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On General Form of $\mathcal{N}$-fold Supersymmetry

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Abstract

We analyze general structure of $\mathcal{N}$-fold supersymmetry which provides a systematic framework to construct weakly quasi-solvable quantum mechanical systems. Main ingredients of our analysis are dimensional analysis and introduction of an equivalent class of linear differential operators associated with $\mathcal{N}$-fold supersymmetry for each $\mathcal{N}$. To illustrate how they work, we construct the most general form of $\mathcal{N}$-fold supersymmetric systems for $\mathcal{N} = 2$, 3, and 4.

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I. INTRODUCTION

$\mathcal{N}$-fold supersymmetry (SUSY) [1–3] is one of the most powerful frameworks for constructing a one-dimensional quantum mechanical (QM) system which admits analytic solutions in closed form in a certain sense. This is due to the fact that $\mathcal{N}$-fold SUSY is essentially equivalent to weak quasi-solvability which is until now the least restrictive concept about the availability of solutions in closed form. The latter crucial fact was first proved in a general fashion in Ref. [2] and was later complemented slightly in Ref. [4]. For a review, see, e.g., Ref. [5].

In general, construction of an $\mathcal{N}$-fold SUSY system get more difficult as the number $\mathcal{N} \in \mathbb{N}$ increases since we must solve coupled nonlinear differential equations for $\mathcal{N}$ unknown functions (see the next section). To bypass the latter difficulty, a systematic algorithm for constructing an $\mathcal{N}$-fold SUSY system based on quasi-solvability (in the strong sense) was proposed in Ref. [6]. A key ingredient of the algorithm is to choose first an $\mathcal{N}$-dimensional linear space of specific functions such that it can be preserved by a second-order linear differential operator. It has been proved to be quite efficient and so far four inequivalent types of $\mathcal{N}$-fold SUSY, namely, type A [4, 7], type B [8], type C [6], and type X_2 [9] are successfully constructed with the algorithm. We note that almost all the models having essentially the same symmetry as $\mathcal{N}$-fold SUSY but called with other terminologies in the literature, such as Pöschl–Teller and Lamé potentials are actually particular cases of type A $\mathcal{N}$-fold SUSY.

It is evident, however, that the algorithm is helpless to construct a *weakly* quasi-solvable system which only admits a finite-dimensional invariant subspace determined by another differential equation. The framework of $\mathcal{N}$-fold SUSY covers such systems and thus provides a more general formalism than higher-derivative generalizations of Darboux transformation such as the Crum's method [10] which relies on a set of exact eigenfunctions of a regular Sturm–Liouville system. To construct a weakly quasi-solvable system, we must in general treat directly the aforementioned coupled nonlinear differential equations. For the simplest case of $\mathcal{N} = 2$, the general result was already studied and reported in Refs. [2, 11, 12]. On the other hand, for the cases of $\mathcal{N} > 2$ until now on there is, at the best of our knowledge, only one paper [13] which studied the $\mathcal{N} = 3$ case. This fact would reflect the difficulty and complexity of the problems for larger $\mathcal{N}$.

In this work, we investigate general strucuture of $\mathcal{N}$-fold SUSY systems to extract relevant clues to construct them which have in particular weak quasi-solvability. For this purpose, we first employ dimensional analysis which is a well-known powerful tool in general physics. It turns out that it is also quite efficient in acquiring deeper understandings of $\mathcal{N}$-fold SUSY. We then introduce equivalent classes of linear differential operators associated with $\mathcal{N}$-fold SUSY. We find that it enables us to deal with operator equalities appeared in $\mathcal{N}$-fold SUSY more systematically and transparently.

We organize the paper as follows. In the next section, we first briefly review the ingredients of $\mathcal{N}$-fold SUSY. Then, we introduce two key concepts for analyzing its general structure, namely, dimensional analysis and equivalent classes of linear differential operators associated with $\mathcal{N}$-fold SUSY. In Sections III–IV, we apply the general arguments to obtain general form of $\mathcal{N}$-fold SUSY for $\mathcal{N} = 2$, 3, and 4, respectively. We show how dimensional analysis enables us to reduce the complexity of the problems on solving the conditions for $\mathcal{N}$-fold SUSY and on finding integral constants of the systems. In the last section, we summarize the paper and provide comments on the future issues.

II. GENERAL CONSIDERATION

To begin with, we shall briefly review ingredients of $\mathcal{N}$-fold SUSY as preliminaries. For details, see the review [5]. An $\mathcal{N}$-fold SUSY QM system in one-dimension is composed of a pair of Hamiltonians $H^{\pm}$ and a pair of $\mathcal{N}$th-order linear differential operators $P_{\mathcal{N}}^{\pm}$

$$H^{\pm} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V^{\pm}(q), \quad P_{\mathcal{N}}^{-} = \frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}q^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} w_k^{[\mathcal{N}]}(q)\frac{\mathrm{d}^k}{\mathrm{d}q^k}, \quad P_{\mathcal{N}}^{+} = (P_{\mathcal{N}}^{-})^{\mathrm{T}}, \tag{2.1}$$

where the superscript T denotes the transposition of a linear operator [3], which satisfy the intertwining relation

$$P_{\mathcal{N}}^{-}H^{-} - H^{+}P_{\mathcal{N}}^{-} = \sum_{k=0}^{\mathcal{N}} I_k^{[\mathcal{N}]}\frac{\mathrm{d}^k}{\mathrm{d}q^k} = 0, \tag{2.2}$$

and its transposed relation $H^{-}P_{\mathcal{N}}^{+} - P_{\mathcal{N}}^{+}H^{+} = 0$. The operators $P_{\mathcal{N}}^{\pm}$ are actually components of $\mathcal{N}$-fold supercharges.

One of the most significant consequences of the intertwining relation (2.2) is *weak quasi-solvability* [2, 4]. That is, each $\mathcal{N}$-fold SUSY Hamiltonian $H^{\pm}$ preserves the linear space $\ker P_{\mathcal{N}}^{\pm}$:

$$H^{\pm}\ker P_{\mathcal{N}}^{\pm} \subset \ker P_{\mathcal{N}}^{\pm}. \tag{2.3}$$

If the differential equation $P_{\mathcal{N}}^{-}\phi = 0$ and/or $P_{\mathcal{N}}^{+}\phi = 0$ admits a number of analytic solutions in closed form, H^{-} and/or H^{+} is not only weakly quasi-solvable but is *quasi-solvable* in the strong sense. But in general, an $\mathcal{N}$-fold SUSY Hamiltonian is merely weakly quasi-solvable and does not admit any analytic local solutions. We also note that $\ker P_{\mathcal{N}}^{\pm}$ is not necessarily a subspace of the linear space, which is usually the Hilbert space $L^2(S)$ ($S \subset \mathbb{R}$), in which the operator $H^{\pm}$ acts.

Another peculiar feature of an $\mathcal{N}$-fold SUSY system is that the product $P_{\mathcal{N}}^{\mp}P_{\mathcal{N}}^{\pm}$ which arises as a component of the anti-commutator of $\mathcal{N}$-fold supercharges is an $\mathcal{N}$th-degree polynomial in the Hamiltonian $H^{\pm}$ [2, 3] and thus has the following form:

$$P_{\mathcal{N}}^{\mp}P_{\mathcal{N}}^{\pm} = 2^{\mathcal{N}}\left[(H^{\pm}+C_0)^{\mathcal{N}} + \sum_{k=1}^{\mathcal{N}-1} C_k(H^{\pm}+C_0)^{\mathcal{N}-k-1}\right], \tag{2.4}$$

where C_k ($k = 0, \dots, \mathcal{N}-1$) are all constants. The $\mathcal{N}$ zeros of the polynomial in the r.h.s. of (2.4) correspond the spectrum of $H^{\pm}$ in the space $\ker P_{\mathcal{N}}^{\pm}$. Hence, they are actually a part of the eigenvalues of $H^{\pm}$ if $\ker P_{\mathcal{N}}^{\pm} \subset L^2(S)$. In the latter case, we can calculate the part of the eigenvalues algebraically from (2.4) even though the corresponding eigenfunctions cannot be obtained in closed form.

One of the most difficult problems on $\mathcal{N}$-fold SUSY is to analyze the condition (2.2) for $\mathcal{N}$-fold SUSY. It is composed of coupled nonlinear differential equations for the so far undetermined functions $w_k^{[\mathcal{N}]}$ ($k = 0, \dots, \mathcal{N}-1$). As is easily expected, its complexity gets terrible as the integer $\mathcal{N}$ increases. Hence, it is quite difficult to solve directly the condition (2.2) for larger $\mathcal{N}$. In the subsequent two sections, we shall investigate general aspects of the complicated structure of $\mathcal{N}$-fold SUSY systems which would provide us a clearer view on them.

A. Dimensional Analysis

Dimensional analysis is one of the powerful methods to make physical consideration and in particular to estimate a physical quantity under consideration without solving equations directly, cf., any textbook on general physics. In this section, we shall see that it supplies us with a valuable guiding principle in solving the condition for $\mathcal{N}$-fold SUSY.

To make dimensional analysis on our system (2.1), we first note that we have implicitly employed the unit system where the ratio of action (the Planck constant) to mass $\hbar/m$ is dimensionless. The only relevant physical dimension is then the *length*, denoted by [L], which is carried by the physical position variable q. It is easy to see from (2.1) and (2.4) that the physical dimensions of $V^{\pm}$, C_k, and $w_k^{[\mathcal{N}]}$ in terms of the length are given by

$$\begin{aligned} &V^{\pm}\ [\mathrm{L}^{-2}], \qquad C_k\ [\mathrm{L}^{-2(k+1)}] \quad (k=0,\dots,\mathcal{N}-1),\\ &w_k^{[\mathcal{N}](m)}\ [\mathrm{L}^{k-\mathcal{N}-m}] \quad (k=0,\dots,\mathcal{N}-1,\ m=0,1,2,\dots), \end{aligned} \tag{2.5}$$

where $w_k^{[\mathcal{N}](m)}(q)$ is the mth derivative of $w_k^{[\mathcal{N}]}(q)$ with respect to q.

