BIASED RANDOM WALK IN POSITIVE RANDOM CONDUCTANCES ON $\mathbb{Z}^d$

ALEXANDER FRIBERGH

ABSTRACT. We study the biased random walk in positive random conductances on $\mathbb{Z}^d$. This walk is transient in the direction of the bias. Our main result is that the random walk is ballistic if, and only if, the conductances have finite mean. Moreover, in the sub-ballistic regime we find the polynomial order of the distance moved by the particle. This extends results obtained by L. Shen in [25], who proved positivity of the speed in the uniformly elliptic setting.

1. Introduction

One of the most fundamental questions in random walks in random media is understanding the long-term behavior of the random walk. This topic has been intensively studied and we refer the reader to [29] for a general survey of the field. An interesting feature of random walks in random environments (RWRE) is that several models exhibit anomalous behaviors. One of the main reasons for such behaviors is trapping, a phenomenon observed by physicists long ago [19] and which is a central topic in RWRE. The importance of trapping in several physical models (including RWRE) motivated the introduction of the Bouchaud trap model (BTM). This is an idealized model that received a lot of mathematical attention. A review of the main results can be found in [2], a survey which conjectures that the type of results obtained in the BTM should extend to a wide variety of models, including RWRE.

One very characteristic behavior associated to trapping is the existence of a zero asymptotic speed for RWRE with directional transience. In the last few years, several articles have analyzed such models from a trapping perspective, such as [13] and [14] on $\mathbb{Z}$ and [4], [5] and [17] on trees. The results on the d-dimensional lattice (with $d \geq 2$) are much more rare, since RWRE on $\mathbb{Z}^d$ are harder to analyze. Among the most natural examples of directionally transient RWRE in $\mathbb{Z}^d$ are biased random walks in random conductances. So far, mathematically, only two models of biased random walks in $\mathbb{Z}^d$ have

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been studied: one is on a supercritical percolation cluster and the other is in environments assumed to be uniformly elliptic.

In the case of biased random walks on a percolation cluster in $\mathbb{Z}^d$, it was shown in [6] (for $d = 2$) and in [26] that the walk is directionally transient and, more interestingly, there exists a zero-speed regime. More recently in [15] a characterization of the zero-speed regime has been achieved. Those results confirmed the predictions of the physicists that trapping occurs in the model, see [8] and [9].

In the case of a biased random walk in random conductances which are uniformly elliptic, it has been shown in [25] that the walk is directionally transient and has always positive speed and verifies an annealed central limit theorem. These results are coherent with the conjecture that, a directionally transient random walk in random environment which is uniformly elliptic, should have positive speed (see [27]). Hence, trapping does not seem to appear under uniform ellipticity conditions.

The results on those two models do not bring any understanding on the behavior of the random walk in positive conductances that might be arbitrarily close to zero. In such a model, we truly lose the uniform elliptic assumption, as opposed to the biased random walk on the percolation cluster, where the walk is still uniformly elliptic on the graph where the walk is restricted.

Our purpose in this paper is to understand the ballistic-regime of a biased random walk in positive i.i.d. conductances and how trapping arises in such a model.

2. Model

We introduce $\mathbf{P}[\,\cdot\,] = P_*^{\otimes E(\mathbb{Z}^d)}$, where P_* is the law of a positive random variable $c_* \in (0,\infty)$. This measure characterizes gives a random environment usually denoted ω.

In order to define the random walk, we introduce a bias $\ell = \lambda\vec{\ell}$ of strength $\lambda > 0$ and a direction $\vec{\ell}$ which is in the unit sphere with respect to the Euclidian metric of $\mathbb{R}^d$. In an environment ω, we consider the Markov chain of law P_x^ω on $\mathbb{Z}^d$ with transition probabilities $p^\omega(x,y)$ for $x, y \in \mathbb{Z}^d$ defined by

(1) $X_0 = x$, P_x^ω-a.s.,

(2) $p^\omega(x,y) = \dfrac{c^\omega(x,y)}{\sum_{z\sim x} c^\omega(x,z)}$,

where $x \sim y$ means that x and y are adjacent in $\mathbb{Z}^d$ and also we set

$$\text{for all } x \sim y \in \mathbb{Z}^d, \qquad c^\omega(x,y) = c_*^\omega([x,y])e^{(y+x)\cdot\ell}. \tag{2.1}$$

This Markov chain is reversible with invariant measure given by

$$\pi^\omega(x) = \sum_{y\sim x} c^\omega(x,y).$$

The random variable $c^\omega(x,y)$ is called the conductance between x and y in the configuration ω. This comes from the links existing between reversible Markov chains and electrical networks. We refer the reader to [10] and [22] for a further background on this relation, which we will use extensively. Moreover for an edge $e=[x,y]\in E(\mathbb{Z}^d)$, we denote $c^\omega(e)=c^\omega(x,y)$.

Finally the annealed law of the biased random walk will be the semi-direct product $\mathbb{P}=\mathbf{P}[\,\cdot\,]\times P_0^\omega[\,\cdot\,]$.

In the case where $c_*\in(1/K,K)$ for some $K<\infty$, the walk is uniformly elliptic and this model is the one previously studied in [25].

3. Results

Firstly, we prove that the walk is directionally transient.

Proposition 3.1. *We have*

$$\lim X_n\cdot\vec{\ell}=\infty,\qquad \mathbb{P}-a.s.$$

This proposition is a consequence of Proposition 7.1.

Our main result is

Theorem 3.1. *For $d\geq 2$, we have*

$$\lim \frac{X_n}{n}=v,\qquad \mathbb{P}-a.s.,$$

where

(1) *if $E_*[c_*]<\infty$, then $v\cdot\vec{\ell}>0$,*
(2) *if $E_*[c_*]=\infty$, then $v=\vec{0}$.*

Moreover, if $\lim\frac{\ln P_*[c_*>n]}{\ln n}=-\gamma$ with $\gamma<1$ then

$$\lim\frac{\ln X_n\cdot\vec{\ell}}{\ln n}=\gamma,\qquad \mathbb{P}-a.s..$$

This theorem follows from Proposition 8.1, Proposition 9.1, Proposition 9.2 and Proposition 9.2.

This result proves that trapping phenomena may occur in an elliptic regime, that is, when all transition probabilities are positive.

Let us rapidly discuss the different main ways the walk may be trapped (see Figure 1):

(1) an edge with high conductance surrounded by normal conductances,
(2) a normal edge surrounded by very small conductances,

Let us discuss how the first type of traps function. Assume that we have an edge e of conductance $c_*(e)$ surrounded by edges of fixed conductances, say 1. A simple computation shows that the walk will need a time of the order of $c_*(e)$ to leave the endpoints of e. Hence, if the expectation of c_* is

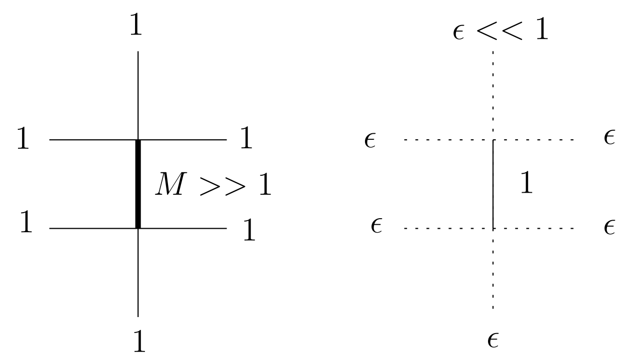

FIGURE 1. The two main types of traps

infinite, then the annealed exit time of the e is infinite. Heuristically, one edge is enough to trap the walk strongly. This phenomenon is enough to explain the zero-speed regime.

At first glance it is surprising that, in Theorem 3.1, there is only a condition on the tail of c_* at infinity. Indeed, if the tail of c_* at 0 is sufficiently big, more precisely such that $E[\min_{i=1,\dots,4d-2} 1/c_*^{(i)}] = \infty$ for $c_*^{(i)}$ i.i.d. chosen under P_*, then the second type of traps are such that the annealed exit time of the central edge is infinite. This condition does not appear in Theorem 3.1. because the walk is unlikely to reach such an edge. Indeed, it needs to cross an edge with extremely low conductance to enter the trap. This type of trapping is barely not strong enough to create a zero-speed regime (see Remark 9.1), nevertheless it forces us to be very careful in our analysis of the model.

One may try to create traps similar to those encountered in the biased random walk on the percolation cluster. In this model, if the bias is high enough, a long dead-end in the direction of the drift can trap the walk strongly enough to force zero-speed. In our context, we are not allowed to use zero conductances, but we may use extremely low conductances, forcing the walk to exit the dead-end at the same place it entered. Nevertheless, this type of traps is very inefficient. Indeed, most edges forming a dead-end have to verify $c_*(e) < \varepsilon$ to be able to contain the walk for a long period, and this for any fixed $\varepsilon > 0$. The probabilistic cost of creating such a trap is way too high.

Hence, small conductances cannot create zero-speed but high conductances can. To conclude, we give an idealized version of the two most important types of traps in this model:

$$X_1 = \text{Geom}((1/c_*) \wedge 1) \text{ or } X_2 = \begin{cases} \text{Geom}(c_*' \wedge 1) & \text{with probability } c_*' \wedge 1 \\ 0 & \text{else.} \end{cases}$$

where c_* is chosen according to the law P_* and c_*' has the law of $\max_{i=1,\dots,4d-2} c_*^{(i)}$ where $c_*^{(i)}$ are i.i.d. chosen under the law P_* and independent of the geometric random variables. Intuitively, one should be able to understand anything related to trapping with biased random walks in an elliptic setting using those idealized traps.

Let us explain the organization of the paper. We begin by studying exit probabilities of large boxes, the main point of Section 5 is to prove Theorem 5.1 which is a property similar to Sznitman's conditions (T) and $(T)_\gamma$, see [27]. This property is one of the key estimates for studying directionally transient RWRE. It allows us to define regeneration times, similar to the ones introduced in [28], and study them; this is done in Section 7. The construction of regeneration times in this model is complicated by the fact that we lack any type of uniform ellipticity. This issue is explained in more details and dealt with in Section 6. The law of large numbers in the positive speed regime is obtained in Section 8. The zero-speed regime is studied in Section 9. The next section is devoted to notations which will be used all along this paper.

4. Notations

Let us denote by $(e_i)_{i=1\dots d}$ an orthonormal basis of $\mathbb{Z}^d$ such that $e_i \cdot \vec{\ell} \geq e_2 \cdot \vec{\ell} \geq \cdots \geq e_d \cdot \vec{\ell} \geq 0$, in particular we have $e_1 \cdot \vec{\ell} \geq 1/\sqrt{d}$. Moreover we complete $f_1 := \vec{\ell}$ into an orthonormal basis $(f_i)_{1\leq i\leq d}$ of $\mathbb{R}^d$.

We set

$$\mathcal{H}^+(k) = \{x \in \mathbb{Z}^d, x \cdot \vec{\ell} > k\} \text{ and } \mathcal{H}^-(k) = \{x \in \mathbb{Z}^d, x \cdot \vec{\ell} \leq k\},$$

and

$$\mathcal{H}^+_x = \mathcal{H}^+(x \cdot \vec{\ell}) \text{ and } \mathcal{H}^-_x = \mathcal{H}^-(x \cdot \vec{\ell}).$$

Let us introduce $d_G(x,y)$ the graph distance in G between x and y. Define for $x \in G$ and $r > 0$

$$B_G(x,r) = \{y \in G,\ d_G(x,y) \leq r\}.$$

Given a set V of vertices of $\mathbb{Z}^d$, we denote by $|V|$ its cardinality, by $E(V) = \{[x,y] \in E(\mathbb{Z}^d) \mid\ x,y \in V\}$ its edges and

$$\partial V = \{x \in V \mid y \in \mathbb{Z}^d \setminus V,\ x \sim y\},\ \partial_E V = \{[x,y] \in E(\mathbb{Z}^d) \mid\ x \in V,\ y \notin V\},$$

its borders.

Given a set E of edges of $\mathbb{Z}^d$, we denote $V(E) = \{x \in \mathbb{Z}^d \mid x \text{ is an endpoint of } e \in E\}$ its vertices.

Denote for any $L, L' \geq 1$

$$B(L,L') = \left\{z \in \mathbb{Z}^d \mid \left|z \cdot \vec{\ell}\right| < L,\ |z \cdot f_i| < L' \text{ for } i \in [2,d]\right\},$$

and

$$\partial^+ B(L,L') = \{z \in \partial B(L,L') \mid z \cdot \vec{\ell} \geq L\}.$$

We introduce the following notations. For any set of vertices A of a certain graph on which a random walk X_n is defined, we denote

$$T_A = \inf\{n \geq 0,\ X_n \in A\},\ T_A^+ = \inf\{n \geq 1,\ X_n \in A\},$$

and

$$T_A^{\text{ex}} = \inf\{n \geq 0,\ X_n \notin A\}.$$

This allows us to define the hitting time of "level" n by

$$\Delta_n = T_{\mathcal{H}^+(n)}.$$

Finally, θ_n will denote the time shift by n units of time.

In this paper constants are denote by $c \in (0, \infty)$ or $C \in (0, \infty)$ without emphasizing their dependence on d and the law P_*. Moreover the value of those constants may change from line to line.

5. Exit probability of large boxes

Our first goal is to obtain estimates on the exit probabilities of large boxes, which will allow us to prove directional transience and is key to analyzing this model. In particular, we aim at showing

Theorem 5.1. *For $\alpha > d + 3$*

$$\mathbb{P}[T_{\partial B(L,L^\alpha)} \neq T_{\partial^+ B(L,L^\alpha)}] \leq e^{-cL}.$$

After this section α will be fixed, greater than $d + 3$.

We will adapt a strategy of proof used in [15]. For the most part, the technical details and notations are simpler in our context. We will go over the parts of the proof which can be simplified, but we will eventually refer the reader to [15] for the conclusion of the proof which is exactly similar in both cases. The notations have been chosen so that the reader can follow the needed proofs in [15] to the word.

Firstly, let us describe the strategy we will follow.

The fundamental idea is to partition the space into a good part where the walk is well-behaved and a bad part consisting of small connected components where we have very little control over the random walk.

The strategy is two-fold.

(1) We may study the behavior of the random walk at times where it is in the good part of the space, in which it can easily be controlled. We will refer to this object as the modified walk. We show that the modified walk behaves nicely, i.e. verifies Theorem 5.1. This is essentially achieved using a combination of spectral gap estimates and the Carne-Varopoulos formula [7].

(2) We need to show that information on the exit probabilities for this modified random walk allows us to derive interesting statements on the actual random walk. This is a natural thing to expect, since the bad parts of the environment are small.

A more detailed discussion of the strategy of proof can be found at the beginning of Section 7 in [15].

5.1. **Bad areas.** We say that an edge e is K-normal if $c_*(e) \in [1/K, K]$, where K will be taken to be very large in the sequel. If an edge is not K-normal, we will say it is K-abnormal which occurs with arbitrarily small probability $\varepsilon(K) := P_*[c_* \notin [1/K, K]]$, since $c_* \in (0, \infty)$.

In relation to this, we will say that a vertex x is K-open if for all $y \sim x$ the edge $[x, y]$ is K-normal. If a vertex is not K-open, we will say it is K-closed. Finally a vertex $x \in \mathbb{Z}^d$ is K-good, if there exists an infinite directed K-open path starting at x, that is: we have $\{x = x_0, x_1, x_2, x_3, \ldots\}$ with $x_0 = x$ such that for all $i \geq 0$

(1) if $x_{2i+1} - x_{2i} = e_1$ then $x_{2i+2} - x_{2i+1} \in \nu \setminus \{-e_1, \ldots, -e_d\}$,
(2) x_i is K-open.

If a vertex is not K-good it is said to be K-bad. By taking K large enough, the probability that a vertex is K-good goes to 1.

To ease the notation, we will not always mention the K-dependences.

The first of these results is stated in terms of the width of a subset $A \subseteq \mathbb{Z}^d$, which we define to be

$$W(A) = \max_{1 \leq i \leq d} \Big(\max_{y \in A} y \cdot e_i - \min_{y \in A} y \cdot e_i \Big).$$

Let us denote $\mathrm{BAD}_K(x)$ the connected component of K-bad vertices containing x, in case x is good then $\mathrm{BAD}_K(x) = \emptyset$.

Lemma 5.1. *There exists K_0 such that, for any $K \geq K_0$ and for any $x \in \mathbb{Z}^d$, we have that the cluster $BAD_K(x)$ is finite $\mathbf{P}_p[/, / \cdot /,]$-a.s. and*

$$\mathbf{P}_p[W(BAD_K(x)) \geq n] \leq C \exp(-\xi_1(K) n),$$

where $\xi_1(K) \to \infty$ as K tends to infinity.

Proof. We call two vertices 2-connected if $||u - v||_1 = 2$, so that we may define the even bad points $\mathrm{BAD}^e_K(x)$ of x as the 2-connected component of bad vertices containing x. Any element of $\mathrm{BAD}(x)$ is a neighbor of $\mathrm{BAD}^e_K(x)$ so that $W(\mathrm{BAD}_K(x)) \leq W(\mathrm{BAD}^e_K(x)) + 2$.

