LIE ALGEBRAS AND YANG–BAXTER EQUATIONS

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Abstract

At the previous congress (CRM 6), we reviewed the constructions of Yang-Baxter operators from associative algebras, and presented some (colored) bialgebras and Yang-Baxter systems related to them.

The current talk deals with Yang-Baxter operators from $(\mathbb{G}, \theta)$-Lie algebras (structures which unify the Lie algebras and Lie superalgebras). Thus, we produce solutions for the constant and the spectral-parameter Yang-Baxter equations, Yang-Baxter systems, etc.

Attempting to present the general framework we review the work of other authors and we propose problems, applications and directions of study.

1 Introduction

Quantum Groups can be identified with quasitriangular Hopf Algebras. This notion is due to Drinfeld, motivated by developments in mathematical physics. The significance of the quasitriangular condition is that it gives an explanation for the Yang-Baxter equation (see [13, 12, 16]). This equation plays a role in Theoretical Physics ([21]), Knot Theory ([15]), Quantum Groups ([17, 19, 18]), etc.

In the next section, we review the constructions of Yang-Baxter operators from associative algebras, the associated bialgebras and some results on Yang-Baxter systems (from [18] and [22]).

Section 3 deals with Yang-Baxter operators from $(\mathbb{G}, \theta)$-Lie algebras (structures which unify the Lie algebras and Lie superalgebras). We produce solutions for the constant and the spectral-parameter Yang-Baxter equations and Yang-Baxter systems (see [22]).

Finally, we present the general framework, results of other authors, and our new results. We discuss about an extension for the duality between Lie algebras and Lie coalgebras, Poisson algebras, and the classical Yang-Baxter equation.

In this paper we propose (open) problems, applications and directions of study.

2 Non-linear equations and bialgebras

This section is a survey on Yang-Baxter operators from algebra structures and some related topics: connections to knot theory, FRT constructions, coloured Yang-Baxter operators and Yang-Baxter systems.

The quantum Yang-Baxter equation (QYBE) first appeared in theoretical physics and statistical mechanics. It plays a crucial role in knot theory, in analysis of integrable systems, in quantum and statistical mechanics and also in the theory of quantum groups. In the quantum group theory, the solutions of the constant QYBE lead to examples of bialgebras via the FRT construction [6, 12]. Non-additive solutions of the two-parameter form of the QYBE are referred to as a *coloured Yang-Baxter operators*. *Yang–Baxter systems* ([8, 9, 10]) emerged from the study of quantum integrable systems, as generalisations of the QYBE related to nonultralocal models.

2.1 The constant QYBE

Throughout this paper k is a field. All tensor products appearing in this paper are defined over k. For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map of the space V.

We use the following notations concerning the Yang-Baxter equation.

If $R : V \otimes V \to V \otimes V$ is a k-linear map, then $R^{12} = R \otimes I, R^{23} = I \otimes R, R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2.1. An invertible k-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{2.1}$$

Remark 2.2. The equation (2.1) is usually called the braid equation. It is a well-known fact that the operator R satisfies (2.1) if and only if $R \circ \tau$ satisfies the constant QYBE (if and only if $\tau \circ R$ satisfies the constant QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2.2}$$

Remark 2.3. (i) An exhaustive list of invertible solutions for (2.2) in dimension 2 is given in [7].

(ii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let A be an associative k-algebra, and $\alpha, \beta, \gamma \in k$. We define the k-linear map:
$R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A, \quad R^A_{\alpha,\beta,\gamma}(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b.$

Theorem 2.4. *(S. Dăscălescu and F. F. Nichita, [3]) Let A be an associative k-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^A_{\alpha,\beta,\gamma}$ is a Yang-Baxter operator if and only if one of the following holds:*

(i) $\alpha = \gamma \neq 0, \quad \beta \neq 0$;

(ii) $\beta = \gamma \neq 0, \quad \alpha \neq 0$;
(iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.
If so, we have $(R^A_{\alpha,\beta,\gamma})^{-1} = R^A_{\frac{1}{\beta},\frac{1}{\alpha},\frac{1}{\gamma}}$ *in cases (i) and (ii), and* $(R^A_{0,0,\gamma})^{-1} = R^A_{0,0,\frac{1}{\gamma}}$ *in case (iii).*

Remark 2.5. The Yang–Baxter equation plays an important role in knot theory. Turaev has described a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". In [15], we considered the problem of applying Turaev's method to the Yang–Baxter operators derived from algebra structures presented in the above theorem.

We concluded that the Alexander polynomial is the knot invariant corresponding to the axioms of associative algebras.

