

NEW IRREDUCIBLE MODULES FOR HEISENBERG AND AFFINE LIE ALGEBRAS

VIKTOR BEKKERT, GEORGIA BENKART, VYACHESLAV FUTORNY, AND IRYNA KASHUBA

ABSTRACT. We study \mathbb{Z} -graded modules of nonzero level with arbitrary weight multiplicities over Heisenberg Lie algebras and the associated generalized loop modules over affine Kac-Moody Lie algebras. We construct new families of such irreducible modules over Heisenberg Lie algebras. Our main result establishes the irreducibility of the corresponding generalized loop modules providing an explicit construction of many new examples of irreducible modules for affine Lie algebras. In particular, to any function $\varphi : \mathbb{N} \rightarrow \{\pm\}$ we associate a φ -highest weight module over the Heisenberg Lie algebra and a φ -imaginary Verma module over the affine Lie algebra. We show that any φ -imaginary Verma module of nonzero level is irreducible.

1. INTRODUCTION

Affine Lie algebras are the most studied among the infinite-dimensional Kac-Moody Lie algebras and have widespread applications. Their representation theory is far richer than that of finite-dimensional simple Lie algebras. In particular, affine Lie algebras have modules containing both finite- and infinite-dimensional weight spaces, something that cannot happen in the finite-dimensional setting. These representations arise from taking non-standard partitions of the root system; that is, partitions which are not equivalent under the Weyl group to the standard partition into positive and negative roots (see [DFG]). For affine Lie algebras, there are always only finitely many equivalence classes of such nonstandard partitions (see [F4]). Corresponding to each partition is a Borel subalgebra, and one can form representations induced from one-dimensional modules for these Borel subalgebras. These modules, often referred to as *Verma-type modules*, were first studied by Jakobsen and Kac [JK], and by Futorny [F3, F4]. Results on the structure of Verma-type modules can also be found in ([Co, F1, FS]).

Let \mathfrak{g} be an affine Lie algebra, \mathfrak{h} its standard Cartan subalgebra, and $\mathfrak{z} = \mathbb{C}c$ its center, where c is the canonical central element. Let \mathbf{V} be a weight \mathfrak{g} -module, that is, $\mathbf{V} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbf{V}_\mu$, where $\mathbf{V}_\mu = \{v \in \mathbf{V} \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. If \mathbf{V} is irreducible, then c acts as a scalar on \mathbf{V} called the *level* of \mathbf{V} . The theory of Verma-type modules is best developed in the case when the level is nonzero [F4]. For example, the so-called imaginary Verma modules induced from the natural Borel subalgebra are always irreducible when the level is nonzero ([JK], [F2]).

The classification of irreducible modules is known only for modules with finite-dimensional weight spaces (see [FT]) and for certain subcategories of induced modules with some infinite-dimensional weight spaces (see for example, [F3], [FKM], [FK]). Our main goal

is to go beyond the modules with finite-dimensional weight spaces and to construct new irreducible modules of nonzero level with infinite-dimensional weight spaces. Examples of such modules have been constructed previously by Chari and Pressley in [CP] as the tensor product of highest and lowest weight modules.

Here we consider different Borel-type subalgebras that do not correspond to partitions of the root system of \mathfrak{g} . Such a subalgebra is determined by a function $\varphi : \mathbb{N} \rightarrow \{\pm\}$ on the set \mathbb{N} of positive integers, and so is denoted \mathfrak{b}_φ . The subalgebra \mathfrak{b}_φ gives rise to a class of \mathfrak{g} -modules called *φ -imaginary Verma modules*. These modules can be viewed as induced from φ -highest weight modules over the Heisenberg subalgebra of \mathfrak{g} . This construction is similar to the construction of imaginary Whittaker modules in [Ch], but unlike the modules in [Ch], our modules over the Heisenberg subalgebra are \mathbb{Z} -graded. If $\varphi(n) = +$ for all $n \in \mathbb{N}$, then \mathfrak{b}_φ is the natural Borel subalgebra of \mathfrak{g} .

We establish a criterion for the irreducibility of φ -imaginary Verma modules. It comes as no surprise that any such module is irreducible if and only if it has a nonzero level.

Next we consider the classification problem for irreducible \mathbb{Z} -graded modules for the Heisenberg subalgebra of \mathfrak{g} . The ones of level zero were determined by Chari [C]. Any such module of nonzero level with a \mathbb{Z} -grading has all its graded components infinite-dimensional by [F1]; otherwise, it is a highest weight module. We classify all locally-finite \mathbb{Z} -graded irreducible modules of nonzero level for an arbitrary infinite-dimensional Heisenberg Lie algebra. We show that these modules have a \mathbb{Z}^∞ -gradation and can be obtained from weight modules over an associated Weyl algebra as in [BBF] by compression of the gradation. This leads to an equivalence between the category of locally-finite \mathbb{Z} -graded modules for Heisenberg Lie algebras and a certain category of weight modules for the Weyl algebra A_∞ . Examples of such modules were considered earlier by Casati [Ca], where they were constructed by means of an action of differential operators on a space of polynomials in infinitely many variables (compare Theorem 4.21 below).

We use parabolic induction to construct generalized loop modules for the affine Lie algebra \mathfrak{g} . The modules are induced from an arbitrary irreducible \mathbb{Z} -graded module of nonzero level for the Heisenberg subalgebra of \mathfrak{g} . This construction extends Chari's construction to the nonzero level case. Our main result establishes the irreducibility of any generalized loop module induced from a locally-finite irreducible module of nonzero level for the Heisenberg subalgebra. By this process, we obtain new families of irreducible modules of nonzero level for any affine Lie algebra. The irreducible modules constructed in [Ca] are "dense" in the sense that they have the maximal possible set of weights and hence are different from the ones studied here.

It should be noted that all results in our paper hold for both the untwisted and twisted affine Lie algebras.

The structure of the paper is as follows. In Section 3, we construct φ -imaginary Verma modules for affine Lie algebras for any function $\varphi : \mathbb{N} \rightarrow \{\pm\}$, and establish a criterion for their irreducibility in Theorem 3.5. In Section 4, we study \mathbb{Z} -graded irreducible modules for the Heisenberg algebra. Theorems 4.5 and 4.15 (see Corollary 4.16) provide the classification of all irreducible locally-finite modules of nonzero level for the Heisenberg algebra. Finally, in Section 5, we introduce generalized loop modules for affine Lie algebras and

study their structure. These modules are induced from irreducible locally-finite modules over the Heisenberg subalgebra of \mathfrak{g} . Our main result is Theorem 5.6 which establishes the irreducibility of any generalized loop module induced from an irreducible locally-finite module of nonzero level for the Heisenberg subalgebra. Hence, we develop a method for constructing new irreducible modules for affine Lie algebras starting from an irreducible locally-finite module over the Heisenberg subalgebra. In general, these modules cannot be obtained by the pseudo-parabolic induction method considered in [FK].

2. PRELIMINARIES

Let \mathfrak{g} denote an affine Lie algebra over the complex numbers \mathbb{C} . Associated to \mathfrak{g} is a finite-dimensional simple Lie subalgebra $\dot{\mathfrak{g}}$ with Cartan subalgebra $\dot{\mathfrak{h}}$ and root system $\dot{\Delta}$. There are elements c, d of \mathfrak{g} (the canonical central element c and the degree derivation d) so that $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Cartan subalgebra of \mathfrak{g} , and $\mathfrak{z} = \mathbb{C}c$ is the center of \mathfrak{g} . The algebra \mathfrak{g} has a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$$

relative to \mathfrak{h} , where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. The set $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is the root system of \mathfrak{g} . Let $\dot{\Delta} = \dot{\Delta}_+ \cup \dot{\Delta}_-$ be a decomposition of the corresponding finite root system $\dot{\Delta}$ into positive and negative roots relative to a base Π of simple roots. When there are two root lengths, let $\dot{\Delta}_l$ and $\dot{\Delta}_s$ denote the long and short roots in $\dot{\Delta}$ respectively. The root system Δ of \mathfrak{g} has a natural partition into positive and negative roots, $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_- = -\Delta_+$. Moreover, $\Delta_+ = \Delta_+^{\text{re}} \cup \Delta_+^{\text{im}}$, where the imaginary positive roots $\Delta_+^{\text{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$ are positive integer multiples of the indivisible imaginary root δ , and the real positive roots Δ_+^{re} are given by

$$(2.1) \quad \Delta_+^{\text{re}} = \begin{cases} \left\{ \begin{array}{ll} \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\}, & \text{if } r = 1 \text{ (the untwisted case),} \\ \{\alpha + n\delta \mid \alpha \in (\dot{\Delta}_s)_+, n \in \mathbb{Z}\} \cup \{\alpha + nr\delta \mid \alpha \in (\dot{\Delta}_l)_+, n \in \mathbb{Z}\} & \text{if } r = 2, 3 \text{ and} \\ & \text{not } A_{2\ell}^{(2)} \text{ type,} \end{array} \right. \\ \left\{ \begin{array}{ll} \{\alpha + n\delta \mid \alpha \in (\dot{\Delta}_s)_+, n \in \mathbb{Z}\} \cup \{\alpha + 2n\delta \mid \alpha \in (\dot{\Delta}_l)_+, n \in \mathbb{Z}\} \\ \cup \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in (\dot{\Delta}_l)_+, n \in \mathbb{Z}\} & \text{if } A_{2\ell}^{(2)} \text{ type.} \end{array} \right. \end{cases}$$

We refer to [K] for basic results on Kac-Moody theory and for the notation used in (2.1).

A subset S of Δ affords a *partition* of Δ if $S \cup (-S) = \Delta$ and $S \cap (-S) = \emptyset$. A partition $\Delta = S \cup (-S)$ is said to be *closed* if whenever α and β are in S and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in S$. For any S giving a closed partition of Δ , the spaces $\mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{-S} = \bigoplus_{\alpha \in -S} \mathfrak{g}_\alpha$ are subalgebras of \mathfrak{g} , and $\mathfrak{g} = \mathfrak{g}_{-S} \oplus \mathfrak{h} \oplus \mathfrak{g}_S$ is a triangular decomposition of \mathfrak{g} .

A *weight module* V with respect to \mathfrak{h} has a decomposition $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$, and we say that the set of weights of V is the *support* of V and write $\text{supp}(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$.