We construct a new percolation model on the even lattice $\{v \in \mathbb{Z}^d,\ ||v||_1 \text{ is even}\}$. The bond between y and $y + e_1 + e_i$ (for $i \leq d$) is even-open if, and only if, in the original model, the vertices y, $y + e_1$ and $y + e_1 + e_i$ are open. This model is a 3-dependent even-open oriented percolation model, which has a measure we denote by $P_{p,\mathrm{orient}}$.

Fix p' close to 1. For K large enough, the probability that a vertex is K-open can be made arbitrarily close to 1, so, by Theorem 0.0 in [23], the law $P_{p,\text{orient}}$ dominates an i.i.d. bond percolation with parameter p'.

We describe how to do the proof for $d = 2$. Consider the outer edge-boundary $\partial_E \text{BAD}^e_K(x)$ of $\text{BAD}^e_K(x)$ (represented dually in Figure 2): by an argument similar to that of [11] (p.1026), we see that $n_{\nearrow} + n_{\searrow} = n_{\swarrow} + n_{\nwarrow}$, where $n_{\nearrow}$, for example, is the number of edges labelled $\nearrow$ in $\partial_E \text{BAD}^e_K(x)$.

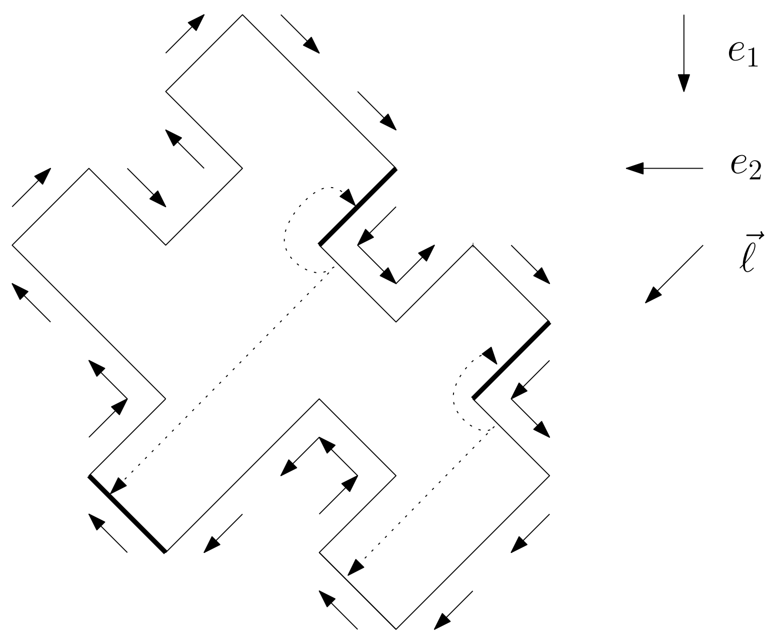

FIGURE 2. The outer edge-boundary $\partial_E \text{BAD}^e_K(x)$ of $\text{BAD}^e_K(x)$ when $d = 2$.

Any $\nwarrow$ edge of $\partial_E \text{BAD}^e_K(x)$ has one endpoint, say y, which is bad, and one, $y + e_1 + e_2$, which is good. This implies that y is even-closed.

(1) Hence, if $n_{\nwarrow} \geq n_{\swarrow}/2$, then at least one sixth of the edges of $\partial_E \text{BAD}^e_K(x)$ have an endpoint which is an even-closed vertex.

(2) Otherwise, let us assume that $n_{\swarrow} \geq 2n_{\nwarrow}$. We may notice (see Figure 2) any $\swarrow$ edge followed (in the sense of the arrows) by an $\searrow$ edge can be mapped in an injective manner to an $\nwarrow$ edge. This means that at least half of the $\swarrow$ edges are not followed by an $\searrow$ edge. So at least one sixth of the edges of $\partial_E \text{BAD}^e_K(x)$ are $\swarrow$ edges that are not followed by an $\searrow$ edge. In this case, we note that there is one endpoint y of $\swarrow$ which is bad and that $y + 2e_1$ is good, and hence y is even-closed.

This means that at least one sixth of the edges of $\partial_E \text{BAD}^e_K(x)$ are adjacent to an even-closed vertex. The outer boundary is a minimal cutset, as described in [1]. The number of such boundaries of size n is bounded (by Corollary 9 in [1]) by $\exp(Cn)$. Hence, if p' is close enough to 1, a counting argument allows us to obtain the desired exponential tail for $W(\text{BAD}_K^e(x))$ under $P_{p'}$, and hence under $P_{p,\text{orient}}$ (since the latter is dominated by the former).

For a general dimensions, we note that there exists $i_0 \in [2,d]$ such that a proportion at least $1/(d-1)$ of the $\partial_E \text{BAD}_K^e(x)$ are edges of the form $[y, y+e_{i_0}]$. We may then apply the previous reasoning in the plane $y+\mathbb{Z}e_1+\mathbb{Z}e_{i_0}$ to show that at least a proportion $1/(6(d-1))$ of the edges $\partial_E \text{BAD}_K^e(x)$ are adjacent to an even-closed vertex. Thus, the same counting argument allows us to infer the lemma. □

Let us define $\text{BAD}_K = \cup_{x\in\mathbb{Z}} \text{BAD}_K(x)$ which is a union of finite sets. Also we set $\text{GOOD}_K = \mathbb{Z}^d \setminus \text{BAD}_K$. We may notice that

$$\text{for any } x \in \text{BAD}_K,\ \partial \text{BAD}_K(x) \subset \text{GOOD}_K, \tag{5.1}$$

since $\text{BAD}_K(x)$ is a connected component of bad points.

For technical reasons, we will need a slightly stronger result. We say that a vertex $x \in \mathbb{Z}^d$ is K-super-open if all nearest neighbors of x are K-open. If a vertex is not super-open, it is say to be K-weakly-closed. We extend the notion of good and bad points to super-good and weakly-bad points.

Remark 5.1. *Any neighbor of a super-good point is good.*

Let us denote $\text{BAD}_K^+(x)$ the connected component of K-weakly-bad vertices containing x. Using those new definitions, we can mimic the proof of Lemma 5.1 to obtain the following result

Lemma 5.2. *There exists K_0 such that, for any $K \geq K_0$ and for any $x \in \mathbb{Z}^d$, we have that the cluster $BAD_K^+(x)$ is finite $\mathbf{P}_p[\,cdot\,]$-a.s. and*

$$\mathbf{P}_p[W(BAD_K^+(x)) \geq n] \leq C\exp(-\xi_2(K)n),$$

where $\xi_2(K) \to \infty$ as K tends to infinity.

In the sequel K will always be large enough so that $\text{BAD}_K(x)$ and $\text{BAD}_K^+(x)$ are finite for any $x \in \mathbb{Z}^d$.

5.2. **A graph transformation to seal off big traps.** Given a certain configuration ω, we construct a graph ω_K in which traps are sealed off and such that the random walk induced outside of large traps by the random walk in ω has the same law as the random walk in ω_K.

We denote ω_K the graph obtained from ω by the following transformation. The vertices of ω_K are the vertices of GOOD_K, and the edges of ω_K are

(1) $\{[x,y],\ x,y \in \text{GOOD}_K,$ with $x \notin \partial\text{BAD}_K$ or $y \notin \partial\text{BAD}_K\}$ and have conductance $c^{\omega_K}([x,y]) := c^{\omega}([x,y])$,

(2) $\{[x,y],\ x,y\in\partial\mathrm{BAD}_K\}$ (including loops) which have conductance

$$\begin{aligned}c^{\omega_K}([x,y]) :&= \pi^\omega(x)P_x^\omega[X_1\in \mathrm{BAD}_K\cup\partial\mathrm{BAD}_K,\ T_y^+=T^+_{\partial\mathrm{BAD}_K}]\\ &= \pi^\omega(y)P_y^\omega[X_1\in \mathrm{BAD}_K\cup\partial\mathrm{BAD}_K,\ T_x^+=T^+_{\partial\mathrm{BAD}_K}],\end{aligned}$$

the last equality being a consequence of reversibility and ensures symmetry for the conductances.

We call the walk induced by X_n on GOOD_K, the walk Y_n defined to be $Y_n=X_{\rho_n}$ where

$$\rho_0=T_{\mathrm{GOOD}_K} \text{ and } \rho_{i+1}=T^+_{\mathrm{GOOD}_K}\circ\theta_{\rho_i}.$$

From [15] Proposition 7.2, we have the two following properties

Proposition 5.1. *The reversible walk defined by the conductances ω_K verifies the two following properties*

1. *It is reversible with respect to $\pi^\omega(\cdot)$.*
2. *If started at $x\in\omega_K$, it has the same law as the walk induced by X_n on $GOOD_K$ started at x.*

and

Lemma 5.3. *For $x,y\in GOOD_K$ which are nearest neighbors in $\mathbb{Z}^d$, we have $c^{\omega_K}([x,y])\geq c^\omega([x,y])$.*

Hence, we may notice that

Remark 5.2. *We may notice that for any $x\in GOOD_K$, we have*

$$\frac{c}{K}e^{-2\lambda x\cdot\vec{\ell}}\leq \pi^{\omega_K}(x)\leq CKe^{-2\lambda x\cdot\vec{\ell}},$$

and for any $y\in GOOD_K$ adjacent, in $\mathbb{Z}^d$, to x

$$\frac{c}{K}e^{-2\lambda x\cdot\vec{\ell}}\leq c^{\omega_K}([x,y])\leq CKe^{-2\lambda x\cdot\vec{\ell}}.$$

5.3. **Spectral gap estimate in ω_K.** The following arguments are heavily inspired from [26] and uses spectral gap estimates. After showing that the spectral gap in ω_K, we can deduce that the walk is likely to exit the box quickly in ω_K. Finally, we need to argue that exiting the box quickly, we should exit it in the direction of the drift. This allows us to obtain Theorem 5.1, once we have argued that the exit probabilities in ω and ω_K are strongly related. This paper only contains the first step of this reasoning, the following ones being treated in [15].

For technical reasons, we introduce the notation

$$\tilde{B}(L,L^\alpha)=\{x\in\mathbb{Z}^d,\ -L\leq x\cdot\vec{\ell}\leq 2L \text{ and } |x\cdot f_i|\leq L^\alpha \text{ for } i\geq 2\}.$$

Let us introduce the principal Dirichlet eigenvalue of $I-P^{\omega_K}$, in $\tilde{B}(L,L^\alpha)\cap\omega_K$

(5.2)

$$\Lambda_{\omega_K}(\tilde{B}(L,L^\alpha)) = \begin{cases} \inf\{\mathcal{E}_{\mathrm{GOOD}_K}(f,f), f_{|(\tilde{B}(L,L^\alpha)\cap\omega_K)^c}=0, ||f||_{L^2(\pi(\omega_K))}=1\}, \\ \qquad \text{when } \tilde{B}(L,L^\alpha)\cap\omega_K\neq\emptyset, \\ \infty, \text{ by convention when } \tilde{B}(L,L^\alpha)\cap\omega_K=\emptyset, \end{cases}$$

where the Dirichlet form is defined for $f,g\in L^2(\pi^{\omega_K})$ by

$$\begin{aligned}\mathcal{E}_{\mathrm{GOOD}_K}(f,g) &= (f,(I-P^{\omega_K})g)_{\pi^{\omega_K}} \\ &= \frac{1}{2}\sum_{x,y \text{ neighbors in } \omega_K}(f(y)-f(x))(g(y)-g(x))c^{\omega_K}([x,y]).\end{aligned}$$

We have

Lemma 5.4. *For ω such that $\tilde{B}(L,L^\alpha)\cap\omega_K\neq\emptyset$, we have*

$$\Lambda_{\omega_K}(\tilde{B}(L,L^\alpha)) \geq c(K)L^{-(d+1)}.$$

Proof. By definition of a good point, from any vertex $x\in\omega_K$, there exists a directed open path $x=p_x(0),p_x(1)\dots,p_x(l_x)$ in ω_K (which are neighbors in $\mathbb{Z}^d$) such that $p_x(i)\in\tilde{B}(L,L^\alpha)$ for $i<l_x$ and $p_x(l_x)\notin\tilde{B}(L,L^\alpha)$. This allows us to say that

$$\max_{i\leq l_x}\frac{\pi_{\omega_K}(x)}{c_{\omega_K}([p_x(i+1),p_x(i)])} \leq C\max_{i\leq l_x}\frac{\pi_\omega(x)}{\pi_\omega(p_x(i))}\leq C, \tag{5.3}$$

where we used Lemma 5.3 and Remark 5.2. Moreover $l_x\leq CL$.

We use a classical argument of Saloff-Coste [24], we write for $||f||_{L^2(\pi^{\omega_K})}=1$

$$\begin{aligned}1 = \sum_x f^2(x)\pi_{\omega_K}(x) &= \sum_x\Big[\sum_i f(p_x(i+1))-f(p_x(i))\Big]^2\pi_{\omega_K}(x) \\ &\leq \sum_x l_x\Big[\sum_i(f(p_x(i+1))-f(p_x(i)))^2\Big]\pi_{\omega_K}(x).\end{aligned}$$

Now by (5.3), we obtain

$$1\leq C\sum_{x,y \text{ neighbors in } \omega_K}(f(z)-f(y))^2c_{\omega_K}([x,y])\times\max_{b\in E(\mathbb{Z}^d)}\sum_{x\in\omega_K\cap\tilde{B}(L,L^\alpha),b\in p_x} l_x$$

where $b\in p_x$ means that $b=[p_x(i),p_x(i+1)]$ for some i. Using that

(1) $l_x\leq CL$ for any $x\in\omega_K$,
(2) $b=[x,y]\in\omega_K$ can only be crossed by paths "p_z" if $b\in E(\mathbb{Z}^d)$ and $z\in B_{\mathbb{Z}^d}(x,CL)$,

we have

$$\max_b \sum_{x\in \tilde{B}(n,n^\alpha), b\in p_x} l_x \leq CL^{d+1},$$

and

$$1 \leq CL^{d+1} \sum_{x,y \text{ neighbors in } \omega_K} (f(z)-f(y))^2 c_{\omega_K}([x,y]),$$

so that using (5.2)

$$\Lambda_{\omega_K}(\tilde{B}(n,n^\alpha)) \geq cL^{-(d+1)}.$$

□

We explained how to obtain Theorem 5.1 at the beginning of subsection 5.3. The proof of Theorem 5.1 is almost completely similar to the end of the proof of Theorem 1.4 in [15]. The reader may read subsection 7.4, 7.5 and 7.6 of [15] for the complete details.

To ease this task, the notations have been chosen so that only two minor changes have to be made: in our case $K_\infty = \mathbb{Z}^d$ and $\mathcal{I} = \Omega$.

The proof in [15] uses some reference to previous results, for the reader's convenience we specify the correspondence. The following results in [15]: Lemma 7.5, Proposition 7.2, Lemma 7.9 and the second part of Lemma 7.6 correspond respectively to Lemma 5.1, Proposition 5.1, Lemma 5.4 and (5.1).

A final remark is that any inequality needed on π^{ω_K} can be found in Remark 5.2.

6. Construction of K-open ladder points

A classical tool for analyzing directional transient RWRE is to use a regeneration structure [28]. The construction of a regeneration structure is rather involved in this model. We call a new maximum of the random walk in the direction $\vec{\ell}$, a ladder-point. The standard way of constructing regeneration times is to consider successive ladder points and arguing, using some type of uniform ellipticity assumption, that there is a positive probability of never backtracking again. Such a ladder point creates a separation between the past and the future of the random walk leading to interesting independence properties. We call this point a regeneration time.

A major issue in our case is that we do not have any type of uniform ellipticity. Ladder points are conditioned parts of the environment and, at least intuitively, the edge that led us to a ladder point should have uncharacteristically high conductance. Those high conductances (without uniform ellipticity) may strongly inder the walk from never backtracking and creating a regeneration time. In some sense, we need to show that the environment seen from the particle at a ladder-point is relatively normal. More precisely, we will prove that we encounter open ladder-points and find tail estimates on the location of the first open ladder-point.

We define the following random variable

$$\mathcal{M}^{(K)} = \inf\{i \geq 0,\ X_i \text{ is } K\text{-open and for } j < i-2,\ X_j \cdot \vec{\ell} < X_{i-2} \cdot \vec{\ell} \\ X_i = X_{i-1} + e_1 = X_{i-2} + 2e_1\} \leq \infty.$$

The dependence on K will be dropped outside of major statements of definitions.

6.1. **Preparatory lemmas.** We need three preparatory lemmas before turning to the study of $\mathcal{M}^{(K)}$. For this, we introduce the inner positive boundary of $B(n, n^\alpha)$

$$\partial_i^+ B(n, n^\alpha) = \{x \in B(n, n^\alpha), x \sim y \text{ with } y \in \partial^+ B(n, n^\alpha)\},$$

and

$$A(n) = \{T_{\partial B(n,n^\alpha)} \geq T_{\partial_i^+ B(n,n^\alpha)}\}.$$

It follows from Theorem 5.1 that

Lemma 6.1. *We have*

$$\mathbb{P}[A(n)^c] \leq Ce^{-cn}.$$

We say that a vertex $x \in B(n, n^\alpha)$ is K-n-closed, if there exists a nearest neighbor $y \in \mathcal{H}^+(n)$ of x such that $c^*([x, y]) \notin [1/K, K]$.