Remark 2.6. In dimension two, the Theorem 2.4 leads to the following R-matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q & q & 0 \\ \eta & 0 & 0 & -q \end{pmatrix} \tag{2.3}$$

where $\eta \in \{0,\ 1\}$, and $q \in k - \{0\}$.

The FRT bialgebras associated to (2.3) have the following independent commutation relations:
(i) the case $\eta = 0$

$$ba = qab, ac = ca, [a,d] = (1-q)cb, (1+q)b^2 = 0,$$

$$bc = qcb, bd = -qdb, (1+q)c^2 = 0, dc = -cd$$

(ii) the case $\eta = 1$

$$ba = qab, ab = dc + cd, [a,c] = db, a^2 - d^2 = (1+q)c^2,$$

$$[a,d] = (1-q)cb, b^2 = 0, bc = qcb, bd = -qdb$$

where $[a,c] = ac - ca, \quad [a,d] = ad - da$.

The comultiplication $\delta(T) = T \otimes T$ and counit $\epsilon(T) = I_2$ form the underlying coalgebra structure, where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The coquasitriangular structure is associated in the standard way.

2.2 The two-parameter form of the QYBE

Formally, a coloured Yang-Baxter operator is defined as a function

$$R : X \times X \to \mathrm{End}_k(V \otimes V),$$

where X is a set and V is a finite dimensional vector space over a field k.

Thus, for any $u, v \in X$, $R(u,v) : V \otimes V \to V \otimes V$ is a linear operator.

We consider three operators acting on a triple tensor product $V\otimes V\otimes V$, $R^{12}(u,v) = R(u,v)\otimes I$, $R^{23}(v,w) = I\otimes R(v,w)$, and similarly $R^{13}(u,w)$ as an operator that acts non-trivially on the first and third factor in $V\otimes V\otimes V$.

R is a coloured Yang-Baxter operator if it satisfies the two-parameter form of the QYBE,

$$R^{12}(u,v)R^{13}(u,w)R^{23}(v,w) = R^{23}(v,w)R^{13}(u,w)R^{12}(u,v) \tag{2.4}$$

for all $u,v,w\in X$.

Below, we present families of solutions for the equations (2.4). We assume that X is equal to (a subset of) the ground field k. The key point of the construction is to suppose that $V = A$ is an associative k-algebra, and then to derive a solution to equation (2.4) from the associativity of the product in A.

Thus, in [19], we sought solutions to equation (2.4) of the following form

$$R(u,v)(a\otimes b) = \alpha(u,v)1\otimes ab + \beta(u,v)ab\otimes 1 - \gamma(u,v)b\otimes a, \tag{2.5}$$

where α,β,γ are k-valued functions on $X\times X$.

Inserting this ansatz into equation (2.4), we obtained the following system of equations (whose solutions produce coloured Yang-Baxter operators):

$$\begin{aligned}&(\beta(v,w)-\gamma(v,w))(\alpha(u,v)\beta(u,w)-\alpha(u,w)\beta(u,v))\\ &\quad+(\alpha(u,v)-\gamma(u,v))(\alpha(v,w)\beta(u,w)-\alpha(u,w)\beta(v,w))=0\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\beta(v,w)(\beta(u,v)-\gamma(u,v))(\alpha(u,w)-\gamma(u,w))\\ &\quad+(\alpha(v,w)-\gamma(v,w))(\beta(u,w)\gamma(u,v)-\beta(u,v)\gamma(u,w))=0\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\alpha(u,w)-\gamma(u,w))+\alpha(v,w)\gamma(u,w)(\gamma(u,v)-\alpha(u,v))\\ &\quad+\gamma(v,w)(\alpha(u,v)\gamma(u,w)-\alpha(u,w)\gamma(u,v))=0\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\beta(u,w)-\gamma(u,w))+\beta(v,w)\gamma(u,w)(\gamma(u,v)-\beta(u,v))\\ &\quad+\gamma(v,w)(\beta(u,v)\gamma(u,w)-\beta(u,w)\gamma(u,v))=0\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\alpha(u,v)(\alpha(v,w)-\gamma(v,w))(\beta(u,w)-\gamma(u,w))\\ &\quad+(\beta(u,v)-\gamma(u,v))(\alpha(u,w)\gamma(v,w)-\alpha(v,w)\gamma(u,w))=0\end{aligned} \tag{2.10}$$

Remark 2.7. (i) The system of equations (2.6–2.10) is rather non-trivial. It is an open problem to classify its solutions.

(ii) We found the following solutions to that system: $\alpha(u,v) = p(u-v)$, $\beta(u,v) = q(u-v)$ and $\gamma(u,v) = pu-qv$. (Thus, we obtained the next theorem.)