2.1. Imaginary Verma modules.

Let $\Delta = \mathbf{S}\mathbf{U}(-\mathbf{S})$ denote a closed partition of Δ . By the Poincaré-Birkhoff-Witt theorem, the triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-\mathbf{S}} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\mathbf{S}}$ of \mathfrak{g} afforded by \mathbf{S} determines a triangular decomposition of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of \mathfrak{g} given by $\mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{g}_{-\mathbf{S}}) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g}_{\mathbf{S}})$. Let $\mathfrak{b}_{\mathbf{S}} = \mathfrak{h} \oplus \mathfrak{g}_{\mathbf{S}}$ be the associated Borel subalgebra. Any $\lambda \in \mathfrak{h}^*$ extends to an algebra homomorphism (also denoted by λ) on the enveloping algebras $\mathbf{U}(\mathfrak{h})$ and $\mathbf{U}(\mathfrak{b}_{\mathbf{S}})$ with zero values on $\mathfrak{g}_{\mathbf{S}}$. Corresponding to any such λ is a one-dimensional $\mathbf{U}(\mathfrak{b}_{\mathbf{S}})$ -module $\mathbb{C}v$ with $xv = \lambda(x)v$ for all $x \in \mathbf{U}(\mathfrak{b}_{\mathbf{S}})$. The induced module

$$\mathbf{M}_{\mathbf{S}}(\lambda) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{b}_{\mathbf{S}})} \mathbb{C}v,$$

is a *Verma type module* as defined in [Co] and [FS]. The canonical central element c acts by multiplication by the scalar $\lambda(c)$ on $\mathbf{M}_{\mathbf{S}}(\lambda)$, and we say that $\lambda(c)$ is the *level* of $\mathbf{M}_{\mathbf{S}}(\lambda)$. Clearly, $\mathbf{M}_{\mathbf{S}}(\lambda) \simeq \mathbf{U}(\mathfrak{g}_{-\mathbf{S}})$ as a $\mathfrak{g}_{-\mathbf{S}}$ -module.

When $\mathbf{S} = \Delta_+$, the module $\mathbf{M}_{\mathbf{S}}(\lambda)$ is an *imaginary Verma module*. It was shown in [F2] that the imaginary Verma module $\mathbf{M}_{\mathbf{S}}(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

3. VERMA MODULES CORRESPONDING TO THE MAP φ

3.1. φ -Verma modules for the Heisenberg subalgebra.

The subspace $\mathbf{L} := \mathbb{C}c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{n\delta}$ forms a Heisenberg Lie subalgebra of the affine algebra \mathfrak{g} . Thus, $[x, y] = \xi(x, y)c$ for all $x \in \mathfrak{g}_{m\delta}, y \in \mathfrak{g}_{n\delta}$, where $\xi(x, y)$ is a certain skew-symmetric bilinear form with $\xi(\mathfrak{g}_{m\delta}, \mathfrak{g}_{n\delta}) = 0$ if $n \neq -m$, and whose restriction to $\mathfrak{g}_{m\delta} \times \mathfrak{g}_{-m\delta}$ is nondegenerate for all $m \neq 0$. The algebra \mathbf{L} has a triangular decomposition $\mathbf{L} = \mathbf{L}^- \oplus \mathbb{C}c \oplus \mathbf{L}^+$, where $\mathbf{L}^{\pm} = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}_{\pm n\delta}$.

Now let $\varphi : \mathbb{N} \rightarrow \{\pm\}$ be an arbitrary function defined on $\mathbb{N} = \{1, 2, \dots\}$. The spaces

$$\mathbf{L}_{\varphi}^{\pm} = \left(\bigoplus_{n \in \mathbb{N}, \varphi(n) = \pm} \mathfrak{g}_{n\delta} \right) \oplus \left(\bigoplus_{m \in \mathbb{N}, \varphi(m) = \mp} \mathfrak{g}_{-m\delta} \right)$$

are abelian subalgebras of \mathbf{L} , and

$$\mathbf{L} = \mathbf{L}_{\varphi}^- \oplus \mathbb{C}c \oplus \mathbf{L}_{\varphi}^+$$

is a triangular decomposition. Of course, if $\varphi(n) = +$ for all $n \in \mathbb{N}$, then $\mathbf{L}_{\varphi}^+ = \mathbf{L}^+$, and this is just the triangular decomposition above.

Let $\mathbb{C}v$ be a one-dimensional representation of $\mathbb{C}c \oplus \mathbf{L}_{\varphi}^+$, where $cv = av$ for some $a \in \mathbb{C}$ and $\mathbf{L}_{\varphi}^+ v = 0$. The corresponding φ -Verma module is the induced module

$$\mathbf{M}_{\varphi}(a) = \mathbf{U}(\mathbf{L}) \otimes_{\mathbf{U}(\mathbb{C}c \oplus \mathbf{L}_{\varphi}^+)} \mathbb{C}v.$$

Clearly, $\mathbf{M}_{\varphi}(a)$ is free as a $\mathbf{U}(\mathbf{L}_{\varphi}^-)$ -module of rank 1 generated by the vector $1 \otimes v$.

When $\varphi(n) = +$ for all $n \in \mathbb{N}$, then $\mathbf{M}_{\varphi}(a)$ is just the usual Verma module for the Heisenberg Lie algebra \mathbf{L} . Note that if $\varphi_1 \neq \varphi_2$ then $\mathbf{M}_{\varphi_1}(a)$ and $\mathbf{M}_{\varphi_2}(a)$ are not isomorphic.

Remark 3.1. Let $\mathbf{S} = \Delta_+$ and consider the imaginary Verma module $\mathbf{M}_{\mathbf{S}}(\lambda)$ where $\lambda \in \mathfrak{h}^*$. This module has both finite- and infinite-dimensional weight spaces relative to \mathfrak{h} . By [F2], the sum of the finite-dimensional weight spaces in $\mathbf{M}_{\mathbf{S}}(\lambda)$ is the Verma module $\mathbf{M}_{\varphi}(\lambda(c))$ for the Heisenberg subalgebra \mathbf{L} , where φ is the function with $\varphi(n) = +$ for all $n \in \mathbb{N}$.

Since $U(\mathfrak{L})$ has a natural \mathbb{Z} -gradation, we obtain for an arbitrary function φ the following:

Proposition 3.2. $M_\varphi(a)$ is a \mathbb{Z} -graded \mathfrak{L} -module, where

$$M_\varphi(a) = \bigoplus_{n \in \mathbb{Z}} M_\varphi(a)_n,$$

and $M_\varphi(a)_n = U(\mathfrak{L}_\varphi^-)_n v$. If $\varphi(k) \neq \varphi(\ell)$ for some $k, \ell \in \mathbb{N}$, then $M_\varphi(a)_n$ is infinite-dimensional for any $n \in \mathbb{Z}$.

Proof. Suppose $n \in \mathbb{Z}_{\geq 0}$ and set $M = M_\varphi(a)$.

If $\varphi(1) = -$, then there is some $r \in \mathbb{N}$ so that $\varphi(r) = +$. Let $x \in \mathfrak{L}_\delta$ and $y \in \mathfrak{L}_{-r\delta}$ be nonzero. Since the vectors $x^{n+kr}y^k v \in M_n$ are linearly independent for all $k \geq 0$, we have that M_n is infinite dimensional. Similarly, the vectors $x^{(rk-1)n}y^{kn} v \in M_{-n}$ are linearly independent for all $k \geq 1$, so that M_{-n} is infinite dimensional as well. The argument when $\varphi(1) = +$ is analogous. \square

Proposition 3.3. $M_\varphi(a)$ is irreducible if and only if $a \neq 0$.

Proof. Let $\{x_i\}_{i \in \mathbb{N}}$ be a basis of \mathfrak{L}_φ^+ of root vectors, and let $\{y_i\}_{i \in \mathbb{N}}$ be the dual basis of \mathfrak{L}_φ^- so that $[x_i, y_j] = \delta_{i,j}c$ for all i, j . Set $\bar{k} = (k_1, k_2, \dots)$ where $k_i \in \mathbb{N} \cup \{0\}$ for each i and only finitely many k_i are nonzero, and say $\bar{k} < \bar{\ell}$ if there is some s such that $k_i = \ell_i$ for $i < s$ and $k_s < \ell_s$. The elements $x(\bar{k}) = \prod_i x_i^{k_i}$ and $y(\bar{k}) = \prod_i y_i^{k_i}$ are well defined since \mathfrak{L}_φ^+ and \mathfrak{L}_φ^- are abelian. Moreover, the vectors $y(\bar{k})v$ as \bar{k} ranges over all such tuples in $(\mathbb{N} \cup \{0\})^\infty$ form a basis for $M_\varphi(a)$.

Now suppose $a \neq 0$, and let $w = \sum_{\bar{k}} \xi(\bar{k})y(\bar{k})v$ be a nonzero element of $M_\varphi(a)$, where only finitely many of the scalars $\xi(\bar{k}) \in \mathbb{C}$ are nonzero. Let \bar{m} be the largest tuple with $\xi(\bar{m}) \neq 0$. Then since $x_i y_i^\ell v = [x_i, y_i^\ell]v = \ell a y_i^{\ell-1} v$ for each $\ell \geq 0$, it follows that

$$x(\bar{m})w = \xi(\bar{m}) \left(\prod_i m_i! \right) a^{\sum_i m_i} v.$$

Since $a \neq 0$, this implies that the submodule generated by w contains v and so is all of $M_\varphi(a)$. But w was an arbitrary nonzero element, so $M_\varphi(a)$ is irreducible in this case.

If $a = 0$, then $\mathfrak{N} := \bigoplus_{\bar{k} \neq \bar{0}} \mathbb{C}y(\bar{k})v$ is a proper submodule. \square

3.2. φ -imaginary Verma modules for \mathfrak{g} . For $\varphi : \mathbb{N} \rightarrow \{\pm\}$, next we construct a \mathfrak{g} -module containing a submodule for the Heisenberg subalgebra \mathfrak{L} isomorphic to $M_\varphi(a)$. Let $\lambda \in \mathfrak{h}^*$ and assume $\lambda(c) = a$. For

$$S_\varphi = \Delta_+^{\text{re}} \cup \{n\delta \mid n \in \mathbb{N}, \varphi(n) = +\} \cup \{-m\delta \mid m \in \mathbb{N}, \varphi(m) = -\},$$

where Δ_+^{re} is as in (2.1), the spaces $\mathfrak{g}_{S_\varphi} = \bigoplus_{\alpha \in S_\varphi} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{-S_\varphi} = \bigoplus_{\alpha \in -S_\varphi} \mathfrak{g}_\alpha$ are subalgebras of \mathfrak{g} affording a triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-S_\varphi} \oplus \mathfrak{h} \oplus \mathfrak{g}_{S_\varphi}$ of \mathfrak{g} . Let $\mathfrak{b}_\varphi = \mathfrak{h} \oplus \mathfrak{g}_{S_\varphi}$ be the Borel subalgebra corresponding to S_φ , and observe that $\mathfrak{b}_\varphi \supset \mathbb{C}c \oplus \mathfrak{L}_\varphi^+$. Let $\mathbb{C}v_\lambda$ be a one-dimensional module for \mathfrak{b}_φ with $\mathfrak{g}_{S_\varphi} v_\lambda = 0$ and $h v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$.