Let us denote $\overline{K}_x(n)$ the K-n-closed connected component of x. This allows us to introduce

$$B(n) = \{\text{for all } x \in \partial_i^+ B(n, n^\alpha),\ \left|\overline{K}_x(n)\right| \leq \ln n\}. \tag{6.1}$$

It is convenient to set $\overline{K}_x(n) = \{x\}$ when $\overline{K}_x(n)$ is empty.

Lemma 6.2. *For any $M < \infty$, we can find K_0 such that for any $K \geq K_0$*

$$\mathbf{P}[B(n)^c] \leq Cn^{-M}.$$

Proof. Obviously, for any $x \in \partial^+ B(n, n^\alpha)$

$$\overline{K}_x(n) \subset \text{CLOSED}_K(x),$$

where $\text{CLOSED}_K(x)$ is the closed connected component of K-closed point containing x.

Using lemma 5.1 in [20], we may notice that there are at most an exponential number of lattice animals. Hence, for any $x \in \partial^+ B(n, n^\alpha)$

$$\mathbf{P}[|\text{CLOSED}_K(x)| \geq \ln n] \leq \sum_{n \geq 0} C(C_1 \varepsilon(K))^{\ln n} = Cn^{-\xi_2(K)},$$

where $\xi_2(K)$ tends to infinity K goes to infinity. The right hand side can be made lower than n^{-M} for any M by choosing K large enough. The result follows from a union bound. □

Lemma 6.3. *Take $G \neq \emptyset$ to be a finite connected subset of $\mathbb{Z}^d$. Assume that each edge e of $\mathbb{Z}^d$ is assigned a positive conductance $c(e)$ and that there exist $x \in \partial G$ and $y \in G$ such that $x \sim y$ and $c([x,y]) \geq c_1 c(e)$ for any $e \in \partial_E G$. We have*

$$P_y[T_x \leq T_{\partial G}] \geq \frac{c_1}{4d} |G|^{-1},$$

where P_y is the law of the random walk in $G \cup \partial G$ started at y arising from the conductances $(c(e))_{e \in E(\mathbb{Z}^d)}$.

Proof. We will be using comparisons to electrical networks and we refer the reader to Chapter 2 of [22] for further background on this topic.

Let us first notice that a walk started at $y \in G$ will reach ∂G before $\mathbb{Z}^d \setminus (G \cup \partial G)$, so this lemma is actually a result on a finite graph $G \cup \partial G$.

To simplify the proof, we will consider the graph $\tilde{G}$ where all edges emanating from x that are not $[x,y]$ will be assigned conductance 0, which corresponds to reflecting the walk on those edges. It is plain to see that

$$P_y[T_x \leq T_{\partial G}] \geq P_y^{\tilde{G} \cup \partial \tilde{G}}[T_x \leq T_{\partial \tilde{G}}],$$

where $P_y^{\tilde{G} \cup \partial \tilde{G}}$ is the law of the random walk started at y in the conductances of the graph $\tilde{G} \cup \partial \tilde{G}$.

Hence, it is enough to prove our statement in the finite graph $\tilde{G} \cup \partial \tilde{G}$. We may see that

$$P_y^{\tilde{G} \cup \partial \tilde{G}}[T_x \leq T_{\partial \tilde{G}}] = u(y), \tag{6.2}$$

where $u(\cdot)$ is the voltage function verifying $u(x) = 1$ and $u(z) = 0$ for $z \in \partial \tilde{G} \setminus \{x\}$. Let us denote $i(\cdot)$ the associate intensity. Since y is the only vertex adjacent to x in $\tilde{G} \cup \partial \tilde{G}$, we know that the current flowing into the circuit at x passes through the edges $[x,y]$, so

$$\frac{1}{R^{\tilde{G} \cup \partial \tilde{G}}(x, \partial \tilde{G} \setminus \{x\})} = i([x,y]),$$

where $R^{\tilde{G} \cup \partial \tilde{G}}(x, \partial \tilde{G} \setminus \{x\})$ is the effective conductance between x and $\partial \tilde{G} \setminus \{x\}$ in $\tilde{G} \cup \partial \tilde{G}$. By Ohm's law we may deduce that

$$u(x) - u(y) = r^{\tilde{G} \cup \partial \tilde{G}}([x,y]) i([x,y]) = \frac{r^{\tilde{G} \cup \partial \tilde{G}}([x,y])}{R^{\tilde{G} \cup \partial \tilde{G}}(x, \partial \tilde{G} \setminus \{x\})}.$$

Now, since x is the only vertex adjacent to y, we can see by an electrical network reduction of resistances in series that $R^{\tilde{G} \cup \partial \tilde{G}}(x, \partial \tilde{G} \setminus \{x\}) = r^{\tilde{G} \cup \partial \tilde{G}}([x,y]) + R^{\tilde{G} \cup \partial \tilde{G} \setminus \{x\}}(y, \partial \tilde{G} \setminus \{x\})$. This means that

$$u(y) = \frac{R^{\tilde{G} \cup \partial \tilde{G} \setminus \{x\}}(y, \partial \tilde{G} \setminus \{x\})}{R^{\tilde{G} \cup \partial \tilde{G} \setminus \{x\}}(y, \partial \tilde{G} \setminus \{x\}) + r^{\omega}([x,y])} = \Big(1 + \frac{r^{\tilde{G} \cup \partial \tilde{G}}([x,y])}{R^{\tilde{G} \cup \partial \tilde{G} \setminus \{x\}}(y, \partial \tilde{G} \setminus \{x\})}\Big)^{-1}.$$

We consider the graph $\tilde{G} \cup \partial\tilde{G} \setminus \{x\}$, by Rayleigh's monotonocity principle, we know that collapsing all vertices $\tilde{G} \cup \partial\tilde{G} \setminus \{x, y\}$ into one vertex δ decreases all effective conductances. In this new graph, y is connected to δ by at most $\left|\tilde{G} \cup \partial\tilde{G}\right|$ edges of resistances at least $c_1 r^{\tilde{G}\cup\partial\tilde{G}}([x, y])$ by our assumptions on the graph. By network reduction of conductances in parallel, this means

$$R^{\tilde{G}\cup\partial\tilde{G}\setminus\{x\}}(y, \partial\tilde{G} \setminus \{x\}) \geq \frac{c_1}{\left|\tilde{G} \cup \partial\tilde{G}\right|} r^{\tilde{G}\cup\partial\tilde{G}}([x, y]).$$

The two last equations imply, with (6.2), that

$$u(y) \geq \frac{c_1}{4d} |G|^{-1},$$

which concludes the proof. □

6.2. **Successive attempts to find an open ladder point.** We will show that an open ladder point can occur shortly after we exit a box $\mathcal{B}_n$. After exiting many such boxes, $\mathcal{B}_n$, $\mathcal{B}_{2n}$, ... we will eventually see an open ladder point with high probability.

Let us denote for $k \leq n$

$$R^{(K)}(nk) = \{\mathcal{M}^{(K)} > T_{\partial\mathcal{B}_{nk}} + 2\}, \tag{6.3}$$

where we used

$$\mathcal{B}_n := B(n, n^\alpha)$$

Moreover we have

Lemma 6.4. *For any $\varepsilon_1 > 0$ and $M < \infty$, we can find $K_0 = K_0(\varepsilon_1, M)$ and $n_0 = n_0(\varepsilon_1, M)$ large enough such that the following holds: for any $n \geq n_0$, $K \geq K_0$ and any $k \in [2, n]$,*

$$\mathbb{P}[R^{(K)}(kn)] \leq (1 - cn^{-\varepsilon_1})\mathbb{P}[R^{(K)}((k-1)n)] + Cn^{-M},$$

where the constants depend on K.

Proof. We set

$$\mathcal{K}(n) = \overline{K}_{X_{T_{\partial_i^+\mathcal{B}_n}}}(n) \subseteq \mathcal{B}_n, \tag{6.4}$$

where we recall that the notation $\overline{K}_x(n)$ was defined above (6.1). In case $T_{\partial_i^+\mathcal{B}_n} = \infty$, we simply set $\mathcal{K}(n) = \emptyset$ and $\partial\mathcal{K}(n) = \emptyset$.

We introduce

$$C(n) = \{x \text{ is open, for } x \in \partial\mathcal{K}(n) \cap \mathcal{H}^+(n)\},$$

as well as

$$D(n) = \{T_{\partial\mathcal{K}(n)} \circ \theta_{T_{\partial_i\mathcal{B}_n}} = T_{\partial\mathcal{K}(n)\cap\mathcal{H}^+(n)} \circ \theta_{T_{\partial_i\mathcal{B}_n}}\}$$

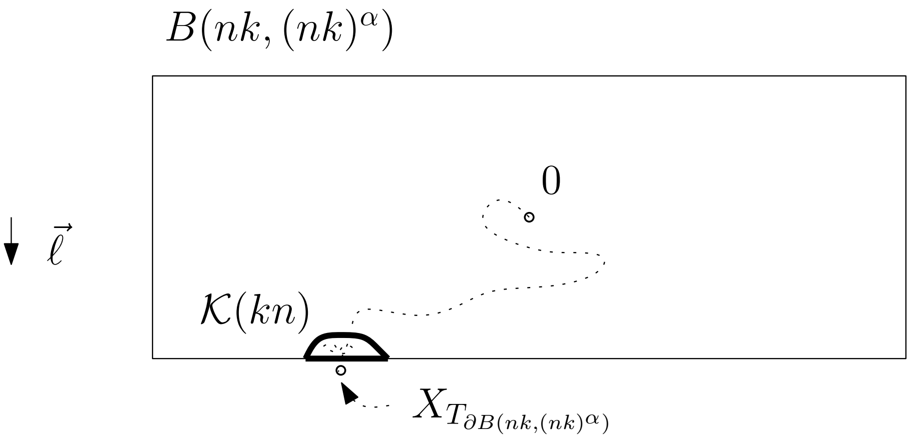

FIGURE 3. A way to find an open ladder points

and

$$E(n)=\{X_{T_{\partial B(n,n^\alpha)}+2}=X_{T_{\partial B(n,n^\alpha)}+1}+e_1=X_{T_{\partial B(n,n^\alpha)}}+2e_1$$
$$\text{and } X_{T_{\partial B(n,n^\alpha)}+2},\ X_{T_{\partial B(n,n^\alpha)}+1} \text{ are open}\}.$$

Let us consider an event in $A(n)\cap C(n)\cap D(n)\cap E(n)$. It verifies all the following conditions

(1) on $A(n)$, we have $T_{\partial_i^+\mathcal{B}_n}\le T_{\partial\mathcal{B}_n}$, so that

$$T_{\partial B(n,n^\alpha)}\ge T_{\partial\mathcal{K}(n)}\circ\theta_{T_{\partial_i\mathcal{B}_n}},$$

(2) on $D(n)$, by (6.4) we have

$$T_{\partial\mathcal{K}(n)}\circ\theta_{T_{\partial_i\mathcal{B}_n}}=T_{\partial\mathcal{K}(n)\cap\mathcal{H}^+(n)}\circ\theta_{T_{\partial_i\mathcal{B}_n}},$$

(3) on $C(n)$, we have $\partial\mathcal{K}(n)\cap\mathcal{H}^+(n)$ is open.

Hence, on $A(n)\cap C(n)\cap D(n)\cap E(n)$, we see that $X_{T_{\partial B(n,n^\alpha)}}$ is a new maximum of the trajectory in the direction $\vec{\ell}$, $X_{T_{\partial B(n,n^\alpha)}+1}=X_{T_{\partial B(n,n^\alpha)}}+e_1$ and $X_{T_{\partial B(n,n^\alpha)}+2}=X_{T_{\partial B(n,n^\alpha)}}+2e_1$ is a K-open point. This means that

$$A(n)\cap C(n)\cap D(n)\cap E(n)\subset\{\mathcal{M}\le T_{\partial\mathcal{B}_n}+2\}.$$

The situation is illustrated in Figure 3.

We have

$$\begin{aligned}
(6.5)\qquad &\mathbb{P}[R(kn)]\\
\le&\mathbb{P}[R((k-1)n),(A(kn)\cap C(kn)\cap D(kn)\cap E(kn))^c]\\
\le&\mathbb{P}[A(kn)^c]+\mathbb{P}[B(kn)^c]\\
&\dots+\mathbb{P}[R((k-1)n),A(kn),B(kn),C(kn)^c]\\
&\dots+\mathbb{P}[R((k-1)n),A(kn),B(kn),C(kn),D(kn)^c]
\end{aligned}$$

$$\dots + \mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn), E(kn)^c]$$

The first term is controlled by Theorem 5.1

$$\mathbb{P}[A(kn)^c] \le C\exp(-ckn) \le C\exp(-cn), \tag{6.6}$$

and for any $M < \infty$, by Lemma 6.2, we can choose K large enough such that

$$\mathbb{P}[B(kn)^c] \le n^{-M} \tag{6.7}$$

Step 1 : Control of the third term

For $k \le n$, on $A(kn) \cap B(kn)$, we see that

$$|\mathcal{K}(kn)| \le \ln(kn) \le 2\ln n, \tag{6.8}$$

in particular

$$\begin{aligned} &\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn)^c] \\ &= \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F, C(kn)^c] \\ &= \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbf{E}[P^\omega[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F] \\ &\qquad\qquad \times \mathbf{1}\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}]. \end{aligned}$$

We recall that $\mathcal{K}(kn)$ was defined at (6.4). We may now see that

(1) on the one hand, the random variable $P^\omega[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F]$ is measurable with respect to $\sigma\{c_*([x,y]),\ \text{with } x, y \notin \mathcal{H}^+_{nk}\}$,
(2) on the other hand, the event $\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}$ is measurable with respect to $\sigma\{c_*([x,y]),\ \text{with } x \in \mathcal{H}^+_{nk}\}$.

Hence, the random variables $P^\omega[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F]$ and $\mathbf{1}\{\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}\}$ are $\mathbf{P}$-independent. This yields

$$\begin{aligned} &\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn)^c] \\ &= \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}[R((k-1)n), A(kn), B(kn), \mathcal{K}(kn) = F] \\ &\qquad\qquad \times \mathbf{P}[\text{some } x \in \partial F \cap \mathcal{H}^+(kn) \text{ is closed}] \\ &\le \sum_{F\subset\mathbb{Z}^d, |F|\le 2\ln n} \mathbb{P}[R((k-1)n), \mathcal{K}(kn) = F] \\ &\qquad\qquad \times (1 - \mathbf{P}[\text{all } x \in \partial F \cap \mathcal{H}^+(kn) \text{ are open}]). \end{aligned}$$

Now, we know by the Harris-inequality [18] that for $F \subset \mathbb{Z}^d$, with $|F| \le 2\ln n$

$$\mathbf{P}[\text{all } x \in \partial F \cap \mathcal{H}(kn) \text{ are open}] \ge \mathbf{P}[x \text{ is open}]^{|F|}$$

$$\geq (1-\varepsilon(K))^{2\ln n} = n^{2\ln(1-\varepsilon(K))}.$$

By choosing K large enough, we can assume that $2\ln(1-\varepsilon(K)) \geq -\varepsilon_1$. This means that the two previous equations imply that

(6.9)
$$\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn)^c] \leq (1-n^{-\varepsilon_1})\mathbb{P}[R((k-1)n), A(kn), B(kn)].$$

Step 2 : Control of the fourth term

We wish to decompose $\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c]$ according to all possible values of $X_{T_{\partial_i \mathcal{B}_{nk}}}$ and $\mathcal{K}(kn)$. For this, we notice that

(1) on $A(kn)$, we have $X_{T_{\partial_i \mathcal{B}_{nk}}} \in \partial_i^+ \mathcal{B}_{nk}$ and by the definition of $\mathcal{K}(kn)$ (see (6.4)), $X_{T_{\partial_i \mathcal{B}_{nk}}} \in \mathcal{K}(kn)$,
(2) moreover, on $A(kn) \cap B(kn)$, we have (6.8).

