

THE GEOMETRY OF EMBEDDED PSEUDO-RIEMANNIAN SURFACES IN TERMS OF POISSON BRACKETS

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ABSTRACT. Arnlin, Hoppe and Huisken showed in [1] how to express the Gauss and mean curvature of a surface embedded in a Riemannian manifold in terms of Poisson brackets of the embedding coordinates. We generalize these expressions to the pseudo-Riemannian setting and derive explicit formulas for the case of surfaces embedded in \mathbb{R}^m with indefinite metric.

1. INTRODUCTION

Motivated by certain equations of Membrane Theory, Arnlin, Hoppe and Huisken found a way to express geometric quantities of a surface Σ embedded in a Riemannian manifold in a purely algebraic way [1]: They showed using the canonical Poisson bracket on $C^\infty(\Sigma)$ that the Gauss curvature of a surface embedded in the euclidean \mathbb{R}^m via the coordinates x^1, \dots, x^m is given by

$$K = -\frac{1}{8(m-3)!} \sum_{L \in \{1, \dots, m\}^{m-3}} \epsilon_{jklL} \epsilon_{irnl} \{x^i, \{x^k, x^l\}\} \{x^j, \{x^r, x^n\}\}, \quad (1.1)$$

thereby also providing a starting point to generalize the notion of curvature to certain non-commutative spaces, e. g. matrix models that relate the Poisson bracket to the commutator of matrices.

The crucial point in the derivation of 1.1 turns out to be the explicit construction of normal vectors to Σ in terms of Poisson brackets of the embedding coordinates. In section 3, we will perform this construction for surfaces Σ embedded in a pseudo-Riemannian manifold. In theorem 3.7, we will then derive a generalization of equation 1.1 to the pseudo-Riemannian setting.

2. PRELIMINARIES

We use the following notation (cf. [1], [3]): Let M be an m -dimensional pseudo-Riemannian manifold and $\Sigma \subset M$ a 2-dimensional orientable pseudo-Riemannian submanifold with codimension $p = m - 2$. The metric tensors of M and Σ are denoted by \bar{g} and g , the Levi-Civita connections by $\bar{\nabla}$ and ∇ , the curvature tensors by \bar{R} and R (with the convention $R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$). Indices a, b, c, d, p, q run from 1 to 2, indices $i, j, k, l, r, s, t, v, w$ from 1 to m and A, B from 1 to p . x^i and u^a are local coordinates of M and Σ , the Christoffel symbols are $\bar{\Gamma}_{ij}^k$ and Γ_{ab}^c , respectively. The metric of M in the coordinates x^i is given by the matrix (\bar{g}_{ij}) , the metric of Σ in the coordinates u^a is given by $(g_{ab}) = \bar{g}(e_a, e_b)$, $e_a = (\partial_a x^i) \partial_i$ being an oriented basis of the tangent space $T\Sigma$. The inverses of

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these matrices are denoted by (\bar{g}^{ij}) and (g^{ab}) . We write $g = \det(g_{ab})$. Furthermore, let $N_A = N_A^m \partial_m$ be a smooth local orthonormal frame of the normal bundle $T\Sigma^\perp \subset TM$, that is, $\bar{g}(N_A, N_B) = \delta_{AB} \sigma_A$ with $\sigma_A = \pm 1$. Finally, let ϵ^{ab} be the totally antisymmetric tensor of rank 2.

$\mathfrak{X}(M)$ denotes the space of vector fields on M , $\Omega^1(M)$ the space of 1-forms, $\mathfrak{T}_s^r(M)$ the space of tensor fields on M of type (r, s) . The space of smooth maps $Z: \Sigma \rightarrow TM$ with $\pi_{TM \rightarrow M} \circ Z = id_\Sigma$ is denoted by $\bar{\mathfrak{X}}(\Sigma)$. $\bar{\Omega}^1(\Sigma)$ and $\bar{\mathfrak{T}}_s^r(\Sigma)$ are defined analogously. For $p \in \Sigma$ we have the orthogonal projections $\tan_p: T_p M \rightarrow T_p \Sigma$ and $\text{nor}_p = 1 - \tan_p: T_p M \rightarrow T_p \Sigma^\perp$.