We say that the \mathfrak{g} -module

$$M_\varphi(\lambda) := M_{S_\varphi}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_\varphi)} \mathbb{C}v_\lambda$$

is a φ -imaginary Verma module. We identify $1 \otimes v_\lambda$ with v_λ . The $U(\mathbf{L})$ -submodule of $M_\varphi(\lambda)$ generated by v_λ is isomorphic to $M_\varphi(a)$. If $\varphi(n) = +$ for all n , then $M_\varphi(\lambda)$ coincides with the imaginary Verma module $M_S(\lambda)$ above with $S = \Delta_+$.

In the proposition below we collect some basic statements about the structure of $M_\varphi(\lambda)$. The proofs are similar to the proofs of corresponding properties for the imaginary Verma modules in [F2, Props. 3.4 and 5.3] and so are omitted.

Proposition 3.4. *Let $\lambda \in \mathfrak{h}^*$ and assume $\lambda(c) = a$. If $a \neq 0$, then $M_\varphi(\lambda)$ has the following properties.*

- $M_\varphi(\lambda)$ is a free $U(\mathfrak{g}_{-S_\varphi})$ -module of rank 1.
- $M_\varphi(\lambda)$ has a unique maximal submodule and hence a unique irreducible quotient.
- $\text{supp}(M_\varphi(\lambda)) = \bigcup_{\beta \in \dot{Q}_+} \{\lambda - \beta + n\delta \mid n \in \mathbb{Z}\}$, where \dot{Q}_+ is the free abelian monoid generated by all the simple roots in the base Π of $\dot{\Delta}_+$ (In the $A_{2\ell}^{(2)}$ -case, \dot{Q}_+ is the free abelian monoid generated by the simple roots $\alpha \in (\dot{\Delta}_s)_+$ and by the $\frac{1}{2}\alpha$ for the simple roots $\alpha \in (\dot{\Delta}_l)_+$.)
- If $\varphi(k) \neq \varphi(\ell)$ for some $k, \ell \in \mathbb{N}$, then $\dim M_\varphi(\lambda)_\mu = \infty$ for any $\mu \in \text{supp}(M_\varphi(\lambda))$.

We have the following irreducibility criterion for modules $M_\varphi(\lambda)$.

Theorem 3.5. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a$. Then $M_\varphi(\lambda)$ is irreducible if and only if $a \neq 0$.*

This theorem will be proved in Section 5.1 as a particular case of the main result (see Corollary 5.7).

4. \mathbb{Z} -GRADED MODULES FOR HEISENBERG ALGEBRAS

In this section we consider \mathbb{Z} -graded modules for the Heisenberg subalgebra \mathbf{L} with nonzero action of the central element c . To simplify the exposition, we will assume we have an infinite-dimensional Heisenberg Lie algebra $\mathbf{H} = \mathbb{C}c \oplus \bigoplus_{i \in \mathbb{Z} \setminus \{0\}} \mathbb{C}e_i$, where $[e_i, e_j] = \delta_{i,-j}c$, and $[e_j, c] = 0$ for all $i \geq 1$ and all j . The case of the Heisenberg subalgebra \mathbf{L} can easily be reduced to \mathbf{H} by choosing an orthogonal basis in each root space $\mathfrak{g}_{k\delta}$ and a dual basis in $\mathfrak{g}_{-k\delta}$ for each $k \geq 1$.

Let \mathcal{K} denote the category of all \mathbb{Z} -graded \mathbf{H} -modules \mathbf{V} such that $\mathbf{V} = \bigoplus_{j \in \mathbb{Z}} \mathbf{V}_j$ and $e_i \mathbf{V}_j \subseteq \mathbf{V}_{i+j}$. The irreducible modules in \mathcal{K} on which c acts as zero (which we say have zero level) have been classified by Chari in [C]. Irreducible modules with nonzero level, but with $0 < \dim_{\mathbb{C}} \mathbf{V}_j < \infty$ for at least one j have been described in [F3].

Definition 4.1. *Let \mathbf{V} be a \mathbb{Z} -graded module for the Heisenberg Lie algebra \mathbf{H} . Then we say*

- (a) \mathbf{V} has i -torsion if $e_i e_{-i}$ has an eigenvector in \mathbf{V} ;
- (b) \mathbf{V} is a torsion module if \mathbf{V} has i -torsion for every i ;
- (c) \mathbf{V} is torsion free if it has no i -torsion for any i ;
- (d) \mathbf{V} is locally finite if for any $i > 0$, $e_i e_{-i}$ is locally finite on \mathbf{V} , that is, $\dim \text{span}_{\mathbb{C}} \{(e_i e_{-i})^k v \mid k \geq 0\} < \infty$ for any $v \in \mathbf{V}$.

Clearly, any locally-finite \mathbf{H} -module is a torsion module, and the converse is true for irreducible modules as the next result states.

Proposition 4.2. *Let \mathbf{V} be an irreducible \mathbb{Z} -graded \mathbf{H} -module.*

- *If \mathbf{V} has i -torsion, then \mathbf{V} has a countable basis which consists of eigenvectors of $e_i e_{-i}$, that is, $e_i e_{-i}$ is diagonalizable on \mathbf{V} .*
- *If \mathbf{V} is a torsion module, then \mathbf{V} is locally finite.*
- *If \mathbf{V} is locally finite, then the $e_i e_{-i}$ are simultaneously diagonalizable on \mathbf{V} for all i .*

Lemma 4.3. *Let V be an \mathbf{H} -module such that c acts by the nonzero scalar a on V , and assume $w \in V$ is an eigenvector for $e_j e_{-j}$ with eigenvalue λ and for $e_i e_{-i}$ with eigenvalue μ for some $i \neq j$. Then $e_j^r w$ and $e_{-j}^s w$ are eigenvectors for $e_j e_{-j}$ with eigenvalues $\lambda - ra$ and $\lambda + sa$ respectively, and they are eigenvectors for $e_i e_{-i}$ corresponding to eigenvalue μ .*

Proof. The Heisenberg relations imply for $r, s \geq 1$ that

$$(4.4) \quad e_{-j} e_j^r w = (\lambda - ra) e_j^{r-1} w \quad \text{and} \quad e_j e_{-j}^s w = (\lambda + (s-1)a) e_{-j}^{s-1} w$$

from which it follows $(e_j e_{-j}) e_j^r w = (\lambda - ra) e_j^r w$ and $(e_j e_{-j}) e_{-j}^s w = (\lambda + sa) e_{-j}^s w$. The remaining assertion is clear from the commuting properties in \mathbf{H} . \square

Next we consider \mathbb{Z}^∞ -graded \mathbf{H} -modules \mathbf{V} . By that we mean $\mathbf{V} = \bigoplus_{\bar{k} \in \mathbb{Z}^\infty} V_{\bar{k}}$ where if $V_{\bar{k}} \neq 0$, then $\bar{k} = (k_1, k_2, \dots)$, $k_i \in \mathbb{Z}$, and $k_N = 0$ for all $N \gg 0$. Moreover, we require that $e_{\pm j} V_{\bar{k}} \subseteq V_{\bar{k} \pm \zeta_j}$ holds for all $j \in \mathbb{Z}$ and $\bar{k} \in \mathbb{Z}^\infty$. Here $\zeta_j \in \mathbb{Z}^\infty$ is the Kronecker multi-index with 1 at the j th place and 0 elsewhere.

Theorem 4.5. *Any \mathbb{Z} -graded irreducible locally-finite \mathbf{H} -module \mathbf{V} with nonzero level has a \mathbb{Z}^∞ -gradation $\mathbf{V} = \bigoplus_{\bar{k} \in \mathbb{Z}^\infty} V_{\bar{k}}$.*

Proof. Let $\mathbf{V} = \bigoplus_{n=0}^\infty V_n$ be a \mathbb{Z} -graded irreducible locally-finite \mathbf{H} -module with c acting by the scalar $a \neq 0$. Then all $e_j e_{-j}$ are simultaneously diagonalizable on \mathbf{V} by Proposition 4.2. In particular, all the homogeneous spaces V_n have a basis consisting of common eigenvectors for the elements $e_j e_{-j}$, $j \in \mathbb{Z} \setminus \{0\}$. For some n , choose $0 \neq w \in V_n$, a common eigenvector for all $e_j e_{-j}$, $j \in \mathbb{Z} \setminus \{0\}$. Let $\mathbf{H}(j)$ be the Heisenberg subalgebra generated by e_j, e_{-j} for each j . Then by (4.4), $\mathbf{U}(\mathbf{H}(j))w$ is spanned by the vectors $e_j^r w, e_{-j}^s w$ for $r, s \geq 0$. Now for $\bar{k} = (k_1, k_2, \dots, k_m, 0, 0, \dots) \in \mathbb{Z}^\infty$ consider $y_1^{|k_1|} y_2^{|k_2|} \dots y_m^{|k_m|} w$, where $y_i = e_i$ if $k_i \geq 0$ and $y_i = e_{-i}$ if $k_i < 0$. These vectors span $\mathbf{V} = \mathbf{U}(\mathbf{H})w$ by the irreducibility of \mathbf{V} . For each such $\bar{k} \in \mathbb{Z}^\infty$, set $V_{\bar{k}} = \mathbb{C} y_1^{|k_1|} y_2^{|k_2|} \dots y_m^{|k_m|} w$. Then $\mathbf{V} = \bigoplus_{\bar{k} \in \mathbb{Z}^\infty} V_{\bar{k}}$. The sum is direct, since by Lemma 4.3, the eigenvalues of the $e_j e_{-j}$ are sufficient to distinguish them. Note that all components $V_{\bar{k}}$ are at most one-dimensional and that \mathbf{V} remains irreducible as \mathbb{Z}^∞ -graded module. \square

4.1. Irreducible locally-finite modules over Heisenberg algebras.

Let \mathbf{H} be a Heisenberg algebra as in Section 4. Denote by $\mathcal{K}_{\mathbf{H},a}$ the category of all \mathbb{Z} -graded locally-finite \mathbf{H} -modules \mathbf{V} where the central element c of \mathbf{H} acts by the scalar $a \in \mathbb{C}$. Also, by $\mathcal{ZK}_{\mathbf{H},a}$ we denote the category of all \mathbb{Z}^∞ -graded \mathbf{H} -modules \mathbf{V} with c acting by a . Theorem 4.5 implies that any irreducible module in $\mathcal{K}_{\mathbf{H},a}$ belongs to $\mathcal{ZK}_{\mathbf{H},a}$. Therefore, to

classify irreducible modules in $\mathcal{K}_{H,a}$ it is sufficient to classify irreducible torsion modules in $\mathcal{Z}\mathcal{K}_{H,a}$.