Theorem 2.8. *(F. F. Nichita and D. Parashar, [19]) For any two parameters $p,q\in k$, the function $R : X\times X\to \mathrm{End}_k(A\otimes A)$ defined by*

$$R(u,v)(a\otimes b) = p(u-v)1\otimes ab + q(u-v)ab\otimes 1 - (pu-qv)b\otimes a, \tag{2.11}$$

satisfies the coloured QYBE (2.4).

Remark 2.9. If $pu \neq qv$ and $qu \neq pv$ then the operator (2.11) is invertible. Moreover, the following formula holds:

$$R^{-1}(u,v)(a \otimes b) = \frac{p(u-v)}{(qu-pv)(pu-qv)} ba \otimes 1 + \frac{q(u-v)}{(qu-pv)(pu-qv)} 1 \otimes ba - \frac{1}{(pu-qv)} b \otimes a$$

Algebraic manipulations of the previous theorem lead to the following result.

Theorem 2.10. *(F. F. Nichita and B. P. Popovici, [22]) Let A be an associative k-algebra with $\dim A \geq 2$ and $q \in k$. Then the operator*

$$S(\lambda)(a \otimes b) = (e^{\lambda} - 1) 1 \otimes ab + q(e^{\lambda} - 1) ab \otimes 1 - (e^{\lambda} - q) b \otimes a \tag{2.12}$$

satisfies the one-parameter form of the Yang-Baxter equation:

$$S^{12}(\lambda_1 - \lambda_2) S^{13}(\lambda_1 - \lambda_3) S^{23}(\lambda_2 - \lambda_3) =$$
$$= S^{23}(\lambda_2 - \lambda_3) S^{13}(\lambda_1 - \lambda_2) S^{12}(\lambda_1 - \lambda_2). \tag{2.13}$$

If $e^{\lambda} \neq q, \ \frac{1}{q}$, then the operator (2.12) is invertible.
Moreover, the following formula holds:

$$S^{-1}(\lambda)(a \otimes b) = \frac{e^{\lambda}-1}{(qe^{\lambda}-1)(e^{\lambda}-q)} ba \otimes 1 + \frac{q(e^{\lambda}-1)}{(qe^{\lambda}-1)(e^{\lambda}-q)} 1 \otimes ba - \frac{1}{e^{\lambda}-q} b \otimes a.$$

Remark 2.11. The operator from Theorem 2.10 can be obtained from Theorem 2.4 and the **Baxterization** procedure from [5] (page 22).

Hint: Consider the operator $R^A_{q,\frac{1}{q},\frac{1}{q}} : A \otimes A \to A \otimes A, \ \ a \otimes b \mapsto qab \otimes 1 + \frac{1}{q} \otimes ab - \frac{1}{q} a \otimes b$ and its inverse, $R^A_{q,\frac{1}{q},q}$.

2.3 Yang-Baxter systems

It is convenient to describe the Yang-Baxter systems in terms of the Yang-Baxter commutators.

Let V, V', V'' be finite dimensional vector spaces over the field k, and let $R : V \otimes V' \to V \otimes V'$, $S : V \otimes V'' \to V \otimes V''$ and $T : V' \otimes V'' \to V' \otimes V''$ be three linear maps. The *constant Yang–Baxter commutator* is a map $[R,S,T] : V \otimes V' \otimes V'' \to V \otimes V' \otimes V''$ defined by

$$[R,S,T] := R^{12} S^{13} T^{23} - T^{23} S^{13} R^{12}. \tag{2.14}$$

Note that $[R,R,R] = 0$ is just a short-hand notation for the constant QYBE (2.2).

A system of linear maps $W : V \otimes V \to V \otimes V$, $Z : V' \otimes V' \to V' \otimes V'$, $X : V \otimes V' \to V \otimes V'$, is called a WXZ–system if the following conditions hold:

$$[W,W,W] = 0 \qquad [Z,Z,Z] = 0 \qquad [W,X,X] = 0 \qquad [X,X,Z] = 0 \tag{2.15}$$

It was observed that WXZ–systems with invertible W, X and Z can be used to construct dually paired bialgebras of the FRT type leading to quantum doubles. The above is one type of a constant Yang–Baxter system that has recently been studied in [19] and also shown to be closely related to entwining structures [2].

