RADIUS OF CLOSE-TO-CONVEXITY OF HARMONIC FUNCTIONS

DAVID KALAJ, SAMINATHAN PONNUSAMY, AND MATTI VUORINEN

Abstract. Let $\mathcal{H}$ denote the class of all normalized complex-valued harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D}$, and let $K = H + \overline{G}$ denote the harmonic Koebe function. Let a_n, b_n, A_n, B_n denote the Maclaurin coefficients of h, g, H, G, and

$$\mathcal{F} = \{f = h + \overline{g} \in \mathcal{H} : \, |a_n| \le A_n \text{ and } |b_n| \le B_n \text{ for } n \ge 1\}.$$

We show that the radius of univalence of the family $\mathcal{F}$ is $0.112903\ldots$. We also show that this number is also the radius of the starlikeness of $\mathcal{F}$. Analogous results are proved for a subclass of the class of harmonic convex functions in $\mathcal{H}$. These results are obtained as a consequence of a new coefficient inequality for certain class of harmonic close-to-convex functions. Surprisingly, the new coefficient condition helps to improve Bloch-Landau constant for bounded harmonic mappings.

1. Introduction and Main Results

Denote by $\mathcal{H}$ the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \, |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. Each f can be decomposed as $f = h + \overline{g}$, where g and h are analytic in $\mathbb{D}$ so that [6, 8]

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Let $\mathcal{S}_H$ denote the class of univalent and orientation-preserving functions $f = h + \overline{g}$ in $\mathcal{H}$. Then the Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. We note that if $f = h + \overline{g} \in \mathcal{S}_H$ and $g(z) \equiv 0$ in $\mathbb{D}$, then $f = h \in \mathcal{S}$, where $\mathcal{S}$ denotes the well-known class of normalized univalent analytic functions in $\mathbb{D}$. A necessary and sufficient condition (see [6] or Lewy [10]) for a harmonic function f to be locally univalent in $\mathbb{D}$ is that $J_f(z) > 0$ in $\mathbb{D}$. The function $\omega(z) = g'(z)/h'(z)$ denotes the complex dilatation of f. Thus, for $f = h + \overline{g} \in \mathcal{S}_H$ with $g'(0) = b_1$ and $|b_1| < 1$ (because $J_f(0) = 1 - |b_1|^2 > 0$), the function

$$F = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

is also in $\mathcal{S}_H$. Thus, it is customary to restrict our attention to the subclass

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : \, f_{\overline{z}}(0) = 0\}.$$

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The family $\mathcal{S}_H^0$ is known to be compact. The uniqueness result of the Riemann mapping theorem does not extend to these classes of harmonic functions, [6, 8]. Several authors have studied the subclass of functions that map $\mathbb{D}$ onto specific domains, eg. starlike domains, convex and close-to-convex domains. Let $\mathcal{S}_H^*$ ($\mathcal{K}_H$, $\mathcal{C}_H$ resp.) consist of all sense-preserving harmonic mappings $f = h+\overline{g} \in \mathcal{H}$ of $\mathbb{D}$ onto starlike (convex, close-to-convex, resp.) domains. Denote by $\mathcal{S}_H^{*0}$ ($\mathcal{K}_H^0$, $\mathcal{C}_H^0$ resp.) the class consists of those functions f in $\mathcal{S}_H^*$ ($\mathcal{K}_H$, $\mathcal{C}_H$ resp.) for which $f_{\overline{z}}(0) = 0$.

In [6, Lemma 5.15], Clunie and Sheil-Small proved the following result.

Lemma A. *If h, g are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each ϵ, $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex in $\mathbb{D}$.*

This lemma has been used to obtain many important results. In the case of $\mathcal{S}_H^0$, we have the harmonic Koebe function $K = H + \overline{G}$ in $\mathcal{S}_H^0$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \ \text{ and } \ G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{1.2}$$

We see that the function K has the dilatation $\omega(z) = z$ and K maps the unit disk $\mathbb{D}$ onto the slit plane $\mathbb{C}\backslash\{u + iv : \ u \leq -1/6, \ v = 0\}$. Moreover,

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \ \text{ and } \ G(z) = \sum_{n=2}^{\infty} B_n z^n,$$

where

$$A_n = \frac{1}{6}(2n+1)(n+1) \ \text{ and } \ B_n = \frac{1}{6}(2n-1)(n-1), \quad n \geq 1. \tag{1.3}$$