Denote by $\mathcal{W}\mathcal{K}_{H,a} \subset \mathcal{K}_{H,a}$ and $\mathcal{W}\mathcal{Z}\mathcal{K}_{H,a} \subset \mathcal{Z}\mathcal{K}_{H,a}$ the full subcategories of the respective categories consisting of all finitely-generated H -modules on which $e_i e_{-i}$ is diagonalizable for all $i \neq 0$ (with a countable basis of eigenvectors). Note that by Proposition 4.2 any irreducible object of $\mathcal{K}_{H,a}$ belongs to $\mathcal{W}\mathcal{K}_{H,a}$ and also to $\mathcal{W}\mathcal{Z}\mathcal{K}_{H,a}$. Moreover, we have

Proposition 4.6. *The categories $\mathcal{W}\mathcal{K}_{H,a}$ and $\mathcal{W}\mathcal{Z}\mathcal{K}_{H,a}$ are equivalent.*

Proof. We construct a functor from $\mathcal{W}\mathcal{K}_{H,a}$ to $\mathcal{W}\mathcal{Z}\mathcal{K}_{H,a}$. Let $\mathbf{V} \in \mathcal{W}\mathcal{K}_{H,a}$ and $\mathbf{V} = \bigoplus_{n \in \mathbb{Z}} \mathbf{V}_n$. We may assume that the generators are homogeneous (say in the spaces \mathbf{V}_{n_i} for $i = 1, \dots, s$) and are common eigenvectors for $e_j e_{-j}$, $j \in \mathbb{Z} \setminus \{0\}$. The module \mathbf{V} is spanned by the vectors $\prod_{j=1}^{\infty} y_j^{|k_j|} w$, where $w \in \mathbf{V}_{n_i}$ is a generator which is a common eigenvector for the $e_j e_{-j}$ for all j , $y_j = e_j$ if $k_j \geq 0$ and $y_j = e_{-j}$ if $k_j < 0$, and $k_N = 0$ for $N \gg 0$. We assign to such a vector $\prod_{j=1}^{\infty} y_j^{|k_j|} w$ the gradation $n_i \zeta_1 + \bar{k} \in \mathbb{Z}^{\infty}$, where ζ_1 has 1 in the first position and 0 elsewhere. This makes \mathbf{V} into a \mathbb{Z}^{∞} -graded module. Denote this \mathbb{Z}^{∞} -graded module by $\mathbf{F}_1(\mathbf{V})$. Hence, we obtain a functor

$$\mathbf{F}_1 : \mathcal{W}\mathcal{K}_{H,a} \rightarrow \mathcal{W}\mathcal{Z}\mathcal{K}_{H,a}.$$

For the reverse functor, consider any $\mathbf{M} \in \mathcal{W}\mathcal{Z}\mathcal{K}_{H,a}$, $\mathbf{M} = \bigoplus_{\bar{k} \in \mathbb{Z}^{\infty}} \mathbf{M}_{\bar{k}}$. Now define a \mathbb{Z} -grading on \mathbf{M} as follows: for any $n \in \mathbb{Z}$ define $\mathbf{M}_n = \bigoplus_{\bar{k}} \mathbf{M}_{\bar{k}}$, where the sum is over $\bar{k} = (k_1, k_2, \dots) \in \mathbb{Z}^{\infty}$, such that $\sum_{j=1}^{\infty} k_j j = n$. Denote this \mathbb{Z} -graded module by $\mathbf{F}_2(\mathbf{M})$. hence we obtain a functor

$$\mathbf{F}_2 : \mathcal{W}\mathcal{Z}\mathcal{K}_{H,a} \rightarrow \mathcal{W}\mathcal{K}_{H,a}.$$

Note that \mathbf{F}_2 preserves irreducibility. It is easy to see that the pair $(\mathbf{F}_1, \mathbf{F}_2)$ establishes the required equivalence of the categories. \square

4.2. Weight modules for Weyl algebras.

Fix a nonzero $a \in \mathbb{C}$. For $n = 1, 2, \dots$, consider the n -th Weyl algebra \mathbf{A}_n with generators x_i, ∂_i , $i = 1, \dots, n$, and defining relations $[\partial_i, \partial_j] = 0 = [x_i, x_j]$ and $[\partial_i, x_j] = \delta_{i,j} a 1$. The algebra \mathbf{A}_n is isomorphic to the tensor product of n copies of the first Weyl algebra \mathbf{A}_1 , which is the associative algebra of differential operators on the affine line. We allow n to be ∞ , in which case \mathbf{A}_{∞} is just the direct limit of the algebras \mathbf{A}_n .

We will classify irreducible modules in $\mathcal{K}_{H,a}$ using the classification of irreducible weight \mathbf{A}_{∞} -modules. By specializing c to a and identifying ∂_i with e_i and x_i with e_{-i} for all $i > 0$ (after a suitable normalization), we obtain an isomorphism of the universal enveloping algebra of H modulo the ideal generated by $c - a$ with \mathbf{A}_{∞} . Hence, any irreducible module $\mathbf{V} \in \mathcal{K}_{H,a}$ becomes a module for \mathbf{A}_{∞} .

Set $[n] = \{1, 2, \dots, n\}$ (where $n = \infty$ is allowed and $[\infty] = \mathbb{N}$). In \mathbf{A}_n , the elements $t_i = \partial_i x_i$, $i \in [n]$, generate the polynomial algebra $\mathbf{D} = \mathbb{C}[t_i \mid i \in [n]]$, which is a maximal commutative subalgebra of \mathbf{A}_n . Denote by \mathcal{G} the group generated by the automorphisms σ_i , $i \in [n]$, of \mathbf{D} , where $\sigma_i(t_j) = t_j - \delta_{i,j} a 1$. Then \mathcal{G} acts on the set $\mathbf{mar}\mathbf{D}$ of maximal ideals of \mathbf{D} .

A module V for A_n is said to be a *weight module* if $V = \bigoplus_{\mathfrak{m} \in \mathfrak{max} D} V_{\mathfrak{m}}$, where $V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}$ (see [DGO], [BB], and [BBF]).

If V is a weight module and $V_{\mathfrak{m}} \neq 0$ for $\mathfrak{m} \in \mathfrak{max} D$, then \mathfrak{m} is said to be a *weight* of V , and the set $\{\mathfrak{m} \in \mathfrak{max} D \mid V_{\mathfrak{m}} \neq 0\}$ of weights of V is the *support* of V . It follows easily from the fact that $x_i d = \sigma_i(d)x_i$ and $y_i d = \sigma_i^{-1}(d)y_i$ for all i and all $d \in D$ that $x_i V_{\mathfrak{m}} \subseteq V_{\sigma_i(\mathfrak{m})}$ and $\partial_i V_{\mathfrak{m}} \subseteq V_{\sigma_i^{-1}(\mathfrak{m})}$.

Each weight module V can be decomposed into a direct sum of A_n -submodules:

$$V = \bigoplus_{\mathcal{O}} V_{\mathcal{O}}, \quad V_{\mathcal{O}} := \bigoplus_{\mathfrak{m} \in \mathcal{O}} V_{\mathfrak{m}}$$

where \mathcal{O} runs over the orbits of \mathcal{G} on $\mathfrak{max} D$. In particular, if V is irreducible, then its support belongs to a single orbit.

Following [BBF], we say that a maximal ideal \mathfrak{m} of D is a *break with respect to $i \in [n]$* if $t_i \in \mathfrak{m}$. We let $I(\mathfrak{m})$ denote the set of breaks of \mathfrak{m} . An orbit \mathcal{O} is *degenerate with respect to i* if $i \in I(\mathfrak{m})$ for some $\mathfrak{m} \in \mathcal{O}$. Often we simply say \mathcal{O} is degenerate without specifying i or \mathfrak{m} . When $I(\mathfrak{m}) = \emptyset$ for all $\mathfrak{m} \in \mathcal{O}$, then \mathcal{O} is said to be nondegenerate.

A maximal ideal \mathfrak{m} of D is a *maximal break with respect to $I \subseteq [n]$* if $t_i \in \mathfrak{m}$ for each $i \in I$, and $t_j \notin \tau(\mathfrak{m})$ for each $j \in I^c := [n] \setminus I$ and each $\tau \in \mathcal{G}$. The *order* of the maximal break is the cardinality of I , which may be infinite.

We will always assume that a degenerate orbit \mathcal{O} for A_n has a maximal break \mathfrak{m} . The only time this assumption is necessary is when $n = \infty$ (see [BBF, Lem. 2.5]). We refer to A_n -modules whose weights belong to a degenerate orbit \mathcal{O} with a maximal break as being *admissible*.

The classification of irreducible admissible weight A_n -modules was obtained in [BB], and in [BBF] for the case $n = \infty$. We briefly recall this classification for the sake of completeness.

For a given orbit \mathcal{O} , we define the set $\mathfrak{B}_{\mathcal{O}}$ as follows. If \mathcal{O} is nondegenerate, then any maximal ideal gives a maximal break with respect to the empty subset of $[n]$. We fix a choice of a maximal ideal \mathfrak{m} in \mathcal{O} , and set $\mathfrak{B}_{\mathcal{O}} = \{\mathfrak{m}\}$ and $\mathcal{O}_{\mathfrak{m}} = \mathcal{O}$. In this case $I(\mathfrak{m}) = \emptyset$, since there are no breaks.

