^\varepsilon \sum_\alpha \Big\| \int_{U_\alpha(\frac{\rho}{K})} g(y) e^{i\varphi(x,y)} dy \Big\|^2_{L^p_{(R)}} \Big\}^{\frac12} \tag{2.45}$$

under the assumption (2.43).

Taking $R = \rho^{-2} K_2$ with $K_2 > K_1^{C(\varepsilon)}$ in (2.45), we obtain

Lemma 6. *Let $K_2 > K_1^{C(\varepsilon)}$. Then*

$$\Big\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_{K_2\rho^{-2}})}$$
$$\ll_\varepsilon \max_{K_1 < K < K_2} \Big\{ K^\varepsilon \sum_\alpha \Big\| \int_{U_\alpha(\frac{c\rho}{K})} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{(K_2\rho^{-2})} \Big\}^{\frac{1}{2}}. \tag{2.46}$$

If $R > K_2\rho^{-2}$, we can partition B_R in cubes of size $K_2\rho^{-2}$ and apply (2.46) on each of them, with $g(\xi)$ replaced by $g(\xi)\ e^{ia.\xi}$ for some $a \in B_R$. Hence

Lemma 6′. *Let $R > K_2\rho^{-2}, K_2 = K_1^{C(\varepsilon)}$. Then*

$$\Big\| \int_{U(\rho)} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_R)}$$
$$\ll_\varepsilon \max_{K_1 < K < K_2} \Big\{ K^\varepsilon \sum_\alpha \Big\| \int_{U_\alpha(\frac{c\rho}{K})} g(\xi) e^{ix.\xi)} \sigma(d\xi) \Big\|^2_{L^p_{(R)}} \Big\}^{\frac{1}{2}}. \tag{2.47}$$

It is now straightforward to iterate Lemma 6′ and derive the following statement

Proposition 1. *Let $0 < \delta \ll 1$ and $R > C(\varepsilon)\delta^{-2}$. Then, with $p = \frac{2n}{n-1}$*

$$\Big\| \int g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p_{(R)}} \ll_\varepsilon \delta^{-\varepsilon} \Big\{ \sum_\alpha \Big\| \int_{U_\alpha(\delta)} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|^2_{L^p_{(R)}} \Big\}^{\frac{1}{2}}. \tag{2.48}$$

(3). L^p-bounds for certain exponential polynomials and applications

We fix a smooth compact hyper-surface S in $\mathbb{R}^n$ with positive definite second fundamental form. We consider exponential polynomials with frequencies on some dilate $D.S$ of S.

Proposition 2. *Let $0 < \rho < D$ and let $\mathcal{E}$ be a discrete set of points on the dilate $D.S$ that are mutually at least ρ separated. Then, for $p = \frac{2n}{n-1}$ and any (fixed) $\varepsilon > 0$*

$$\Big[\int_{B_R} \Big| \sum_{z \in \mathcal{E}} a_z e^{ix.z)} \Big|^p dx \Big]^{\frac{1}{p}} \ll_\varepsilon \Big(\frac{D}{\rho}\Big)^\varepsilon \Big(\sum_{z \in \mathcal{E}} |a_z|^2 \Big)^{\frac{1}{2}} \tag{3.1}$$

provided

$$R > C(\varepsilon) D \rho^{-2}. \tag{3.2}$$

Proof.

By rescaling, we may clearly assume $D = 1$.