Theorem 2.12. *(F. F. Nichita and D. Parashar, [19]) Let A be a k-algebra, and $\lambda, \mu \in k$. The following is a WXZ–system:*
$W : A\otimes A \to A\otimes A, \quad W(a\otimes b) = ab\otimes 1 + \lambda 1\otimes ab - b\otimes a,$
$Z : A\otimes A \to A\otimes A, \quad Z(a\otimes b) = \mu ab\otimes 1 + 1\otimes ab - b\otimes a,$
$X : A\otimes A \to A\otimes A, \quad X(a\otimes b) = ab\otimes 1 + 1\otimes ab - b\otimes a.$

Remark 2.13. Let R be a solution for the two-parameter form of the QYBE, i.e. $R^{12}(u, v)R^{13}(u, w)R^{23}(v, w) = R^{23}(v, w)R^{13}(u, w)R^{12}(u, v) \ \ \forall\ u, v, w \in X$.
Then, if we fix $s, t \in X$, we obtain the following WXZ–system:
$W = R(s, s)$, $X = R(s, t)$ and $Z = R(t, t)$.

Remark 2.14. The Section 5 of [18] provides connections between the constant and coloured Yang-Baxter operators and Yang-Baxter systems from algebra structures, which were discovered while presenting the poster [20] in Cambridge (2006).

3 YANG–BAXTER OPERATORS FROM $(\mathbb{G}, \theta)$-LIE ALGEBRAS

The $(\mathbb{G}, \theta)$-Lie algebras are structures which unify the Lie algebras and Lie superalgebras. We use them to produce solutions for the quantum Yang–Baxter equation. The spectral-parameter Yang-Baxter equations and Yang-Baxter systems are also studied. The following authors constructed Yang-Baxter operators from Lie (co)algebras and Lie superalgebras before: [14], [1], [16], etc.

3.1 Lie superalgebras

Definition 3.1. A Lie superalgebra is a (nonassociative) $\mathbb{Z}_2$-graded algebra, or superalgebra, over a field k with the Lie superbracket, satisfying the two conditions:

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

$$(-1)^{|z||x|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0$$

where x, y and z are pure in the $\mathbb{Z}_2$-grading. Here, $|x|$ denotes the degree of x (either 0 or 1). The degree of $[x, y]$ is the sum of degree of x and y modulo 2.

Let $(L, [,])$ be a Lie superalgebra over k, and $Z(L) = \{z \in L : [z, x] = 0 \ \ \forall\ x \in L\}$.
For $z \in Z(L)$, $|z| = 0$ and $\alpha \in k$ we define:

$$\phi^L_\alpha \ : \ L\otimes L \ \longrightarrow \ L\otimes L$$

$$x\otimes y \mapsto \alpha[x, y]\otimes z + (-1)^{|x||y|} y\otimes x \ .$$

Its inverse is:

$$\phi^{L\,-1}_\alpha \ : \ L\otimes L \ \longrightarrow \ L\otimes L$$

$$x \otimes y \mapsto \alpha z \otimes [x, y] + (-1)^{|x||y|} y \otimes x$$

Theorem 3.2. *(F. F. Nichita and B. P. Popovici, [22])*
Let $(L, [,])$ be a Lie superalgebra and $z \in Z(L), |z| = 0$, and $\alpha \in k$. Then: ϕ^L_α is a YB operator.

Theorem 3.3. *(F. F. Nichita and B. P. Popovici, [22]) Let $(L, [,])$ be a Lie superalgebra, $z \in Z(L), |z| = 0$, $X \subset k$, and $\alpha, \beta : X \times X \to k$. Then, $R : X \times X \to \mathrm{End}_k(L \otimes L)$ defined by*

$$R(u, v)(a \otimes b) = \alpha(u, v)[a, b] \otimes z + \beta(u, v)(-1)^{|a||b|} a \otimes b, \tag{3.16}$$

satisfies the colored QYBE (2.4) $\iff \beta(u, w)\alpha(v, w) = \alpha(u, w)\beta(v, w)$.

Remark 3.4. $\alpha(u, v) = f(v)$ and $\beta(u, v) = g(v)$ is a solution for the above condition.
Letting $u = v$, we obtain that:

$$\phi^L_{\alpha,\beta} \ : \ L{\otimes}L \ \longrightarrow \ L{\otimes}L$$

$$x{\otimes}y \mapsto \alpha[x, y]{\otimes}z + (-1)^{|x||y|}\beta y{\otimes}x \ .$$

and its inverse:

$${\phi^L_{\alpha,\beta}}^{-1} \ : \ L{\otimes}L \ \longrightarrow \ L{\otimes}L$$

$$x \otimes y \mapsto \frac{\alpha}{\beta^2} z \otimes [x, y] + (-1)^{|x||y|}\frac{1}{\beta} y \otimes x$$

are Yang-Baxter operators.