Now let \mathcal{O} be a degenerate orbit, and assume \mathfrak{m} is a fixed maximal break in \mathcal{O} . Let $I = I(\mathfrak{m})$ be the set of breaks for \mathfrak{m} . Define

$$(4.7) \quad \mathfrak{B}_{\mathcal{O}} = \left\{ \left(\prod_j \sigma_j^{\delta_j} \right) (\mathfrak{m}) \mid \delta_j \in \{0, 1\} \text{ if } j \in I, \delta_j = 0 \text{ if } j \in I^c, \text{ and only finitely many } \delta_j \neq 0 \right\}.$$

The σ_i commute, so the product is well-defined. For each $\mathfrak{p} = \left(\prod_j \sigma_j^{\delta_j} \right) (\mathfrak{m}) \in \mathfrak{B}_{\mathcal{O}}$, we set

$$(4.8) \quad \mathcal{O}_{\mathfrak{p}} := \left\{ \left(\prod_j \sigma_j^{\gamma_j} \right) (\mathfrak{p}) \mid \gamma_j = (-1)^{\delta_j+1} k, k \in \mathbb{Z}_{\geq 0} \text{ if } j \in I, \text{ and } \gamma_j \in \mathbb{Z} \text{ if } j \in I^c \right\},$$

where only finitely many γ_j are nonzero.

Suppose first that \mathcal{O} is a nondegenerate orbit of \mathcal{G} on $\mathfrak{max} D$. Set $A = A_n$ and

$$(4.9) \quad \mathcal{S}(\mathcal{O}) = \bigoplus_{\mathfrak{n} \in \mathcal{O}} \mathbb{D}/\mathfrak{n},$$

and define a left \mathbf{A} -module structure on $\mathcal{S}(\mathcal{O})$ by specifying for $i \in [n]$ and $d \in \mathbb{D}$ that

$$(4.10) \quad x_i(d + \mathfrak{n}) := \sigma_i(d) + \sigma_i(\mathfrak{n}), \quad \partial_i(d + \mathfrak{n}) := t_i \sigma_i^{-1}(d) + \sigma_i^{-1}(\mathfrak{n}).$$

As $\mathcal{S}(\mathcal{O})$ is generated by $1 + \mathfrak{m}$, we have that $\mathcal{S}(\mathcal{O}) \cong \mathbf{A}/\mathbf{A}\mathfrak{m}$ where $1 + \mathfrak{m} \mapsto 1 + \mathbf{A}\mathfrak{m}$.

Now assume that \mathcal{O} is degenerate, and \mathfrak{m} is the fixed maximal break. For $\mathfrak{p} \in \mathfrak{B}_{\mathcal{O}}$ set

$$(4.11) \quad \mathcal{S}(\mathcal{O}, \mathfrak{p}) := \bigoplus_{\mathfrak{n} \in \mathcal{O}_{\mathfrak{p}}} \mathbb{D}/\mathfrak{n},$$

where $\mathcal{O}_{\mathfrak{p}}$ is as in (4.8). One can define a structure of a left \mathbf{A} -module on $\mathcal{S}(\mathcal{O}, \mathfrak{p})$ by the same formulae as in (4.10), but with the understanding that when the image is not in $\mathcal{S}(\mathcal{O}, \mathfrak{p})$, the result is 0. Assuming $\mathfrak{p} = \left(\prod_j \sigma_j^{\delta_j}\right)(\mathfrak{m})$, we have in this case $\mathcal{S}(\mathcal{O}, \mathfrak{p}) \cong \mathbf{A}/\mathbf{A}\langle \mathfrak{p}, z_i, i \in \mathbf{I} \rangle$ where $z_i = x_i$ if \mathfrak{p} is a break with respect to i , and $z_i = \partial_i$ otherwise. The isomorphism is given by $1 + \mathfrak{p} \mapsto 1 + \langle \mathfrak{p}, z_i, i \in \mathbf{I} \rangle$. It follows from the construction that $\mathcal{S}(\mathcal{O})$ and $\mathcal{S}(\mathcal{O}, \mathfrak{p})$ are irreducible \mathbf{A} -modules.

Theorem 4.12. [BBF, Thm. 4.7] *Let \mathcal{O} be an orbit of $\mathfrak{max}\mathbb{D}$ under the group \mathcal{G} . Then the modules $\mathcal{S}(\mathcal{O})$ and $\mathcal{S}(\mathcal{O}, \mathfrak{p})$, where $\mathfrak{p} \in \mathfrak{B}_{\mathcal{O}}$, constitute an exhaustive list of pairwise nonisomorphic irreducible admissible weight \mathbf{A}_n -modules with support in \mathcal{O} .*

Let $\mathcal{W}(\mathbf{A}_n)$ be the category of all admissible weight \mathbf{A}_n -modules, $n \leq \infty$, and let \mathbf{H}_n denote the $(2n + 1)$ -dimensional Heisenberg Lie algebra with basis $c, e_i, e_{-i}, i \in [n]$. Then as an immediate consequence of Proposition 4.6, we have the following.

Corollary 4.13.

- The categories $\mathcal{W}\mathcal{K}_{\mathbf{H}_n, a}$ and $\mathcal{W}(\mathbf{A}_n)$ are equivalent.
- The categories $\mathcal{W}\mathcal{K}_{\mathbf{H}, a}$, $\mathcal{W}\mathcal{Z}\mathcal{K}_{\mathbf{H}, a}$ and $\mathcal{W}(\mathbf{A}_{\infty})$ are equivalent.

Example 4.14. To illustrate Corollary 4.13, suppose \mathcal{O} is nondegenerate, and \mathfrak{m} is the designated maximal ideal. Consider the irreducible weight module for $\mathbf{A} = \mathbf{A}_{\infty}$ given by $\mathcal{S}(\mathcal{O}) = \mathbf{A}/\mathbf{A}\mathfrak{m} = \bigoplus_{\mathfrak{n} \in \mathcal{O}} \mathbb{D}/\mathfrak{n}$, where $\mathbb{D} = \mathbb{C}[t_i \mid i \in \mathbb{N}]$, and $\mathbb{D}/\mathfrak{n} \cong \mathbb{C}$ for any $\mathfrak{n} \in \mathcal{O}$. Then $\mathcal{S}(\mathcal{O})$ becomes a \mathbb{Z} -graded irreducible \mathbf{H} -module if we set

$$\mathcal{S}(\mathcal{O}) = \sum_{k \in \mathbb{Z}} \mathcal{S}^k(\mathcal{O}),$$

where

$$\mathcal{S}^k(\mathcal{O}) = \sum_{\eta_j \in \mathbb{Z}, \sum_j j \eta_j = -k} \mathbb{D} / \left(\left(\prod_j \sigma_j^{\eta_j} \right) (\mathfrak{m}) \right),$$

where only finitely many η_j are nonzero. The homogeneous space $\mathcal{S}^0(\mathcal{O}) = \mathbb{D}/\mathfrak{m}$.

As a consequence of Corollary 4.13 we have the following stronger version of Theorem 4.5.

Theorem 4.15. *For nonzero $a \in \mathbb{C}$, there is one-to-one correspondence between the isomorphism classes of irreducible modules in the categories $\mathcal{K}_{\mathfrak{H},a}$, $\mathcal{WZ}\mathcal{K}_{\mathfrak{H},a}$ and $\mathcal{W}(\mathfrak{A}_\infty)$.*

Combining Theorem 4.15 and Theorem 4.12, we obtain the classification of irreducible modules in $\mathcal{K}_{\mathfrak{H},a}$.

4.3. Irreducible locally-finite modules over \mathfrak{L} . Now consider the Heisenberg subalgebra $\mathfrak{L} = \mathbb{C}c \oplus \bigoplus_{k \neq 0} \mathfrak{g}_{k\delta}$ of the affine Lie algebra \mathfrak{g} . For each $k \in \mathbb{Z}$, $k \neq 0$, assume $\mathbf{d}_k = \dim \mathfrak{g}_{k\delta}$ and write $[\mathbf{d}_k] = \{1, \dots, \mathbf{d}_k\}$. Choose a basis $\{x_{k,i} \mid i \in [\mathbf{d}_k]\}$ for $\mathfrak{g}_{k\delta}$ so that $[x_{k,i}, x_{-k,j}] = \delta_{i,j}kc$ for all i, j . Then for every i , $x_{k,i}$ and $x_{-k,i}$ generate a Lie subalgebra isomorphic \mathfrak{H}_1 .

We will consider \mathbb{Z} -graded \mathfrak{L} -modules $\mathbf{V} = \bigoplus_{j \in \mathbb{Z}} \mathbf{V}_j$ where $\mathfrak{g}_{k\delta}\mathbf{V}_j \subseteq \mathbf{V}_{k+j}$ for all k and j . One can easily extend Definition 4.1 to the algebra \mathfrak{L} substituting basis elements $\{e_\ell\}$ by $\{x_{k,i}\}$.

Fix $a \in \mathbb{C} \setminus \{0\}$. Let $\mathcal{K}_{\mathfrak{L},a}$ be the category of all \mathbb{Z} -graded locally-finite \mathfrak{L} -modules \mathbf{V} where the central element c of \mathfrak{L} acts by the scalar $a \in \mathbb{C}$. Similarly one defines category $\mathcal{WZ}\mathcal{K}_{\mathfrak{L},a}$. Clearly, all statements from the previous sections can be generalized to the setup of the Heisenberg algebra \mathfrak{L} . In particular, there exists the Weyl algebra $\tilde{\mathfrak{A}}$, generalizing \mathfrak{A}_∞ , which takes into account the dimensions of the spaces $\mathfrak{g}_{k\delta}$. Then Theorem 4.15 has the following straightforward generalization for \mathfrak{L} .

Corollary 4.16. *For $a \in \mathbb{C} \setminus \{0\}$, there is one-to-one correspondence between the isomorphism classes of irreducible modules in the categories $\mathcal{K}_{\mathfrak{L},a}$, $\mathcal{WZ}\mathcal{K}_{\mathfrak{L},a}$ and $\mathcal{W}(\tilde{\mathfrak{A}})$.*

Example 4.17. Let $a \in \mathbb{C} \setminus \{0\}$ and $\varphi : \mathbb{N} \rightarrow \{\pm\}$ be any function. Then the φ -Verma module $\mathbf{M}_\varphi(a)$ is an irreducible object in the categories $\mathcal{K}_{\mathfrak{L},a}$ and $\mathcal{WZ}\mathcal{K}_{\mathfrak{L},a}$. If $\varphi(k) = +$ and $\varphi(\ell) = -$ for some $k, \ell \in \mathbb{N}$, then all the homogeneous components of $\mathbf{M}_\varphi(a)$ in the \mathbb{Z} -grading are nonzero and infinite dimensional.