Dimensional analysis relies on the obvious fact that all the terms which appear in a single formula under consideration must have the same physical dimension. For instance, a potential has the physical dimension $[\mathrm{L}^{-2}]$ and thus must be expressed as a sum of terms all of which have the same physical dimension $[\mathrm{L}^{-2}]$. Hence, if we only consider a polynomial of C_k and $w_k^{[\mathcal{N}](m)}$ $(m=0,1,2,\dots)$, it must have the following form

$$V=\alpha_0 w_{\mathcal{N}-2}^{[\mathcal{N}]}+\alpha_1 w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}+\alpha_2\bigl(w_{\mathcal{N}-1}^{[\mathcal{N}]}\bigr)^2-C_0\ [\mathrm{L}^{-2}], \tag{2.6}$$

where α_k $(k=0,1,2)$ are all dimensionless parameters. In fact, we can see that a pair of $\mathcal{N}$-fold SUSY potentials $V^{\pm}$ satisfying (2.2) does have the form (2.6). The l.h.s. of (2.2) is a linear differential operator of at most $\mathcal{N}$th order, as is indicated in (2.2), and it is evident that the identity (2.2) holds if and only if all the coefficients $I_k^{[\mathcal{N}]}$ of $\partial^k=\mathrm{d}^k/\mathrm{d}q^k$ $(k=0,\dots,\mathcal{N})$ vanish. The latter requirement for the coefficients of $\partial^{\mathcal{N}}$ and $\partial^{\mathcal{N}-1}$ reads as

$$I_{\mathcal{N}}^{[\mathcal{N}]}=w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}-(V^+-V^-)=0\ [\mathrm{L}^{-2}], \tag{2.7a}$$

$$2I_{\mathcal{N}-1}^{[\mathcal{N}]}=w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime}+2w_{\mathcal{N}-2}^{[\mathcal{N}]\prime}+2\mathcal{N}V^{-\prime}-2w_{\mathcal{N}-1}^{[\mathcal{N}]}(V^+-V^-)=0\ [\mathrm{L}^{-3}]. \tag{2.7b}$$

The set of conditions (2.7) can be easily solved as

$$V^{\pm}=-\frac{1}{\mathcal{N}}w_{\mathcal{N}-2}^{[\mathcal{N}]}+\left(\frac{\mathcal{N}-1}{2\mathcal{N}}\pm\frac{1}{2}\right)w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}+\frac{1}{2\mathcal{N}}\bigl(w_{\mathcal{N}-1}^{[\mathcal{N}]}\bigr)^2-C_0\ [\mathrm{L}^{-2}], \tag{2.8}$$

which indeed has the form of (2.6). The formula (2.8) provides a general expression for a pair of $\mathcal{N}$-fold SUSY potentials $V^{\pm}$ for an arbitrary $\mathcal{N}\in\mathbb{N}$. One of its characteristic features is that they are expressible solely in terms of the two functions $w_{\mathcal{N}-1}^{[\mathcal{N}]}$ and $w_{\mathcal{N}-2}^{[\mathcal{N}]}$ irrespective of what additional conditions $w_k^{[\mathcal{N}]}$ $(k=0,\dots,\mathcal{N}-1)$ should satisfy.

The remaining conditions for $\mathcal{N}$-fold SUSY coming from the coefficients of ∂^k for $k=0,\dots,\mathcal{N}-2$ in (2.2) are in general algebraic equations consisting of $w_k^{[\mathcal{N}](m)}$ $(m=0,1,2,\dots)$ after the substitution of (2.8) into (2.2). Dimensional analysis tells us that the operator in the l.h.s. of (2.2) has the physical dimension $[\mathrm{L}^{-\mathcal{N}-2}]$ and thus $I_k^{[\mathcal{N}]}$ $(k=0,\dots,\mathcal{N}-2)$ has the physical dimension $[\mathrm{L}^{k-\mathcal{N}-2}]$. For instance, $I_{\mathcal{N}-2}^{[\mathcal{N}]}$ and $I_{\mathcal{N}-3}^{[\mathcal{N}]}$ are calculated as

$$\begin{aligned}-4\mathcal{N}I_{\mathcal{N}-2}^{[\mathcal{N}]}=&\,\mathcal{N}(\mathcal{N}-1)w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime\prime}+2\mathcal{N}(\mathcal{N}-2)w_{\mathcal{N}-2}^{[\mathcal{N}]\prime\prime}-4\mathcal{N}w_{\mathcal{N}-3}^{[\mathcal{N}]\prime}-2(\mathcal{N}-1)^2w_{\mathcal{N}-1}^{[\mathcal{N}]}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime}\\&-2\mathcal{N}(\mathcal{N}-1)\big(w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}\big)^2+4\mathcal{N}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}w_{\mathcal{N}-2}^{[\mathcal{N}]}+4(\mathcal{N}-1)w_{\mathcal{N}-1}^{[\mathcal{N}]}w_{\mathcal{N}-2}^{[\mathcal{N}]\prime}\\&-4(\mathcal{N}-1)\big(w_{\mathcal{N}-1}^{[\mathcal{N}]}\big)^2w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}\ [\mathrm{L}^{-4}],\end{aligned}\tag{2.9}$$

$$\begin{aligned}-12\mathcal{N}I_{\mathcal{N}-3}^{[\mathcal{N}]}=&\,\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)\big(w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime\prime\prime}+2w_{\mathcal{N}-2}^{[\mathcal{N}]\prime\prime\prime}\big)-6\mathcal{N}\big(w_{\mathcal{N}-3}^{[\mathcal{N}]\prime\prime}+2w_{\mathcal{N}-4}^{[\mathcal{N}]\prime}\big)\\&-(\mathcal{N}-1)(\mathcal{N}-2)\Big[(2\mathcal{N}-3)w_{\mathcal{N}-1}^{[\mathcal{N}]}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime\prime}+6\mathcal{N}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime}\Big]\\&+6(\mathcal{N}-2)\Big[w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime}w_{\mathcal{N}-2}^{[\mathcal{N}]}+(\mathcal{N}-1)w_{\mathcal{N}-1}^{[\mathcal{N}]}w_{\mathcal{N}-2}^{[\mathcal{N}]\prime\prime}\Big]+12\mathcal{N}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}w_{\mathcal{N}-3}^{[\mathcal{N}]}\\&+12(\mathcal{N}-2)w_{\mathcal{N}-2}^{[\mathcal{N}]}w_{\mathcal{N}-2}^{[\mathcal{N}]\prime}-6(\mathcal{N}-1)(\mathcal{N}-2)\Big[\big(w_{\mathcal{N}-1}^{[\mathcal{N}]}\big)^2w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime}+w_{\mathcal{N}-1}^{[\mathcal{N}]}\big(w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}\big)^2\Big]\\&-12(\mathcal{N}-2)w_{\mathcal{N}-1}^{[\mathcal{N}]}w_{\mathcal{N}-1}^{[\mathcal{N}]\prime}w_{\mathcal{N}-2}^{[\mathcal{N}]}\ [\mathrm{L}^{-5}],\end{aligned}\tag{2.10}$$

and consist of the terms which are consistent with the dimensional analysis.

Ideally, one can obtain a general form of $\mathcal{N}$-fold SUSY systems if one succeeds in expressing all the $\mathcal{N}$ functions $w_k^{[\mathcal{N}]}$ $(k=0,\dots,\mathcal{N}-1)$, which characterize the systems, in terms of a single function, say, u and its derivatives u', u'', ... by solving the set of the $\mathcal{N}-1$ constraints $I_k^{[\mathcal{N}]}=0$ $(k=0,\dots,\mathcal{N}-1)$. If it is eventually the case, we have a set of $\mathcal{N}$ functionals $u_k^{[\mathcal{N}]}$ $(k=0,\dots,\mathcal{N}-1)$ such that

$$w_k^{[\mathcal{N}]}=u_k^{[\mathcal{N}]}[u]\ [\mathrm{L}^{k-\mathcal{N}}]\quad(k=0,\dots,\mathcal{N}-1).\tag{2.11}$$

One of the most important aspects of (2.11) is that each $u_k^{[\mathcal{N}]}$ $(k=0,\dots,\mathcal{N}-1)$ has the same physical dimension as the one of $w_k^{[\mathcal{N}]}$. It in particular means that there would be a set of transformations $w_k^{[\mathcal{N}]}\to u_k^{[\mathcal{N}]}$ which preserve all the physical dimensions. Conversely, if we can find a set of dimension-preserving transformations $w_k^{[\mathcal{N}]}\to u_k^{[\mathcal{N}]}$, we may solve the set of the constraints more easily. In Sections III–V, we will employ this strategy to see how drastically we can reduce the complexity of the constraints.