Let $0 < \tau < \rho/10$ and let g be the function on S defined by

$$\begin{aligned} g(\xi) &= \frac{a_z}{\sigma(U(z,\tau))} \text{ if } \xi \in U(z,\tau) \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{3.3}$$

Here $U(z,\tau) \subset S$ denotes a τ-neighborhood of z on S. Thus

$$\int g(\xi) e^{ix.\xi} \sigma(d\xi) = \sum_{z\in\mathcal{E}} a_z \int_{U(z,\tau)} e^{ix.\xi} \sigma(d\xi). \tag{3.4}$$

Applying (2.48) with $\delta = \rho$, it follows from (3.3), (3.4) that

$$\Big\{ \int_{B_R} \Big| \sum_{z\in\mathcal{E}} a_z \int_{U(z,\tau)} e^{ix.\xi} \sigma(d\xi) \Big|^p dx \Big\}^{\frac{1}{p}} \ll_\varepsilon \rho^{-\varepsilon} \Big(\sum_z |a_z|^2 \Big)^{1/2} \tag{3.5}$$

letting $\tau \to 0$, (3.1) clearly follows.

Next, observe that if $\mathcal{E}$ is contained in a lattice, then $\sum_{z\in\mathcal{E}} a_z e^{ix.\xi}$ is a periodic function. Hence Proposition 2 implies

Proposition 3. *Let S be as above and $\mathcal{E} = \mathbb{Z}^n \cap DS$, $D \to \infty$.*

Then, with $p = \frac{2n}{n-1}$

$$\Big[\int_{\mathbb{T}^n} \Big| \sum_{z\in\mathcal{E}} a_z e^{2\pi i x.z} \Big|^p dx \Big]^{\frac{1}{p}} \ll_\varepsilon D^\varepsilon \Big(\sum |a_z|^2 \Big)^{1/2} \tag{3.6}$$

where $\mathbb{T}^n$ stands for the n-dimensional torus.

Corollary 4. *Let $\varphi = \varphi_E, -\Delta\varphi_E = E\varphi_E$ be an eigenfunction of $\mathbb{T}^n$, $n \geq 2$. Then for $p = \frac{2n}{n-1}$ and any $\varepsilon > 0$, we have*

$$\|\varphi\|_{L^p(\mathbb{T}^n)} \leq C(\varepsilon) E^\varepsilon \|\varphi\|_{L^2(\mathbb{T}^n)}. \tag{3.7}$$

Remark. Corollary 4 should be compared with the result from [B1]. It is conjectured that for eigenfunctions of $\mathbb{T}^n, n \geq 2$, there is a uniform bound

$$\|\varphi\|_p \leq C(p)\|\varphi\|_2 \text{ for } p < \frac{2n}{n-2}. \tag{3.8}$$

If $n = 2$, (3.8) is known to hold for $p \leq 4$ (due to Zygmund-Cook) but for no exponent $p > 4$.

If $n = 3$, (3.7) is valid for $p \leq 4$. This is a consequence of the following observation. One clearly has the estimate

$$\|\varphi\|_4 \leq K^{1/4}\|\varphi\|_2$$

denoting

$$K = \max_{\xi \in Z^3} \big(\#\{(\xi_1, \xi_2) \in \mathbb{Z}^3 \times \mathbb{Z}^3; |\xi_1|^2 = E = |\xi_2|^2 \text{ and } \xi_1 + \xi_2 = \xi\}\big).$$

Projecting on one of the coordinate planes reduces the issue to bounding the number $|\mathcal{E} \cap \mathbb{Z}^2|$ with $\mathcal{E} \subset \mathbb{R}^2$ some ellipse of size at most $E^{1/2}$. It is well known that

$$|\mathcal{E} \cap \mathbb{Z}^2| \ll E^\varepsilon \tag{3.9}$$

(cf. [B-R]) and hence $K \ll E^\varepsilon$.

For $n \geq 4$, no estimates of the type (3.7) for some $p > 2$, seemed to be previously known. Recall that for $n \geq 4$ and R a large positive integer

$$|RS^{(n-1)} \cap \mathbb{Z}^n| \sim R^{n-2}. \tag{3.10}$$

Thus Corollary 4 provides for any $p = \frac{2n}{n-1}$ an *explicit* construction of an 'almost' Λ_p-set which is not a Λ_q-set for $q \geq \frac{2n}{n-2}$. No explicit constructions of proper Λ_p-sets for $2 < p < 4$ seem to be known and their existence results from probabilistic arguments (see [B2], [B4]).