Remark 3.5. Let us consider the above data and apply it to Remark 2.13. Then, if we let $s, t \in X$, we obtain the following WXZ–system:
$W(a \otimes b) = R(s, s)(a \otimes b) = f(s)[a, b] \otimes z + g(s)(-1)^{|a||b|} a \otimes b$, and
$Z(a{\otimes}b) = R(t, t)(a{\otimes}b) = X(a{\otimes}b) = R(s, t)(a{\otimes}b) = f(t)[a, b]{\otimes}z{+}g(t)(-1)^{|a||b|}a{\otimes}b$.

Remark 3.6. The results presented in this section hold for Lie algebras as well. This is a consequence of the fact that these operators restricted to the first component of a Lie superalgebra have the same properties.

3.2 $(\mathbb{G}, \theta)$-Lie algebras

We now consider the case of $(\mathbb{G}, \theta)$-Lie algebras as in [11]: a generalization of Lie algebras and Lie superalgebras.

A $(\mathbb{G}, \theta)$-Lie algebras consists of a $\mathbb{G}$-graded vector space L, with $L = \oplus_{g\in\mathbb{G}} L_g$, $\mathbb{G}$ a finite abelian group, a non associative multiplication $\langle .., .. \rangle : L \times L \to L$ respecting the graduation in the sense that $\langle L_a, L_b \rangle \subseteq L_{a+b}, \ \forall a, b \in \mathbb{G}$ and a function $\theta : \mathbb{G} \times \mathbb{G} \to C^*$ taking non-zero complex values. The following conditions are imposed:

- θ-braided (G-graded) antisymmetry: $\langle x, y \rangle = -\theta(a, b)\langle y, x \rangle$
- θ-braided (G-graded) Jacobi id: $\theta(c, a)\langle x, \langle y, z \rangle\rangle + \theta(b, c)\langle z, \langle x, y \rangle\rangle + \theta(a, b)\langle y, \langle z, x \rangle\rangle = 0$

- $\theta : G \times G \to C^*$ color function $\begin{cases} \theta(a+b,c) = \theta(a,c)\theta(b,c) \\ \theta(a,b+c) = \theta(a,b)\theta(a,c) \\ \theta(a,b)\theta(b,a) = 1 \end{cases}$

for all homogeneous $x \in L_a, y \in L_b, z \in L_c$ and $\forall a,b,c \in \mathbb{G}$.

Theorem 3.7. *(F. F. Nichita and B. P. Popovici, [22]) Under the above assumptions,*

$$R(x \otimes y) = \alpha[x,y] \otimes z + \theta(a,b) x \otimes y, \tag{3.17}$$

with $z \in Z(L)$, satisfies the equation (2.2) $\iff$ $\theta(g,a) = \theta(a,g) = \theta(g,g) = 1$, $\forall x \in L_a$ and $z \in L_g$.

The inverse operator reads: $R^{-1}(x \otimes y) = \alpha[y,x] \otimes z + \theta(b,a) x \otimes y$

Proof. If we consider the homogeneous elements $x \in L_a$, $y \in L_b$, $t \in L_c$,

$$R^{12}R^{13}R^{23}(x \otimes y \otimes t) = R^{23}R^{13}R^{12}(x \otimes y \otimes t)$$

is equivalent to

$$\theta(a,g)[x,[y,t]] \otimes z \otimes z + \theta(b,c)[[x,t],y] \otimes z \otimes z = \theta(g,g)[[x,y],c] \otimes z \otimes z \tag{3.18}$$

$$\theta(a,g)\theta(a,b+c) x \otimes [y,t] \otimes z = \theta(a,b)\theta(a,c) x \otimes \ [y,t] \otimes z \tag{3.19}$$

$$\theta(b,c)\theta(a+c,b)[x,t] \otimes y \otimes z = \theta(a,b)\theta(b,g)[x,t] \otimes y \otimes z \tag{3.20}$$

$$\theta(b,c)\theta(a,c)[x,y] \otimes z \otimes t = \theta(a+b,c)\theta(g,c)[x,y] \otimes z \otimes t \tag{3.21}$$

Due to the conditions $\langle L_a, L_b \rangle \subseteq L_{a+b}$ the above relations are true if $\theta(a,g) = \theta(b,g) = \theta(g,c) = \theta(g,g) = 1$ is assumed. □

4 Applications. Problems. Directions of study

4.1 A Duality Theorem for (Co)Algebras

Our aim in this subsection is to present an extension of the duality of finite dimensional algebras and coalgebras to the category of finite dimensional Yang-Baxter structures, denoted **f.d. YB str**.