To solve the constraints to get (2.11) is in principle possible unless some of the constraints automatically imply others since we have $\mathcal{N}-1$ constraints for the $\mathcal{N}$ unknown functions. However, the task would get drastically harder as the integer $\mathcal{N}$ increases. One of the clues to circumvent the situation is in (2.4). All the coefficients of derivative operators (except for the highest $2\mathcal{N}$th-order and including the lowest 0th-order) in the l.h.s. of (2.4) are quadratic forms of $w_k^{[\mathcal{N}](m)}$ while the r.h.s. depends on, in addition to the potentials $V^{\pm}$, the $\mathcal{N}$ constants C_k $(k=0,\dots,\mathcal{N}-1)$ which are absent in the l.h.s. The latter fact indicates the existence of $\mathcal{N}$ integral constants of any $\mathcal{N}$-fold SUSY system which would be functionals of $w_j^{[\mathcal{N}]}$ $(j=0,\dots,\mathcal{N}-1)$ and $V^{\pm}$ whose physical dimensions are the same as the ones of C_k:

$$C_k=J_k[w^{[\mathcal{N}]},V]\ [\mathrm{L}^{-2(k+1)}]\quad(k=0,\dots,\mathcal{N}-1).\tag{2.12}$$

They must emerge from the integration of the set of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}q}J_k[w^{[\mathcal{N}]},V]=0\ [\mathrm{L}^{-(2k+3)}]\quad(k=0,\dots,\mathcal{N}-1).\tag{2.13}$$

The latter equations are another set of constraints. On the other hand, the set of equalities $I_k^{[\mathcal{N}]} = 0$ $(k = 0, \dots, \mathcal{N})$ are the only constraints which come from the condition for $\mathcal{N}$-fold SUSY. Hence, the differential equations (2.13) must be such equations that hold whenever all the conditions $I_k^{[\mathcal{N}]} = 0$ $(k = 0, \dots, \mathcal{N})$ are satisfied. This means that all the quantities $\mathrm{d}J_k/\mathrm{d}q$ $(k = 0, \dots, \mathcal{N} - 1)$ in the l.h.s. of (2.13) would be expressible in terms of $I_j^{[\mathcal{N}]}$ in a way such that the identities $I_j^{[\mathcal{N}]} = 0$ $(j = 0, \dots, \mathcal{N})$ apparently imply (2.13). The most general form of such a kind would be

$$\frac{\mathrm{d}J_k}{\mathrm{d}q} = \sum_{j=0}^{\mathcal{N}} L_{kj} I_j^{[\mathcal{N}]} \ [\mathrm{L}^{-(2k+3)}] \quad (k = 0, \dots, \mathcal{N} - 1), \tag{2.14}$$

where L_{kj} are all linear differential operators whose coefficients consist of only $w_k^{[\mathcal{N}]}$, $V^{\pm}$, and their derivatives. Each L_{kj} must has the physical dimension $[L^{\mathcal{N}-2k-j-1}]$ since the ones of $\mathrm{d}J_k/\mathrm{d}q$ and $I_j^{[\mathcal{N}]}$ are $[\mathrm{L}^{-(2k+3)}]$ and $[\mathrm{L}^{\mathcal{N}-2-j}]$, respectively.

To see the validity of the above argument, let us consider the constant C_0. From (2.8), we immediately know the form of J_0 as

$$C_0 = J_0[w^{[\mathcal{N}]}, V] = -V^- - \frac{1}{\mathcal{N}} w_{\mathcal{N}-2}^{[\mathcal{N}]} + \frac{1}{2\mathcal{N}} w_{\mathcal{N}-1}^{[\mathcal{N}]\prime} + \frac{1}{2\mathcal{N}} \left(w_{\mathcal{N}-1}^{[\mathcal{N}]}\right)^2 \ [\mathrm{L}^{-2}]. \tag{2.15}$$

Hence, the differential equation which leads to the latter equation is

$$0 = \frac{\mathrm{d}J_0}{\mathrm{d}q} = -V^{-\prime} - \frac{1}{\mathcal{N}} w_{\mathcal{N}-2}^{[\mathcal{N}]\prime} + \frac{1}{2\mathcal{N}} w_{\mathcal{N}-1}^{[\mathcal{N}]\prime\prime} + \frac{1}{\mathcal{N}} w_{\mathcal{N}-1}^{[\mathcal{N}]} w_{\mathcal{N}-1}^{[\mathcal{N}]\prime} \ [\mathrm{L}^{-3}]. \tag{2.16}$$

We then check by using (2.7) that the r.h.s. of (2.16) can be in fact expressed in terms of $I_j^{[\mathcal{N}]}$ as

$$\frac{\mathrm{d}J_0}{\mathrm{d}q} = \frac{1}{\mathcal{N}} w_{\mathcal{N}-1}^{[\mathcal{N}]} I_{\mathcal{N}}^{[\mathcal{N}]} - \frac{1}{\mathcal{N}} I_{\mathcal{N}-1}^{[\mathcal{N}]} \ [\mathrm{L}^{-3}], \tag{2.17}$$

which indeed has the form of (2.14) with

$$L_{0\mathcal{N}} = \frac{1}{\mathcal{N}} w_{\mathcal{N}-1}^{[\mathcal{N}]} \ [\mathrm{L}^{-1}], \quad L_{0\,\mathcal{N}-1} = -\frac{1}{\mathcal{N}} \ [\mathrm{L}^0], \quad L_{0k} = 0 \ (k = 0, \dots, \mathcal{N} - 2), \tag{2.18}$$

all having the correct physical dimensions $[\mathrm{L}^{\mathcal{N}-j-1}]$ for L_{0j} $(j = 0, \dots, \mathcal{N})$.

In practice, we already solved the two conditions (2.7) to obtain the general form of $V^{\pm}$ as (2.8), and thus we can totally eliminate $V^{\pm}$ in the remaining conditions $I_k^{[\mathcal{N}]} = 0$ $(k = 0, \dots, \mathcal{N} - 2)$. As a result, the remaining integral constants C_k $(k = 1, \dots, \mathcal{N} - 1)$ would be functionals of only $w_j^{[\mathcal{N}]}$ $(j = 0, \dots, \mathcal{N} - 1)$:

$$C_k = J_k[w^{[\mathcal{N}]}, V[w^{[\mathcal{N}]}]] := J_k[w^{[\mathcal{N}]}] \ [\mathrm{L}^{-2(k+1)}] \quad (k = 1, \dots, \mathcal{N} - 1). \tag{2.19}$$

Accordingly, $\mathrm{d}J_k/\mathrm{d}q$ $(k = 1, \dots, \mathcal{N} - 1)$ would be expressed in terms of the remaining $I_j^{[\mathcal{N}]}$ $(j = 0, \dots, \mathcal{N} - 2)$ as

$$0 = \frac{\mathrm{d}J_k}{\mathrm{d}q} = \sum_{j=0}^{\mathcal{N}-2} L_{kj} I_j^{[\mathcal{N}]} \ [\mathrm{L}^{-(2k+3)}] \quad (k = 1, \dots, \mathcal{N} - 1). \tag{2.20}$$

We will later see in Sections III–V that the above analysis is actually valid and helps us to obtain general forms of $\mathcal{N}$-fold SUSY for $\mathcal{N} = 2$, 3, and 4.

To summarize, the existence of $\mathcal{N}$ constants C_k $(k = 0, \ldots, \mathcal{N} - 1)$ in the r.h.s. of (2.4) has the direct relation to the existence of $\mathcal{N}$ constraints $I_k^{[\mathcal{N}]} = 0$ $(k = 0, \ldots, \mathcal{N} - 1)$ which are differential equations after eliminating the algebraic constraint $I_{\mathcal{N}}^{[\mathcal{N}]} = 0$. The physical dimensions of the relevant quantities are

$$I_k^{[\mathcal{N}]}\ [\mathrm{L}^{k-\mathcal{N}-2}], \quad u_k^{[\mathcal{N}]}\ [\mathrm{L}^{k-\mathcal{N}}], \quad J_k\ [\mathrm{L}^{-2(k+1)}], \quad L_{kj}\ [\mathrm{L}^{\mathcal{N}-2k-j-1}]. \tag{2.21}$$

B. Equivalent Classes of Linear Differential Operators

The existence of $\mathcal{N} - 1$ constraints $I_j^{[\mathcal{N}]} = 0$ $(j = 0, \ldots, \mathcal{N} - 2)$ after the determination of the potential pair (2.8) results in the existence of null operators associated with the $\mathcal{N}$-fold SUSY system under consideration. Let us first introduce a linear space of linear differential operators, denoted by $\mathcal{K}^{[\mathcal{N}]}$, whose coefficients are all functionals of only $w_k^{[\mathcal{N}]}$ $(k = 0, \ldots, \mathcal{N} - 1)$. Let $K_{ij}[w^{[\mathcal{N}]}] \in \mathcal{K}^{[\mathcal{N}]}$ $(i = 0, 1, 2, \ldots)$ and define a set of functionals $f_i^{[\mathcal{N}]}[w^{[\mathcal{N}]}]$ by

$$f_i^{[\mathcal{N}]}[w^{[\mathcal{N}]}] = \sum_{j=0}^{\mathcal{N}-2} K_{ij}[w^{[\mathcal{N}]}] I_j^{[\mathcal{N}]}. \tag{2.22}$$

We then define a subspace of $\mathcal{K}^{[\mathcal{N}]}$, denoted by $\mathcal{K}_0^{[\mathcal{N}]}$, which consists of linear differential operators whose coefficients are all given by $f_i^{[\mathcal{N}]}$ introduced in (2.22). That is, $K_0 \in \mathcal{K}_0^{[\mathcal{N}]}$ means that there exists a set of linear differential operators $K_{ij}[w^{[\mathcal{N}]}] \in \mathcal{K}^{[\mathcal{N}]}$ such that

$$K_0 = \sum_i f_i^{[\mathcal{N}]}[w^{[\mathcal{N}]}]\partial_i = \sum_i \left(\sum_{j=0}^{\mathcal{N}-2} K_{ij}[w^{[\mathcal{N}]}] I_j^{[\mathcal{N}]} \right) \partial_i. \tag{2.23}$$

It is obvious by definition that any element of $\mathcal{K}_0^{[\mathcal{N}]}$ is a null operator so long as all the $\mathcal{N}$-fold SUSY constraints $I_j^{[\mathcal{N}]} = 0$ $(j = 0, \ldots, \mathcal{N} - 2)$ are satisfied. Hence, the linear space in which an $\mathcal{N}$-fold SUSY system is considered is actually the quotient space $\mathcal{K}^{[\mathcal{N}]}/\mathcal{K}_0^{[\mathcal{N}]}$. It naturally leads us to introduce an equivalence class of linear differential oerators in $\mathcal{K}^{[\mathcal{N}]}$. We shall say that two linear differential operators $L_1, L_2 \in \mathcal{K}^{[\mathcal{N}]}$ belong to *equivalent class associated with $\mathcal{N}$-fold supersymmetry* and express the equivalence as $L_2 \overset{\mathcal{N}}{\sim} L_1$ if $L_2 - L_1 \in \mathcal{K}_0^{[\mathcal{N}]}$. Any equality between operators $L_2 = L_1$ appeared in an $\mathcal{N}$-fold SUSY system should be thus regarded as an equivalent relation $L_2 \overset{\mathcal{N}}{\sim} L_1$ of the latter equivalent class. In particular, any explicit expression for a specific operator such as $H^{\pm}$ and $P_{\mathcal{N}}^{\pm}$ should be considered as a representative of it with respect to the equivalent class. In what follows, we will employ the equivalence relation $\overset{\mathcal{N}}{\sim}$ only when we would like to stress that the left and the right hand sides of the formula under consideration is identical if and only if (some of) the constraints $I_j^{[\mathcal{N}]} = 0$ $(j = 0, \cdots, \mathcal{N} - 2)$ are satisfied.