A well-known coefficient conjecture of Clunie and Sheil-Small [6], is that if $f = h+\overline{g} \in \mathcal{S}_H^0$ then the Taylor coefficients of the series of h and g satisfy the inequality

$$|a_n| \leq A_n \ \text{ and } \ |b_n| \leq B_n \text{ for all } n \geq 1. \tag{1.4}$$

Although, the coefficients conjecture remains an open problem for the full class $\mathcal{S}_H^0$, the same has been verified for certain subclasses, namely, the class $\mathcal{T}_H$ (see [8, Section 6.6]) of harmonic univalent typically real functions, the class of harmonic convex functions in one direction, harmonic starlike functions in $\mathcal{S}_H^0$ (see [8, Section 6.7]), and the class of harmonic close-to-convex functions (see [17]).

It is interesting to know to what extent do the conditions (1.4) influence the univalency of the normalized harmonic function $f(z)$ and of all of its partial sums, namely, $f_n(z)$ and $f_{\overline{m}}(z)$, where

$$f_n(z) = h_n(z) + \overline{g_m(z)} \text{ if } n \geq m; \ f_{\overline{m}}(z) = h_n(z) + \overline{g_m(z)} \text{ if } m \geq n.$$

Here $h_n(z)$ and $g_m(z)$ represent the n-th section/partial sums of h and g given by

$$h_n(z) = z + \sum_{k=2}^{n} a_k z^k \ \text{ and } \ g_m(z) = \sum_{k=1}^{m} b_k z^k,$$

respectively. According to our notation, the degree of the polynomials $f_n(z)$ and $f_{\overline{m}}(z)$ is n if $n = m$.

Theorem 1.5. *Let h and g have the form* (1.1) *and the coefficients of the series satisfy the conditions* (1.4). *Then $f = h + \overline{g}$ is close-to-convex* (*univalent*), *and starlike in the disk $|z| < r_S$, where*

$$r_S = 1 + \frac{\sqrt{2}}{4} - \sqrt{\sqrt{2} + \frac{1}{8}} \approx 0.112903$$

is the root of the quadratic equation

$$\sqrt{2}\, r^2 - (1 + 2\sqrt{2})\, r + \sqrt{2} - 1 = 0$$

in the interval $(0, 1)$. *The result is sharp.*

The radii problems for various subclasses of univalent harmonic mappings are open [2, Problem 3.3] (see also [6, 8, 15, 14]). However, Theorem 1.5 quickly yields

Corollary 1.6. *The radius of close-to-convexity and the radius of starlikeness for mappings in $\mathcal{S}_H^{*0}$ (resp. $\mathcal{C}_H^0$ and $\mathcal{T}_H$) is at least* 0.112903.

Under the hypotheses of Theorem 1.5, all the partial sums of f are close-to-convex (univalent), and starlike in $|z| < r_S$. Similar comments apply to the next two results.

Another well-known result due to Clunie and Sheil-Small [6] states that the coefficients of the series of h and g of every convex function $f = h + \overline{g} \in \mathcal{K}_H^0$ satisfy the inequalities

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2} \quad \text{for all } n \geq 1. \tag{1.7}$$

Equality occurs for the function $L = M + \overline{N} \in \mathcal{K}_H^0$, where

$$M(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) \quad \text{and} \quad N(z) = \frac{1}{2}\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right). \tag{1.8}$$

We observe that

$$L(z) = \mathrm{Re}\left(\frac{z}{1-z}\right) + \mathrm{Im}\left(\frac{z}{(1-z)^2}\right) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n - \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} z^n}.$$

At this place it is worth recalling that the convexity (resp. starlikness) property is not a hereditary property in the harmonic case, unlike the analytic case. For instance, the convex function L maps the subdisk $|z| < r$ onto a convex domain for $r \leq \sqrt{2} - 1$, but onto a non-convex domain for $\sqrt{2} - 1 < r < 1$.

Theorem 1.9. *Let h and g have the form* (1.1) *and the coefficients of the series satisfy the conditions* (1.7). *Then $f = h + \overline{g}$ is close-to-convex* (*univalent*), *and starlike in the disk $|z| < r_S$, where*

$$r_S = 1 + \frac{\sqrt[3]{-18 + \sqrt{330}}}{6^{2/3}} - \frac{1}{\sqrt[3]{6(-18 + \sqrt{330})}} \approx 0.164878$$

is the real root of the cubic equation

$$2r^3 - 6r^2 + 7r - 1 = 0$$

in the interval $(0, 1)$. *The result is sharp.*

Theorem 1.9 easily gives the following corollary although Theorem 1.9 is much more stronger.