Hence,

$$\begin{aligned}&\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c]\\ \leq &\sum_{y,F} \mathbb{P}[R((k-1)n), X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F, C(kn), D(kn)^c],\end{aligned}$$

where $\sum_{y,F}$ stands for $\sum_{y\in\partial_i^+\mathcal{B}_{nk}} \sum_{F\subset\mathbb{Z}^d, |F|\leq 2\ln n, y\in F}$

Let us notice that, for a fixed ω, the events $R((k-1)n)$, $\{X_{T_{\partial_i \mathcal{B}_{nk}}} = y\}$ and $\{\mathcal{K}(kn) = F\}$ are P^ω-measurable with respect to $\{X_i, i \leq T_{\partial_i \mathcal{B}_{nk}}\}$. Thus, we may use the Markov property at $T_{\partial_i \mathcal{B}_{nk}}$ to see that

(6.10)
$$\begin{aligned}&\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c]\\ \leq &\sum_{y,F} \mathbf{E}[P^\omega[R((k-1)n), X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F]\\ &\qquad \times P_y^\omega[T_{\partial F} < T_{\partial F \cap \mathcal{H}^+(kn)}]\mathbf{1}\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}].\end{aligned}$$

We claim that if $\{\mathcal{K}(kn) = F\}$ and $\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}$ then $\partial_E F$ is composed of normal edges. Indeed,

(1) notice that the definition of $\mathcal{K}(kn)$ at (6.4) (which is a K-(kn)-closed component) implies that e is normal for all $e \in \partial_E \mathcal{K}(kn)$ when e has no endpoint in $\mathcal{H}^+(kn)$.
(2) Moreover, if for any $x \in \partial F \cap \mathcal{H}^+(kn)$ the vertex x is open, then any edge $e \in \partial_E F$ with one endpoint in $\mathcal{H}^+(kn)$ is normal.

Now, for any $y \in F \cap \partial_i^+ \mathcal{B}_{nk}$, there exists $x \in \mathcal{H}^+(kn)$ adjacent to y. Since $F \subset \mathcal{B}_{nk}$, we can see that for any $z \in \partial F$ we have $(x-z)\cdot\vec{\ell} \geq -1$. Using this, along with the conclusion of the previous paragraph, we see with (2.1) that

$$\text{for all } e \in \partial_E F,\ c^\omega(e) \leq K^2 e^{2\lambda} c^\omega([x,y]).$$

We can apply Lemma 6.3 to F, and we see that if $\{\mathcal{K}(kn) = F\}$ and $\{x \text{ is open, for } x \in \partial F \cap \mathcal{H}^+(kn)\}$, then we obtain

$$P_y^\omega[T_{\partial F} < T_{\partial F \cap \mathcal{H}(kn)}] \le P_y^\omega[T_{\partial F} < T_x] \le (1 - c\,|F|^{-1}) \le (1 - c\ln^{-1} n),$$

since $|F| \le 2\ln n$.

This turns (6.10) into

$$\begin{aligned}(6.11)\qquad &\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)^c] \\ \le& \sum_{y,F} \mathbf{E}[P^\omega[R((k-1)n), X_{T_{\partial_i \mathcal{B}_{nk}}} = y, \mathcal{K}(kn) = F](1 - c\ln^{-1} n) \\ \le&(1 - c\ln n^{-1})\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn)].\end{aligned}$$

Step 3: Control the fifth term

On $A(kn) \cap B(kn) \cap C(kn) \cap D(kn)$, we know that $X_{\partial B(n,n^\alpha)} \in \partial^+ B(n, n^\alpha)$ is an open point. So introducing

$$R'((k-1)n) = R((k-1)n) \cap A(kn) \cap B(kn) \cap C(kn) \cap D(kn)$$

we see

$$\begin{aligned}&\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn), E(kn)^c] \\ \le& \sum_{x \in \partial^+ \mathcal{B}_{nk}} \mathbb{P}[R'((k-1)n),\ X_{\mathcal{B}_{nk}} = x,\ x \text{ is open},\ E(kn)^c] \\ \le& \sum_{x \in \partial^+ \mathcal{B}_{nk}} \mathbf{E}[P^\omega[R'((k-1)n),\ X_{\mathcal{B}_{nk}} = x]\mathbf{1}\{x \text{ is open}\}, \\ &\qquad (\mathbf{1}\{x + e_1 \text{ or } x + 2e_1 \text{ is not } x\text{-open}\} \\ &\qquad + P_x^\omega[X_1 \ne x + e_1 \text{ or } X_2 \ne x + 2e_1]\mathbf{1}\{x + e_1, x + 2e_1 \text{ are } x\text{-open}\})]\end{aligned}$$

where a vertex is said to be x-open if it is open in ω_x coinciding with ω on all edges but those that are adjacent to x which are normal in ω_x.

On $\{x + e_1, x + 2e_1 \text{ are } x\text{-open}\} \cap \{x \text{ is open}\}$, we see that $P_x^\omega[X_1 = x + e_1,\ X_2 = x + 2e_1] \ge c > 0$ by Remark 5.2.

$$\begin{aligned}&\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn), E(kn)^c] \\ \le& \sum_{x \in \partial^+ \mathcal{B}_{nk}} \mathbf{E}[P^\omega[R'((k-1)n),\ X_{\partial B(n,n^\alpha)} = x]\mathbf{1}\{x \text{ is open}\}, \\ &\quad (\mathbf{1}\{x + e_1 \text{ or } x + 2e_1 \text{ is not } x\text{-open}\} + (1 - c)\mathbf{1}\{x + e_1, x + 2e_1 \text{ are } x\text{-open}\})].\end{aligned}$$

We may also see that $\{R'((k-1)n),\ X_{\partial \mathcal{B}_{nk}} = x,\ x \text{ is open}\}$ is measurable with respect to $\sigma\{c_*(e), e \in E(\mathcal{B}_{nk}) \text{ or } e \in x + \nu\}$, whereas $\{x + e_1, x + 2e_1 \text{ are } x\text{-open}\}$ is measurable with respect to $\sigma\{c_*(e), e \notin E(\mathcal{B}_{nk}) \text{ and } e \notin x + \nu\}$. So these random variables are independent, which yields

$$\begin{aligned}
(6.12)\qquad & \mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn), E(kn)^c] \\
& \leq \mathbb{P}[R'((k-1)n)](\mathbf{P}[x+e_1 \text{ or } x+e_2 \text{ is not } x\text{-open}] \\
(6.13)\qquad & \qquad + (1-c)\mathbf{P}[x+e_1, x+e_2 \text{ are } x\text{-open}]) \\
& \leq (1-c)\mathbb{P}[R((k-1)n), A(kn), B(kn), C(kn), D(kn)],
\end{aligned}$$

since $\mathbf{P}[x+e_1, x+e_2 \text{ are } x\text{-open}] > 0$.

Step 4: Conclusion

For any $\varepsilon_1 > 0$, we see using (6.6), (6.7), (6.9) (6.11) and (6.12), (which are valid K chosen larger than some K_0 depending only on $M < \infty$), that we have for any $k \in [2, n]$

$$\mathbb{P}[R(kn)] \leq \mathbb{P}[R((k-1)n)](1 - c\ln n^{-1} n^{-\varepsilon_1}) + Cn^{-M},$$

which implies the result. □

We now prove the following

Lemma 6.5. *For any M, there exists K_0 such that, for any $K \geq K_0$*

$$\mathbb{P}[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} \geq n] \leq C(K) n^{-M}.$$

Proof. For any $M < \infty$, by Lemma 6.4, there exists K_0 such that, for any $K \geq K_0$ such that

$$\mathbb{P}[R(nk)] \leq (1 - cn^{-1/2})\mathbb{P}[R(n(k-1))] + n^{-M}.$$

By a simple induction, this means that for

$$\mathbb{P}[R(n^2)] \leq (1 - cn^{-1/2})^n + n^{-M+1} \leq 2n^{-M+1}.$$

Recalling the definition of $R(n)$ at (6.3), we see by Borel-Cantelli's Lemma that $\mathcal{M}^{(K)} < \infty$

Also this implies that

$$\mathbb{P}[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} > n^2] \leq 2n^{-M+1},$$

and

$$\mathbb{P}[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} > n] \leq 2n^{-(M+1)/2},$$

which proves the lemma, since M is arbitrary. □

6.3. **Consequence of our estimates on $\mathcal{M}$.** Let us introduce the ladder times

$$W_0 = 0 \text{ and } W_{k+1} = \inf\{n \geq 0, X_n \cdot \vec{\ell} > X_{W_k} \cdot \vec{\ell}\}. \tag{6.14}$$

We set

$$M^{(K)}(n) = \{\text{for } k \text{ with } W_k \leq \Delta_n,\ X_{\mathcal{M}^{(K)} \circ \theta_{W_k} + W_k} \cdot \vec{\ell} - X_{W_k} \cdot \vec{\ell} \leq n^{1/2}\}, \tag{6.15}$$

Lemma 6.6. *For any $M < \infty$, there exists K_0 such that, for any $K \geq K_0$ we have*

$$\mathbb{P}[M^{(K)}(n)^c] \leq n^{-M}.$$

Proof. Denote

$$M_j(n) = \{\text{for } k \leq j-1,\ X_{\mathcal{M}^{(K)} \circ \theta_{W_k} + W_k} \cdot \vec{\ell} - X_{W_k} \cdot \vec{\ell} \leq n^{1/2}\},$$

and

$$N_j(n) = \{X_{\mathcal{M} \circ \theta_{W_j} + W_j} \cdot \vec{\ell} - X_{W_j} \cdot \vec{\ell} > n^{1/2},\ X_{W_j} \in B(n, n^\alpha)\}.$$

On $M(n)^c \cap \{T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}\}$, there is k such that $X_{W_k} \leq \Delta_n$. Moreover, all X_{W_i} are different necessarily so $k \leq n^{2d\alpha}$. By decomposing along the smallest such k, we see that

$$\mathbb{P}[M(n)^c] \leq \sum_{k=1}^{n^{2d\alpha}} \mathbb{P}[M_k(n), N_k(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}] + \exp(-cn), \tag{6.16}$$

by Theorem 5.1.

Now, we see that on $\{M_j(n), N_j(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}\}$ we have $X_{W_{j-1}} \in B(n, n^\alpha)$, so by a simple union bound argument

$$\begin{aligned}
&\mathbb{P}[M_j(n), N_j(n), T_{\partial B(n,n^\alpha)} = T_{\partial^+ B(n,n^\alpha)}] \\
\leq &\sum_{x \in B(n,n^\alpha)} \mathbb{P}[X_{W_j} = x, X_{\mathcal{M}^{(K)} \circ \theta_{W_j} + W_j} \cdot \vec{\ell} - x \cdot \vec{\ell} > n^{1/2}] \\
\leq &\sum_{x \in B(n,n^\alpha)} \mathbf{E}[P_x^\omega[X_{\mathcal{M}^{(K)}} \cdot \vec{\ell} - x \cdot \vec{\ell} \geq n^{1/2}]] \\
\leq &|B(n, n^\alpha)|\, \mathbb{P}[\mathcal{M}^{(K)} \geq n^{1/2}],
\end{aligned}$$

by Markov's property at W_j and translation invariance of the environment. Hence, by the two last equations, Lemma 6.5 and using (6.16), we may see that

$$\mathbb{P}[M(n)^c] \leq n^{-M}.$$

□

7. Regeneration times

The aim of this section is to define regeneration times and prove some standard properties on regeneration times. These properties are summed up in subsection 7.6.

We define

$$D_K = \mathbf{1}\{X_0 \text{ is good}\}D',$$

which we will often abbreviate D, where

$$D' = \inf\{n > 0,\ X_n \cdot \vec{\ell} \leq X_0 \cdot \vec{\ell}\}.$$

Also we introduce

$$M^1 = \sup_{n \leq D'} X_n \cdot \vec{\ell},$$

and

$$M^2(x) = \sup\{(y - x) \cdot \vec{\ell}, y \in K_{\text{good}}(x)\},$$

where

$$K_{\text{good}}(x) = \{y,\ \text{where } x \text{ is connected to } y \text{ using a directed open path}\}, \tag{7.1}$$

where directed open path was defined at the beginning of Section 5

We define the configuration dependent stopping times S_k, $k \geq 0$ and the levels M_k, $k \geq 0$:

$$S_0 = 0,\ M_0 = X_0 \cdot \vec{\ell},\ \text{and for } k \geq 0, \tag{7.2}$$

$$S_{k+1} = \mathcal{M}^{(K)} \circ \theta_{T_{\mathcal{H}^+(M_k)}} + T_{\mathcal{H}^+(M_k)},$$

where

$$M_k = \sup\{X_m \cdot \vec{\ell},\ 0 \leq m \leq R_k\}.$$

with

$$R_k = \begin{cases} D' \circ \theta_{S_k} + S_k, & \text{if } X_{S_k} \text{ is good,} \\ T_{\mathcal{H}^+(M^2(X_{S_k}) + X_{S_k} \cdot \vec{\ell})} & \text{if } X_{S_k} \text{ is not good.} \end{cases}$$

These definitions imply that if $S_{i+1} < \infty$, then

$$X_{S_{i+1}} \cdot \vec{\ell} - X_{S_i} \cdot \vec{\ell} \geq 2. \tag{7.3}$$

If X_{S_k} is good, then R_k can be infinite so that $M_k = \infty$.

Finally we define the basic regeneration time

$$\tau_1 = S_N,\ \text{with } N = \inf\{k \geq 1,\ S_k < \infty, M_k = \infty\}. \tag{7.4}$$

Let us give some intuition about those definitions. Assume S_k is constructed, it is, by definition, an open-point, so there is a positive probability that it is a good point. Then, if it is a good point, it is natural to expect there is a lower-bounded chance of never backtracking again. Hence, there is

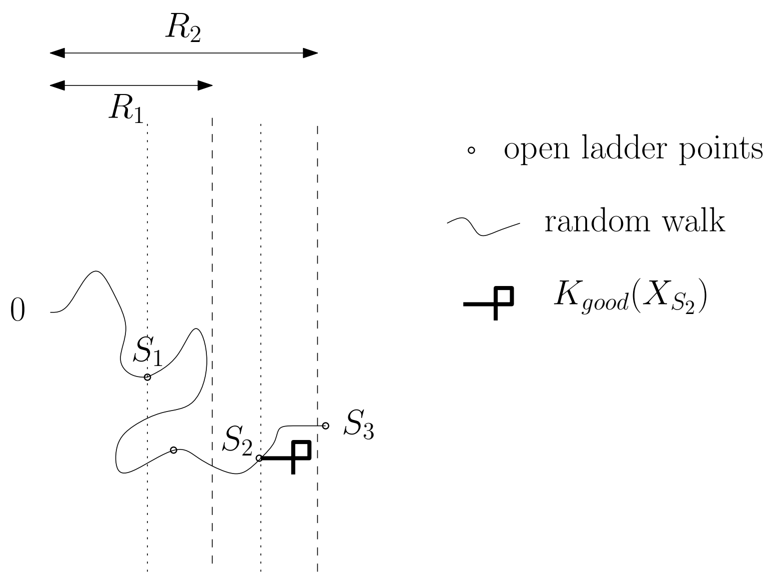

FIGURE 4. The construction of regeneration times

a positive probability of creating a point separating the past and the future of the random walk: a regeneration point called τ_1.

In case this does not occur, the future of the random walk and the environment ahead of us may be conditioned. A conditioning may be induced

(1) by the fact that S_k is not good, that conditioning is limited to the value of conductances in the finite set $K_{\text{good}}(X_{S_k})$,
(2) and by the fact that the walk will eventually backtrack: a conditioning limited to the conductances of the edges adjacent to the trajectory of the walk before it backtracks and the walk itself (before backtracking).

We may notice that by our definitions, all those edges have one endpoint in $\mathcal{H}^-(M_k)$, and that the environment in $\mathcal{H}^+(M_k)$ and the walk once it reaches this set are largely unconditioned. This property will give us the opportunity to construct another open ladder point for a walk free of any constraints from its past. Figure 4 illustrates the construction of regeneration times.

7.1. **Control the variables M^1 and M^2.**

Lemma 7.1. *We have*

$$\mathbb{P}[M^1 \geq n \mid D' < \infty] \leq C \exp(-cn).$$

Proof. We have

$$\begin{aligned} &\mathbb{P}[2^k \leq M^1 < 2^{k+1}] \\ \leq &\mathbb{P}[T_{\partial B(2^k,2^{\alpha k})} \neq T_{\partial^+ B(2^k,2^{\alpha k})}] \end{aligned}$$

$$+\mathbb{P}[X_{T_{\partial B(2^k,2^{\alpha k})}}\in\partial^+B(2^k,2^{\alpha k}),T^+_{\mathcal{H}^-(0)}\circ\theta_{T_{\partial^+B(2^k,2^{\alpha k})}}<T_{\mathcal{H}^+(2^{k+1})}\circ\theta_{T_{\partial^+B(2^k,2^{\alpha k})}}],$$

and by a union bound on the $2^{\alpha k}$ possible positions of $X_{T_{\partial^+B(2^k,2^{\alpha k})}}$ and using translation invariance arguments

$$\mathbb{P}[2^k\le M^1<2^{k+1}]\le 2^{\alpha k}\mathbb{P}[T_{\partial B(2^{k+1},2^{\alpha(k+1)})}\neq T_{\partial^+B(2^{k+1},2^{\alpha(k+1)})}]+e^{-(2^k)},$$

as a consequence of Theorem 5.1.