In view of (3.10), Corollary 4 also provides explicit almost Euclidean subspaces of dimension $\sim N^{\frac{4}{p}-1}$ in ℓ^p_N, for p of the form $\frac{2n}{n-1}, n \geq 4$ (while their maximal dimension is $\sim N^{\frac{2}{p}}$ for $2 < p < \infty$). To be compared with the result from [G-L-R] on explicit almost Euclidean subspaces of ℓ^1_n.

Returning to Proposition 3, we have more generally

Proposition 3$'$. *Let S be as in Proposition 3 and $T \in GL_n(\mathbb{R}), \|T\| > 1$, an arbitrary invertible linear transformation. Let $\mathcal{E} = \mathbb{Z}^n \cap T(S)$. Then, letting $p = \frac{2n}{n-1}$, we have the inequality*

$$\Big[\int_{\mathbb{T}^n} \Big|\sum_{z \in \mathcal{E}} a_z e^{2\pi i x.z}\Big|^p dx\Big]^{\frac{1}{p}} \ll \|T\|^\varepsilon \Big(\sum_{x \in \mathcal{E}} |a_z|^2\Big)^{\frac{1}{2}}. \tag{3.11}$$

Proof. Consider the set

$$\mathcal{E}' = \{T^{-1}z; z \in \mathcal{E}\} \subset S$$

which elements are at least $\frac{1}{\|T\|}$-separated. Applying Proposition 2 with $D = 1$ and $\rho = \frac{1}{\|T\|}$, we obtain

$$\lim_{R\to\infty} \Big| \int\!\!\!\!\!\!-_{B_R} \Big| \sum_{z\in\mathcal{E}} a_z e^{2\pi i x'.T^{-1}z} \Big|^p dx' \Big]^{\frac{1}{p}} \ll \|T\|^\varepsilon \Big(\sum_{z\in\mathcal{E}} |a_z|^2 \Big)^{\frac{1}{2}}. \tag{3.12}$$

By change of variables $x = (T^{-1})^* x'$, it follows that

$$\lim_{R\to\infty} \Big[\int\!\!\!\!\!\!-_{(T^{-1})^*(B_R)} \Big| \sum_{z\in\mathcal{E}} a_z e^{2\pi i x.z} \Big|^p dx \Big]^{\frac{1}{p}} \ll \|T\|^\varepsilon \Big(\sum_{z\in\mathcal{E}} |a_z|^2 \Big)^{\frac{1}{2}} \tag{3.13}$$

which, by periodicity, is equivalent to (3.11).

Take $S = \{(y, |y|^2); y \in \mathbb{R}^n, |y| < 1\}$ the truncated paraboloid in $\mathbb{R}^{n+1}$ and let $T(x,t) = (Rx, R^2 t), R > 1$. From Proposition 3′, we immediately derive the following Strichatz' type inequality for the periodic Schrödinger group $e^{it\Delta}$.

Corollary 5. *Denote Δ the Laplacian on $\mathbb{T}^n$. Then, for $p = \frac{2(n+1)}{n}$, we have the inequality*

$$\|e^{it\Delta} f\|_{L^p(\mathbb{T}^n\times\mathbb{T})} \ll R^\varepsilon \|f\|_{L^2(\mathbb{T})} \tag{3.14}$$

assuming supp $\hat{f} \subset B(0, R)$.

This bound should be compared with the following result established in [B3].

Proposition 6. *Let $f \in L^2(\mathbb{T}^n)$, $\|f\|_2 = 1$ and such that supp $\hat{f} \subset B(0, R)$. Then, for $\lambda > R^{\frac{n}{4}}$ and $q > \frac{2(n+2)}{n}$, the following inequality holds*

$$\text{mes}\,[(x,t) \in \mathbb{T}^{n+1}; |e^{it\Delta} f|(x) > \lambda] < C_q R^{\frac{n}{2}q-(n+2)} \lambda^{-q}. \tag{3.15}$$

Combining Corollary 5, Proposition 6, we obtain the following improvement over Proposition 3.110 in [B3].