III. 2-FOLD SUPERSYMMETRY

It is quite instructive to see how the general consideration in the previous section make sense in the case of 2-fold SUSY though its general form was already obtained by the direct integrations of the constraints [2, 11, 12]. Components of 2-fold supercharges are given by

$$P_2^- = \partial^2 + w_1\partial + w_0, \qquad P_2^+ = \partial^2 - w_1\partial + w_0 - w_1', \tag{3.1}$$

where and hereafter we shall omit the superscript $[\mathcal{N}]$ of $w_k^{[\mathcal{N}]}$ etc. for the simplicity unless the omission would not cause any ambiguity or confusion. The condition for 2-fold SUSY $P_2^- H^- - H^+ P_2^- = 0$ is satisified if and only if the following three equalities hold:

$$V^+ - V^- = w_1', \tag{3.2}$$
$$w_1'' + 2w_0' + 4V^{-\prime} - 2w_1(V^+ - V^-) = 0, \tag{3.3}$$
$$w_0'' + 2V^{-\prime\prime} + 2w_1 V^{-\prime} - 2w_0(V^+ - V^-) = 0. \tag{3.4}$$

Substituting (3.2) into (3.3) and (3.4), and integrating the resulting equation from (3.3), we obtain

$$4V^+ = 3w_1' - 2w_0 + (w_1)^2 - 4C_0, \tag{3.5}$$
$$4V^- = -w_1' - 2w_0 + (w_1)^2 - 4C_0, \tag{3.6}$$
$$-4I_0 = w_1''' - w_1 w_1'' - 2(w_1')^2 + 4w_1' w_0 + 2w_1 w_0' - 2(w_1)^2 w_1' = 0, \tag{3.7}$$

where C_0 is an integral constant. To integrate the third equation (3.7), we shall first make a dimension-preserving transformation $w_0 \to u_0$. We choose it such that it will convert simultaneously both the pairs $P_2^\pm$ and $V^\pm$ into symmetric forms. The most general transformation of polynomial type preserving the physical dimension $[\mathrm{L}^{-2}]$ of w_0 would be

$$w_0 = u_0 + \frac{1}{2}w_1' - \alpha_0(w_1)^2 \ [\mathrm{L}^{-2}], \tag{3.8}$$

where u_0 $[\mathrm{L}^{-2}]$ and α_0 is a dimensionless parameter. We note that the latter transformation indeed renders both the pair of 2-fold supercharge components and the pair of potentials of symmetric forms as

$$P_2^\pm = \partial^2 \mp w_1\partial + u_0 - \alpha_0(w_1)^2 \mp \frac{1}{2}w_1', \tag{3.9}$$
$$4V^\pm = -2u_0 + (2\alpha_0 + 1)(w_1)^2 \pm 2w_1' - 4C_0. \tag{3.10}$$

With the transformation (3.8), the condition (3.7) reads as

$$-4I_0 = w_1''' + 4w_1' u_0 + 2w_1 u_0' - 2(4\alpha_0 + 1)(w_1)^2 w_1' = 0 \ [\mathrm{L}^{-4}]. \tag{3.11}$$

Hence, it gets simplest when

$$\alpha_0 = -1/4. \tag{3.12}$$

The next task we should do is to construct the total differential $\mathrm{d}J_1/\mathrm{d}q$ in (2.20). We note that $\mathrm{d}J_1/\mathrm{d}q$ and I_0 have the physical dimensions $[\mathrm{L}^{-5}]$ and $[\mathrm{L}^{-4}]$, respectively. Hence, the operator L_{10} defined in (2.20) in this case must have the physical dimension $[\mathrm{L}^{-1}]$. Except for the differential operator $\mathrm{d}/\mathrm{d}q$, there is essentially only one multiplicative operator of polynomial type which have that dimension, namely, $L_{10} \propto w_1$ $[\mathrm{L}^{-1}]$. In fact, we can easily check that $w_1 I_0$ is of a total differential form and thus we put

$$16\frac{\mathrm{d}J_1}{\mathrm{d}q} = -8w_1 I_0 = 2w_1 w_1''' + 8w_1 w_1' u_0 + 4(w_1)^2 u_0' = 0 \ [\mathrm{L}^{-5}]. \tag{3.13}$$

The latter differential equation is integrated to yield

$$16J_1[w] = 2w_1 w_1'' - (w_1')^2 + 4(w_1)^2 u_0 = 16C_1 \ [\mathrm{L}^{-4}], \tag{3.14}$$

where C_1 is another integral constant having the correct physical dimension $[\mathrm{L}^{-4}]$ listed in (2.5). Hence, we can express u_0, and thus w_0 as well, in terms of w_1 as

$$u_0 = w_0 - \frac{w_1'}{2} - \frac{(w_1)^2}{4} = -\frac{w_1''}{2w_1} + \frac{(w_1')^2}{4(w_1)^2} + \frac{4C_1}{(w_1)^2}. \tag{3.15}$$

Substituting it into (3.9) and (3.10), we finally get the general form of 2-fold SUSY systems as

$$P_2^{\pm} \overset{2}{\sim} \partial^2 \mp w_1\partial + \frac{(w_1)^2}{4} - \frac{w_1''}{2w_1} + \frac{(w_1')^2}{4(w_1)^2} + \frac{4C_1}{(w_1)^2} \mp \frac{w_1'}{2}, \tag{3.16}$$

$$V^{\pm} \overset{2}{\sim} \frac{(w_1)^2}{8} + \frac{w_1''}{4w_1} - \frac{(w_1')^2}{8(w_1)^2} - \frac{2C_1}{(w_1)^2} \pm \frac{w_1'}{2} - C_0. \tag{3.17}$$

Finally, products of the components of 2-fold supercharges $P_2^{\mp}P_2^{\pm}$ are calculated as

$$P_2^- P_2^+ = 4[(H^+ + C_0)^2 + C_1] - 2I_0, \tag{3.18}$$

$$P_2^+ P_2^- = 4[(H^- + C_0)^2 + C_1] + 2I_0, \tag{3.19}$$

where I_0 and C_1 are given by (3.7) and (3.14), respectively. Hence, we obtain the equality (2.4) for $\mathcal{N} = 2$ as an equivalent relation associated with 2-fold SUSY:

$$P_2^{\mp} P_2^{\pm} \overset{2}{\sim} 4[(H^{\pm} + C_0)^2 + C_1]. \tag{3.20}$$

IV. 3-FOLD SUPERSYMMETRY

Next, we shall reexamine the case of 3-fold SUSY, which was once investigated briefly in Ref. [13], by utilizing our general analysis. Components of 3-fold supercharges are given by

$$\begin{aligned} P_3^- &= \partial^3 + w_2\partial^2 + w_1\partial + w_0, \\ P_3^+ &= -\partial^3 + w_2\partial^2 - (w_1 - 2w_2')\partial + w_0 - w_1' + w_2''. \end{aligned} \tag{4.1}$$

The condition for 3-fold SUSY $P_3^- H^- - H^+ P_3^- = 0$ is satisfied if and only if the following four equalities hold:

$$V^+ - V^- = w_2', \tag{4.2}$$

$$w_2'' + 2w_1' + 6V^{-\prime} - 2w_2(V^+ - V^-) = 0, \tag{4.3}$$

$$w_1'' + 2w_0' + 6V^{-\prime\prime} + 4w_2 V^{-\prime} - 2w_1(V^+ - V^-) = 0, \tag{4.4}$$

$$w_0'' + 2V^{-\prime\prime\prime} + 2w_2 V^{-\prime\prime} + 2w_1 V^{-\prime} - 2w_0(V^+ - V^-) = 0. \tag{4.5}$$