Hence, using Theorem 5.1, again

$$\mathbb{P}[2^k\le M^1<2^{k+1}]\le c2^{\alpha k}e^{-c(2^k)},$$

and since $M^1<\infty$ on $D'<\infty$, we see that

$$\mathbb{P}[M^{(1)}\ge n\mid D'<\infty]\le\frac{1}{\mathbb{P}[D'<\infty]}\sum_{k,\ 2^k\ge n}\mathbb{P}[2^k\le M^1<2^{k+1}]\le Ce^{-cn}.$$

□

Also

Lemma 7.2. *We have*

$$\mathbb{P}[M^2\ge n\mid 0\text{ is not good}]\le C\exp(-cn).$$

Proof. Hence, by Remark 5.1, on {0 is not good}

$$\{x\text{ where 0 is connected to }x\text{ using a directed open path}\}\subset\mathrm{BAD}^+(0),$$

so, in particular

$$M^2\le\sup_{x\in\mathrm{BAD}^+(0)}\{x\cdot\vec{\ell}+1\}\le W(\mathrm{BAD}^+(0))+1,$$

and by Lemma 5.2, we are done. □

We introduce

$$S(n)=\{\text{ for }i\text{ with }S_i\le\Delta_n\text{ and }M_i<\infty,\ M_i-X_{S_i}\cdot\vec{\ell}\le n^{1/2}\}. \tag{7.5}$$

Let us prove

Lemma 7.3. *We have*

$$\mathbb{P}[S(n)^c]\le\exp(-n^{1/2}).$$

Proof. By (7.3), we know that card$\{i;S_i\le\Delta_n\}\le n$. So, by Theorem 5.1, we see that

(7.6)

$$\mathbb{P}[S(n)^c]\le\mathbb{P}[T_{B(n,n^\alpha)}\neq T_{\partial^+B(n,n^\alpha)}]+\sum_{i\le n}\sum_{x\in B(n,n^\alpha)}\mathbb{P}[M_i-X_{S_i}\cdot\vec{\ell}>n^{1/2},\ M_i<\infty,\ X_{S_i}=x].$$

Now, we may see that

$$\begin{aligned}(7.7)\qquad &\mathbb{P}[M_i - X_{S_i}\cdot\vec{\ell} > n^{1/2},\ M_i<\infty,\ X_{S_i}=x]\\ \le&\mathbb{P}\Big[\sup_{n\le D'\circ\theta_{S_i}+S_i}(X_n-x)\cdot\vec{\ell}>n^{1/2}, M_i<\infty, X_{S_i}=x,\ x\text{ is good}\Big]\\ &+\mathbb{P}\big[\sup\{(y-x)\cdot\vec{\ell}, y\in K_{\text{good}}(x)\}>n^{1/2}\mid x\text{ is not good}\big],\end{aligned}$$

where $K_{\text{good}}(x)$ was defined at (7.1).

By translation invariance of $\mathbf{P}$ and Lemma 7.2, we have

(7.8)

$$\mathbb{P}[\sup\{(y-x)\cdot\vec{\ell}, y\in K_{\text{good}}(x)\}>n^{1/2}\mid x\text{ is not good}]\le C\exp(-cn^{1/2}).$$

We may see that, if $M_i<\infty$ and X_{S_i} is good, then $D'\circ\theta_{S_i}+S_i<\infty$, hence

$$\begin{aligned}(7.9)\qquad &\mathbb{P}\big[\sup_{n\le D'\circ\theta_{S_i}+S_i}(X_n-x)\cdot\vec{\ell}\ge n^{1/2}, M_i<\infty, X_{S_i}=x,\ x\text{ is good}\big]\\ \le&\mathbb{P}\big[\sup_{n\le D'\circ\theta_{S_i}+S_i}(X_n-x)\cdot\vec{\ell}\ge n^{1/2}, D'\circ\theta_{S_i}+S_i<\infty, X_{S_i}=x\big]\end{aligned}$$

By using Markov's property at the time S_i, we see that

$$\begin{aligned}(7.10)\qquad &\mathbb{P}\Big[\sup_{n\le D'\circ\theta_{S_i}+S_i}(X_n-x)\cdot\vec{\ell}\ge n^{1/2}, D'\circ\theta_{S_i}+S_i<\infty, X_{S_i}=x\Big]\\ \le&\mathbf{E}\Big[P_x^\omega\big[\sup_{n\le D}X_n\cdot\vec{\ell}\ge n^{1/2}, D'<\infty\big]\Big]\\ =&\mathbb{P}\Big[\sup_{n\le D'}X_n\cdot\vec{\ell}\ge n^{1/2}\mid D'<\infty\Big]\le C\exp(-cn^{1/2}),\end{aligned}$$

where we used and translation invariance and Lemma 7.1. The result follows from putting together (7.6), (7.7), (7.8), (7.9) and (7.10). □

7.2. **Exponential tails for backtracking.** We can deduce that

Lemma 7.4. *We have*

$$\mathbb{P}[T_{\mathcal{H}^-(-n)}<\infty]\le\exp(-cn).$$

Proof. Fix $n>0$. For this proof, we will use the notation

$$A(n)=\{T_{\partial B(2^n,2^{n\alpha})}=T_{\partial^+B(2^n,2^{n\alpha})}\}.$$

For any $k\ge n$, let us denote $B_X(2^{k+1},2^{(k+1)\alpha})=\{z\in\mathbb{Z}^d,\ z=X_{T_{\partial B(2^k,2^{k\alpha})}}+y$ with $y\in B(2^{k+1},2^{(k+1)\alpha})\}$ and

$$C(k)=\{T_{\partial^+B_X(2^{k+1},2^{(k+1)\alpha})}\circ\theta_{T_{\partial B(2^k,2^{k\alpha})}}=T_{\partial B_X(2^{k+1},2^{(k+1)\alpha})}\circ\theta_{T_{\partial B(2^k,2^{k\alpha})}}\}.$$

A simple induction shows that on $\cap_{k\in[n,m]}C(k)\cap A(n)$, we have $\{T_{\mathcal{H}^-(-2^n-1)}\ge T_{B(2^m,m2^{\alpha m})}\}$, hence, we see that

$$(7.11)\qquad \cap_{k\ge n}C(k)\cap A(n)\subseteq\{T_{\mathcal{H}^-(-2^n-1)}=\infty\}.$$

Denote for $m > n$,

$$D(n,m) = \{A(n),\ \text{for } n \le k < m,\ C(k),\ C(m)^c\},$$

so that

$$\Big(\cap_{k\ge n} C(k) \cap A(n)\Big)^c \subset \cup_{m\ge n} D(n,m) \cup A(n)^c.$$

which implies with (7.11) that

(7.12)

$$\mathbb{P}[T_{\mathcal{H}^-(-2^n)} < \infty] \le \mathbb{P}[A(n)^c] + \sum_{m\ge n} \mathbb{P}[D(n,m)] \le \exp(-c2^n) + \sum_{m\ge n} \mathbb{P}[D(n,m)],$$

by Theorem 5.1.

We may notice that on $D(n,m)$, we have $\{T_{\partial B(2^m, m2^{m\alpha})} = T_{\partial^+ B(2^m, m2^{m\alpha})}\}$ (note that is different from $A(m)$). Hence, when using Markov's property at $T_{\partial B(2^m, m2^{m\alpha})}$ the random walk is located in $\partial^+ B(2^m, m2^{m\alpha})$, so

$$\begin{aligned}
&\mathbb{P}[D(n,m)]\\
\le & \sum_{x\in\partial^+ B(2^m, m2^{m\alpha})} \mathbf{E}[P^\omega[X_{T_{\partial B(2^m, m2^{m\alpha})}} = x] P_x^\omega[T_{x+\partial B(2^{m+1}, 2^{(m+1)\alpha})} \neq T_{x+\partial^+ B(2^{m+1}, 2^{(m+1)\alpha})}]]\\
\le & Cm^d 2^{dm\alpha} \max_{x\in\partial^+ B(2^m, m2^{m\alpha})} \mathbf{E}[P_x^\omega[T_{\partial B(2^{m+1}, 2^{(m+1)\alpha})} \neq T_{\partial^+ B(2^{m+1}, 2^{(m+1)\alpha})}]]\\
\le & Cm^d 2^{dm\alpha} \mathbb{P}[A(m+1)^c] \le \exp(-c2^m).
\end{aligned}$$

by translation invariance and Theorem 5.1.

The lemma follows from the previous and (7.12). $\square$

7.3. Uniformly bounded chance of never backtracking at open points.

We denote $\mathcal{C} = \{x > 0\}^\nu$. For any $a \in \mathcal{C}$, we define the environment ω_x^a to have the same conductances as in ω on any edge non adjacent to x and where all edges adjacent to $c_*^{\omega_x^a}([x, x+e]) = a(e)$ for any $e \in \nu$.

We say that $a \in \mathcal{C}$ is K-open if $a(e) \in [1/K, K]$ for any $e \in \nu$. Furthermore a vertex $x \in \mathbb{Z}^d$ is called open-good if x is good in a configuration ω_x^a where a is open. Note that we do not need to specify the value of the conductances of the edges since the event $\{x \text{ is good}\}$ is measurable with respect to $(\{c_*(e) \in [1/K, K]\})_{e\in\mathbb{Z}^d}$.

Lemma 7.5. *We have*

$$\mathbf{E}\Big[\max_{a\in\mathcal{C} \text{ is open}} P^{\omega_0^a}[D' < \infty] \mid 0 \text{ is open good}\Big] < 1.$$

Proof. Fix $n > 0$. On the event that $\{0 \text{ is open good}\}$, we denote $\mathcal{P}(i)$ a directed path starting at 0 where all points, except maybe 0, are open. We denote $L_{\partial^+ B(n,n^\alpha)} = \inf\{i, \mathcal{P}(i) \in \partial^+ B(n, n^\alpha)\}$. Now, we see that if the two following conditions are verified

(1) $X_i = \mathcal{P}(i)$ for $i \le L_{\partial^+ B(n,n^\alpha)}$,

(2) $T_{\mathcal{H}^-(2)} \circ \theta_{T_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}} = \infty$,

then $D' = \infty$.

We can see that {0 is open good}

$$\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[X_i = \mathcal{P}(i) \text{ for } i \le L_{\partial^+ B(n,n^\alpha)}] \ge \kappa_0^{2n},$$

by Remark 5.2.

In particular, we have

$$\begin{aligned} &\mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[D' = \infty] \mid 0 \text{ is open good}\Big] \\ \ge&\mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[X_i = \mathcal{P}(i) \text{ for } i \le L_{\partial^+ B(n,n^\alpha)}] \\ &\qquad \times P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] \mid 0 \text{ is open good}\Big] \\ \ge&\kappa_0^n \mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] \mid 0 \text{ is open good}\Big]. \end{aligned}$$

Moreover, we see that

$$P^{\omega_0^a}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] = P^{\omega}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty],$$

so that for any n

$$\begin{aligned} &\mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[D' = \infty] \mid 0 \text{ is open good}\Big] \\ \ge&\kappa_0^{2n} \mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} = \infty] \mid 0 \text{ is open good}\big]. \end{aligned}$$

Now,

$$\begin{aligned} &\mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} < \infty] \mid 0 \text{ is open good}\big] \\ \le&\mathbf{P}[0 \text{ is open good}]^{-1} \mathbf{E}\big[P^{\omega}_{\mathcal{P}(L_{\partial^+ B(n,n^\alpha)})}[T_{\mathcal{H}^-(2)} < \infty]\big] \\ \le& C\mathbb{P}[T_{\mathcal{H}^-(-n+2)} < \infty], \end{aligned}$$

where we use translation invariance.

Now, by Lemma 7.4, we see that the previous quantity is less than 1/2 for $n \ge n_0$. Hence combining the last two equations

$$\mathbf{E}\Big[\min_{a \in \mathcal{C} \text{ open}} P^{\omega_0^a}[D' = \infty] \mid 0 \text{ is open good}\Big] \ge (1/2)\kappa_0^{n_0} > 0,$$

which implies the result. □

7.4. **Number of trials before finding an open ladder point which is a regeneration time.** Let us introduce the collection of edges with maximum scalar product with $\vec{\ell}$

$$\mathcal{E} = \{e \in \nu \text{ such that } e \cdot \vec{\ell} = e_1 \cdot \vec{\ell}\},$$

and

$$\mathcal{B}_x = \{e \in \mathbb{E}(\mathbb{Z}^d),\ e = [-e_1, f - e_1] \text{ with } f \text{ any unit vector of } \mathcal{E}\}. \tag{7.13}$$

Imagining the bias is oriented to the right, the set of edges to the "left"of x is defined to be

$$\mathcal{L}^x := \{[y,z] \in E(\mathbb{Z}^d),\ y \cdot \ell \le x \cdot \ell \text{ or } z \cdot \ell \le x \cdot \ell\} \cup \mathcal{B}_x, \tag{7.14}$$

and the edges to the "right"are

$$\mathcal{R}^x := \{[y,z] \in E(\mathbb{Z}^d),\ y \cdot \ell > x \cdot \ell \text{ and } z \cdot \ell > x \cdot \ell\} \cup \mathcal{B}_x. \tag{7.15}$$

We recall that N was defined at (7.4). Let us prove

Lemma 7.6. *We have*

$$\mathbb{P}[N \ge n] \le \exp(-cn).$$

Proof. We have
(7.16)

$$\{N \ge n\} \subseteq C(n) := \{k \le n,\ S_k < \infty, X_{S_k} \text{ is not good or } D' \circ S_k + S_k < \infty\}.$$

Because of the way our regeneration times are constructed, we can see that $C(n)$ is P^ω-measurable with respect to $\sigma\{X_k \text{ with } k \le S_{n+1}\}$ (see the discussion related to Figure 4). Using Markov's property at S_{n+1},

$$\begin{aligned}
&\mathbb{P}[C(n+1)] \\
\le& \sum_{x \in \mathbb{Z}^d} \mathbf{E}[\mathbf{1}\{x \text{ is not good}\} P^\omega[X_{S_{n+1}} = x, C(n)]] \\
&\qquad + \mathbf{E}[\mathbf{1}\{x \text{ is good}\} P^\omega[X_{S_{n+1}} = x, C(n)] P_x^\omega[D' < \infty]] \\
\le& \sum_{x \in \mathbb{Z}^d} \mathbf{E}[\mathbf{1}\{x \text{ is not open good}\} P^\omega[X_{S_{n+1}} = x, C(n)]] \\
&\qquad + \mathbf{E}[P^\omega[X_{S_{n+1}} = x, C(n)] \mathbf{1}\{x \text{ is open good}\} \max_{a \text{ open}} P_x^{\omega^{x,a}}[D' < \infty]],
\end{aligned}$$

where we used the fact that $X_{S_{n+1}}$ is open. Furthermore

(1) $P^\omega[X_{S_{n+1}} = x, C(n)]$ is measurable with respect to $\sigma\{c_*(e) \text{ with } e \in \mathcal{L}_x\}$,
(2) $\{x \text{ is not open good}\}$, $\{x \text{ is open good}\}$ and $\max_{a \text{ open}} P_x^{\omega_x^a}[D < \infty]$ are measurable with respect to $\sigma\{c_*(e) \text{ with } e \notin \mathcal{L}_x\}$.

So we have independence between the random variable in (1) and those in (2). Hence

$$\begin{aligned}&\mathbb{P}[C(n+1)]\\ \le&\mathbb{P}[C(n)](\mathbf{P}[x \text{ is not open good}]+\mathbf{E}\big[\mathbf{1}\{x \text{ is open good}\}\max_{a\in\mathcal{C}\text{ open}} P^{\omega_x^a}[D<\infty]\big])\\ \le&\mathbb{P}[C(n)]\Big(1-\mathbf{P}[0 \text{ is open good}]\big(1-\mathbf{E}\big[\max_{a\in\mathcal{C}\text{ open}} P^{\omega_0^a}[D<\infty]\mid 0 \text{ is open good}\big]\big)\Big)\end{aligned}$$

where we used translation invariance. It is clear that $\mathbf{P}[0 \text{ is open good}] \ge \mathbf{P}[x \text{ is good}] > 0$ and further we use Lemma 7.5 to see that

$$\mathbb{P}[C(n+1)] \le (1-c)\mathbb{P}[C(n)] \le \ldots \le (1-c)^n,$$

hence, the result by (7.16). □

7.5. **Tails of regeneration times.** Now

Theorem 7.1. *For any $M<\infty$, there exists K_0 such that, for any $K\ge K_0$ we have $\tau_1^{(K)}<\infty$, $\mathbb{P}$-a.s. and*

$$\mathbb{P}[X_{\tau_1^{(K)}}\cdot\vec{\ell}\ge n]\le C(M)n^{-M}.$$

Proof. Recalling the definitions (7.2) and (6.14), we may see that $\{T_{\mathcal{H}^+(M_k)}, k\ge 0\}\subset\{W_k, k\ge 0\}$. This means that on $M(n)$, defined at (6.15),

$$\text{for } k \text{ with } S_k\le\Delta_n,\ X_{S_{k+1}}\cdot\vec{\ell}-X_{T_{\mathcal{H}^+(M_k)}}\cdot\vec{\ell}\le n^{1/2}.$$

Moreover, on $S(n)$ (defined at (7.5)) we have

$$\text{for } k \text{ with } S_k\le\Delta_n \text{ and } M_k<\infty,\ M_k-X_{S_k}\cdot\vec{\ell}\le n^{1/2}.$$

Noticing that $X_{T_{\mathcal{H}^+(M_k)}}\cdot\vec{\ell}\le M_k+1$, we may see that, on $S(n)\cap M(n)$

$$X_{S_{k+1}}\cdot\vec{\ell}-X_{S_k}\cdot\vec{\ell}\le 2n^{1/2}+1,$$

for any k with $S_k<\Delta_n$ and $M_k<\infty$. By induction, this means that if $k\le n^{1/3}$, $S_k<\Delta_n$ and $M_k<\infty$, then

$$X_{S_{k+1}}\cdot\vec{\ell}\le k(2n^{1/2}+1)<n,$$

and so that $S_{k+1}\le\Delta_n$, since $X_{S_{k+1}}$ is a new maximum for the random walk in the direction $\vec{\ell}$.