For $X, Y \in \mathfrak{X}(\Sigma)$, the Gauss and Weingarten equations

$$\bar{\nabla}_X Y = \nabla_X Y + II(X, Y) \quad (2.1)$$

$$\bar{\nabla}_X N_A = -W_A(X) + D_X N_A \quad (2.2)$$

hold with $W_A(X) = -\tan \bar{\nabla}_X N_A$ and $D_X N_a = \text{nor} \bar{\nabla}_X N_a$. The orthonormal decomposition of the shape tensor II is

$$II(X, Y) = \sum_{A=1}^p \sigma_A \bar{g}(II(X, Y), N_A) N_A = \sum_{A=1}^p \sigma_A h_A(X, Y) N_A \quad (2.3)$$

with

$$h_A(X, Y) = \bar{g}(II(X, Y), N_A) = -\bar{g}(X, \bar{\nabla}_Y N_A).$$

Setting $h_{A,ab} = h_A(e_a, e_b)$, we have

$$h_{A,ab} = h_{A,ba} \quad (2.4)$$

$$(W_A)_b^a = g^{ac} W_{A,bc} = g^{ac} g(e_b, W_A e_c) = g^{ac} h_{A,bc} = g^{ac} h_{A,cb}. \quad (2.5)$$

Using (2.3), we can write the Gauss curvature of Σ as

$$\begin{aligned} K &= \frac{1}{g} [\bar{g}(\bar{R}(e_1, e_2)e_1, e_2) + \bar{g}(II(e_1, e_1), II(e_2, e_2)) - \bar{g}(II(e_1, e_2), II(e_1, e_2))] \\ &= \frac{1}{g} \left(\bar{g}(\bar{R}(e_1, e_2)e_1, e_2) + \sum_{A=1}^p \sigma_A \det(h_{A,ab}) \right). \end{aligned} \quad (2.6)$$

The mean curvature vector is given by

$$H = \frac{1}{2} g^{ab} II(e_a, e_b) = \frac{1}{2} \sum_{A=1}^p \sigma_A (W_A)_a^a N_A = \frac{1}{2} \sum_{A=1}^p \sigma_A (\text{tr} W_A) N_A. \quad (2.7)$$

2.1. The Poisson algebra $C^\infty(\Sigma)$. We have the following standard construction: Let $\omega = \rho du_1 \wedge du_2$ be a symplectic form on Σ . There is a unique map $X: C^\infty(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$, $f \mapsto X_f$, with $\omega(X_f, Y) = df(Y)$ for all $Y \in \mathfrak{X}(\Sigma)$. Then

$$\{f, g\} := \omega(X_f, X_g) = \frac{1}{\rho} \epsilon^{ab} (\partial_a f) (\partial_b g) \quad (2.8)$$

is a Poisson bracket on $C^\infty(\Sigma)$.

We continue by recalling some of the results obtained in [1] which are still valid in the pseudo-Riemannian setting. Define the functions

$$\mathcal{P}^{ij} = \{x^i, x^j\} \quad (2.9)$$

$$\mathcal{S}_A^{ij} = \frac{1}{\rho} \epsilon^{ab} (\partial_a x^i) (\bar{\nabla}_b N_A)^j = \{x^i, N_A^j\} + \{x^i, x^k\} \bar{\Gamma}_{kl}^j N_A^l. \quad (2.10)$$

They are components of $\bar{\mathfrak{X}}_0^2(\Sigma)$ tensors because by lowering the second index we obtain maps $\mathcal{P}, \mathcal{S}_A \in \text{End}(\bar{\mathfrak{X}}(\Sigma))$,

$$\begin{aligned}\mathcal{P}(X) &= \mathcal{P}^{ik} \bar{g}_{kj} X^j \partial_i = -\frac{1}{\rho} \bar{g}(X, e_a) \epsilon^{ab} e_b \\ \mathcal{S}_A(X) &= \mathcal{S}_A^{ik} \bar{g}_{kj} X^j \partial_i = -\frac{1}{\rho} \bar{g}(X, \bar{\nabla}_a N_A) \epsilon^{ab} e_b.\end{aligned}$$

Proposition 3.4 in [1] states that

$$\text{tr } \mathcal{P}^2 = -2 \frac{g}{\rho^2} \quad (2.11)$$

$$\text{tr } \mathcal{S}_A^2 = -\frac{2}{\rho^2} \det(h_{A,ab}). \quad (2.12)$$

Defining $\mathcal{B}_A \in \text{End}(\bar{\mathfrak{X}}(\Sigma))$ to be the composition $\mathcal{B}_A = \mathcal{P} \mathcal{S}_A$, one can check (cf. proposition 3.3 in [1]) that for $X \in \bar{\mathfrak{X}}(\Sigma)$,

$$\mathcal{B}_A(X) = -\frac{g}{\rho^2} \bar{g}(X, \bar{\nabla}_a N_A) g^{ab} e_b. \quad (2.13)$$

In particular, for $Y \in \bar{\mathfrak{X}}(\Sigma)$, $\mathcal{B}_A(Y) = \frac{g}{\rho^2} W_A(Y)$. Consequently, $\text{tr } \mathcal{B}_A = \frac{g}{\rho^2} \text{tr } W_A$.