Substituting (4.2) into (4.3)–(4.5) and integrating the resulting equation from (4.3), we obtain

$$6V^{+} = 5w_2' - 2w_1 + (w_2)^2 - 6C_0, \tag{4.6}$$
$$6V^{-} = -w_2' - 2w_1 + (w_2)^2 - 6C_0, \tag{4.7}$$
$$-6I_1 = 3w_2''' + 3w_1'' - 6w_0' - 4w_2w_2'' - 6(w_2')^2 + 6w_2'w_1 + 4w_2w_1' - 4(w_2)^2w_2' = 0, \tag{4.8}$$
$$\begin{aligned}-6I_0 = {} & w_2'''' + 2w_1''' - 3w_0'' - w_2w_2''' - 6w_2'w_2'' + w_2''w_1 + 2w_2w_1'' + 6w_2'w_0 + 2w_1w_1' \\ & - 2(w_2)^2w_2'' - 2w_2(w_2')^2 - 2w_2w_2'w_1 = 0,\end{aligned} \tag{4.9}$$

where C_0 is an integral constant. To integrate the remaining equations (4.8) and (4.9), we shall first construct a set of dimension-preserving transformations $w_k \to u_k$ $(k = 0, 1)$ which will convert simultaneously both the pairs $P_3^{\pm}$ and $V^{\pm}$ into symmetric forms, like (3.8) in the case of 2-fold SUSY. The most general transformations of polynomial type preserving the physical dimensions $[\mathrm{L}^{k-3}]$ of w_k would be

$$\begin{aligned}w_1 &= 6u_1 + w_2' - \alpha_1(w_2)^2 \ [\mathrm{L}^{-2}], \\ w_0 &= u_0 + 3u_1' - \beta_1 w_2'' - \alpha_1 w_2 w_2' - 6\beta_2 w_2 u_1 - \beta_3(w_2)^3 \ [\mathrm{L}^{-3}],\end{aligned} \tag{4.10}$$

where u_k $[\mathrm{L}^{k-3}]$ and α_0, β_k $(k = 1, 2, 3)$ are all dimensionless parameters. They actually render both the pair of 3-fold supercharge components and the pair of potentials of symmetric forms as

$$\begin{aligned}P_3^{\pm} = {} & \mp\partial^3 + w_2\partial^2 \mp [6u_1 - \alpha_1(w_2)^2 \mp w_2']\partial \\ & + u_0 - \beta_1 w_2'' - 6\beta_2 w_2 u_1 - \beta_3(w_2)^3 \mp (3u_1' - \alpha_1 w_2 w_2'),\end{aligned} \tag{4.11a}$$
$$6V^{\pm} = -12u_1 + (2\alpha_1 + 1)(w_2)^2 \pm 3w_2' - 6C_0. \tag{4.11b}$$

With the transformations (4.10), the two constraints (4.8) and (4.9) are equivalent to the following new set of conditions:

$$\begin{aligned}-3\bar{I}_1 &= 9I_1 \\ &= 3(\beta_1 + 1)w_2''' - 3u_0' + 18(\beta_2 + 1)w_2'u_1 + 6(3\beta_2 + 2)w_2u_1' \\ &\quad - (7\alpha_1 - 9\beta_3 + 2)(w_2)^2w_2' = 0 \ [\mathrm{L}^{-4}],\end{aligned} \tag{4.12}$$
$$\begin{aligned}3\bar{I}_0 &= -9(2I_0 - \partial I_1) \\ &= 3u_1''' - (\alpha_1 - 1)w_2w_2''' - 3(2\beta_1 + \alpha_1 + 1)w_2'w_2'' + 6w_2'u_0 \\ &\quad + 72u_1u_1' - 12(2\alpha_1 + 3\beta_2 + 1)w_2w_2'u_1 - 12\alpha_1(w_2)^2u_1' \\ &\quad + 2[2(\alpha_1)^2 + \alpha_1 - 3\beta_3](w_2)^3w_2' = 0 \ [\mathrm{L}^{-5}].\end{aligned} \tag{4.13}$$

Hence, they would get simplest when

$$\alpha_1 = 1, \quad \beta_1 = -1, \quad \beta_2 = -1, \quad \beta_3 = 1. \tag{4.14}$$

With the latter choice of parameter values, the new set of conditions (4.12) and (4.13) reads as

$$\bar{I}_1 = u_0' + 2w_2u_1' = 0 \ [\mathrm{L}^{-4}], \tag{4.15}$$
$$\bar{I}_0 = u_1''' + 2w_2'u_0 + 24u_1u_1' - 4(w_2)^2u_1' = 0 \ [\mathrm{L}^{-5}]. \tag{4.16}$$

The first equation (4.15) enables us to express u_0 in terms of u_1 and w_2 as an indefinite integral

$$u_0 = -2 \int \mathrm{d}q \, w_2 u_1' \; [\mathrm{L}^{-3}]. \tag{4.17}$$

Next, let us construct the total differential $\mathrm{d}J_1/\mathrm{d}q$ in (2.20). We note that $\mathrm{d}J_1/\mathrm{d}q$ and $\bar{I}_k$ $(k = 1, 2)$ have the physical dimensions $[\mathrm{L}^{-5}]$ and $[\mathrm{L}^{k-5}]$, respectively. Hence, the operators L_{1k} defined in (2.20) in this case must have the physical dimension $[\mathrm{L}^{-k}]$. Thus, L_{10} is just a dimensionless constant while $L_{11} \propto w_2$ if we restrict L_{1k} to multiplicative operators of polynomial type. Indeed, we can easily find that one of the latter choices leads to a total differential

$$-4\frac{\mathrm{d}J_1}{\mathrm{d}q} = \bar{I}_0 + 2w_2\bar{I}_1 = u_1''' + 2(w_2'u_0 + w_2u_0') + 24u_1u_1' = 0 \; [\mathrm{L}^{-5}], \tag{4.18}$$

which can be easily integrated as

$$-4J_1[w] = u_1'' + 2w_2u_0 + 12(u_1)^2 = -4C_1 \; [\mathrm{L}^{-4}], \tag{4.19}$$

where C_1 is another integral constant having the correct physical dimension $[\mathrm{L}^{-4}]$ listed in (2.5). From (4.16) and (4.19), we can express u_0 in terms of u_1 and w_2 without recourse to any indefinite integral as

$$u_0 = -\frac{u_1''' + 24u_1u_1' - 4(w_2)^2u_1'}{2w_2'} = -\frac{u_1'' + 12(u_1)^2 + 4C_1}{2w_2}. \tag{4.20}$$

To construct the second integral C_2 of 3-fold SUSY systems, we first note that $\mathrm{d}J_2/\mathrm{d}q$ and $\bar{I}_k$ $(k = 0, 1)$ have the physical dimensions $[\mathrm{L}^{-7}]$ and $[\mathrm{L}^{k-5}]$, respectively. Hence, the operators L_{2k} defined in (2.20) in this case must have the physical dimension $[\mathrm{L}^{-k-2}]$. Thus, candidates for L_{20} are w_2', u_1, and $(w_2)^2$, while those for L_{21} are w_2'', u_1', u_0, w_2w_2', w_2u_1, and $(w_2)^3$, if we restrict L_{2k} to multiplicative operators of polynomial type. With the choice of $L_{20} = 0$ and $L_{21} \propto u_0$, we can construct a total differential as

$$\begin{aligned} 4\frac{\mathrm{d}J_2}{\mathrm{d}q} = u_0\bar{I}_1 &= u_0u_0' + 2w_2u_1'u_0 \\ &= u_0u_0' - u_1'u_1'' - 12(u_1)^2u_1' - 4C_1u_1' = 0 \; [\mathrm{L}^{-7}], \end{aligned} \tag{4.21}$$

where (4.19) has been used. The latter relation is indeed easily integrated as

$$8J_2[w] = (u_0)^2 - (u_1')^2 - 8(u_1)^3 - 8C_1u_1 = 8C_2 \; [\mathrm{L}^{-6}], \tag{4.22}$$

with another integral constant C_2 having the correct physical dimension $[\mathrm{L}^{-6}]$ listed in (2.5). With the use of (4.22) we can express u_0 solely in terms of u_1. Then, substituting the obtained expression for u_0 into (4.15), we can also express w_2 solely in terms of u_1. Hence, we can eventually have an expression for $V^{\pm}$ and $P_3^{\pm}$ in terms of only a single arbitrary function u_1. However, the latter expression is relatively complicated and thus it would be more convenient in practice to express them in terms of two of the three functions w_2, u_1, and u_0. If we eliminate u_0 in (4.11) by using (4.20), we have the expression in terms of w_2 and u_1 as

$$
\begin{aligned}
P_3^{\pm} \overset{3}{\sim}\ & \mp \partial^3 + w_2\partial^2 \mp \left[6u_1 - (w_2)^2 \mp w_2'\right] \partial \\
& + w_2'' + 6w_2u_1 - (w_2)^3 - \frac{u_1''}{2w_2} - \frac{6(u_1)^2}{w_2} - \frac{2C_1}{w_2} \mp (3u_1' - w_2w_2'), && \text{(4.23a)}\\
V^{\pm} =\ & -2u_1 + \frac{1}{2}(w_2)^2 \pm \frac{1}{2}w_2' - C_0. && \text{(4.23b)}
\end{aligned}
$$

On the other hand, if we eliminate w_2 in (4.11) by using (4.15), we have the expression in terms of u_1 and u_0 as

$$
\begin{aligned}
P_3^{\pm} \overset{3}{\sim}\ & \mp \partial^3 - \frac{u_0'}{2u_1'}\partial^2 \mp \left[6u_1 - \frac{(u_0')^2}{4(u_1')^2} \pm \left(\frac{u_0''}{2u_1'} - \frac{u_1''u_0'}{2(u_1')^2}\right)\right] \partial + u_0 - \frac{u_0'''}{2u_1'} + \frac{u_1''u_0''}{(u_1')^2} \\
& + \left(\frac{u_1'''}{2(u_1')^2} - \frac{(u_1'')^2}{(u_1')^3} - \frac{3u_1}{u_1'}\right)u_0' + \frac{(u_0')^3}{8(u_1)^3} \mp \left(3u_1' - \frac{u_0'u_0''}{4(u_1')^2} + \frac{u_1''(u_0')^2}{4(u_1')^3}\right), && \text{(4.24a)}\\
V^{\pm} \overset{3}{\sim}\ & -2u_1 + \frac{(u_0')^2}{8(u_1')^2} \mp \left(\frac{u_0''}{4u_1'} - \frac{u_1''u_0'}{4(u_1')^2}\right) - C_0. && \text{(4.24b)}
\end{aligned}
$$