4.4. $\tilde{\varphi}$ -imaginary Verma modules for \mathfrak{g} . We will generalize the construction of the φ -Verma modules for \mathfrak{L} as follows. Set

$$(4.18) \quad \mathbf{J} = \bigcup_{k \in \mathbb{N}} \{(k, i) \mid i \in [\mathbf{d}_k]\}.$$

Consider a function $\tilde{\varphi} : \mathbf{J} \rightarrow \{\pm\}$, and define Lie subalgebras $\mathfrak{L}_{\tilde{\varphi}}^\pm$ of \mathfrak{L} by the following rule. For any nonzero $k \in \mathbb{Z}$, we say that $x_{k,i} \in \mathfrak{L}_{\tilde{\varphi}}^\pm$ if $k > 0$ and $\tilde{\varphi}(k, i) = \pm$, or if $k < 0$ and $\tilde{\varphi}(-k, i) = \mp$. Then

$$\mathfrak{L} = \mathfrak{L}_{\tilde{\varphi}}^- \oplus \mathbb{C}c \oplus \mathfrak{L}_{\tilde{\varphi}}^+,$$

where $\mathfrak{L}_{\tilde{\varphi}}^\pm$ are abelian subalgebras of \mathfrak{L} .

Remark 4.19. *The function $\tilde{\varphi}$ clearly depends on the initial choice of orthogonal bases in imaginary root spaces $\mathfrak{g}_{k\delta}$ with respect to the nondegenerate form on \mathfrak{g} . On the other hand, for each positive integer k , the number of $+$ and $-$ in the image of $\tilde{\varphi}$ does not depend on the choice of bases.*

Let $\mathbb{C}v$ be a one-dimensional representation of $\mathbb{C}c \oplus \mathbb{L}_\varphi^+$ with $cv = av$ for $a \in \mathbb{C}$ and $\mathbb{L}_\varphi^+v = 0$. Then we construct the corresponding $\tilde{\varphi}$ -Verma module

$$M_{\tilde{\varphi}}(a) = \mathbf{U}(\mathbb{L}) \otimes_{\mathbf{U}(\mathbb{C}c \oplus \mathbb{L}_\varphi^+)} \mathbb{C}v.$$

If $\tilde{\varphi}(n, i) = \tilde{\varphi}(n, j)$ for all n and all $i, j \in [d_n]$, then $M_{\tilde{\varphi}}(a)$ is just a φ -Verma module for \mathbb{L} , where $\varphi(n) = \tilde{\varphi}(n, i)$ for each n and any i . For any function $\tilde{\varphi}$, the $\tilde{\varphi}$ -Verma module $M_{\tilde{\varphi}}(a)$ is an object in the categories $\mathcal{K}_{\mathbb{L}, a}$ and $\mathcal{WZK}_{\mathbb{L}, a}$. One can easily see that $M_{\tilde{\varphi}}(a)$ is irreducible if and only if $a \neq 0$.

For any such function $\tilde{\varphi}$ and any $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a$, one can construct the $\tilde{\varphi}$ -imaginary Verma module $M_{\tilde{\varphi}}(\lambda)$ over \mathfrak{g} generalizing the construction of $M_\varphi(\lambda)$ in the case of the function $\varphi : \mathbb{N} \rightarrow \{\pm\}$:

$$M_{\tilde{\varphi}}(\lambda) := M_{S_{\tilde{\varphi}}}(\lambda) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{b}_{\tilde{\varphi}})} \mathbb{C}v_\lambda.$$

Theorem 4.20. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a$ and assume $\tilde{\varphi} : \mathbb{J} \rightarrow \{\pm\}$ is any function. Then $M_{\tilde{\varphi}}(\lambda)$ is irreducible if and only if $a \neq 0$.*

This theorem also will be proved in Section 5.1 as a particular case of the main result (see Corollary 5.7).

4.5. Realization of locally-finite modules.

Locally-finite \mathbb{L} -modules have also been considered by Casati [Ca]. Following Casati's work we will construct realizations of irreducible locally-finite \mathbb{L} -modules.

Let $K \subseteq \mathbb{N}$ and set $V = \mathbb{C}[x_i, x_{-k} \mid i \in \mathbb{N}, k \in K]$. Then it is easy to verify that the following formulas define a representation of $\mathbf{A} = \mathbf{A}_\infty$ on V :

$$\begin{aligned} \partial_i &\rightarrow \begin{cases} \frac{\partial}{\partial x_i} & \text{if } i \in \mathbb{N} \setminus K \\ \frac{\partial}{\partial x_i} + x_{-i} & \text{if } i \in K, \end{cases} \\ x_i &\rightarrow x_i & \text{for all } i \in \mathbb{N}, \\ c &\rightarrow 1. \end{aligned}$$

This module is isomorphic to the universal module V_K over \mathbf{A} generated by a vacuum vector v , where $\partial_i v = 0$ for any $i \in \mathbb{N} \setminus K$. Hence $V_K \simeq \mathbf{A}/\mathbf{B}_K$, where \mathbf{B}_K is left ideal of \mathbf{A} generated by $\partial_i, i \in \mathbb{N} \setminus K$. Clearly, this module is not irreducible.

Now for each $k \in K$, fix $\vartheta_k \in \mathbb{C}$, and let $\vartheta = \{\vartheta_k \mid k \in K\}$. Then we can construct the following quotient of V_K . Let $\mathbf{B}_{K, \vartheta}$ denote the left ideal of \mathbf{A} generated by the elements $\partial_i, i \in \mathbb{N} \setminus K$, and $x_k \partial_k - \vartheta_k, k \in K$, and denote the quotient by $V_{K, \vartheta} = \mathbf{A}/\mathbf{B}_{K, \vartheta}$.

Suppose that $\vartheta_k \in \mathbb{C} \setminus \mathbb{Z}$ for all $k \in K$. In this case, ∂_k and x_i act injectively on $V_{K, \vartheta}$ for all $k \in K$ and all $i \in \mathbb{N}$. Let $\mathbf{A}(i)$ denote the rank one Weyl algebra generated by ∂_i and x_i . Using the irreducibility of the Verma module over $\mathbf{A}(i)$ generated by a vacuum vector v such that $\partial_i v = 0$, we conclude that $V_{K, \vartheta}$ is an irreducible \mathbf{A} -module.

This construction can be immediately generalized to the Heisenberg algebra \mathbf{L} . In doing this, we will adopt the notation from Section 4.3. Then an \mathbf{L} -module \mathbf{V} is locally finite if for any $k \in \mathbb{N}$ and any $i \in [d_k]$, we have $\dim \text{span}_{\mathbb{C}}\{(x_{k,j}x_{-k,j})^m v \mid m \geq 0\} < \infty$ for any $v \in \mathbf{V}$. If \mathbf{V} is irreducible, this is equivalent to the vectors $x_{k,j}x_{-k,j}$ being simultaneously diagonalizable for all $k \in \mathbb{N}$, $j \in [d_k]$.

Fix a nonzero $a \in \mathbb{C}$, and take $\vartheta_{k,j} \in \mathbb{C}$ for $k \in \mathbb{K}$, $j \in [d_k]$. Set $\vartheta = \{\vartheta_{k,j} \mid k \in \mathbb{K}, j \in [d_k]\}$. Consider the \mathbf{L} -module $\mathbf{V}_{\mathbb{K},\vartheta,a} = \mathbf{U}(\mathbf{L})/\mathbf{B}_{\mathbb{K},\vartheta,a}$, where $\mathbf{B}_{\mathbb{K},\vartheta,a}$ is the left ideal of $\mathbf{U}(\mathbf{L})$ generated by $x_{-k,j}$, $k \in \mathbb{N} \setminus \mathbb{K}$, $x_{k,j}x_{-k,j} - \vartheta_{k,j}$ for $k \in \mathbb{K}$, $j \in [d_k]$ and $c - a$. Then $\mathbf{V}_{\mathbb{K},\vartheta,a}$ is an irreducible \mathbf{L} -module if and only if for any $k \in \mathbb{K}$ and any $j \in [d_k]$, $\vartheta_{k,j}$ is not an integer multiple of ka . Moreover, applying Corollary 4.16 we have

Theorem 4.21. *Up to an automorphism of \mathbf{L} , for any irreducible locally-finite \mathbf{L} -module \mathbf{M} of level $a \neq 0$ there exist a set $\mathbb{K} \subseteq \mathbb{N}$ and scalars $\vartheta_{k,j}$, $k \in \mathbb{K}$, $j \in [d_k]$ with $\vartheta_{k,j}$ not an integer multiple of ka such that \mathbf{M} is isomorphic to $\mathbf{V}_{\mathbb{K},\vartheta,a}$.*

Corollary 4.22. *Assume \mathbf{M} is an irreducible \mathbf{L} -module isomorphic to $\mathbf{V}_{\mathbb{K},\vartheta,a}$. For $\mathbb{K} \subseteq \mathbb{N}$, set $y_{k,j} = x_{-k,j}$ if $k \in \mathbb{N} \setminus \mathbb{K}$ for all $j \in [d_k]$ and set $y_{k,j} = x_{k,j}$ or $x_{-k,j}$ if $k \in \mathbb{K}$ for all $j \in [d_k]$. Then for any nonzero vector $v \in \mathbf{M}$, the vectors*

$$(4.23) \quad \cdots y_{2,d_2}^{p_{2,d_2}} \cdots y_{2,1}^{p_{2,1}} y_{1,d_1}^{p_{1,d_1}} \cdots y_{1,1}^{p_{1,1}} v$$

with exponents $p_{k,j} \in \mathbb{N} \cup \{0\}$ for all k, j and only finitely many of them nonzero form a basis for \mathbf{M} .

5. GENERALIZED LOOP MODULES

In this section, we study modules over the affine Lie algebra \mathfrak{g} induced from modules in the category $\mathcal{K}_{\mathbf{L},a}$ via a construction analogous to that of the loop modules in [C]. Let \mathbf{S} denote the set given by

$$\mathbf{S} = \mathbf{R} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}.$$

where $\mathbf{R} = \Delta_+^{\text{re}}$, the positive real roots as in (2.1). Set $\mathbf{P} = \mathfrak{h} \oplus \mathfrak{g}_{\mathbf{S}}$, where $\mathfrak{g}_{\mathbf{S}} = \bigoplus_{\beta \in \mathbf{S}} \mathfrak{g}_{\beta}$. Then $\mathbf{P} = (\mathfrak{h} + \mathbf{L}) \oplus \mathfrak{g}_{\mathbf{R}}$, where $\mathfrak{g}_{\mathbf{R}} = \bigoplus_{\beta \in \mathbf{R}} \mathfrak{g}_{\beta}$, and \mathbf{P} is a parabolic subalgebra of \mathfrak{g} with Levi factor $\mathfrak{h} + \mathbf{L}$.