The equations (2.6) and (2.7) now yield the formulas

$$K = \frac{1}{g} \bar{g}(\bar{R}(e_1, e_2)e_1, e_2) - \frac{\rho^2}{2g} \sum_{A=1}^p \sigma_A \text{tr } \mathcal{S}_A^2 \quad (2.14)$$

$$H = \frac{\rho^2}{2g} \sum_{A=1}^p \sigma_A (\text{tr } \mathcal{B}_A) N_A. \quad (2.15)$$

(Note the occurrence of the signs σ_A in the pseudo-Riemannian setting as opposed to the Riemannian setting.) In the case $M = \mathbb{R}_\nu^m$ with metric $\bar{g}_{ij} = \delta_{ij} \bar{g}_j$, $\bar{g}_1 = \dots = \bar{g}_\nu = -1$, $\bar{g}_{\nu+1} = \dots = \bar{g}_m = 1$, these formulas become

$$K = -\frac{\rho^2}{2g} \sum_{A=1}^p \sigma_A \bar{g}_i \bar{g}_j \{x^i, N_A^j\} \{x^j, N_A^i\} \quad (2.16)$$

$$H = \frac{\rho^2}{2g} \sum_{A=1}^p \sigma_A \bar{g}_i \bar{g}_j \{x^i, x^j\} \{x^j, N_A^i\} N_A^k \partial_k. \quad (2.17)$$

3. CONSTRUCTION OF NORMAL VECTORS

The formulas (2.16) and (2.17) can be written solely in terms of Poisson brackets provided that normal vectors can be expressed by Poisson brackets of the embedding coordinates x^i . We now generalize the construction of normal vectors in [1] to the pseudo-Riemannian case.

First, we introduce further notation: For multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ let $\bar{g}^{IJ} := \prod_{l=1}^k \bar{g}^{i_l j_l}$, $\bar{g}_{IJ} := \prod_{l=1}^k \bar{g}_{i_l j_l}$ and $dx^I := dx^{i_1} \otimes \dots \otimes dx^{i_k} \in \Omega^1(M) \otimes \dots \otimes \Omega^1(M) \cong \mathfrak{T}_k^0(M)$. Since on $\Omega^1(M)$ we have the scalar product (to be understood as a scalar product at each point) $\hat{g}(\eta_i dx^i, \omega_j dx^j) = \eta_i \omega_j \bar{g}^{ij}$, we obtain a scalar product on $\mathfrak{T}_{p-1}^0(M)$ (again, pointwise),

$$\bar{g}_\otimes(\eta_I dx^I, \omega_J dx^J) = \eta_I \omega_J \bar{g}^{IJ}. \quad (3.1)$$

Now, proposition 4.2 in [1] states that for any multi-index $J \in \{1, \dots, m\}^{p-1}$,

$$Z_J = \frac{\rho}{2\sqrt{|g|(p-1)!}} \bar{g}^{ij} \epsilon_{jklJ} \mathcal{P}^{kl} \partial_i \in T\Sigma^\perp, \quad (3.2)$$

where ϵ_{jklJ} denotes the Levi-Civita tensor of M .

By raising the $p-1$ indices, we can also regard the tensor $Z \in \bar{\mathfrak{T}}_{p-1}^1(\Sigma)$ as a tensor of type $((p-1)+1, 0)$. Explicitly,

$$Z^J = \bar{g}^{JK} Z_K = \frac{\rho}{2\sqrt{|g|(p-1)!}} \bar{g}_{kr} \bar{g}_{ln} \epsilon^{irnJ} \{x^k, x^l\} \partial_i. \quad (3.3)$$

From now on, we use $\delta := \text{ind } \bar{g} - \text{ind } g$, ind being the index of the respective bilinear form.

Lemma 3.1. $Z_K^i Z^{Kj} = (-1)^\delta \left(\bar{g}^{ij} + \frac{\rho^2}{g} (\mathcal{P}^2)^{ij} \right)$.