Finally, products of the components of 3-fold supercharges $P_3^{\mp}P_3^{\pm}$ are calculated as

$$
\begin{aligned}
P_3^-P_3^+ =\ & 8[(H^+ + C_0)^3 + C_1(H^+ + C_0) + C_2] + 3I_1\partial^2 + 2[(2\partial + w_2)I_1 - I_0]\partial \\
& + (2\partial^2 + 2w_2\partial - 2w_2' + w_1)I_1 - 2(\partial + w_2)I_0, && \text{(4.25)}\\
P_3^+P_3^- =\ & 8[(H^- + C_0)^3 + C_1(H^- + C_0) + C_2] - 3I_1\partial^2 - 2[(\partial - w_2)I_1 + I_0]\partial \\
& - w_1I_1 - 2(\partial - w_2)I_0, && \text{(4.26)}
\end{aligned}
$$

where I_1, I_0, C_1, and C_2 are given by (4.8), (4.9), (4.19), and (4.22), respectively. Hence, we obtain the equality (2.4) for $\mathcal{N} = 3$ as an equivalent relation associated with 3-fold SUSY:

$$
P_3^{\mp}P_3^{\pm} \overset{3}{\sim} 8[(H^{\pm} + C_0)^3 + C_1(H^{\pm} + C_0) + C_2]. \tag{4.27}
$$

V. 4-FOLD SUPERSYMMETRY

In this section, we shall study the case of 4-fold supersymmetry. Components of 4-fold supercharges are given by

$$
\begin{aligned}
P_4^- &= \partial^4 + w_3\partial^3 + w_2\partial^2 + w_1\partial + w_0,\\
P_4^+ &= \partial^4 - w_3\partial^3 + (w_2 - 3w_3')\partial^2 - (w_1 - 2w_2' + 3w_3'')\partial + w_0 - w_1' + w_2'' - w_3'''.
\end{aligned} \tag{5.1}
$$

The condition for 4-fold supersymmetry $P_4^-H^- - H^+P_4^- = 0$ is satisfied if and only if the following five equalities hold:

$$
\begin{aligned}
& V^+ - V^- = w_3', && \text{(5.2)}\\
& w_3'' + 2w_2' + 8V^{-\prime} - 2w_3(V^+ - V^-) = 0, && \text{(5.3)}\\
& w_2'' + 2w_1' + 12V^{-\prime\prime} + 6w_3V^{-\prime} - 2w_2(V^+ - V^-) = 0, && \text{(5.4)}\\
& w_1'' + 2w_0' + 8V^{-\prime\prime\prime} + 6w_3V^{-\prime\prime} + 4w_2V^{-\prime} - 2w_1(V^+ - V^-) = 0, && \text{(5.5)}\\
& w_0'' + 2V^{-\prime\prime\prime\prime} + 2w_3V^{-\prime\prime\prime} + 2w_2V^{-\prime\prime} + 2w_1V^{-\prime} - 2w_0(V^+ - V^-) = 0. && \text{(5.6)}
\end{aligned}
$$

Substituting (5.2) into (5.3)–(5.6) and integrating the resulting equation from (5.3), we obtain

$$8V^{+} = 7w_3' - 2w_2 + (w_3)^2 - 8C_0, \tag{5.7}$$

$$8V^{-} = -w_3' - 2w_2 + (w_3)^2 - 8C_0, \tag{5.8}$$

$$-8I_2 = 6w_3''' + 8w_2'' - 8w_1' - 9w_3w_3'' - 12(w_3')^2 + 8w_3'w_2 + 6w_3w_2' - 6(w_3)^2w_3' = 0, \tag{5.9}$$

$$\begin{aligned}-8I_1 = {}& 4w_3'''' + 8w_2''' - 4w_1'' - 8w_0' - 5w_3w_3''' - 24w_3'w_3'' + 2w_3''w_2 + 6w_3w_2'' \\ & + 8w_3'w_1 + 4w_2w_2' - 6(w_3)^2w_3'' - 6w_3(w_3')^2 - 4w_3w_3'w_2 = 0,\end{aligned} \tag{5.10}$$

$$\begin{aligned}-8I_0 = {}& w_3''''' + 2w_2'''' - 4w_0'' - w_3w_3'''' - 8w_3'w_3''' - 6(w_3'')^2 + w_3'''w_2 + 2w_3w_2''' \\ & + w_3''w_1 + 8w_3'w_0 + 2w_2w_2'' + 2w_2'w_1 - 2(w_3)^2w_3''' - 6w_3w_3'w_3'' \\ & - 2w_3w_3''w_2 - 2(w_3')^2w_2 - 2w_3w_3'w_1 = 0.\end{aligned} \tag{5.11}$$

where C_0 is an integral constant. To integrate the remaining equations (5.9)–(5.11), let us first look for a set of dimension-preserving transformations $w_k \to u_k$ $(k = 0, 1, 2)$ which will convert simultaneously both the pairs $P_4^{\pm}$ and $V^{\pm}$ into symmetric forms as in the cases of 2- and 3-fold SUSY. The most general transformations of polynomial type preserving the physical dimensions $[\mathrm{L}^{k-4}]$ of w_k would be

$$w_2 = u_2 + \frac{3}{2}w_3' - \alpha_1(w_3)^2 \ [\mathrm{L}^{-2}], \tag{5.12}$$

$$w_1 = u_1 + u_2' - \beta_1 w_3'' - 2\alpha_1 w_3w_3' - \beta_2 w_3u_2 - \beta_3(w_3)^3 \ [\mathrm{L}^{-3}], \tag{5.13}$$

$$\begin{aligned}w_0 = {}& u_0 + \frac{1}{2}u_1' - \gamma_1 u_2'' - \left(\frac{\beta_1}{2} + \frac{1}{4}\right) w_3''' - \gamma_2 w_3w_3'' - \gamma_3(w_3')^2 - \frac{\beta_2}{2}(w_3u_2)' \\ & - \gamma_4 w_3u_1 - \gamma_5(u_2)^2 - \frac{3\beta_3}{2}(w_3)^2w_3' - \gamma_6(w_3)^2u_2 + \gamma_7(w_3)^4 \ [\mathrm{L}^{-4}],\end{aligned} \tag{5.14}$$

where u_k $[\mathrm{L}^{k-4}]$ and α_0, β_k $(k = 1, 2, 3)$, and γ_k $(k = 1, \ldots, 7)$ are all dimensionless parameters. They indeed make both the pair of 4-fold supercharge components and the pair of potentials symmetric as

$$\begin{aligned}P_4^{\pm} = {}& \partial^4 \mp w_3\partial^3 + \left[u_2 - \alpha_1(w_3)^2 \mp \frac{3}{2}w_3'\right]\partial^2 \mp \Big[u_1 - \beta_1 w_3'' - \beta_2 w_3u_2 - \beta_3(w_3)^3 \\ & \mp (u_2' - 2\alpha_1 w_3w_3')\Big]\partial + u_0 - \gamma_1 u_2'' - \gamma_2 w_3w_3'' - \gamma_3(w_3')^2 - \gamma_4 w_3u_1 - \gamma_5(u_2)^2 \\ & - \gamma_6(w_3)^2u_2 - \gamma_7(w_3)^4 \mp \left[\frac{1}{2}u_1' - \left(\frac{\beta_1}{2} + \frac{1}{4}\right) w_3''' - \frac{\beta_2}{2}(w_3u_2)' - \frac{3\beta_3}{2}(w_3)^2w_3'\right],\end{aligned} \tag{5.15a}$$

$$8V^{\pm} = -2u_2 + (2\alpha_1 + 1)(w_3)^2 \pm 4w_3' - 8C_0. \tag{5.15b}$$

With the transformations (5.14), the remaining three constraints (5.9)–(5.11) are equivalent to the following new set of conditions:

$$\begin{aligned}\bar{I}_2 &= 4I_2 \\ &= -(4\beta_1 + 9)w_3''' + 4u_1' - 4(\beta_2 + 1)w_3'u_2 - (4\beta_2 + 3)w_3u_2' \\ &\quad + (10\alpha_1 - 12\beta_3 + 3)(w_3)^2w_3' = 0 \ [\mathrm{L}^{-4}],\end{aligned} \tag{5.16}$$

$$
\begin{aligned}
4\bar{I}_1 &= 4(I_1 - \partial I_2) \\
&= -2(2\gamma_1+1)u_2''' + 4u_0' - (4\beta_1\gamma_4 - 4\alpha_1 + 4\gamma_2 + 9\gamma_4 + 2)w_3 w_3''' \\
&\quad + 2(6\alpha_1 + 2\beta_1 - 2\gamma_2 - 4\gamma_3 + 3)w_3' w_3'' - 4(\gamma_4+1)w_3' u_1 \\
&\quad - 2(4\gamma_5+1)u_2 u_2' - 2(2\beta_2\gamma_4 - 2\alpha_1 - 2\beta_2 + 2\gamma_4 + 4\gamma_6 - 1)w_3 w_3' u_2 \\
&\quad - (4\beta_2\gamma_4 - 2\alpha_1 + 3\gamma_4 + 4\gamma_6)(w_3)^2 u_2' \\
&\quad - [4(\alpha_1)^2 - 10\alpha_1\gamma_4 + 12\beta_3\gamma_4 + 2\alpha_1 - 3\gamma_4 - 4\beta_3 + 16\gamma_7](w_3)^3 w_3' = 0\ [\mathrm{L}^{-5}], \quad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_0 &= -4(4I_0 - 2\partial I_1 + \partial^2 I_2) \\
&= w_3''''' + 4w_3''' u_2 - (4\beta_1+3)w_3'' u_2' - 4(4\gamma_1+1)w_3' u_2'' + w_3 u_2''' + 16 w_3' u_0 \\
&\quad + 4u_2' u_1 - (6\alpha_1+1)(w_3)^2 w_3''' + 4(2\alpha_1\beta_1 + 2\alpha_1 + \beta_1 - 4\gamma_2)w_3 w_3' w_3'' \\
&\quad + 8(\alpha_1 - 2\gamma_3)(w_3')^3 - 4(2\alpha_1 + 4\gamma_4 + 1)w_3 w_3' u_1 - 16\gamma_5 w_3'(u_2)^2 \\
&\quad - 4\beta_2 w_3 u_2 u_2' + 4(2\alpha_1\beta_2 + \beta_2 - 4\gamma_6)(w_3)^2 w_3' u_2 - 4\beta_3 (w_3)^3 u_2' \quad (5.18) \\
&\quad + 4(2\alpha_1\beta_3 + \beta_3 - 4\gamma_7)(w_3)^4 w_3' = 0\ [\mathrm{L}^{-6}]. \quad (5.19)
\end{aligned}
$$