Let $\mathbf{V} \in \mathcal{K}_{\mathbf{L},a}$, $\mathbf{V} = \bigoplus_{k \in \mathbb{Z}} \mathbf{V}_k$, and assume $a \neq 0$. Suppose $\lambda \in \mathfrak{h}^*$ is such that $\lambda(c) = a$. Extend the module structure to \mathbf{P} by setting $\mathfrak{g}_{\mathbf{R}}\mathbf{V} = 0$, and $hv = \lambda(h)v$ for any $v \in \mathbf{V}_0$ and any $h \in \mathfrak{h}$. Here \mathbf{V}_0 is the 0-component of \mathbf{V} in the \mathbb{Z} -grading. The action of \mathfrak{h} on the other components of \mathbf{V} in the \mathbb{Z} -grading differs only in the value of the degree derivation; that is, for any $w \in \mathbf{V}_k$, $hw = (\lambda + k\delta)(h)w$ for each $h \in \mathfrak{h}$. (Recall that δ is zero on $\mathfrak{h} \oplus \mathbb{C}c$ and $\delta(d) = 1$.)

Now consider the induced \mathfrak{g} -module given by

$$(5.1) \quad \mathbf{M}(\lambda, \mathbf{V}) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathbf{P})} \mathbf{V}.$$

When \mathbf{V} is an irreducible module in the category $\mathcal{K}_{\mathbf{L},a}$, then $\mathbf{M}(\lambda, \mathbf{V})$ is said to be a *generalized loop module*. When \mathbf{V} is a φ -Verma module of \mathbf{L} for some function $\varphi : \mathbb{N} \rightarrow \{\pm\}$, which has been extended to a module for \mathbf{P} by setting $\mathfrak{g}_{\mathbf{R}}\mathbf{V} = 0$, then $\mathbf{M}(\lambda, \mathbf{V}) \cong \mathbf{M}_{\varphi}(\lambda)$, the φ -imaginary Verma module for \mathfrak{g} as in Section 3.2.

Proposition 5.2. *Let $\lambda \in \mathfrak{h}^*$ and suppose that $\lambda(c) = a \neq 0$. Let \mathbf{V} be an irreducible module in $\mathcal{K}_{L,a}$. Then*

- $\mathbf{M}(\lambda, \mathbf{V})$ is a free $\mathbf{U}(\mathfrak{g}_{-R})$ -module of rank 1.
- $\text{supp}(\mathbf{M}(\lambda, \mathbf{V})) = \bigcup_{\beta \in \dot{\mathbf{Q}}_+} \{\lambda - \beta + n\delta \mid n \in \mathbb{Z}\}$ where $\dot{\mathbf{Q}}_+$ is as in Proposition 3.4.
- $\dim \mathbf{M}(\lambda, \mathbf{V})_\mu = \infty$ for any μ of the form $\mu = \lambda - \beta + k\delta$ for some $\beta \neq 0$ and $k \in \mathbb{Z}$.
- $\dim \mathbf{M}(\lambda, \mathbf{V})_\mu < \infty$ only if \mathbf{V} is a φ -imaginary Verma module for some $\varphi : \mathbb{N} \rightarrow \{\pm\}$, $\varphi(m) = \varphi(n)$ for all $m, n \in \mathbb{N}$ and $\mu = \lambda - (\varphi(m)m)\delta$ for some $m \in \mathbb{N} \cup \{0\}$.

5.1. Irreducibility of generalized loop modules.

Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a \neq 0$, and assume \mathbf{V} is a module in $\mathcal{K}_{L,a}$. Set

$$\widehat{\mathbf{M}}(\lambda, \mathbf{V}) = \bigoplus_{k \in \mathbb{Z}} \mathbf{M}(\lambda, \mathbf{V})_{\lambda + k\delta}.$$

Then $\widehat{\mathbf{M}}(\lambda, \mathbf{V})$ is an L -submodule of $\mathbf{M}(\lambda, \mathbf{V})$ isomorphic to \mathbf{V} .

Lemma 5.3. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a \neq 0$, and suppose that \mathbf{V} is an irreducible module in $\mathcal{K}_{L,a}$. Then for any nonzero submodule $\mathbf{N} \subset \mathbf{M}(\lambda, \mathbf{V})$ we have $\widehat{\mathbf{N}} = \mathbf{N} \cap \widehat{\mathbf{M}}(\lambda, \mathbf{V}) \neq 0$.*

Proof. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the base of simple roots for $\dot{\Delta}$, and assume $\alpha = \sum_{j=1}^n k_j \alpha_j$ where each k_j is in $\mathbb{Z}_{\geq 0}$ (or in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ for any $\alpha_j \in (\dot{\Delta}_l)_+$ in the $A_{2\ell}^{(2)}$ case). Set $\text{ht}(\alpha) = \sum_{j=1}^n k_j$, the height of α . The argument will follow the general lines of the proof of [F4, Lem. 5.4] and will proceed by induction on the height. Say $\alpha \leq \beta = \sum_{j=1}^n \ell_j \alpha_j$ if $\text{ht}(\alpha) < \text{ht}(\beta)$ or if $\text{ht}(\alpha) = \text{ht}(\beta)$ and $k_1 = \ell_1, \dots, k_s = \ell_s$, but $k_{s+1} < \ell_{s+1}$.

By Theorem 4.21, we may assume $\mathbf{V} \cong \mathbf{V}_{K, \vartheta, a}$ for some $K \subseteq \mathbb{N}$ and some ϑ . Let $\mathbf{N} \neq 0$ be a submodule of $\mathbf{M}(\lambda, \mathbf{V})$. Assume v is a homogeneous generator of \mathbf{V} , and let $w \in \mathbf{N}$ be a nonzero homogeneous element. Then

$$(5.4) \quad w = \sum_{i \in \mathcal{J}} u_i v_i,$$

where we may suppose that the v_i are distinct monomial basis elements of the form (4.23) and the u_i are linearly independent homogeneous elements of $\mathbf{U}(\mathfrak{g}_{-R})$. We may suppose that for each $i \in \mathcal{J}$ there is some $\ell_i \in \mathbb{Z}$ such that $u_i \in \mathbf{U}(\mathfrak{g}_{-R})_{-\beta + \ell_i \delta}$.

Initially assume $\text{ht}(\beta) \leq 1$, so that β is a simple root in $\dot{\Delta}_+$ (or is $\frac{1}{2}\alpha_j$ for some simple root $\alpha_j \in (\dot{\Delta}_l)_+$ in the $A_{2\ell}^{(2)}$ case). Suppose $0 \neq x \in \mathfrak{g}_{\beta + m\delta}$ for some $m \in \mathbb{Z}$. Then $xv_i = 0$ for any i and

$$(5.5) \quad xw = \sum_{i \in \mathcal{J}} [x, u_i] v_i.$$

Here $[x, u_i] \in \mathfrak{g}_{(m+\ell_i)\delta}$ and $[x, u_i] \neq 0$ for all i (which can be seen from the loop realization of \mathfrak{g}). Since the u_i are linearly independent, and $u_i \in \mathfrak{g}_{-\beta + \ell_i \delta}$, which is one-dimensional, we have that $\ell_i \neq \ell_j$ if $i \neq j$. Fix $i_\bullet \in \mathcal{J}$. Now using the notation of Section 4.3, we have that $[x, u_{i_\bullet}]$ is a linear combination of the basis elements $x_{m+\ell_{i_\bullet}, j}$. We may suppose that m was chosen with $|m|$ sufficiently large so that $m + \ell_{i_\bullet}$ is not equal to k for any $y_{k,j}$ occurring in

any of the v_i , and so that at least one of the $x_{m+\ell_{i_\bullet, j}}$ appearing in $[x, u_{i_\bullet}]$ equals $y_{m+\ell_{i_\bullet, j}}$ in Corollary 4.22 (any x -term not equal to a corresponding y -term will annihilate v_{i_\bullet}). Then $[x, u_{i_\bullet}]v_{i_\bullet} \neq 0$, and we have found a nonzero element xv in $\widehat{\mathbf{N}}$, which gives the starting point for induction on the height.

Suppose now that $\text{ht}(\beta) > 1$. A basis for $\mathbf{U}(\mathfrak{g}_{-\mathbf{R}})$ consists of monomials of the form $z_{\beta_t, n_t}^{p_t} \cdots z_{\beta_2, n_2}^{p_2} z_{\beta_1, n_1}^{p_1}$, where $0 \neq z_{\beta_j, n_j} \in \mathfrak{g}_{-\beta_j + n_j \delta}$, $\beta_1 \leq \beta_2 \leq \dots$, and if $\beta_i = \beta_{i+1}$, then $n_i < n_{i+1}$. Thus, we can assume for w in (5.4) that

$$w = \sum_{i \in \mathcal{J}} u_i v_i = \sum_{i \in \mathcal{J}} z_{\beta_{i, t(i)}, n_{i, t(i)}}^{p_{i, t(i)}} \cdots z_{\beta_{i, 2}, n_{i, 2}}^{p_{i, 2}} z_{\beta_{i, 1}, n_{i, 1}}^{p_{i, 1}} v_i,$$

where for each i we have $u_i = z_{\beta_{i, t(i)}, n_{i, t(i)}}^{p_{i, t(i)}} \cdots z_{\beta_{i, 2}, n_{i, 2}}^{p_{i, 2}} z_{\beta_{i, 1}, n_{i, 1}}^{p_{i, 1}}$; each factor $z_{\beta_{i, j}, n_{i, j}}$ is basis element for $\mathfrak{g}_{-\beta_{i, j} + n_{i, j} \delta}$ and these basis elements are ordered as in the monomials above; $\sum_{j=1}^{t(i)} \beta_{i, j} = \beta$; and $\sum_{j=1}^{t(i)} n_{i, j} = \ell_i$. We may suppose that we have indexed the summands so $\beta_{1, 1} \leq \beta_{2, 1} \leq \dots$ and that among the $\beta_{i, j}$ with height equal to $\text{ht}(\beta_{1, 1})$, $p_{1, 1}$ is minimal.