Corollary 1.10. *The radius of close-to-convexity and the radius of starlikeness for convex mappings in* $\mathcal{S}_H^0$ *is at least* 0.164878.

Theorem 1.11. *Let* h *and* g *have the form* (1.1) *with* $|b_1| = |g'(0)| < 1$, *and the coefficients of the series satisfy the conditions*

$$|a_n| + |b_n| \le c \quad \text{for all } n \ge 2.$$

Then $f = h + \overline{g}$ *is close-to-convex* (*univalent*)*, and starlike in the disk* $|z| < r_S$, *where*

$$r_S = 1 - \sqrt{\frac{c}{c + 1 - |b_1|}}.$$

The result is sharp.

Theorem 1.11 helps to improve the Bloch-Landau's theorem for bounded harmonic functions. Consider the class $\mathcal{B}_H^M$ of a harmonic mapping f of the unit disk $\mathbb{D}$ with $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$. There are two important constants one is relative to the domain of the function while the other one, namely the Bloch constant, is defined relative to the range. In [3], authors proved that if $f \in \mathcal{B}_H^M$ then f is univalent in $|z| < \rho_0$ and $f(|z| < \rho_0)$ contains a disk $|w| < R_0$, where

$$\rho_0 \approx \frac{1}{11.105M} \quad \text{and} \quad R_0 = \frac{\rho_0}{2} \approx \frac{1}{22.21M}.$$

Better estimates were given in [7, 9, 11, 12] and later in [5], see Table 1 in which the functions ϕ and ψ are explicitly given by

$$\phi(x) = \frac{x}{\sqrt{2}(x^2 + x - 1)} \quad \text{and} \quad \psi(x) = \frac{1}{\sqrt{2}}\left[1 + \left(\frac{x^2 - 1}{x}\right)\log\left(\frac{x^2 - 1}{x^2 + x - 1}\right)\right].$$

This result is the best known but not sharp.

The purpose the next theorem is to give a new proof of one of these results. Indeed our method of proof is simple and improves the best known result. In fact our distortion estimate for $f \in \mathcal{B}_H^M$ provides the radius of close-to-convexity and the radius starlikeness of $\mathcal{B}_H^M$.

Theorem 1.12. *Let* $f \in \mathcal{B}_H^M$. *Then* $f = h + \overline{g}$ *is close-to-convex* (*univalent*) *in the disk* $|z| < r_0$, *where*

$$r_S = 1 - \sqrt{\frac{4M}{4M + \pi}}$$

and $f(\mathbb{D}_{r_0})$ *contains a univalent disk of radius at least*

$$R_S = r_S - \frac{4M}{\pi}\frac{r_S^2}{1 - r_S}.$$

M	$r=\phi(8M/\pi)$	$R_0=\psi(8M/\pi)$	M	r_S	R_S
1	0.22421	0.12629	1	0.251602	0.143904
2	0.11992	0.06367	2	0.152633	0.082622
3	0.08311	0.04328	3	0.109765	0.0580693

TABLE 1. The left side columns refer to Theorem 4 in [5] and the right side columns refer to Theorem 1.12.

2. Useful Lemmas and their Proofs

We need the following two lemmas to prove our main results.

Lemma 2.1. *Let h and g have the form (1.1) with $|b_1|<1$, $f=h+\overline{g}$, and satisfy the condition*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1. \tag{2.2}$$

Then $f\in \mathcal{C}_H^2$, where $\mathcal{C}_H^2=\{f\in \mathcal{S}_H:\, |f_z(z)-1|<1-|f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}$. The bound in (2.2) is sharp as the harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{\epsilon_n}{n}z^n+\sum_{n=1}^{\infty}\frac{\epsilon_n'}{n}\overline{z^n},$$

for which $\sum_{n=2}^{\infty}|\epsilon_n|+\sum_{n=1}^{\infty}|\epsilon_n'|=1$, shows.