Hence, they would get simplest when[1]

$$
\begin{aligned}
&\alpha_1 = 3/2, \quad \beta_1 = -9/4, \quad \beta_2 = -1, \quad \beta_3 = 3/2, \quad \gamma_1 = -1/2, \quad \gamma_2 = 1, \\
&\gamma_3 = 11/8, \quad \gamma_4 = -1, \quad \gamma_5 = -1/4, \quad \gamma_6 = 1/2, \quad \gamma_7 = -3/8.
\end{aligned} \quad (5.20)
$$

With the latter choice of parameter values, the new set of conditions (5.16)–(5.19) reads as

$$
\begin{aligned}
\bar{I}_2 &= 4u_1' + w_3 u_2' = 0\ [\mathrm{L}^{-4}], \quad (5.21) \\
\bar{I}_1 &= u_0' = 0\ [\mathrm{L}^{-5}], \quad (5.22) \\
\bar{I}_0 &= w_3''''' + 4w_3''' u_2 + 6w_3'' u_2' + 4w_3' u_2'' + w_3 u_2''' + 16 w_3' u_0 + 4u_2' u_1 \\
&\quad - 10(w_3)^2 w_3''' - 40 w_3 w_3' w_3'' - 10(w_3')^3 + 4w_3'(u_2)^2 + 4w_3 u_2 u_2' \\
&\quad - 24(w_3)^2 w_3' u_2 - 6(w_3)^3 u_2' + 30(w_3)^4 w_3' = 0\ [\mathrm{L}^{-6}]. \quad (5.23)
\end{aligned}
$$

The first equation (5.21) enables us to express u_1 in terms of u_2 and w_3 as an indefinite integral

$$
4u_1 = -\int \mathrm{d}q\, w_3 u_2'\ [\mathrm{L}^{-3}]. \quad (5.24)
$$

The second equation (5.22) just means that u_0 is a constant. On the other hand, C_k ($k = 0, \ldots, \mathcal{N}-1$) are the only constants which appear in general form of $\mathcal{N}$-fold SUSY. The physical dimension of u_0 is $[\mathrm{L}^{-4}]$ and thus we can put

$$
u_0 = 2C_1\ [\mathrm{L}^{-4}], \quad (5.25)
$$

[1] Another possible choice could be $\alpha_1 = 0$, $\beta_1 = -9/4$, $\beta_2 = -3/4$, $\beta_3 = 1/4$, $\gamma_1 = -1/2$, $\gamma_2 = -1/2$, $\gamma_3 = -1/8$, $\gamma_4 = -1$, $\gamma_5 = -1/4$, $\gamma_6 = 0$, $\gamma_7 = 1/16$. With the latter choice, it turns out that the third integral C_3 admits a simpler expression than (5.31) but the expression for the second integral C_2 contains terms which are linear in u_1, in contrast with (5.27) and (5.29).

where C_1 is an integral constant having the correct physical dimension [L^{-4}] listed in (2.5). We have now obtained the first integral C_1 and thus can skip over the step to construct the quantity J_1.

Next, to construct the second integral C_2 of 4-fold SUSY systems, we first note that $\mathrm{d}J_2/\mathrm{d}q$ and $\bar{I}_k$ ($k = 0, 1, 2$) have the physical dimensions [L^{-7}] and [L^{k-6}], respectively. Hence, the operators L_{2k} defined in (2.20) in this case must have the physical dimension [L^{-k-1}]. Thus, candidates for L_{20} are w_3, those for L_{21} are w_3', u_2, and $(w_3)^2$, while those for L_{22} are w_3'', u_2', u_1, $w_3 w_3'$, $w_3 u_2$, and $(w_3)^3$, if we restrict L_{2k} to multiplicative operators of polynomial type. With the choice of $L_{20} \propto w_3$ and $L_{21} = L_{22} = 0$, we can construct a total differential as

$$-128\frac{\mathrm{d}J_2}{\mathrm{d}q} = 2w_3\bar{I}_0 = 0 \ [\mathrm{L}^{-7}], \tag{5.26}$$

where we have omitted the explicit expression. The integration of the latter yields

$$\begin{aligned}-128J_2[w] = {}& 2w_3w_3'''' - 2w_3'w_3''' + (w_3'')^2 - 16(u_1)^2 + 8w_3w_3''u_2 - 4(w_3')^2u_2 \\ & + 4w_3w_3'u_2' + 2(w_3)^2u_2'' - 20(w_3)^3w_3'' - 10(w_3)^2(w_3')^2 + 4(w_3)^2(u_2)^2 \\ & - 12(w_3)^4u_2 + 10(w_3)^6 + 32C_1(w_3)^2 = -128C_2 \ [\mathrm{L}^{-6}],\end{aligned} \tag{5.27}$$

where (5.25) has been applied, and C_2 is another integral constant having the correct physical dimension [L^{-6}] listed in (2.5). From (5.23) and (5.25), we can express u_1 in terms of u_2 and w_3 without recourse to any indefinite integral as

$$\begin{aligned}u_1 = {}& [-w_3''''' - 4w_3'''u_2 - 6w_3''u_2' - 4w_3'u_2'' - w_3u_2''' + 10(w_3)^2w_3''' \\ & + 40w_3w_3'w_3'' + 10(w_3')^3 - 4w_3'(u_2)^2 - 4w_3u_2u_2' + 24(w_3)^2w_3'u_2 \\ & + 6(w_3)^3u_2' - 30(w_3)^4w_3' - 32C_1w_3']/(4u_2').\end{aligned} \tag{5.28}$$

Instead, using (5.27), we can express u_1 in terms of u_2 and w_3 as a solution to the following quadratic equation

$$\begin{aligned}16(u_1)^2 = {}& 2w_3w_3'''' - 2w_3'w_3''' + (w_3'')^2 + 8w_3w_3''u_2 - 4(w_3')^2u_2 + 4w_3w_3'u_2' \\ & + 2(w_3)^2u_2'' - 20(w_3)^3w_3'' - 10(w_3)^2(w_3')^2 + 4(w_3)^2(u_2)^2 \\ & - 12(w_3)^4u_2 + 10(w_3)^6 + 32C_1(w_3)^2 + 128C_2.\end{aligned} \tag{5.29}$$

To obtain the third integral C_3 of 4-fold SUSY systems, we note that $\mathrm{d}J_3/\mathrm{d}q$ and $\bar{I}_k$ ($k = 0, 1, 2$) have the physical dimensions [L^{-9}] and [L^{k-6}], respectively. Hence, the operators L_{3k} defined in (2.20) in this case must have the physical dimension [L^{-k-3}]. With a similar dimensonal analysis, we find that the choice of $L_{30} \propto w_3'' + 4u_1 + 2w_3u_2 - 2(w_3)^3$ and $L_{31} = L_{32} = 0$ leads to a total differential:

$$512\frac{\mathrm{d}J_3}{\mathrm{d}q} = [w_3'' + 4u_1 + 2w_3u_2 - 2(w_3)^3]\bar{I}_0 = 0 \ [\mathrm{L}^{-9}], \tag{5.30}$$

where we have omitted the explicit expression. The integral of the latter equation reads as

$$\begin{aligned}
1024J_3[w] &= 2w_3''w_3'''' - (w_3''')^2 + 8w_3''''u_1 + 4w_3w_3''''u_2 - 4w_3'w_3'''u_2 + 6(w_3'')^2u_2 \\
&- 2w_3w_3'''u_2' + 6w_3'w_3''u_2' + 2w_3w_3''u_2'' + 32w_3''u_2u_1 + 24w_3'u_2'u_1 + 8w_3u_2''u_1 \\
&+ 32u_2(u_1)^2 - 4(w_3)^3w_3'''' + 12(w_3)^2w_3'w_3''' - 16(w_3)^2(w_3'')^2 - 24w_3(w_3')^2w_3'' \\
&+ (w_3')^4 - 80(w_3)^2w_3''u_1 - 80w_3(w_3')^2u_1 + 16w_3w_3''(u_2)^2 - 4(w_3')^2(u_2)^2 \\
&+ 8w_3w_3'u_2u_2' + 4(w_3)^2u_2u_2'' - (w_3)^2(u_2')^2 + 32w_3(u_2)^2u_1 - 56(w_3)^3w_3''u_2 \\
&- 20(w_3)^2(w_3')^2u_2 - 4(w_3)^4u_2'' - 64(w_3)^3u_2u_1 + 8(w_3)^2(u_2)^3 + 40(w_3)^5w_3'' \\
&+ 10(w_3)^4(w_3')^2 + 48(w_3)^5u_1 - 28(w_3)^4(u_2)^2 + 36(w_3)^6u_2 - 15(w_3)^8 + 32C_1(w_3')^2 \\
&+ 256C_1w_3u_1 + 64C_1(w_3)^2u_2 - 32C_1(w_3)^4 + 256(C_1)^2 = 1024C_3\ [\mathrm{L}^{-8}], \qquad (5.31)
\end{aligned}$$

with another integral constant C_3 having the correct physical dimension $[\mathrm{L}^{-8}]$ listed in (2.5). By using (at least) two of the equalities (5.28), (5.29), and (5.31), we can eliminate u_1 to obtain the relation between w_3 and u_2. Hence, we are now, in principle, able to express a 4-fold SUSY system solely in terms of a single arbitrary function. As in the case of 3-fold SUSY, however, it would be more convenient for a practical purpose to have an expression in terms of two of the four functions w_3, u_2, u_1, and u_0. For example, if we eliminate u_1 and u_0 in the system by using (5.28) and (5.25), respectively, the system (5.15) with the parameter values (5.20) can be represented in terms of w_3 and u_2 as