Hence, if $\{N\le n^{1/3}\}$ and $M(n)\cap S(n)$, then for n large enough

$$X_{\tau_1}\cdot\vec{\ell}\le n^{1/3}(2n^{1/2}+1)<n.$$

Thus

$$\begin{aligned}\mathbb{P}[X_{\tau_1}\cdot\vec{\ell}\ge n]\le&\mathbb{P}[N\ge n^{1/3}]+\mathbb{P}[M(n)^c]+\mathbb{P}[S(n)^c]\\ \le&\exp(-cn^{1/3})+2n^{-M}\le 3n^{-M},\end{aligned}$$

by Lemma 7.3, Lemma 6.6 and Lemma 7.6. This concludes the proof. □

7.6. **Fundamental property of regeneration times.** Then let us define the sequence $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_k < \ldots$, via the following procedure

$$\tau_{k+1} = \tau_1 + \tau_k(X_{\tau_1+\cdot} - X_{\tau_1},\ \omega(\ \cdot + X_{\tau_1})),\ k \geq 0, \tag{7.17}$$

meaning we look at the $k+1$-th regeneration time is the k-th regeneration time after the first one.

We set

$$\mathcal{G}_k := \sigma\{\tau_1, \ldots, \tau_k;\ (X_{\tau_k \wedge m})_{m\geq 0};\ c_*(e), e \in \mathcal{L}^{X_{\tau_{k+1}}}\},$$

Let us introduce for any $x \in \mathbb{Z}^d$

$$a_x = (c_*([x-e_1, x-e_1+e]))_{e\in\mathcal{E}} = (c_*(e))_{e\in\mathcal{B}_x} \in [1/K, K]^{\mathcal{E}},$$

recalling notation from (7.13). For $a \in [1/K, K]^{\mathcal{E}}$,

$$\mathbf{P}_x^a = \delta_a\Big((c_*([x-e_1, x-e_1+e]))_{e\in\mathcal{E}}\Big) \otimes \int_{e\in E(\mathbb{Z}^d)\setminus\mathcal{B}_x} \otimes d\mathbf{P}(c_*(e)),$$

and the associated annealed measure

$$\mathbb{P}_x^a = \mathbf{P}_x^a \times P_x^\omega.$$

We may notice that Theorem 7.1, can easily be generalized to become

Theorem 7.2. *For any $M < \infty$, there exists K_0 such that, for any $K \geq K_0$, we have $\tau_1^{(K)} < \infty$, $\mathbb{P}_0^a$-a.s. for $a \in [1/K, K]^{\mathcal{E}}$ and*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[X_{\tau_1^{(K)}} \cdot \vec{\ell} \geq n] \leq Cn^{-M},$$

Similarly, we can turn Theorem 5.1 into

Theorem 7.3. *For $\alpha > d+3$*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[T_{\partial B(L,L^\alpha)} \neq T_{\partial^+ B(L,L^\alpha)}] \leq e^{-cL}.$$

The fundamental properties of regeneration times are that

(1) the past and the future of the random walk that has arrived X_{τ_k} are only linked by the conductances of the edges in $a_{X_{\tau_k}}$,
(2) the law of the future of the random walk has the same law as a random walk under $\mathbb{P}_0^{a_{X_{\tau_k}}}[\,\cdot \mid D_K = \infty]$.

Now, let us state a theorem corresponding to the previous heuristic.

Theorem 7.4. *Let f, g, h_k be bounded and respectively, $\sigma\{X_n : n \geq 0\}$-,$\sigma\{c_*(e), e \in \mathcal{R}^0\}-$ and $\mathcal{G}_k$-measurable functions. Then for $a \in [1/K, K]^{\mathcal{E}}$,*

$$\mathbb{E}^a[f(X_{\tau_k+\cdot} - X_{\tau_k}) g \circ t_{X_{\tau_k}} h_k] = \mathbb{E}^a[h_k \mathbb{E}_0^{a_{X_{\tau_k}}}[fg \mid D_K = \infty]].$$

A similar theorem was proved in [25] (as Theorems 3.3 and 3.5). In our context, the random variables studied: τ_1, D etc. are defined differently from the corresponding ones in [25]. Nevertheless, our notations were chosen so that we may prove Theorem 7.4 simply by following word for word the proofs of Theorems 3.3 and 3.5 in [25]. The reader should start reading after Remark 3.2 in [25] to have all necessary notations. To avoid any possible confusion we point out that in [25], the measures $\mathbb{P}$, $P_{x,\omega}$ and P correspond respectively to the environment, the quenched random walk and the annealed measure and that $\omega(b)$ denotes the conductances of an edge b.

We bring the reader's attention to the fact that we have not proved yet that $\tau_k < \infty$, $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]$). We only know this for $k = 1$. This is enough to prove Theorem 7.4 for $k = 1$. Using Theorem 7.4 for $k = 1$ and Theorem 7.2, we may show that $\tau_2 < \infty$, $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]$) and thereafter obtain Theorem 7.4 for $k = 2$. Hence, we may proceed by induction to prove Theorem 7.4 alongside the following result.

Proposition 7.1. *For any $k \geq 1$, we have $\tau_k^{(K)} < \infty$ $\mathbb{P}$-a.s. (or $\mathbb{P}^a$-a.s. for any $a \in [1/K, K]$).*

We see that this implies Proposition 3.1, which states directional transience in the direction $\vec{\ell}$ for the random walk.

As in [25], we may notice that a consequence of Theorem 7.4 is

Proposition 7.2. *Let*

$$\Gamma := \mathbb{N} \times \mathbb{Z}^d \times [1/K, K]^{\mathcal{E}},$$

with its canonical product σ-algebra and let $y_i = (j^i, z^i, a^i) \in \Gamma$, $i \geq 0$. For $a \in [1/K, K]^{\mathcal{E}}$ and $G \subset \Gamma$ measurable let also

$$\tilde{R}_K(a; G) := \mathbb{P}_0^a[(\tau_1^{(K)}, X_{\tau_1^{(K)}}, a_{X_{\tau_1^{(K)}}}) \in G \mid D_K = \infty].$$

Then under $\mathbb{P}$ the Γ-valued random variables (with $\tau_0 = 0$),

$$Y_i^K := (J_i, Z_i, A_i) := (\tau_{i+1}^{(K)} - \tau_i^{(K)}, X_{\tau_{i+1}^{(K)}} - X_{\tau_i^{(K)}}, a_{\tau_{i+1}^{(K)}}), \qquad i \geq 0, \tag{7.18}$$

define a Markov chain on the state space Γ, which has transition kernel

$$\mathbb{P}[Y_{i+1} \in G \mid Y_0 = y_0, \ldots, Y_i = y_i] = \tilde{R}_K(a^i; G),$$

and initial distribution

$$\tilde{\Lambda}_K(G) := \mathbb{P}[(\tau_1^{(K)}, X_{\tau_1^{(K)}}, a_{X_{\tau_1^{(K)}}}) \in G].$$

Similarly, on the state space $[1/K, K]^{\mathcal{E}}$, the random variables

$$A_i = a_{X_{\tau_{i+1}^{(K)}}}, \qquad k \geq 0, \tag{7.19}$$

also define a Markov chain under $\mathbb{P}$. *With* $a \in [1/K,K]^{\mathcal{E}}$ *and* $B \subset [1/K,K]^{\mathcal{E}}$ *measurable, its transition kernel is*

$$R_K(a;B) := \mathbb{P}_0^a[a_{X_{\tau_1^{(K)}}} \in B \mid D_K = \infty] = \sum_{j\in\mathbb{N}, z\in\mathbb{Z}^d} \tilde{R}_K(a;(j,z,B)),$$

and the initial distribution is

$$\Lambda_K(B) := \mathbb{P}[a_{X_{\tau_1^{(K)}}} \in B] = \tilde{\Lambda}_K((j,z,B)). \tag{7.20}$$

Now let us quote Lemma 3.7 and Theorem 3.8 from [25],

Theorem 7.5. *There exists a unique invariant distribution* ν_K *for the transition kernel* R_K. *It verifies*

$$\sup_{a\in[1/K,K]^{\mathcal{E}}} ||R_K^m(a;\cdot) - \nu_K(\cdot)||_{var} \leq Ce^{-cm}, \qquad m \geq 0,$$

where $||\cdot||_{var}$ *denotes the total variation distance.*

Further, this probability measure ν_K *is invariant with respect to the transition kernel R; that is,* $\nu_K R_K = \nu_K$, *and the Markov chain* $(A_k)_{k\geq 0}$, *defined in (7.19) with transition kernel* R_K *and initial distribution* ν_K *on the state space* $[1/K,K]^{\mathcal{E}}$ *is ergodic. Moreover, the initial distribution* $\Lambda_K(\cdot)$ *given in (7.20) is absolutely continuous with respect to* $\nu_K(\cdot)$.

Theorem 7.6. *The distribution* $\tilde{\nu}_K := \nu_K \tilde{R}_K$ *is the unique invariant distribution for the transition kernel* $\tilde{R}_K$. *It verifies*

$$\sup_{a\in[1/M,M]^{\mathcal{E}}} \left|\left|\tilde{R}_K^m(a;\cdot) - \tilde{\nu}_K(\cdot)\right|\right|_{var} \leq Ce^{-cm}, \qquad m \geq 0.$$

With initial distribution equal $\tilde{\nu}_K$, *the Markov chain* $(Y_k)_{k\geq 0}$ *defined in (7.18) is ergodic. Moreover, the law of the Markov chain* $(Y_{k+1})_{k\geq 0}$ *under* $\mathbb{P}$ *is absolutely continuous with respect to the law of the chain with initial distribution* $\tilde{\nu}_K$.

The proofs in [25] carry over to our context simply, once we have shown the following Doeblin condition: there exists $c > 0$ such that for any $a \in [1/K,K]^{\mathcal{E}}$, we have

$$R_K(a,B) \geq c \otimes_{\mathcal{E}} \mathbf{P}[c_* \in B \mid c_* \in [1/K,K]].$$

Let us prove this condition,

$$\begin{aligned} R_K(a;B) &= \mathbb{P}_0^a[a_{X_{\tau_1}} \in B \mid D = \infty] \\ &= \frac{1}{\mathbb{P}_0^a[D' = \infty, 0 \text{ is good}]} \mathbf{E}^a[P_0^\omega[a_{X_{\tau_1}} \in B, D' = \infty, 0 \text{ is good}]]. \end{aligned}$$

This means, using Remark 5.2

$$R_K(a;B) \geq c\mathbf{E}^a[P_0^\omega[X_1 = e_1,\ X_2 = 2e_1,\ D' \circ \theta_2 = \infty]$$

$$
\begin{aligned}
&\dots a_{X_2} \in B, e_1 \text{ is open and } 2e_1 \text{ is good}]\\
\geq & c\kappa_0^2 \mathbf{E}^a[P^{\omega}_{2e_1}[D' = \infty], a_{2e_1} \in B, e_1 \text{ is open and } 2e_1 \text{ is good}]\\
\geq & c\kappa_0^2 \mathbf{E}^a\big[\min_{a \text{ open}} P^{\omega_0^a}_{2e_1}[D' = \infty], a_{2e_1} \in B, e_1, 2e_1 \text{ are open}\\
&\text{and } 2e_1 \text{ is open good}\big],
\end{aligned}
$$

and seeing that

1. $\min_{a \text{ open}} P^{\omega_0^a}_{2e_1}[D' = \infty]$ and $\{2e_1 \text{ is open good}\}$ are measurable with respect to $\sigma\{c_*([y,z]) \text{ with } (y-2e_1)\cdot\vec{\ell} > 0, z \neq 2e_1\}$,
2. $\{a_{2e_1} \in B, e_1, 2e_1 \text{ are open}\}$ is measurable with respect to $\sigma\{c_*([y,z]) \text{ with } (y-2e_1)\cdot\vec{\ell} \leq 0 \text{ or } 2e_1 = z\}$

hence they are $\mathbf{P}$-independent so that

$$
\begin{aligned}
R_K(a;B) \geq & c\mathbf{E}^a\big[\min_{a \text{ open}} P^{\omega_0^a}_{2e_1}[D = \infty], 2e_1 \text{ is open good}\big]\\
&\times \mathbf{P}_0^a[a_{2e_1} \in B, e_1, 2e_1 \text{ are open}]\\
\geq & c\mathbf{P}_0^a[a_{2e_1} \in B, e_1, 2e_1 \text{ are open}],
\end{aligned}
$$

by Lemma 7.5. Now by simple combinatorics we see that

$$
R_K(a;B) \geq c \otimes_{\mathcal{E}} \mathbf{P}[c_* \in B \mid c_* \in [1/K.K]],
$$

which is the Doeblin condition we were looking for.

8. Positive speed regime

Our aim for this section is to show that

Theorem 8.1. *If $E_*[c_*] < \infty$, we have,*

$$
\max_{a \in [1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\Delta_n \mid D = \infty] \leq C(K)n,
$$

for any $K \geq K_0$ for some K_0.

Used in combination with the existence of a law of large numbers provided by the existence of a regeneration structure this will allow us to prove the positivity of the speed if $E_*[c_*] < \infty$. Let us enumerate the keep points for proving the previous result.

1. The number en visits to a good point is bounded (see Lemma 8.1). This limits the expected number of entries in a trap to, roughly, the size of its border.
2. The time spent during one visit to a trap is linked to its size and the conductances in that trap (see Lemma 8.3). It is already know by Lemma 5.1 that the size of traps is extremely small, so we may neglect this effect.

(3) The conductances in traps are, relatively, similar to usual conductances. In particular, they do not have infinite expectation and cannot force zero-speed (see Lemma 8.4).

This reasoning allows us to say that, essentially, Δ_n should be of the same order as the number of sites visited before Δ_n, since there is no local trapping. More precisely, we get an upper-bound of $\mathbb{E}[\Delta_n]$ in terms of the number of sites visited in a regeneration box (see Lemma 8.5). The last step of the proof is to estimate the probability that we reach x during the first regeneration time, see Lemma 8.6.

We proceed to give the details associated with the previous outline. Firstly, we notice

Lemma 8.1. *For any $x \in GOOD_K(\omega)$ we have*

$$E_x^\omega\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big] \leq C(K) < \infty.$$

Proof. We see that

$$E_x^\omega\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big] = \frac{1}{P_x^\omega[T_x^+ = \infty]} = \frac{\pi^\omega(x)}{C^\omega(x \leftrightarrow \infty)},$$

where $C^\omega(x \leftrightarrow \infty)$ is the effective conductance between x and infinity in ω. Since $x \in \mathrm{GOOD}_K$, we can upper-bound $\pi^\omega(x)$ using Remark 5.2 and we may use Rayleigh's monotonicity principle (see [22])to see that

$$C^\omega(x \leftrightarrow \infty) \geq \frac{c}{K} \sum_{i \geq 0} c^\omega(p_i) \geq c \exp(2\lambda x \cdot \vec{\ell}),$$

where $(p_i)_{i\geq 0}$ is a directed path of open points starting at x. This yields the result. □

8.1. **Time spent in traps.** For $x \in \partial\mathrm{BAD}(\omega)$, we define $\mathrm{BAD}_x^{\mathrm{s}}(K) = \{x\} \cup \cup_{y\sim x} \mathrm{BAD}_K(y)$ the union of all bad areas adjacent to x. It follows from Lemma 5.1 that

Lemma 8.2. *For $x \in \partial BAD(\omega)$, we have that $BAD_x^s(K)$ is finite $\mathbf{P}$-a.s. and*

$$\mathbf{P}_p[W(BAD_x^s(K)) \geq n] \leq C \exp(-\xi_1(K) n),$$

where $\xi_1(K) \to \infty$ as K tends to infinity.

We have

Lemma 8.3. *For any $x \in \partial BAD(\omega)$ we have that*

$$E_x^\omega[T^+_{GOOD(\omega)}] \leq C(K) \exp(3\lambda\, |\partial BAD_x^s(\omega)|) \Big(1 + \sum_{e \in E(BAD_x^s)} c_*^\omega(e)\Big).$$

Proof. The first remark to be made is that since $x \in \partial \mathrm{BAD}(\omega) \subset \mathrm{GOOD}(\omega)$, all $y \sim x$ then $c_*([x,y]) \in [1/K, K]$.

We introduce the notation $\mathrm{BAD}_x^{\mathrm{ss}}(K) = \mathrm{BAD}_x^{\mathrm{s}}(K) \setminus \{x\}$.