Proof. In [13], it was shown that $\operatorname{Re} f_z(z)>|f_{\overline{z}}(z)|$ whenever (2.2) holds. The proof of this lemma follows from an easy modification of the proof of the corresponding result from [13]. For the sake of completeness, we include the detail. Note that the coefficient inequality implies that both h and g are analytic in $\mathbb{D}$. Thus, $f=h+\overline{g}$ is harmonic in $\mathbb{D}$. Without loss of generality, we may assume that f is not affine. Then, as $f_z=h'$ and $f_{\overline{z}}=\overline{g'}$, it follows from the hypotheses that

$$\begin{aligned}|h'(z)-1| &\leq \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1}\\ &\leq \sum_{n=2}^{\infty} n|a_n| \leq 1-\sum_{n=1}^{\infty} n|b_n|\\ &\leq 1-|g'(z)|\end{aligned}$$

implying that $f\in\mathcal{C}_H^2$ (since strict inequality occurs either at the second or fourth inequality). In particular, $\operatorname{Re} h'(z)>|g'(z)|$ in $\mathbb{D}$ and hence, f is locally univalent in $\mathbb{D}$. □

For example, the functions

$$f_n(z)=z+\frac{n+1}{2n^2}z^n+\frac{n-1}{2n^2}\overline{z^n} \quad \text{for } n\geq 2$$

satisfy the condition (2.2) and hence, belong to the class $\mathcal{C}_H^2$. In the following lemma, we show that functions in $\mathcal{C}_H^2$ are indeed close-to-convex in $\mathbb{D}$.

Lemma 2.3. *Let h and g have the form (1.1) with $|b_1| < 1$, $f = h + \overline{g}$. Suppose $f \in \mathcal{C}_H^2$. Then, we have the following*

- **(a)** *f is close-to-convex in $\mathbb{D}$.*
- **(b)** *$\big| |a_n| - |b_n| \big| \leq 1/n$ for $n \geq 2$ whenever $b_1 = 0$. The equality occurs, for example, for the function*

$$f(z) = z + \frac{e^{i\theta}}{n} z^n \quad \text{or} \quad f(z) = z + \frac{e^{i\theta}}{n} \overline{z^n} \quad \text{for } n \geq 2 \text{ and } \theta \text{ real.}$$

- **(c)** $\displaystyle\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2.$

Proof. First we prove part **(a)**. Let $f = h + \overline{g} \in \mathcal{C}_H^2$ and $F = h + \epsilon g$, where $|\epsilon| = 1$. Then,

$$|F'(z) - 1| \leq |h'(z) - 1| + |g'(z)| < 1$$

showing that F is analytic and close-to-convex in $\mathbb{D}$. According to Lemma A, it follows that the harmonic function f is also close-to-convex (and univalent) in $\mathbb{D}$.

Next, set $\omega(z) = F'(z) - 1$. Then, as $b_1 = g'(0) = 0$, we have $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. It is well-known property that the coefficients of such an analytic function ω satisfy the inequality $|\omega^{(n)}(0)| \leq n!$ for each $n \geq 1$. This gives the estimate

$$|na_n + \epsilon n b_n| \leq 1 \quad \text{for each } n \geq 2.$$

As $|\epsilon| = 1$, triangle inequality gives the proof for part **(b)**.

For the proof of part **(c)**, we observe that

$$|F'(z) - 1| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} + \epsilon \sum_{n=1}^{\infty} n b_n z^{n-1} \right| < 1, \quad z \in \mathbb{D}.$$

Therefore, with $z = re^{i\theta}$ for $r \in (0,1)$ and $0 \leq \theta \leq 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) r^{2(n-1)} + |b_1|^2 = \frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta}) - 1|^2 \, d\theta \leq 1.$$

Letting $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2$$

and the proof is complete. □

In [13], under the hypotheses of Lemma 2.1, it was actually shown that $f \in \mathcal{C}_H^1$, where

$$\mathcal{C}_H^1 = \{f \in \mathcal{S}_H : \operatorname{Re} f_z(z) > |f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}.$$

Clearly, Lemma 2.1 improves this result because of the strict inclusion $\mathcal{C}_H^2 \subsetneqq \mathcal{C}_H^1$. Later, in [1], it was also shown that if $b_1 = g'(0) = 0$, then the coefficient condition

(2.2) ensures that $f\in \mathcal{S}_H^{*0}$ (see also [16]). In view of Lemma 2.1, the result of [1, 16] may be stated in an improved form.