$$\begin{aligned}
P_4^{\pm} &\overset{4}{\sim} \partial^4 \mp w_3\partial^3 + \left[u_2 - \frac{3}{2}(w_3)^2 \mp \frac{3}{2}w_3'\right]\partial^2 \mp \left[\frac{3}{4}w_3'' - \frac{w_3'''''}{4u_2'} - \frac{w_3'''u_2}{u_2'} - \frac{w_3'u_2''}{u_2'} - \frac{w_3u_2'''}{4u_2'}\right. \\
&+ \frac{5(w_3)^2w_3'''}{2u_2'} + \frac{10w_3w_3'w_3''}{u_2'} + \frac{5(w_3')^3}{2u_2'} - \frac{w_3'(u_2)^2}{u_2'} + \frac{6(w_3)^2w_3'u_2}{u_2'} - \frac{15(w_3)^4w_3'}{2u_2'} \\
&\left. - \frac{8C_1w_3'}{u_2'} \mp (u_2' - 3w_3w_3')\right]\partial + \frac{1}{2}u_2'' - \frac{5}{2}w_3w_3'' - \frac{11}{8}(w_3')^2 + \frac{1}{4}(u_2)^2 - \frac{3}{2}(w_3)^2u_2 \\
&+ \frac{15}{8}(w_3)^4 - \frac{w_3w_3'''''}{4u_2'} - \frac{w_3w_3'''u_2}{u_2'} - \frac{w_3w_3'u_2''}{u_2'} - \frac{(w_3)^2u_2'''}{4u_2'} + \frac{5(w_3)^3w_3'''}{2u_2'} \\
&+ \frac{10(w_3)^2w_3'w_3''}{u_2'} + \frac{5w_3(w_3')^3}{2u_2'} - \frac{w_3w_3'(u_2)^2}{u_2'} + \frac{6(w_3)^3w_3'u_2}{u_2'} - \frac{15(w_3)^5w_3'}{2u_2'} \\
&- \frac{8C_1w_3w_3'}{u_2'} + 2C_1 \mp \left[\frac{7}{8}w_3''' + \frac{1}{2}w_3'u_2 + \frac{3}{8}w_3u_2' - \frac{9}{4}(w_3)^2w_3'\right], \qquad (5.32) \\
V^{\pm} &= -\frac{1}{4}u_2 + \frac{1}{2}(w_3)^2 \pm \frac{1}{2}w_3' - C_0. \qquad (5.33)
\end{aligned}$$

Finally, products of the components of 4-fold supercharges $P_4^{\mp}P_4^{\pm}$ are calculated as

$$P_4^-P_4^+ = 16[(H^+ + C_0)^4 + C_1(H^+ + C_0)^2 + C_2(H^+ + C_0) + C_3] + \sum_{i=0}^{4} f_i^+[w]\partial^i, \qquad (5.34)$$

$$P_4^+P_4^- = 16[(H^- + C_0)^4 + C_1(H^- + C_0)^2 + C_2(H^- + C_0) + C_3] + \sum_{i=0}^{4} f_i^-[w]\partial^i, \qquad (5.35)$$

C_1, C_2, and C_3 are given by (5.25), (5.27), and (5.31), respectively. Both of the second terms in the above are elements of $\mathcal{K}_0^{[4]}$ introduced in Section II B and have the form of (2.23) with

$$\begin{aligned}
f_4^+ =&\ -4I_2, \qquad f_3^+ = 4I_1 - 4(3\partial + w_3)I_2,\\
f_2^+ =&\ -4I_0 + (8\partial + 3w_3)I_1 - \frac{1}{2}[28\partial^2 + 15w_3\partial - 9w_3' + 4w_2 + (w_3)^2]I_2,\\
f_1^+ =&\ -2(2\partial + w_3)I_0 + 2(3\partial^2 + 2w_3\partial - 2w_3' + w_2)I_1\\
&\ -2[4\partial^3 + 3w_3\partial^2 - 2(2w_3' - w_2)\partial - 4w_3'' + w_1 - 2w_3w_3']I_2,\\
f_0^+ =&\ -2(\partial^2 + w_3\partial - 2w_3' + w_2)I_0 + [2\partial^3 + 2w_3\partial^2 - 2(2w_3' - w_2)\partial - 4w_3'' + w_1\\
&\ -2w_3w_3']I_1 - \frac{1}{16}[32\partial^4 + 32w_3\partial^3 - 32(2w_3' - w_2)\partial^2 - 4(31w_3'' - 2w_2' - 6w_1\\
&\ + 14w_3w_3')\partial - 60w_3''' + 8w_2'' + 16w_0 - 68w_3w_3'' + 7(w_3')^2 - 4w_3'w_2 + 8w_3w_2'\\
&\ -4(w_2)^2 - 22(w_3)^2w_3' + 4(w_3)^2w_2 - (w_3)^4]I_2,
\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}
f_4^- =&\ 4I_2, \qquad f_3^- = 4I_1 + 4(\partial - w_3)I_2,\\
f_2^- =&\ 4I_0 + (4\partial - 3w_3)I_1 + \frac{1}{2}[4\partial^2 - 3w_3\partial - 3w_3' + 4w_2 - (w_3)^2]I_2,\\
f_1^- =&\ 2(2\partial - w_3)I_0 + 2(\partial^2 - w_3\partial - w_3' + w_2)I_1 + 2(w_3'' + 2w_2' - w_1 - 2w_3w_3')I_2,\\
f_0^- =&\ 2(\partial^2 - w_3\partial - w_3' + w_2)I_0 + (w_3'' + 2w_2' - w_1 - 2w_3w_3')I_1 - \frac{1}{16}[4(w_3'' + 2w_2'\\
&\ -2w_1 - 2w_3w_3')\partial - 12w_3''' - 24w_2'' + 16w_0 + 28w_3w_3'' + 23(w_3')^2 - 4w_3'w_2\\
&\ + 8w_3w_2' - 4(w_2)^2 - 6(w_3)^2w_3' + 4(w_3)^2w_2 - (w_3)^4]I_2,
\end{aligned} \tag{5.37}$$

where I_2, I_1, and I_0 are given by (5.9), (5.10), and (5.11), respectively. Hence, we obtain the equality (2.4) for $\mathcal{N} = 4$ as an equivalent relation associated with 4-fold SUSY:

$$P_4^{\mp}P_4^{\pm} \overset{4}{\sim} 16[(H^{\pm} + C_0)^4 + C_1(H^{\pm} + C_0)^2 + C_2(H^{\pm} + C_0) + C_3]. \tag{5.38}$$

VI. DISCUSSION AND SUMMARY

In this work, we have clarified general structure of $\mathcal{N}$-fold SUSY systems by considering dimensional analysis and introducing the equivalent classes of linear differential operators associated with them. We have then shown that the latter general consideration is in fact effective in constructing the most general $\mathcal{N}$-fold SUSY systems and their integral constants for $\mathcal{N} = 2$, 3, and 4. Application to systems for $\mathcal{N} > 4$ would be straightforward and the problems would get more transparent even though still remain highly complicated. Finally, some remarks on the future issues are in order.

1. Although we have only considered ordinary one-dimensional Schrödinger operators, generalization to other operators would be possible. In physical applications, one of the interesting extensions is to a quantum system with position-dependent mass for which $\mathcal{N}$-fold SUSY was successfully formulated in Ref. [14]. In the latter case, there is an additional freedom of mass function and it is particularly interesting to see how its existence would force us to modify or generalize the general considerations made in Section II.

2. In this work, we have restricted the dimension-preserving transformations (3.8), (4.10), and (5.14) to polynomial type, namely, transformations which are polynomials in $w_k^{[\mathcal{N}](m)}$ $(m = 0, 1, 2, \ldots)$. On the other hand, the obtained results such as (3.15), (4.20), and (5.28) indicate that the relations among $w_k^{[\mathcal{N}](m)}$ are in general expressed by rational functions. Hence, we may be able to reduce further the complexity of the conditions for $\mathcal{N}$-fold SUSY by extending the type of transformations from polynomial to rational function. However, the number of admissible forms of rational functions which preserve physical dimension would drastically increase. For instance, any sum of rational functions of the form $f_n^{[\mathcal{N}]}[w]/g_n^{[\mathcal{N}]}[w]$ $(n \in \mathbb{Z})$

$$w_k^{[\mathcal{N}]} = u_k^{[\mathcal{N}]} + \sum_{n\in\mathbb{Z}} \frac{f_n^{[\mathcal{N}]}[w]}{g_n^{[\mathcal{N}]}[w]} \; [\mathrm{L}^{k-\mathcal{N}}],$$

where $f_n^{[\mathcal{N}]}[w]$ and $g_n^{[\mathcal{N}]}[w]$ are polynomials in $w_k^{[\mathcal{N}](m)}$ having the physical dimensions $[\mathrm{L}^{n+k-\mathcal{N}}]$ and $[\mathrm{L}^n]$, respectively, can serve as a (part of) transformation which preserves the physical dimension $[\mathrm{L}^{k-\mathcal{N}}]$ of $w_k^{[\mathcal{N}]}$. As a result, we may need additional guidelines to restrict the forms of transformations to make an efficient analysis. We are curious to know how to get such guidelines systematically for the purpose.

3. As was pointed out in Section II, an $\mathcal{N}$-fold SUSY system is in general only weakly quasi-solvable but is not quasi-solvable in the strong sense and thus does not necessarily admit analytic local solutions in closed form. In the case of $\mathcal{N} = 2$, it was proved in Ref. [15] that type A 2-fold SUSY is a necessary and sufficient condition for a one-dimensional quantum mechanical system to have quasi-solvability in the strong sense with two independent analytic local solutions. So, it is interesting to see what kind of condition is necessary and sufficient for a quantum system to admit three or four independent analytic local solutions. We will report on the latter subjects in subsequent publications.

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