Definition 4.1. We define the category **YB str** (respective **f.d.YB str**) whose objects are 4-tuples $(V, \varphi, e, \varepsilon)$, where
i) V is a (finite dimensional) k-space;
ii) $\varphi : V \otimes V \to V \otimes V$ is a YB operator;
iii) $e \in V$ such that $\varphi(x \otimes e) = e \otimes x$, $\varphi(e \otimes x) = x \otimes e$ $\forall x \in V$;
iv) $\varepsilon \in V \to k$ is a k-map such that $(I \otimes \varepsilon) \circ \varphi = \varepsilon \otimes I$, $(\varepsilon \otimes I) \circ \varphi = I \otimes \varepsilon$.
A morphism $f : (V, \varphi, e, \varepsilon) \to (V', \varphi', e', \varepsilon')$ in the category **YB str** is a k-linear map $f : V \to V'$ such that:
v) $(f \otimes f) \circ \varphi = \varphi' \circ (f \otimes f)$;
vi) $f(e) = e'$;
vii) $\varepsilon' \circ f = \varepsilon$.

Remark 4.2. The following are examples of objects from the category **YB str**:

(i) Let $R : V \otimes V \to V \otimes V$ is a YB operator. Then $(V,\, R,\, 0,\, 0)$ is an object in the category **YB str**.

(ii) Let V be a two dimensional k-space generated by the vectors e_1 and e_2. Then $(V,\, T, e_1\,, e_2^*\,)$ is an object in the category **f.d. YB str**.

Theorem 4.1. *(F. F. Nichita and S. D. Schack, [23]) i) There exists a functor:*

$F : \mathbf{k-alg} \longrightarrow \mathbf{YB\ str}$

$(\,A,\ M,\ u) \mapsto (A, \varphi_A,\ u(1) = 1_A,\ 0 \in A^*)$ *where* $\varphi_A(a \otimes b) = ab \otimes 1 + 1 \otimes ab - a \otimes b$.

Any k-algebra map f is simply mapped into a k-map.

ii) F is a full and faithful embbeding.

Theorem 4.2. *(F. F. Nichita and S. D. Schack, [23]) i) There exists a functor:*

$G : \mathbf{k-coalg} \longrightarrow \mathbf{YB\ str}$

$(C, \Delta, \varepsilon) \mapsto (C,\ \psi_C,\ 0 \in C,\ \varepsilon \in C^*)$ *where* $\psi_C = \Delta \otimes \varepsilon + \varepsilon \otimes \Delta - I_2$.

Any k-coalgebra map f is simply mapped into a k-map.

ii) G is a full and faithful embbeding.

Theorem 4.3. *(F. F. Nichita and S. D. Schack, [23])* (**Duality Theorem**)

i) The following is a duality functor: $\mathbf{D}\ :\ \mathbf{f.d.\ YB\ str} \longrightarrow \mathbf{f.d.\ YB\ str^{op}}$

$(V,\, \varphi,\, e,\, \varepsilon) \mapsto (V^*,\, i_{V,V}^{-1} \circ \varphi^* \circ i_{V,V},\, \varepsilon,\, \zeta_e)$ *where* $\zeta_e : V^* \to k,\ \zeta_e(g) = g(e)\quad \forall g \in V^*$.

Note that: $D(f) = f^*$, *for* $f : (V,\, \varphi,\, e,\, \varepsilon) \to (V',\, \varphi',\, e',\, \varepsilon')$.

ii) The following relations hold:

$D(\,(A,\, \varphi_A,\, 1_A,\, 0)\,) \;=\; (A^*,\, \psi_{A^*},\, 0,\, \zeta_{1_A})$

$D(\,(C,\, \psi_C,\, 0,\, \varepsilon)\,) = (C^*,\, \varphi_{C^*},\, \varepsilon = 1_{C^*},\, 0)$

Remark 4.3. We extended the duality between finite dimensional algebras and coalgebras to the category **f.d. YB str**. This can be seen bellow, in the following diagram:

$$\begin{tikzcd}
\mathbf{f.d.\ YB\ str} \arrow[r, shift left, "D = (\ )^*"] & \mathbf{f.d.\ YB\ str^{opp}} \arrow[l, shift left, "D = (\ )^*"] \\
\mathbf{f.d.\ k-alg} \arrow[u, "F"] \arrow[r, shift left, "(\ )^*"] & \mathbf{f.d.\ k-coalg^{opp}} \arrow[u, "G"'] \arrow[l, shift left, "(\ )^*"]
\end{tikzcd}$$

4.2 A Duality Theorem for Lie (Co)Algebras

[4] considered the constructions of Yang-Baxter operators from Lie (co)algebras, suggesting an extension (to a bigger category with a self-dual functor acting on it) for the duality between the category of finite dimensional Lie algebras and the category of finite dimensional Lie coalgebras. This duality extension is explained using the terminology of [16] below.