Lemma 2.4. *Let h and g have the form (1.1) with $b_1=g'(0)=0$, $f=h+\overline{g}$, and satisfy the condition*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=2}^{\infty} n|b_n| \leq 1. \tag{2.5}$$

Then $f\in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$.

The following generalization of Lemma 2.1 is easy to obtain and so we omit its details.

Corollary 2.6. *Let h and g have the form (1.1) with $|b_1|<1-\beta$ for some $\beta\in[0,1)$, and $f=h+\overline{g}$. Then we have the following:*

(a) *If the coefficients of h and g satisfy the condition*

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1-\beta, \tag{2.7}$$

then $f\in \mathcal{C}_H^2(\beta)$, where

$$\mathcal{C}_H^2(\beta)=\{f\in \mathcal{S}_H:\, |f_z(z)-1|<1-\beta-|f_{\overline{z}}(z)| \text{ in } \mathbb{D}\}.$$

In particular, f is close-to-convex in $\mathbb{D}$. The bound here is sharp as the harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{\epsilon_n}{n}z^n+\sum_{n=1}^{\infty}\frac{\epsilon_n'}{n}\overline{z^n},$$

for which $\sum_{n=2}^{\infty}|\epsilon_n|+\sum_{n=1}^{\infty}|\epsilon_n'|=1-\beta$, shows.

(b) *If $f\in \mathcal{C}_H^2(\beta)$, then one has*

$$\big|\,|a_n|-|b_n|\,\big|\leq (1-\beta)/n \quad \text{for } n\geq 2 \text{ whenever } b_1=0.$$

The equality occurs, for example, for the function

$$f(z)=z+(1-\beta)\frac{e^{i\theta}}{n}z^n \quad \text{or} \quad f(z)=z+(1-\beta)\frac{e^{i\theta}}{n}\overline{z^n} \quad \text{for } n\geq 2 \text{ and } \theta \text{ real.}$$

We also have

$$\sum_{n=2}^{\infty} n^2(|a_n|^2+|b_n|^2)\leq (1-\beta)^2-|b_1|^2.$$

It is a matter of routine checking to see that the coefficient condition (2.7) is necessary for $f=h+\overline{g}$ to belong to $\mathcal{C}_H^2(\beta)$ whenever the Taylor coefficients $a_n\leq 0$ for all $n\geq 2$, and $b_n\leq 0$ for all $n\geq 1$.

3. Proofs of Main Theorems

Proof of Theorem 1.5. Let h and g have the form (1.1) satisfying the coefficient conditions (1.4). First we observe that $b_1 = g'(0) = 0$. The conditions (1.4) implies that the series (1.1) are convergent in the unit disk $|z| < 1$, and hence, the sum h and g are analytic in $\mathbb{D}$. Thus, $f = h + \overline{g}$ is harmonic in $\mathbb{D}$. Let $0 < r < 1$, we let

$$f_r(z) := r^{-1}f(rz) = r^{-1}h(rz) + r^{-1}\overline{g(rz)}$$

so that $f_r(z) = h_r(z) + \overline{g_r(z)}$ and

$$f_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \;\; z \in \mathbb{D}.$$

By hypotheses, $|a_n| \leq A_n$ and $|b_n| \leq B_n$ for $n \geq 2$, where A_n and B_n are given by (1.3). Using these coefficient estimates, we obtain

$$\begin{aligned} S &= \sum_{n=2}^{\infty} n|a_n|r^{n-1} + \sum_{n=2}^{\infty} n|b_n|r^{n-1} \\ &\leq \sum_{n=2}^{\infty} nA_n r^{n-1} + \sum_{n=2}^{\infty} nB_n r^{n-1}. \end{aligned}$$

We show that $f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$. According to Lemma 2.4, it suffices to show that $S \leq 1$. By the last inequality, $S \leq 1$ if r satisfies the inequality

$$\sum_{n=2}^{\infty} nA_n r^{n-1} \leq 1 - \sum_{n=2}^{\infty} nB_n r^{n-1},$$

or equivalently (as $A_n + B_n = (2n^2 + 1)/3$),

$$2\sum_{n=2}^{\infty} n^3 r^{n-1} + \sum_{n=2}^{\infty} n r^{n-1} \leq 3. \tag{3.1}$$