Proof. Use $\mathcal{P}^{ij} = -\mathcal{P}^{ji}$, equation (2.11) and $\epsilon_{rstK} \epsilon^{jvwK} = (-1)^{\text{ind } \bar{g}} (p-1)! (\delta_r^j \delta_s^v \delta_t^w + \delta_s^j \delta_t^v \delta_r^w - \delta_s^j \delta_r^v \delta_t^w - \delta_t^j \delta_s^v \delta_r^w - \delta_r^j \delta_t^v \delta_s^w)$. \square

Define the tensor

$$\mathcal{Z}^{IJ} = (-1)^\delta \bar{g}(Z^I, Z^J). \quad (3.4)$$

By lowering the multi-index J , we can regard \mathcal{Z} as an endomorphism of $\bar{\mathfrak{T}}_{p-1}^0(\Sigma)$,

$$\omega_I dx^I \mapsto \mathcal{Z}_J^I \omega_I dx^J. \quad (3.5)$$

Proposition 3.2. *The map $\mathcal{Z} \in \text{End}(\bar{\mathfrak{T}}_{p-1}^0(\Sigma))$ satisfies*

- (1) $\mathcal{Z}^2 = \mathcal{Z}$.
- (2) $\text{Tr } \mathcal{Z} = p$.
- (3) For $\eta, \omega \in \bar{\mathfrak{T}}_{p-1}^0$, $\bar{g}_\otimes(\mathcal{Z}\eta, \omega) = \bar{g}_\otimes(\eta, \mathcal{Z}\omega)$.

Proof. The first two equations follow from lemma 3.1 (cf. lemma 4.3 in [1]). To prove the third equation, we calculate

$$\begin{aligned} \bar{g}_\otimes(\mathcal{Z}\eta, \omega) &= \bar{g}_\otimes(\mathcal{Z}_J^I \eta_I dx^J, \omega_K dx^K) = (-1)^\delta \eta_I \omega_K \bar{g}^{JK} \bar{g}_{ij} Z_J^i Z^I{}^j \\ &= (-1)^\delta \eta_I \omega_K \bar{g}_{ij} Z^{Kj} Z^I{}^i = (-1)^\delta \eta_I \omega_K \bar{g}_{ij} Z^I{}^i Z^{Kj} = \dots \\ &= \bar{g}_\otimes(\eta, \mathcal{Z}\omega). \end{aligned} \quad \square$$

Proposition 3.3. *Let (V, g) be a scalar product space. Let $P: V \rightarrow V$ be linear with $P^2 = P$ and $g(Pv, w) = g(v, Pw)$ for all $v, w \in V$. Then $W = \text{im } P$ is a non-degenerate subspace of V .*

Proof. This follows from $v = Pv + (v - Pv)$ for $v \in V$, i. e. $V = W + W^\perp$. \square

We can now construct an orthonormal basis of $T\Sigma^\perp$. The main ingredient of the construction is the non-degeneracy of the image of \mathcal{Z} .

Proposition 3.4. (1) *The vector space $\bar{\mathfrak{T}}_{p-1}^0(\Sigma)$ with the metric \bar{g}_\otimes has an orthonormal basis $E^I = E_J^I dx^J$ of eigenvectors of $\mathcal{Z} \in \text{End}(\bar{\mathfrak{T}}_{p-1}^0(\Sigma))$.*
 (2) *Define the normal vectors $\hat{N}^I = Z^J E_J^I$. Then exactly $p = \dim T\Sigma^\perp$ of the \hat{N}^I are different from 0. The E^I can be chosen in such a way that the non-vanishing \hat{N}^I are a smooth orthonormal frame of $T\Sigma^\perp$.*

Proof. (1) follows immediately from propositions 3.2 and 3.3.

(2) Let μ^I be the eigenvalue of the eigenvector E^I of \mathcal{Z} . Then

$$\begin{aligned}\bar{g}(\hat{N}^I, \hat{N}^J) &= (-1)^\delta E_L^J \bar{g}^{LM} \mathcal{Z}_M^K E_K^I = (-1)^\delta E_L^J \bar{g}^{LM} \mu^I E_M^I \\ &= (-1)^\delta \mu^I \bar{g}_\otimes(E^I, E^J) = \pm \mu^I \delta^{IJ}.\end{aligned}$$

As $\mu^I \in \{0, 1\}$ and $\text{Tr } \mathcal{Z} = p$, exactly p of the μ^I are 1. The corresponding \hat{N}^I span $T\Sigma^\perp$. Since $T\Sigma^\perp$ is non-degenerate, the other $\hat{N}^J \in T\Sigma^\perp$ vanish. \square