Let $(L, [,])$ be a Lie algebra over k. Then we can equip $L' \;=\; L \oplus kx_0$ with a Lie algebra structure such that $[x,\ x_0] = 0\ \ \forall x \in L'$. We define:

$$\phi = \phi_{L'}\ :\ (L \oplus kx_0) \otimes (L \oplus kx_0) \quad\longrightarrow\quad (L \oplus kx_0) \otimes (L \oplus kx_0)$$

$$x{\otimes}y \mapsto [x,y]{\otimes}x_0 + y{\otimes}x \ .$$

Theorem 4.4. *i) There exists a functor:*
$F:\mathbf{f.d.\ Lie\ alg} \longrightarrow \mathbf{f.d.\ YB\ str}$
$(L,\ [,]) \mapsto ((L \oplus kx_0), \phi,\ x_0,\ 0)$.
Any Lie algebra map f is simply mapped into a k-map.
ii) F is a full and faithful embbeding.

Proof:
i) First, we show that $(L',\ \phi_{L'}, x_0,\ 0)$ is an object in the category **YB str** :
$\phi_{L'}(x \otimes x_0) = x_0{\otimes}x, \quad \phi_{L'}(x_0 \otimes x) = x{\otimes}x_0,$
$(I\ \otimes 0) \circ \phi_{L'} = 0 = 0 \otimes I, \qquad (0 \otimes I) \circ \phi_{L'} = 0 = I\ \otimes 0\,.$
Now, for $f : L_1 \to L_2$ a morphism of Lie algebras, we prove that
$f : (L'_1,\ \phi_{L'_1},\ x_0,\ 0) \to (L'_2,\ \phi_{L'_2}, x_0,\ 0)$ is a morphism in the category **YB str**.
We extend f such that $f(x_0) = x_0$. Now, $0 \circ f = 0$. It only remains to prove that $(f \otimes f) \circ \phi_{L'_1} = \phi_{L'_2} \circ (f \otimes f)$.
$((f \otimes f) \circ \phi_{L'_1})\,(x{\otimes}y) = (f \otimes f)([x,y] \otimes x_0 + y \otimes x) = f([x,y]) \otimes f(x_0) + f(y) \otimes f(x) = f([x,y]) \otimes x_0 + f(y) \otimes f(x)$
$(\phi_{L'_2} \circ (f \otimes f))(x{\otimes}y) = [f(x), f(y)] \otimes x_0 + f(y) \otimes f(x)$.
Since $f : L_1 \to L_2$ is a morphism of Lie algebras, it follows that $(f \otimes f) \circ \phi_{L'_1} = \phi_{L'_2} \circ (f \otimes f)$.
ii) If two Lie algebras $(L_1,\ [,]_1)$ and $(L_2,\ [,]_2)$ project in the same object in the category **YB str** (i.e., $F\,[(L_1,\ [,]_1)] = F\,[(L_2,\ [,]_2)]$) then they have the same ground vector space and the same operation. So, F is an embedding.
Obviously, for two distinct Lie algebra maps $f, g : L_1 \to L_2$ we get two distinct **YB str** maps.
Now, for $f : (L'_1,\ \phi_{L'_1},\ x_0,\ 0) \to (L'_2,\ \phi_{L'_2}, x_0,\ 0)$ a morphism in **YB str** it follows $((f \otimes f) \circ \phi_{L'_1})\,(x \otimes y) = (\phi_{L'_2} \circ (f \otimes f))(x{\otimes}y)$; so,
$(f \otimes f)([x,y] \otimes x_0 + y \otimes x) = [f(x), f(y)] \otimes x_0 + f(y) \otimes f(x)$.
Thus, $f([x,y]) = [f(x), f(y)]$. □

A Lie coalgebra is a dual notion to a Lie algebra. It has a comultiplication, called "cobraket". We refer to [16] for more details and references.
Let (M, Δ) be a Lie coalgebra over k. Then we can equip $M' = M \oplus kx_0$ with a Lie coalgebra structure such that $\Delta(x_0) = 0 \in M'{\otimes}M'$. Observe that for $\nu = (x_0)^* : M' \to k$ the following relation holds: $(\nu{\otimes}I) \circ \Delta = 0 = (I{\otimes}\nu) \circ \Delta$.