Now suppose $x \in \mathfrak{g}_{\beta_{1, 1} - m \delta}$ is nonzero for some $m \in \mathbb{Z}$, and observe that $xv_i = 0$ for each i so that xw is as in (5.5). Since $[x, z_{\beta_{1, 1}, n_{1, 1}}] \in \mathfrak{g}_{(m+n_{1, 1})\delta}$, it is a linear combination of the $x_{m+n_{1, 1}, j}$. We assume that m has been chosen with $|m|$ sufficiently large so that $m + n_{1, 1}$ is distinct from all the k with $y_{k, j}$ occurring in some v_i , and so that at least one of the $x_{m+n_{1, 1}, j}$ equals $y_{m+n_{1, 1}, j}$. Since $[x, u_1]$ has $z_{\beta_{1, t(1)}, n_{1, t(1)}}^{p_{1, t(1)}} \cdots z_{\beta_{1, 2}, n_{1, 2}}^{p_{1, 2}} z_{\beta_{1, 1}, n_{1, 1}}^{p_{1, 1}-1} [x, z_{\beta_{1, 1}, n_{1, 1}}] v_i \neq 0$ appearing in it, we will have $0 \neq xw \in \mathbf{N}$. Because $\text{ht}(\beta - \beta_{1, 1}) < \text{ht}(\beta)$, we may apply induction to xw to find a nonzero element of $\widehat{\mathbf{N}}$. □

Lemma 5.3 immediately implies our main result about the structure of generalized loop modules.

Theorem 5.6. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a \neq 0$, and assume \mathbf{V} is an irreducible module in $\mathcal{K}_{\mathbf{L}, a}$. Then $\mathbf{M}(\lambda, \mathbf{V})$ is an irreducible \mathfrak{g} -module.*

As a consequence of this result, any irreducible module \mathbf{V} from the category $\mathcal{K}_{\mathbf{L}, a}$ with $a \neq 0$ and any $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) = a$ will determine an irreducible module $\mathbf{M}(\lambda, \mathbf{V})$ for the affine Lie algebra \mathfrak{g} . Let \mathbf{V} and \mathbf{W} be irreducible modules from $\mathcal{K}_{\mathbf{L}, a}$ with $a \neq 0$, and suppose $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda(c) = \mu(c) = a$. Then the modules $\mathbf{M}(\lambda, \mathbf{V})$ and $\mathbf{M}(\mu, \mathbf{W})$ are isomorphic if and only if \mathbf{V} and \mathbf{W} are isomorphic as \mathbf{L} -modules and $\lambda = \mu$ (up to a shift of gradation).

Corollary 5.7. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(c) \neq 0$, $\varphi : \mathbb{N} \rightarrow \{\pm\}$ any function, and let $\tilde{\varphi} : \mathcal{J} \rightarrow \{\pm\}$ be as in Section 4.4. Then the φ -imaginary Verma module $\mathbf{M}_\varphi(\lambda)$ and the $\tilde{\varphi}$ -imaginary Verma module $\mathbf{M}_{\tilde{\varphi}}(\lambda)$ are irreducible.*

5.2. Partial generalized loop modules.

Now we consider particular examples of generalized loop modules. Assume $I \subset \mathbb{N}$ and let $\varphi : \mathbb{N} \setminus I \rightarrow \{\pm\}$ be any function. Set $K_I = \mathbb{C}c \oplus \left(\bigoplus_{k \in I} \mathfrak{g}_{\pm k\delta}\right)$ and let

$$K_\varphi^\pm = \left(\bigoplus_{n \in \mathbb{N} \setminus I, \varphi(n) = \pm} \mathfrak{g}_{n\delta} \right) \oplus \left(\bigoplus_{m \in \mathbb{N} \setminus I, \varphi(m) = \mp} \mathfrak{g}_{-m\delta} \right).$$

Then $K := K_I \oplus K_\varphi^+$ is a parabolic subalgebra of L . Let N be an irreducible locally-finite \mathbb{Z} -graded module over the Heisenberg Lie algebra K_I with nonzero level a . Extend the action to a module structure over K by setting $K_\varphi^+ N = 0$. With these ingredients, we construct an induced locally-finite L -module

$$V := V_{I,\varphi}(N) = U(L) \otimes_{U(K)} N.$$

Then V is the tensor product of the vector space N with the Verma module over the Heisenberg Lie algebra $K_\varphi := K_\varphi^- \oplus \mathbb{C}c \oplus K_\varphi^+$. Standard arguments (compare Props. 3.2 and 3.3) show

Lemma 5.8. *The L -module $V = V_{I,\varphi}(N)$ is \mathbb{Z} -graded and irreducible when $a \neq 0$.*

Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda(c) = a$, and suppose that $M(\lambda, V)$ is the generalized loop \mathfrak{g} -module associated with λ and $V = V_{I,\varphi}(N)$. Alternately, we can construct an induced module directly from N by first making N into a module for $K \oplus \mathfrak{g}_\mathbb{R}$ by having $K_\varphi^+ \oplus \mathfrak{g}_\mathbb{R}$ be in the annihilator subalgebra of N . Then we can induce to a module $U(\mathfrak{g}) \otimes_{U(K \oplus \mathfrak{g}_\mathbb{R})} N$ for \mathfrak{g} .

Corollary 5.9. *The \mathfrak{g} -module $U(\mathfrak{g}) \otimes_{U(K \oplus \mathfrak{g}_\mathbb{R})} N$ is isomorphic to $M(\lambda, V)$ for $V = V_{I,\varphi}(N)$, and hence it is irreducible.*

Proof. It is sufficient to note that $V = U(L)N$ and to apply Theorem 5.6. \square

6. ACKNOWLEDGMENT

Part of this work was done while V. Futorny and I. Kashuba visited the University of Murcia. The support of the Seneca Foundation and the hospitality of the University of Murcia are greatly appreciated. This paper was discussed when G. Benkart, V. Futorny, and I. Kashuba visited the Mathematical Sciences Research Institute (MSRI) in Spring 2008. They acknowledge with gratitude the hospitality and stimulating research environment of MSRI. G. Benkart is grateful to the Simons Foundation for the support she received as a Simons Visiting Professor at MSRI. V. Futorny was supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2005/60337-2).

REFERENCES

- [BB] V. Bavula and V. Bekkert, *Indecomposable representations of generalized Weyl algebras*, Comm. Algebra **28** (2000), 5067–5100.
- [BBF] V. Bekkert, G. Benkart, and V. Futorny, *Weight modules for Weyl algebras*, Kac-Moody Lie Algebras and Related Topics, 17–42, Contemp. Math. **343**, Amer. Math. Soc., Providence, RI, 2004.
- [Ca] P. Casati, *A vertex operator construction of some not highest or lowest weight modules of the affine Kac-Moody Lie algebras*, preprint.
- [C] V. Chari, *Integrable representations of affine Lie algebras*, Invent. Math. **85** (1986), 317–335.

- [CP] V. Chari and A. Pressley, *New unitary representations of loop groups*, Math. Ann. **275** (1986), 87-104.
- [Ch] K. Christodouloupoulou, *Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras*, J. Algebra **320** (2008), 2871-2890.
- [Co] B. Cox, *Verma modules induced from nonstandard Borel subalgebras*, Pacific J. Math. **165** (1994), 269-294.
- [DFG] I. Dimitrov, V. Futorny, and D. Grantcharov, *Parabolic sets of roots*, Groups, Rings and Group Rings, 61-73, Contemp. Math., **499** Amer. Math. Soc., Providence, RI, 2009.
- [DGO] Yu. A. Drozd, B.L. Guzner, S.A. Ovsienko, *Weight modules over generalized Weyl algebras*, J. Algebra **184** (1996), 491-504.
- [F1] V. Futorny, *The parabolic subsets of root systems and corresponding representations of affine Lie algebras*, Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989), 45-52, Contemp. Math., **131**, Part 2, Amer. Math. Soc., Providence, RI, 1992.
- [F2] V. Futorny, *Imaginary Verma modules for affine Lie algebras*, Canad. Math. Bull. **37** (1994), 213-218.
- [F3] V. Futorny, *Irreducible non-dense $A_1^{(1)}$ -modules*, Pacific J. Math. **172** (1996), 83-99.
- [F4] V. Futorny, *Representations of Affine Lie Algebras*, Queen's Papers in Pure and Applied Math., **106** Queen's University, Kingston, ON, 1997.
- [FK] V. Futorny and I.Kashuba, *Induced modules for Kac-Moody Lie algebras*, SIGMA - Symmetry, Integrability and Geometry: Methods and Applications **5** (2009), Paper 026.
- [FKM] V. Futorny, S. Koenig and V. Mazorchuk, *Categories of induced modules for Lie algebras with triangular decomposition*, Forum Math. **13** (2001), 641-661.
- [FS] V. Futorny and H. Saifi, *Modules of Verma type and new irreducible representations for affine Lie algebras*, Representations of Algebras (Ottawa, ON, 1992), 185-191, CMS Conf. Proc. **14** Amer. Math. Soc., Providence, RI, 1993.
- [FT] V. Futorny and A. Tsylyke, *Classification of irreducible nonzero level modules with finite-dimensional weight spaces for affine Lie algebras*, J. Algebra **238** (2001), 426-441.
- [JK] H.P. Jakobsen and V. Kac, *A new class of unitarizable highest weight representations of infinite dimensional Lie algebras*, Nonlinear Equations in Classical and Quantum Field Theory (Meudon/Paris, 1983/1984), 1-20, Lecture Notes in Phys. **226** Springer, Berlin, 1985.
- [K] V. Kac, *Infinite-dimensional Lie Algebras*. Third edition. Cambridge University Press, Cambridge, 1990.

DEPARTAMENTO DE MATEMÁTICA, ICEX, UNIVERSIDADE FEDERAL DE MINAS GERAIS, AV. ANTÔNIO CARLOS, 6627, CP 702, CEP 30123-970, BELO HORIZONTE-MG, BRASIL

E-mail address: bekkert@mat.ufmg.br

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706, USA

E-mail address: benkart@math.wisc.edu

INSTITUTE OF MATHEMATICS, UNIVERSITY OF SÃO PAULO, CAIXA POSTAL 66281 CEP 05314-970, SÃO PAULO, BRASIL

E-mail address: futorny@ime.usp.br

E-mail address: kashuba@ime.usp.br