Let us consider the finite network obtained by taking $\mathrm{BAD}_x^{\mathrm{ss}}(\omega) \cup \partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)$ and merging all points of $\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)$ (which contains x) to one point δ. We denote ω_δ the resulting graph which is obviously finite by Lemma 8.2 and connected. We may apply the mean return formula at δ (see [22] exercise 2.33) to obtain that

$$E_\delta^{\omega_\delta}[T_\delta^+] = 2\frac{\sum_{e\in E(\omega_\delta)} c(e)}{\pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{ss}}(\delta)}} = 2\frac{\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{ss}})} c(e) + \pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{ss}}(\delta)}}{\pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{ss}}(\delta)}}.$$

For y a neighbor of δ in ω_δ, we have by Remark 5.2

$$c\exp\Big(2\lambda \min_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot \vec{\ell}\Big) \le c^{\omega_\delta}([\delta,y]) \le C\exp\Big(2\lambda \max_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot \vec{\ell}\Big), \tag{8.1}$$

so that we know that

$$c\exp\Big(2\lambda \max_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot \vec{\ell}\Big) \le \pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{s}}}(\delta) \le C\,|\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)|\exp\Big(2\lambda \max_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot \vec{\ell}\Big). \tag{8.2}$$

Using (8.1) and (8.2) and Remark 5.2

$$\text{for } e\in E(\mathrm{BAD}_x^{\mathrm{ss}}),\ \frac{c(e)}{\pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{ss}}(\delta)}} \le Cc_*(e),$$

which means that

$$E_\delta^{\omega_\delta}[T_\delta^+] \le C\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{ss}})} c_*(e) + C. \tag{8.3}$$

The transition probabilities of the random walk in ω_δ at any point different from δ are the same as that of the walk in ω. This implies that

$$\begin{aligned} E_\delta^{\omega_\delta}[T_\delta^+] &= \sum_{y\sim\delta} P_\delta^{\omega_\delta}[X_1=y]E_y^\omega[T_{\partial \mathrm{BAD}_x^{\mathrm{ss}}}] \\ &= \sum_{y\in \mathrm{BAD}_x^{\mathrm{ss}},\ y\sim\partial \mathrm{BAD}_x^{\mathrm{ss}}} P_\delta^{\omega_\delta}[X_1=y]E_y^\omega[T_{\partial \mathrm{BAD}_x^{\mathrm{ss}}}]. \end{aligned} \tag{8.4}$$

Moreover by (8.1) and (8.2), we have

$$P_\delta^{\omega_\delta}[X_1=y] = \frac{c^{\omega_\delta}([\delta,y])}{\pi^{\omega_\delta}_{\mathrm{BAD}_x^{\mathrm{ss}}}(\delta)} \ge c\frac{\exp(2\lambda \min_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell})}{|\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)|\exp(2\lambda \max_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell})},$$

and,since $\mathrm{BAD}_x^{\mathrm{ss}} \cup \partial \mathrm{BAD}_x^{\mathrm{ss}}$ is connected, we have

$$\max_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}(\delta)} y\cdot\vec{\ell} - \min_{y\in\partial \mathrm{BAD}_x^{\mathrm{ss}}} y\cdot\vec{\ell} \le |\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)|\,,$$

so that

$$P_\delta^{\omega_\delta}[X_1=y]\geq c\,|\partial \mathrm{BAD}_x^{\mathrm{ss}}(\omega)|^{-1}\exp(-2\lambda\,|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)|).$$

This, with (8.3) and (8.4), and considering the fact that $\partial\mathrm{BAD}_x^{\mathrm{ss}}\subset\mathrm{GOOD}$ yields

$$\max_{y\in\mathrm{BAD}_x^{\mathrm{ss}},\ y\sim\partial\mathrm{BAD}_x^{\mathrm{ss}}} E_y^\omega[T_{\mathrm{GOOD}(\omega)}]\leq C\exp(3\lambda\,|\partial\mathrm{BAD}_x^{\mathrm{ss}}(\omega)|)\Big(1+\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{ss}})}c_*(e)\Big).$$

So

$$E_x^\omega[T^+_{\mathrm{GOOD}(\omega)}]\leq C\exp(3\lambda\,|\partial\mathrm{BAD}_x^{\mathrm{s}}(\omega)|)\Big(1+\sum_{e\in E(\mathrm{BAD}_x^{\mathrm{s}})}c_*(e)\Big).$$

□

8.2. **Conductances in traps.** Let us understand, partially, how the conductances in traps are conditioned.

Lemma 8.4. *Take $n\geq 0$ and $K\geq 1$, $F\subset E(\mathbb{Z}^d)$ such that $0\notin V(F)$, and $e\in F$. If $E_*[c_*]<\infty$ then*

$$\begin{aligned}&\mathbf{E}[\mathbf{1}\{E(BAD_x^s(K))=F\}P^\omega[T_{V(F)}\leq\Delta_n]c_*(e)]\\ \leq&C(K)\mathbf{E}[\mathbf{1}\{E(BAD_x^s(K))=F\}P^\omega[T_{V(F)}\leq\Delta_n]].\end{aligned}$$

If $\lim\frac{\ln P_*[c_*>n]}{\ln n}=-\gamma$ with $\gamma<1$ then for any $\varepsilon>0$ we have

$$\mathbf{E}[\mathbf{1}\{E(BAD_x^s(K))=F\}c_*(e)^{\gamma-\varepsilon}]\leq C(K)\mathbf{E}[\mathbf{1}\{E(BAD_x^s(K))=F\}].$$

We introduce the notation $\omega^{e\text{ an}}$ to signify that for $e'\in E(\mathbb{Z}^d)\setminus\{e\}$,

$$\mathbf{1}\{e'\text{ is abnormal}\}(\omega^{e\text{ an}})=\mathbf{1}\{e'\text{ is abnormal}\}(\omega)$$

and

$$\mathbf{1}\{e\text{ is abnormal}\}(\omega^{e\text{ an}})=1.$$

Proof. Firstly, let us notice that if there exists M such that $P[c_*<M]=1$, then we may obtain the first part of the lemma with $C=M$. We will now assume that $P[c_*>M]>0$ for any M.

Take $F\subset E(\mathbb{Z}^d)$ such that $x\in F$ and $0\notin F$. For any $e\in F$.

(8.5)

$$\begin{aligned}&\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\leq\Delta_n]c_*(e)]\\
\leq&K\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\leq\Delta_n]]\\
&\qquad+\mathbf{E}[c_*(e)\mathbf{1}\{c_*(e)>K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\leq\Delta_n]]\\
=&K\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\leq\Delta_n]]\\
&\qquad+\mathbf{E}[c_*(e)\mathbf{1}\{c_*(e)>K\}\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}}(\omega^{e\text{ an}}))=F\}P^\omega[T_{V(F)}\leq\Delta_n]]\\
\leq&K\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}_x^{\mathrm{s}})=F\}P^\omega[T_{V(F)}\leq\Delta_n]]\end{aligned}$$

$$+\mathbf{E}[c_*(e)\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}P^\omega[T_{V(F)}\le\Delta_n]].$$

Using the fact that $c_*(e)$ is independent of $\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}$ and $P^\omega[T_{V(F)}\le\Delta_n]$ since $0\notin V(F)$ and $e\in F$. Hence

(8.6)
$$\begin{aligned}&\mathbf{E}[c_*(e)\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}P^\omega[T_{V(F)}\le\Delta_n]]\\ \le&\mathbf{E}[c_*(e)]\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}P^\omega[T_{V(F)}\le\Delta_n]].\end{aligned}$$

We can use this same independence property again to write

(8.7)
$$\begin{aligned}&\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}P^\omega[T_{V(F)}\le\Delta_n]]\\ \le&\frac{1}{P_*[c_*(e)>K]}\mathbf{E}[\mathbf{1}\{c_*(e)>K\}\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x(\omega^{e\ \mathrm{an}}))=F\}P^\omega[T_{V(F)}\le\Delta_n]]\\ \le&\frac{1}{P_*[c_*(e)>K]}\mathbf{E}[\mathbf{1}\{c_*(e)>K\}\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x)=F\}P^\omega[T_{V(F)}\le\Delta_n]]\\ \le&\frac{1}{P_*[c_*(e)>K]}\mathbf{E}[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x)=F\}P^\omega[T_{V(F)}\le\Delta_n]].\end{aligned}$$

Putting together (8.5), (8.6) and (8.7) proves the first part of the result. The second part can be handled using exactly the same techniques. □

8.3. **Proof of Theorem 8.1.**

Lemma 8.5. *There exists K_0 such that, for any $K\ge K_0$, we have*

$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{E}[\Delta_n\mid D=\infty]\le C(K)n\max_{a\in[1/K,K]^{\mathcal{E}}}\Big(\sum_{x\in\mathbb{Z}^d}\mathbb{P}^a[T_x\le\tau_1]^{1/2}\Big).$$

Proof. If $0\in\mathrm{GOOD}(\omega)$, then a walk started at 0 can only be in a vertex of $\mathrm{BAD}(\omega)$ between visits to $\partial\mathrm{BAD}(\omega)$. Hence, on $\{0\in\mathrm{GOOD}(\omega)\}$,

$$\sum_{x\in\mathrm{BAD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i=x\}\le\sum_{x\in\partial\mathrm{BAD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i.$$

Hence, on $\{0\in\mathrm{GOOD}(\omega)\}$, since $\mathbb{Z}^d$ is partitioned in two parts $\mathrm{GOOD}(\omega)$ and $\mathrm{BAD}(\omega)$ we have

(8.8)
$$\begin{aligned}\Delta_n&\le\sum_{x\in\mathbb{Z}^d}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i=x\}\\ &\le\sum_{x\in\mathrm{GOOD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i=x\}+\sum_{x\in\mathrm{BAD}(\omega)}\sum_{i=0}^{\Delta_n}\mathbf{1}\{X_i=x\}\end{aligned}$$

$$\leq \sum_{x\in \mathrm{GOOD}(\omega)} \sum_{i=0}^{\Delta_n} \mathbf{1}\{X_i = x\} + \sum_{x\in \partial \mathrm{BAD}(\omega)} \sum_{i=0}^{\Delta_n} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i.$$

Hence, on $\{0 \in \mathrm{GOOD}(\omega)\}$,

$$\Delta_n \leq \sum_{x\in \mathrm{GOOD}(\omega)} \mathbf{1}\{T_x \leq \Delta_n\} \sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\} + \sum_{x\in \partial \mathrm{BAD}(\omega)} \mathbf{1}\{T_x \leq \Delta_n\} \sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i. \tag{8.9}$$

We can use Markov's property to say that for any $x \in \mathbb{Z}^d$, on $\{0 \in \mathrm{GOOD}(\omega)\}$

$$E^\omega\Big[\mathbf{1}\{T_x \leq \Delta_n\} \sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big] = P^\omega[T_x \leq \Delta_n] E^\omega_x\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big],$$

and

$$E^\omega\Big[\mathbf{1}\{T_x \leq \Delta_n\} \sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i\Big] = P^\omega[T_x \leq \Delta_n] E^\omega_x\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i\Big].$$

This implies, using (8.9), that on $\{0 \in \mathrm{GOOD}(\omega)\}$

$$E^\omega[\Delta_n] \leq \sum_{x\in \mathrm{GOOD}(\omega)} P^\omega[T_x \leq \Delta_n] E^\omega_x\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big] + \sum_{x\in \partial \mathrm{BAD}(\omega)} P^\omega[T_x \leq \Delta_n] E^\omega_x\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\mathrm{GOOD}(\omega)} \circ \theta_i\Big],$$

Now, we have

$$\begin{aligned} &\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\Delta_n \mid D = \infty] \\ \leq &\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[D = \infty]^{-1} \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}[\mathbf{1}\{0 \in \mathrm{GOOD}(\omega)\}\Delta_n] \\ \leq &C \max_{a\in[1/K,K]^{\mathcal{E}}} \Big[\mathbf{E}^a\Big[\sum_{x\in \mathrm{GOOD}(\omega)} P^\omega[T_x \leq \Delta_n] E^\omega_x\Big[\sum_{i=0}^{\infty} \mathbf{1}\{X_i = x\}\Big]\Big] \end{aligned} \tag{8.10}$$

$$+\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{0\notin \mathrm{BAD}\}\sum_{x\in\partial\mathrm{BAD}(\omega)}P^\omega[T_x\le\Delta_n]E^\omega_x\Big[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Big]\Big]\Big],$$

by Lemma 7.5. So that, using Lemma 8.1,

$$\begin{aligned}&\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{E}^a[\Delta_n\mid D=\infty]\\ \le& C\Big[\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{E}^a\Big[\sum_{x\in\mathbb{Z}^d}\mathbf{1}\{T_x\le\Delta_n\}\Big]\\ &+\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\sum_{x\in\partial\mathrm{BAD}(\omega)}P^\omega[T_x\le\Delta_n]E^\omega_x\Big[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Big]\Big]\Big].\end{aligned}$$

Let us focus, for now, on the second term. By Markov's property for $x\in\partial\mathrm{BAD}(\omega)$

$$\begin{aligned}E^\omega_x\Big[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Big]&=\sum_{i=0}^\infty E^\omega_x[\mathbf{1}\{X_i=x\}]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\\ &=\Big(\sum_{i=0}^\infty E^\omega_x[\mathbf{1}\{X_i=x\}]\Big)E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\\ &\le CE^\omega_x[T^+_{\mathrm{GOOD}(\omega)}],\end{aligned}$$

where we used Lemma 8.1. Hence

(8.11)

$$\begin{aligned}&\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{0\notin \mathrm{BAD}\}\sum_{x\in\partial\mathrm{BAD}(\omega)}P^\omega[T_x\le\Delta_n]E^\omega_x\Big[\sum_{i=0}^\infty \mathbf{1}\{X_i=x\}T^+_{\mathrm{GOOD}(\omega)}\circ\theta_i\Big]\Big]\\ \le& C\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{0\notin \mathrm{BAD}\}\sum_{x\in\partial\mathrm{BAD}(\omega)}P^\omega[T_x\le\Delta_n]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\Big]\\ \le& C\sum_{x\in\mathbb{Z}^d}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}\mathbf{1}\{0\notin \mathrm{BAD}\}P^\omega[T_x\le\Delta_n]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\Big]\\ \le& C\sum_{x\in\mathbb{Z}^d}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}\mathbf{1}\{0\notin \mathrm{BAD}\}P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\le\Delta_n]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\Big],\end{aligned}$$

where we used that for $x\in\partial\mathrm{BAD}(\omega)$, we have $x\in\mathrm{BAD}^{\mathrm{s}}_x$ by definition.

When $x\notin\partial\mathrm{BAD}$ we use the notation $\mathrm{BAD}^{\mathrm{s}}_x=\partial\mathrm{BAD}^{\mathrm{s}}_x=\{x\}$. Using Lemma 8.3

(8.12)

$$\begin{aligned}&\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\big[\mathbf{1}\{0\notin \mathrm{BAD}\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\le\Delta_n]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\big]\\ \le& C\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{0\notin \mathrm{BAD}\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\le\Delta_n]\exp(3\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|)\end{aligned}$$

$$\Big(1+\sum_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x)} c_*(e)\Big)\Big]$$

$$\leq \sum_{\substack{F\subset E(\mathbb{Z}^d)\\ x\in V(F),0\notin V(F)}} C\exp(3\lambda\,|\partial V(F)|)\sum_{e\in E(F)}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\Big[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x)=F\}P^\omega[T_{V(F)}\leq\Delta_n]$$

$$\big(1+c_*(e)\big)\Big]$$

now by Lemma 8.4 and using the fact that $E_*[c_*]<\infty$

(8.13)
$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x)=F\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}P^\omega[T_{V(F)}\leq\Delta_n]c_*(e)\big]$$
$$\leq C\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\big[\mathbf{1}\{E(\mathrm{BAD}^{\mathrm{s}}_x)=F\}P^\omega[T_{V(F)}\leq\Delta_n]\big],$$

hence with (8.12) we have

(8.14)
$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\big[\mathbf{1}\{0\notin\mathrm{BAD}\}\mathbf{1}\{x\in\partial\mathrm{BAD}(\omega)\}P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n]E^\omega_x[T^+_{\mathrm{GOOD}(\omega)}]\big]$$
$$\leq C\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbf{E}^a\big[\exp(3\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|^d\,P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n]\big],$$

where we used that $|E(\mathrm{BAD}^{\mathrm{s}}_x)|\leq C\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|^d$.

Using (8.11), (8.14) and (8.10) this means

$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{E}^a[\Delta_n\mid D=\infty]$$
$$\leq C\max_{a\in[1/K,K]^{\mathcal{E}}}\Big[\sum_{x\in\mathbb{Z}^d}\mathbb{P}^a[T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n]+\sum_{x\in\mathbb{Z}^d}\mathbf{E}^a\big[\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)P^\omega[T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n]\big]\Big]$$
$$\leq C\max_{a\in[1/K,K]^{\mathcal{E}}}\sum_{x\in\mathbb{Z}^d}\mathbb{E}^a\big[\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n\}\big]$$
$$\leq C\max_{a\in[1/K,K]^{\mathcal{E}}}\sum_{x\in\mathbb{Z}^d}\mathbb{E}^a\big[|\partial\mathrm{BAD}^{\mathrm{s}}_x|^d\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_x\leq\Delta_n\}\big],$$

where we used that $\sum_{x\in\mathrm{BAD}^{\mathrm{s}}_x}\mathbf{1}\{T_{\mathrm{BAD}^{\mathrm{s}}_x}\leq\Delta_n\}\leq|\mathrm{BAD}^{\mathrm{s}}_x|\sum_{x\in\mathrm{BAD}^{\mathrm{s}}_x}\mathbf{1}\{T_x\leq\Delta_n\}$.