As

$$\frac{r}{(1-r)^2} = \sum_{n=1}^{\infty} nr^n \;\;\text{and}\;\; \frac{r(1+r)}{(1-r)^3} = \sum_{n=1}^{\infty} n^2 r^n,$$

it follows that

$$\frac{(1-r)(1+2r) + 3r(1+r)}{(1-r)^4} = \sum_{n=1}^{\infty} n^3 r^{n-1}$$

and (3.1) reduces to the inequality,

$$\frac{2(r^2+4r+1)}{(1-r)^4} + \frac{1}{(1-r)^2} \leq 6, \;\; \text{i.e.} \;\; 2(1-r)^4 - (1+r)^2 \geq 0.$$

This gives

$$\sqrt{2}(1-r)^2 - (1+r) = \sqrt{2}\, r^2 - (1+2\sqrt{2})\, r + \sqrt{2} - 1 \geq 0.$$

Thus, from Lemma 2.4, f_r is close-to-convex (univalent) in $\mathbb{D}$ and starlike in $\mathbb{D}$ for all $0 < r \leq r_S$, where r_S is the root of the quadratic equation

$$\sqrt{2}\,r^2 - (1+2\sqrt{2})\,r + \sqrt{2} - 1 = 0$$

in the interval $(0,1)$. In particular, f is close-to-convex (univalent) and starlike in $|z| < r_S$.

Next, to prove the sharpness part of the statement of the theorem, we consider the function

$$F_0(z) = H_0(z) + \overline{G_0(z)}$$

with

$$H_0(z) = 2z - H(z) \ \text{ and } \ G_0(z) = -\overline{G(\bar{z})}.$$

Here H and G are defined by (1.2). We note that

$$F_0(z) = z - \sum_{n=2}^{\infty} A_n z^n - \overline{\sum_{n=2}^{\infty} B_n z^n}.$$

As F_0 has real coefficients we obtain.

$$\begin{aligned} J_{F_0}(r) &= (H_0'(r) + G_0'(r))\,(H_0'(r) - G_0'(r)) \\ &= \left(1 - \sum_{n=2}^{\infty} nA_n r^{n-1} - \sum_{n=2}^{\infty} nB_n r^{n-1}\right)\left(1 - \sum_{n=2}^{\infty} n(A_n - B_n) r^{n-1}\right) \\ &= \left(1 - \sum_{n=2}^{\infty} \frac{n(2n^2+1)}{3} r^{n-1}\right)\left(1 - \sum_{n=2}^{\infty} n^2 r^{n-1}\right) \\ &= \left(1 - \frac{-4r^2+3r^3-r^4}{(-1+r)^3 r}\right)\left(1 + \frac{-6r^2+5r^3-4r^4+r^5}{(-1+r)^4 r}\right) \\ &= \frac{(-1+7r-6r^2+2r^3)(1-10r+11r^2-8r^3+2r^4)}{(-1+r)^7}. \end{aligned}$$

Thus $J_{F_0}(r) = 0$, $0 < r < 1$ if and only if

$$r = r_S = \frac{1}{4}\left(4 + \sqrt{2} - \sqrt{2 + 16\sqrt{2}}\right) \approx 0.112903$$

or

$$r = r_S' = 1 + \left(-18 + \sqrt{330}\right)^{1/3} 6^{-2/3} - \left(6\big(-18 + \sqrt{330}\big)\right)^{-1/3} \approx 0.164878.$$

Moreover for $r_S < r < r_S'$ we have $J_{F_0}(r) < 0$. The graph of the function $J_{F_0}(r)$ for $r \in (0, 0.25)$ is shown in Figure 1.

This observation together with Lewy's theorem gives that (as the Jacobian changes sign), the function $F_0(z)$ is not univalent in $|z| < r$ if $r > r_{\mathcal{S}}$ and thus, $r_{\mathcal{S}}$ cannot be replaced by a larger number. □

Proof of Theorem 1.9. Following the notation and the method of the proof of Theorem 1.5, it suffices to show that $f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$. According to Lemma 2.4,