Corollary 7. *Let $n \geq 4$ (for $n < 4$, better result may be obtained by arithmetical means, cf. [B3]).*

Let f be as in Proposition 6. Then, for $q > \frac{2(n+3)}{n}$

$$\|e^{it\Delta} f\|_{L^q(\mathbb{T}^{n+1})} < C_q R^{\frac{n}{2} - \frac{n+2}{q}} \tag{3.16}$$

holds.

Note that (3.16) is optimal.

Proof.

Denote $q_0 = \frac{2(n+1)}{n}$ and q_1 some exponent $> \frac{2(n+2)}{n}$. Let $F(x,t) = (e^{it\Delta} f)(x)$ and estimate for $q > q_1$

$$\begin{aligned}
\int_{\mathbb{T}^{n+1}} |F|^q &\leq \int_{|F| > R^{\frac{n}{4}}} |F|^q + R^{\frac{n}{4}(q-q_0)} \int |F|^{q_0} \\
&< C_{q_1} R^{\frac{n}{2} q_1 - (n+2)} \int_{R^{\frac{n}{4}}}^{R^{\frac{n}{2}}} \lambda^{q-1-q_1} d\lambda + C_\varepsilon R^{\frac{n}{4}(q-q_0)+\varepsilon} \\
& C_{q_1} \frac{1}{q-q_1} R^{\frac{n}{2} q - (n+2)} + C_\varepsilon R^{\frac{n}{4}(q-q_0)+\varepsilon} < C_q R^{\frac{n}{2} q - (n+2)}
\end{aligned}$$

for q as above.

Corollary 5 admits a generalization that we discuss next. Assume $\psi : \cup \to \mathbb{R}, U \subset \mathbb{R}^n$ a neighborhood of 0, is a smooth function such that $D^2\psi$ is positive (or negative) definite. Then one has

Proposition 8. *Let $p = \frac{2(n+1)}{n}$ and $N \to \infty$. Then for all $\varepsilon > 0$,*

$$\begin{aligned}
&\Big[\int_{[0,1]^{n+1}} \Big| \textstyle\sum_{z \in \mathbb{Z}^n, \frac{z}{N} \in U} a_z e^{2\pi i (x.z + N^2 t \psi(\frac{z}{N}))} \Big|^p dx dt \Big]^{\frac{1}{p}} \ll \\
&N^\varepsilon \Big(\sum |a_z|^2 \Big)^{1/2}.
\end{aligned} \tag{3.17}$$

Note that a coordinate change $x \mapsto x + Nt\nabla\psi(0)$ permits to assume $\psi(0) = \nabla\psi(0) = 0$. Let $S = [(x, \psi(x), x \in U]$ and

$$\mathcal{E} = \Big\{ \Big(\frac{z}{N}, \psi\Big(\frac{z}{N}\Big) \Big); z \in \mathbb{Z}^n, \frac{z}{N} \in U \Big\} \subset S.$$

Application of Proposition 2 with $\rho \sim \frac{1}{N}$ implies that

$$\Big[\int_{[0,1]^{n+1}} \Big| \sum_{z\in\mathbb{Z}^n, \frac{z}{N}\in U} a_z e^{2\pi i(Nz.x+N^2\psi(\frac{z}{N})t)}\Big|^p dxdt\Big]^{\frac{1}{p}} \ll \\ N^\varepsilon \Big(\sum |a_z|^2\Big)^{1/2} \tag{3.18}$$

and (3.17) follows by exploiting periodicity in x. This proves Proposition 8.

Finally, observe that by taking $\psi(x) = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2$ with $\alpha_1, \dots, \alpha_n > 0$, Corollary 5 generalizes to a Strichartz inequality for irrational tori, as considered in [B]. Applications to nonlinear Schrödinger type equations will not be discussed in this paper.

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