Theorem 4.5. *i) There exists a functor:*
$G:\mathbf{f.d.\ Lie\ coalg} \longrightarrow \mathbf{f.d.\ YB\ str}$
$(\ M,\ \Delta) \mapsto ((M \oplus kx_0), \psi,\ 0,\ \nu)$, *where*
$\psi\ :\ (M{\oplus}kx_0){\otimes}(M{\oplus}kx_0) \ \longrightarrow\ (M{\oplus}kx_0){\otimes}(M{\oplus}kx_0), \quad x{\otimes}y \mapsto \Delta(x)\nu(y) + y{\otimes}x\ .$
Any Lie coalgebra map f is simply mapped into a k-map.
ii) G is a full and faithful embbeding.

Proof: The proof is dual to the previous proof, and we will briefly explain only its key points. By Theorem 5.2.1 of [16], it follows that ψ is an Yang-Baxter operator.
$(I \otimes\nu){\circ}\psi_{M'} = \nu{\otimes}I, \quad (\nu{\otimes}I){\circ}\psi_{M'} = I \otimes\nu$ follow from $(\nu{\otimes}I){\circ}\Delta = 0 = (I{\otimes}\nu){\circ}\Delta$.
The proof of ii) follows by direct computations.

Otherwise, it can be viewed as a consequence of the Section 2 of [3]. Thus, the theory of Yang-Baxter operators from (Lie)algebras can be transfered to the Yang-Baxter operators from (Lie) coalgebras. $\square$

Remark 4.4. We extend the duality between finite dimensional Lie algebras and Lie coalgebras to the category **f.d. YB str**. This can be seen in the following diagram:

$$\begin{tikzcd}
\mathbf{f.d.\ YB\ str} \arrow[r, shift left, "D=(\ )^*"] & \mathbf{f.d.\ YB\ str^{opp}} \arrow[l, shift left, "D=(\ )^*"] \\
\mathbf{f.d.\ Lie\ alg} \arrow[u, "F"] \arrow[r, shift left, "(\ )^*"] & \mathbf{f.d.\ Lie\ coalg^{opp}} \arrow[u, "G"'] \arrow[l, shift left, "(\ )^*"]
\end{tikzcd}$$

4.3 Poisson algebras

Poison algebras appear in quantum groups, Hamiltonian mechanics, the theory of Simplectic manifolds, etc.

Definition 4.5. A Poisson algebra is a vector space over k, V, equipped with two bilinear products, $*$ and $\{\ ,\ \}$, having the following properties:
- the product $*$ forms an associative k-algebra;
- the product $\{\ ,\ \}$, called the Poisson bracket, forms a Lie algebra;
- the Poisson bracket acts as a derivation on the product $*$, i.e.
$\{x,\ y * z\} = \{x,\ y\} * z + y * \{x,\ z\} \quad \forall x, y, z \in V.$

Examples.
1. Any associative algebra with the commutator $[x,\ y] = xy - yx$ turns into a Poisson algebra.
2. For a vertex operator algebra, a certain quotient becomes a Poisson algebra.

Remark 4.6. A Lie algebra $(L,\ [,])$ has a Poisson algebra structure such that the Poisson bracket equals the associative product (i.e., $[x, y] = x * y \ \ \forall x, y \in L) \iff [x, y] \in Z(L)\ \forall x, y \in L$.

Theorem 4.6. *Let A be a Poisson algebra with a unity, $1 = 1_A$, for the product *, such that $\{x,\ 1_A\} = 0 \ \ \forall x \in A$. Then, we have the following WXZ-system:*
$W(x \otimes y) = \{x,\ y\}\otimes 1 \ + \ x \otimes y;$
$X(x \otimes y) = 1 \otimes \{x,\ y\} \ + \ x \otimes y;$
$Z(x \otimes y) = 1 \otimes x * y \ + \ x * y \otimes 1 \ - \ y \otimes x.$

Proof. It follows by direct computations.

4.4 Other results and comments

Motivated by the need to create a better frame for the study of Lie (super)algebras than that presented in [24], this paper generalizes the constructions from [14] (to (G, θ)-Lie algebras). Other applications of these results could be in constructions of: FRT bialgebras and knot invariants, solutions for the classical Yang-Baxter equation (see below), etc.

Theorem 4.7. *Let $(L, [,])$ be a Lie algebra, $z \in Z(L)$ and $\alpha \in k$. Then:*
$r : L \otimes L \quad \longrightarrow \quad L \otimes L, \quad x \otimes y \mapsto [x, y] \otimes z + \alpha x \otimes y$
satisfies the classical Yang-Baxter equation:
$[r^{12},\ r^{13}] \ + \ [r^{12},\ r^{23}] \ + \ [r^{13},\ r^{23}] = 0.$

Proof. It follows by direct computations.

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