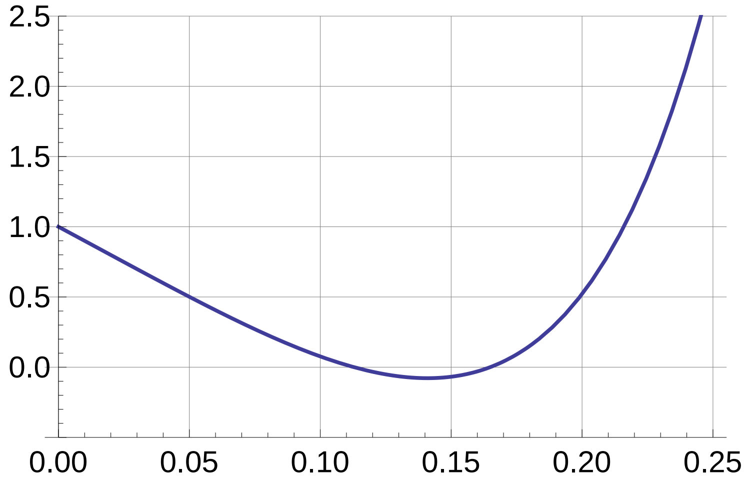

FIGURE 1. The graph of the Jacobian $J_{F_0}(r)$ for $r \in (0, 0.25)$.

$f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^{*0}$ whenever $S \leq 1$, where

$$S = \sum_{n=2}^{\infty} n|a_n|r^{n-1} + \sum_{n=2}^{\infty} n|b_n|r^{n-1}$$

when a_n and b_n satisfy the coefficient inequalities given by (1.7). Finally, using (1.7), we see that $S \leq 1$ if r satisfies the inequality

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2} r^{n-1} \leq 1 - \sum_{n=2}^{\infty} \frac{n(n-1)}{2} r^{n-1}.$$

The last inequality is easily seen to be equivalent to

$$\frac{1}{2}\left[\frac{1}{(1-r)^2} + \frac{1+r}{(1-r)^3} - 1\right] \leq 1 + \frac{1}{2}\left[\frac{1}{(1-r)^2} - \frac{1+r}{(1-r)^3} - 1\right]$$

which upon simplification reduces to

$$2(1-r)^3 - 1 - r = -(2r^3 - 6r^2 + 7r - 1) \geq 0.$$

The first part of the conclusion easily follows as in the proof of Theorem 1.5.

The sharpness part of the statement of Theorem 1.9 follows if we consider the function

$$L_0(z) = 2z - M(z) - \overline{N(z)},$$

where M and N are defined by (1.8). We note that

$$L_0(z) = z - \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} z^n}.$$

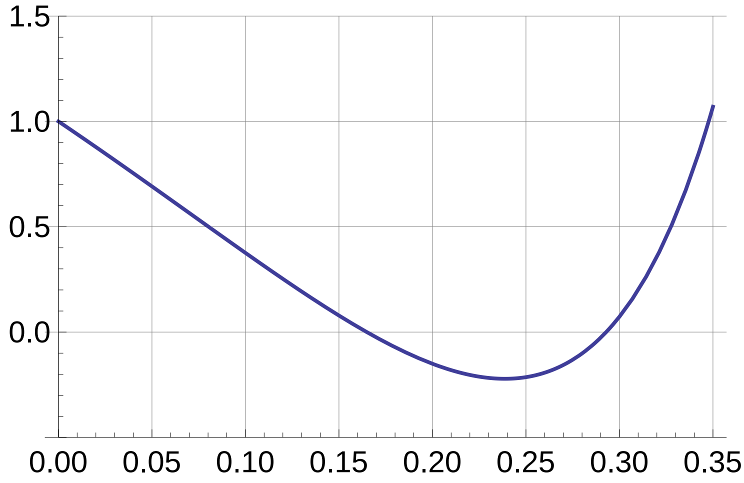

FIGURE 2. The graph of the Jacobian $J_{L_0}(r)$ for $r \in (0, 0.35)$.

Again, as L_0 has real coefficients, we can easily obtain that for $r \in (0,1)$

$$\begin{aligned} J_{L_0}(r) &= (2 - M'(r))^2 - (N'(r))^2 \\ &= (2 - M'(r) + N'(r))\left((2 - M'(r) - N'(r)\right) \\ &= \left(2 - \frac{1+r}{(1-r)^3}\right)\left(2 - \frac{1}{(1-r)^2}\right) \\ &= \frac{2}{(1-r)^5}\left(2(1-r)^3 - (1+r)\right)\left(r - 1 - \frac{\sqrt{2}}{2}\right)\left(r - 1 + \frac{\sqrt{2}}{2}\right). \end{aligned}$$

We see that $J_{L_0}(r_S) = 0$, $0 < r < 1$ if and only if

$$r = r_S \approx 0.16487$$

or

$$r = r'_S = \frac{2 - \sqrt{2}}{2} \approx 0.292893.$$

Moreover for $r_S < r < r'_S$ we have $J_{L_0}(r) < 0$. The graph of the function $J_{L_0}(r)$ for $r \in (0, 0.35)$ is shown in Figure 2.