To obtain explicit formulas for the curvature of Σ , we need to slightly extend lemma 4.4 in [1], the proof essentially being the same:

Lemma 3.5. *For $X \in \tilde{\mathfrak{X}}(\Sigma)$, define $\mathcal{S}^{ij}(X) = \frac{1}{\rho} \epsilon^{ab} (\partial_a x^i) (\bar{\nabla}_b X)^j$. Then for all $N, N' \in \tilde{\mathfrak{X}}^\perp(\Sigma)$ and $f, h \in C^\infty(\Sigma)$,*

$$\mathcal{S}_j^i(fN) \mathcal{S}_i^j(hN') = fh \mathcal{S}_j^i(N) \mathcal{S}_i^j(N'). \quad (3.6)$$

If $f = \text{const}$, then (3.6) holds for arbitrary $N \in \tilde{\mathfrak{X}}(\Sigma)$.

We concentrate on the case $M = \mathbb{R}_y^m$ with metric as above. Lemma 3.5 becomes

$$\bar{g}_i \bar{g}_j \{x^i, fN^j\} \{x^j, hN'^i\} = fh \bar{g}_i \bar{g}_j \{x^i, N^j\} \{x^j, N'^i\}$$

for f, h, N, N' as above. As a computational device, we need

Lemma 3.6. *Let e_1, \dots, e_n be a basis of the scalar product space (V, g) , $\hat{g} = (g_{ij})$ the matrix of g in this basis and $\hat{g}^{-1} = (g^{ij})$ its inverse. Let $v_i = v_i^j e_j$ be an orthonormal basis of V , $\sigma_i = g(v_i, v_i)$. Then*

$$\sum_{i=1}^n v_i^k g(v_i, v_i) v_i^l = g^{kl}.$$

Proof. $V_{ij} := v_j^i$, $D := \text{diag}(\sigma_i) \Rightarrow V^T \hat{g} V = D$. $D^2 = 1$ implies $VDV^T = \hat{g}^{-1}$. \square

Theorem 3.7. *The Gauss and mean curvature of a surface Σ embedded in \mathbb{R}_y^m are given by*

$$K = -\frac{\rho^4}{8g^2(p-1)!} \sum_L \bar{g}_i \bar{g}_r \bar{g}_n \epsilon_{jklL} \epsilon_{irnL} \{x^i, \{x^k, x^l\}\} \{x^j, \{x^r, x^n\}\} \quad (3.7)$$

$$H = \frac{\rho^4}{8g^2(p-1)!} \sum_{L, k'} \bar{g}_j \bar{g}_k \bar{g}_l \epsilon_{irnL} \epsilon_{k'lL} \{x^i, x^j\} \{x^j, \{x^r, x^n\}\} \{x^k, x^l\} \partial_{k'}, \quad (3.8)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket (2.8) on Σ .

Proof. Let E^I, μ^I, \hat{N}^I be as in proposition 3.4 and its proof, $\sigma^I := \bar{g}(\hat{N}^I, \hat{N}^I) = (-1)^\delta \mu^I \bar{g}_\otimes(E^I, E^I)$. Using equation (2.16) and lemma 3.5,

$$K = -\frac{\rho^2}{2g} \sum_I (-1)^\delta \mu^I \bar{g}_\otimes(E^I, E^I) E_K^I E_L^I \bar{g}_i \bar{g}_j \{x^i, Z^{Kj}\} \{x^j, Z^{Li}\}.$$

Since for $\mu^I = 0$ we have $\hat{N}^I = 0$, the factor μ^I may be omitted. Lemma 3.6 implies $\sum_I E_K^I \bar{g}_\otimes(E^I, E^I) E_L^I = \bar{g}_{KL} = \delta_{KL} \bar{g}_K$, so we get with equation (3.3)

$$\begin{aligned}K &= (-1)^{\delta+1} \frac{\rho^2}{2g} \bar{g}_i \bar{g}_j \{x^i, Z_L^j\} \{x^j, Z^{Li}\} \\ &= (-1)^{\text{ind } \bar{g}+1} \frac{\rho^4}{8g^2(p-1)!} \bar{g}_i \bar{g}_r \bar{g}_n \epsilon_{jklL} \epsilon^{irnL} \{x^i, \{x^k, x^l\}\} \{x^j, \{x^r, x^n\}\}.\end{aligned}$$

Using $\epsilon^{irrnL} = \det(\bar{g}_{ij})^{-1} \epsilon_{irrnL}$, we obtain equation (3.7). Equation (3.8) is proved similarly. \square

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