Thus, according to Lewy's theorem, $L_0(z)$ is not univalent in $|z| < r$ if $r > r_{\mathcal{S}}$ and this observation shows that $r_{\mathcal{S}}$ cannot be replaced by a larger number. $\square$

Proof of Theorem 1.11. This time we apply Lemma 2.1 and show that f_r defined by $f_r(z) := r^{-1}f(rz) = r^{-1}h(rz) + r^{-1}\overline{g(rz)}$ belongs to $\mathcal{C}_H^2$.

As in the proof of previous two theorems, it suffices to show the corresponding coefficient inequality (2.2), namely,

$$S = \sum_{n=2}^{\infty} n(|a_n| + |b_n|)r^{n-1} + |b_1| \leq 1.$$

By the hypothesis, $|a_n| + |b_n| \leq c$ for all $n \geq 2$ and so, the last inequality $S \leq 1$ clearly holds if r satisfies the inequality

$$c\left(\frac{1}{(1-r)^2} - 1\right) \leq 1 - |b_1|, \quad \text{i.e.} \quad r \leq r_S = 1 - \sqrt{\frac{c}{c+1-|b_1|}}.$$

Thus, by Lemma 2.1,

$$|h_r'(z) - 1| < 1 - |g_r'(z)|$$

holds for all $z \in \mathbb{D}$ whenever $r \leq r_S$. Thus, $f \in \mathcal{C}_H^2$.

The function $f_0(z) = h_0(z) + \overline{g_0(z)}$, where

$$h_0(z) = z - \frac{c}{2}\left(\frac{z^2}{1-z}\right) \text{ and } g_0(z) = -|b_1|z - \frac{c}{2}\left(\frac{z^2}{1-z}\right),$$

shows that the result is sharp. Indeed, it is easy to compute that

$$J_{f_0}(r) = |h_0'(r)|^2 - |g_0'(r)|^2 = (1 + |b_1|)\left(1 + c - |b_1| - \frac{c}{(1-r)^2}\right)$$

which shows that $J_{f_0}(r_S) = 0$ and $J_{f_0}(r) < 0$ for $r > r_S$. The proof of the theorem is complete. □

Proof of Theorem 1.12. Let $f = h + \overline{g}$ be a harmonic mapping defined on the unit disk $\mathbb{D}$ with $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$, where h and g have the form (1.1) with $b_1 = 0$. According to [4, Lemma 1] (see also [5]), we obtain the sharp estimates

$$|a_n| + |b_n| \leq \frac{4M}{\pi} \text{ for any } n \geq 1. \tag{3.2}$$

As $b_1 = 0$ and $a_1 = 1$, it follows that $M \geq \pi/4 \approx 0.785398$. By Theorem 1.11 with $c = 4M/\pi$, we conclude that f is close-to-convex and starlike (because $b_1 = 0$) for $|z| < 1 - \sqrt{c/(c+1)} = r_S$.

In particular, f is univalent for $|z| < r_S$ and furthermore, we have for $|z| = r_S$,

$$\begin{aligned}
|f(z)| &= \left| z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right) \right| \\
&\geq |z| - \left| \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right) \right| \\
&\geq r_S - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r_S^n \\
&\geq r_S - \frac{4M}{\pi} \sum_{n=2}^{\infty} r_S^n \\
&= r_S - \frac{4M}{\pi} \frac{r_S^2}{1 - r_S} = R_S
\end{aligned}$$

and the proof is complete. □

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David Kalaj, University of Montenegro, faculty of natural sciences and mathematics, Cetinjski put b.b. 81000, Podgorica, Montenegro
E-mail address: `davidk@t-com.me`

S. Ponnusamy, Department of Mathematics, Indian Institute of Technology Madras, Chennai–600 036, India.
E-mail address: `samy@iitm.ac.in`

Matti Vuorinen Department of Mathematics, University of Turku, FIN-20014 Turku, Finland.
E-mail address: `vuorinen@utu.fi`