Since $\tau_n\geq\Delta_n$ by (7.3), using the notation $\tau_0=0$, we have

$$\max_{a\in[1/K,K]^{\mathcal{E}}}\sum_{x\in\mathbb{Z}^d}\mathbb{E}^a\big[|\partial\mathrm{BAD}^{\mathrm{s}}_x|^d\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_x\leq\Delta_n\}\big]$$
$$\leq\max_{a\in[1/K,K]^{\mathcal{E}}}\sum_{x\in\mathbb{Z}^d}\mathbb{E}^a\big[|\partial\mathrm{BAD}^{\mathrm{s}}_x|^d\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_x\leq\tau_n\}\big]$$

$$\leq C \max_{a\in[1/K,K]^{\mathcal{E}}} \sum_{i=0}^{n-1} \mathbb{E}^a\Big[\sum_{x\in\mathbb{Z}^d} \exp(5\lambda\,|\partial \mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_x\in[\tau_i,\tau_{i+1}]\}\Big].$$

We can use Minkowski's inequality which implies for any $a \in [1/K,K]^{\mathcal{E}}$ that

$$\begin{aligned}&\mathbb{E}^a\Big[\exp(5\lambda\,|\partial \mathrm{BAD}^{\mathrm{s}}_x|)\mathbf{1}\{T_x\in[\tau_i,\tau_{i+1}]\}\Big]\\ \leq &\mathbf{E}^a\Big[\exp(10\lambda\,|\partial \mathrm{BAD}^{\mathrm{s}}_x|)\Big]^{1/2}\mathbb{P}^a[T_x\in[\tau_i,\tau_{i+1}]^{1/2}\\ \leq &C\mathbb{P}^a[T_x\in[\tau_i,\tau_{i+1}]^{1/2},\end{aligned}$$

where we used Lemma 8.2.

The three last equations and Theorem 7.4 imply that

$$\begin{aligned}\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\Delta_n \mid D=\infty] &\leq C\sum_{i=0}^{n-1}\sum_{x\in\mathbb{Z}^d}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[T_x\in[\tau_i,\tau_{i+1}]]^{1/2}\\ &\leq Cn\sum_{x\in\mathbb{Z}^d}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[T_x\leq\tau_1]]^{1/2},\end{aligned}$$

where we used Lemma 7.5. □

Let us estimate $\mathbb{P}^a[T_x \leq \tau_1^{(K)}]$.

Lemma 8.6. *We have for any $x\in\mathbb{Z}^d$ then for any $M<\infty$, there exists K_0 such that for any $K\geq K_0$*

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_x\leq \tau_1^{(K)}] \leq Cn^{-M}.$$

Proof. Denote χ the smallest integer so that $\{X_i,\ i\in[0,\tau_1]\}\subset B(\chi,\chi^\alpha)$. First let us notice that

(8.15)

$$\begin{aligned}\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[\chi\geq k]\leq &\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[X_{\tau_1}\cdot\vec{\ell}\geq k]\\ &+\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[X_{\tau_1}\cdot\vec{\ell}\leq k,\ \max_{0\leq i,j\leq\tau_1}\max_{l\in[2,d]}|(X_j-X_i)\cdot f_l|\geq k^\alpha].\end{aligned}$$

We can upper-bound the first term as follows

$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[X_{\tau_1^{(K)}}\cdot\vec{\ell}\geq k]\leq Cn^{-M},$$

for any M by choosing K large enough by Theorem 7.2.

The second term can be upper-bounded with the following reasoning: on the event that $X_{\tau_1}\cdot\vec{\ell}\leq k$ and $\max_{j\neq 1}\max_{0\leq j_1,j_2\leq\tau_1}|(X_{j_1}-X_{j_2})\cdot f_j|\geq k^\alpha$, X_n does not exit the box $B(k,k^\alpha)$ through $\partial^+B(k,k^\alpha)$, this means

$$\max_{a\in[1/K,K]^{\mathcal{E}}}\mathbb{P}^a[X_{\tau_1}\cdot\vec{\ell}\leq k,\ \max_{0\leq i,j\leq\tau_1}\max_{l\in[2,d]}|(X_j-X_i)\cdot f_l|]$$

$$\leq \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_{\partial B(k,k^\alpha)} \neq T_{\partial^+ B(k,k^\alpha)}] \leq ce^{-ck},$$

by Theorem 7.3.

This turns (8.15) into

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[\chi \geq k] \leq Ck^{-M}, \tag{8.16}$$

for any M for K large enough.

Now assume that $\{T_x \leq \tau_1\}$ then $B(\chi,\chi^\alpha) \not\subseteq B_{\mathbb{Z}^d}(0,|x|)$, so

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[T_{\text{BAD}_x^s} \leq \tau_1] \leq \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}[B(\chi,\chi^\alpha) \not\subseteq B_{\mathbb{Z}^d}(0,|x|)]$$
$$\leq C\,|x|^{-M/\alpha},$$

which proves the lemma, since α is fixed and M is arbitrary. □

We can now prove Theorem 8.1

Proof. By choosing $K \geq K_0$, we may apply Lemma 8.5 and Lemma 8.6 with $M > 2d$ to see that

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\Delta_n] \leq Cn \sum_{x\in\mathbb{Z}^d} C\,|x|^{-M/2} < Cn,$$

which proves Theorem 8.1. □

8.4. **Law of large numbers.** We can use exactly the same type of proof as in [25] to obtain

Proposition 8.1. *If $E_*[c_*] < \infty$, then there exists K_0 such that for any $K \geq K_0$ we have*

$$\frac{X_n}{n} \to v = \frac{\mathbb{E}^{\Pi_K}[X_{\tau_1^{(K)}}]}{\mathbb{E}^{\Pi_K}[\tau_1^{(K)}]} \qquad \mathbb{P}\text{-a.s. with } v\cdot\vec{\ell} > 0,$$

where

$$\Pi_K[\cdot] = \int \nu_K(da)\mathbb{P}_0^a[\,\cdot \mid D_K = \infty] \text{ and } \mathbb{E}^{\Pi_K}[\cdot] := \int \nu_K(da)\mathbb{E}^a[\,\cdot \mid D_K = \infty],$$

where ν_K is the unique invariant distribution on $[1/K,K]^{\mathcal{E}}$ given in Theorem 7.5.

Proof. Firstly, we notice that by (8.16), we have $\mathbb{E}^{\Pi}[|X_{\tau_1}|] < \infty$. Then, one may follow the strategy of proof of Theorem 5.1 in [25] to obtain the result. For the convenience of the reader, we recall that the notations necessary to understand the proof in [25] were defined in Proposition 7.2, Theorem 7.5 and

Theorem 7.6 of this paper. This allows us to prove that $\mathbb{P}-a.s.$ (or $\mathbb{P}^a$-a.s. for $a \in [1/K, K]$)

$$\lim_{n\to\infty} \frac{X_n}{n} = v = \frac{\mathbb{E}^\Pi[X_{\tau_1}]}{\mathbb{E}^\Pi[\tau_1]}, \tag{8.17}$$

and in a similar fashion

$$\lim_{n\to\infty} \frac{\Delta_n}{n} = \lim_{n\to\infty} \frac{\Delta_n \circ \tau_1}{n} = \frac{\mathbb{E}^\Pi[\tau_1]}{\mathbb{E}^\Pi[X_{\tau_1} \cdot \vec{\ell}]}. \tag{8.18}$$

where the two previous equations remain true even if $\mathbb{E}^\Pi[\tau_1] = \infty$.

We may see that (8.18) and Theorem 8.1 imply that $\mathbb{E}^\Pi[\tau_1] < \infty$, if $E_*[c_*] < \infty$. Since $X_{\tau_1} \cdot \vec{\ell} > 2/\sqrt{d}$, this means that (8.17) implies that $v \cdot \vec{\ell} > 0$. □

Remark 8.1. *Interestingly, we do not know of any direct way of showing that* $\mathbb{E}^\Pi[\tau_1] < \infty$.

9. Zero-speed regime

9.1. **Characterization of the zero-speed regime.** We set $\mathcal{A}$ to be the set of vertices: 0, e_1, e_1+e_i, $2e_1+e_i$, $2e_1+2e_i$, $3e_1+2e_i$, $3e_1+e_i$, $4e_1+e_i$, for all $i \in [2, 2d-1]$ and

$$A = \{\text{any } x \in \mathcal{A} \text{ is } 4e_1\text{-open}\} \text{ and } B = \{4e_1 \text{ is good}\},$$

where a vertex is x-open if it is open in ω_x coinciding with ω on all edges but those that are adjacent to x which are normal in ω_x.

Note that

(1) on $A \cap B$, the vertex 0 is good,
(2) A and B are independent of $c_*([2e_1, 3e_1])$.

Lemma 9.1. *If* $E_*[c_*] = \infty$, *then* $\min_{a\in[1/K,K]^\varepsilon} E^a[\tau_1 \mid D = \infty] = \infty$.

The typical configuration that will slow the walk down is depicted in Figure 5: the walk is likely to reach the edge $[2e_1, 3e_1]$ and then stay there for a long time. Moreover, we may notice that this picture is compatible with the conditioning $\{D = \infty\}$ when $4e_1$ is good.

Proof. Since, on $A \cap B$, the vertex 0 is good, we have

$$\mathbb{E}^a[\tau_1 \mid D = \infty] \geq \mathbb{E}^a[\mathbf{1}\{D = \infty\}\tau_1] \geq c\mathbb{E}^a[\mathbf{1}\{A, B\}\mathbf{1}\{D' = \infty\}\tau_1]. \tag{9.1}$$

On A, we see, by Remark 5.2, that we have

$$P^\omega[X_1 = e_1, X_2 = 2e_1, X_3 = e_1, X_4 = 2e_1] \geq \kappa_0^4.$$

Using Remark 5.2 again, we may see that on A, we have $P^\omega_{2e_1}[X_1 \neq 3e_1] \leq C/c_*([2e_1, 3e_1])$ and $P_{3e_1}[X_1 \neq 2e_1] \leq C/c_*([2e_1, 3e_1])$. This implies that

$$P^\omega_{2e_1}[T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}} \geq n] \geq (1 - C/c_*([2e_1, 3e_1]))^n, \tag{9.2}$$

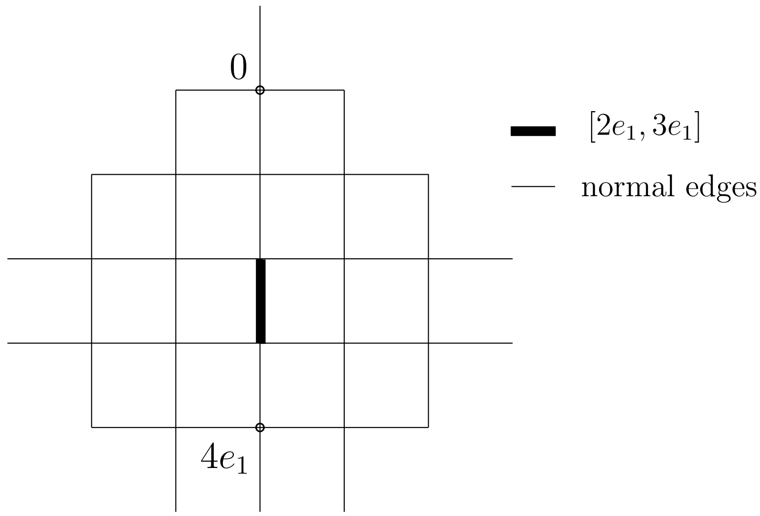

FIGURE 5. The typical configurations slowing the walk down

so

$$E^{\omega}_{2e_1}[T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}] \geq cc_*([2e_1,3e_1]). \tag{9.3}$$

Also on A, we may notice that for any neighbor of $2e_1$ or $3e_1$, there exists an open nearest-neighbor path of length at most 5 in $\mathcal{A}\setminus\{0\}$ to $4e_1$. Using Remark 5.2, this implies that

$$P^{\omega}_{2e_1}[T_{4e_1}\circ\theta_{T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}\circ\theta_4} < T_{\partial\mathcal{A}\setminus\{0\}}\circ\theta_{T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}\circ\theta_4}] \geq \kappa_0^5. \tag{9.4}$$

We may notice that on A, if

(1) $X_1 = e_1, X_2 = 2e_1, X_3 = e_1, X_4 = 2e_1$, (hence $\tau_1 \geq 4$)
(2) $T_{4e_1}\circ\theta_{T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}\circ\theta_4} < T_{\partial\mathcal{A}\setminus\{0\}}\circ\theta_{T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}\circ\theta_4}$,
(3) $D'\circ\theta_{T_{4e_1}} = \infty$,

then we have $D' = \infty$ and $\tau_1 \geq T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}\circ\theta_4$. Using this, Markov's property twice along with (9.4) and (9.3) we may see that

(9.5)
$$\begin{aligned}\mathbb{E}^a[\mathbf{1}\{A,B\}\mathbf{1}\{D'=\infty\}\tau_1] \geq & c\mathbf{E}^a[\mathbf{1}\{A,B\}E^{\omega}_{2e_1}[T_{\mathbb{Z}^d\setminus\{2e_1,3e_1\}}]P^{\omega}_{4e_1}[D'=\infty]] \\ \geq & c\mathbf{E}^a[\mathbf{1}\{A,B\}c_*([2e_1,3e_1])P^{\omega}_{4e_1}[D'=\infty]].\end{aligned}$$

We may now notice that $\mathbf{1}\{A\}$, $c_*([2e_1,3e_1])$ and $\mathbf{1}\{B\}P^{\omega}_{4e_1}[D'=\infty]$ are $\mathbf{P}^a$-independent, so that

$$\mathbb{E}^a[\mathbf{1}\{A,B\}\mathbf{1}\{D'=\infty\}\tau_1] \geq \mathbf{P}^a_0[A]\mathbf{E}^a[c_*([2e_1,3e_1])]\mathbf{E}^a[\mathbf{1}\{B\}P^{\omega}_{4e_1}[D'=\infty]].$$

We have $\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbf{P}^a[A] \geq c > 0$, $\mathbf{E}^a[c_*([2e_1,3e_1])] = \mathbf{E}[c_*([2e_1,3e_1])] = \infty$ by translation invariance and

$$\mathbf{E}^a[\mathbf{1}\{B\}P^{\omega}_{4e_1}[D' = \infty]] = \mathbf{E}[\mathbf{1}\{0 \text{ is good}\}P^{\omega}[D' = \infty]] > 0,$$

by Lemma 7.5 and the fact that $\mathbf{P}[0 \text{ is good}] > 0$.

Hence, by (9.1) and (9.5), we have

$$\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[\tau_1 \mid 0-\text{regen}] = \infty.$$

□

Remark 9.1. *Using a reasoning similar to the previous proof but using a normal edge surrounded by edges with small conductances (see Figure 1), we may show that if* $P_*[c_* \leq x] \geq c\ln(x)^{-\varepsilon}$ *for any* $\varepsilon > 0$*, then*

$$\mathbb{E}_0[\tau_1 \ln(\tau_1)^{\varepsilon} \mid 0 - \textit{regen}] = \infty,$$

for any $\varepsilon > 0$*. Essentially, without any assumption on the tail of* c_* *at 0, we cannot expect any stronger integrability of regeneration times than the first moment being finite.*

This implies

Proposition 9.1. *If* $E_*[c_*] = \infty$*, then* $\lim X_n/n = \vec{0}$ $\mathbb{P}$*-a.s.*

Proof. Using Theorem 7.2

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{E}^a[X_{\tau_1^{(K)}} \cdot \vec{\ell}] < \infty.$$

which implies

$$\mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}] < \infty.$$

Because of Theorem 7.6, we may Birkhoff's ergodic theorem (p.341 in [12]) to see that

$$\frac{(X_{\tau_n} - X_{\tau_1}) \cdot \vec{\ell}}{n} \to \mathbb{E}^{\Pi}[X_{\tau_1} \cdot \vec{\ell}] < \infty. \tag{9.6}$$

Now, by Lemma 9.1, we see that

$$\mathbb{E}^{\Pi}[\tau_1] = \infty,$$

which implies, by Birkhoff's ergodic theorem (p. 341 of [12]),that

$$\frac{\tau_n - \tau_1}{n} \to \infty.$$

From here, we may argue as in Theorem 5.1 in [25] to see that

$$\lim \frac{X_n}{n} = \vec{0}.$$

□

9.2. **Lower-bound on the fluctuations of the random walk.**

Lemma 9.2. *If* $-\lim \frac{\ln P_*[c_*>n]}{\ln n} = \gamma < 1$, *we have*

$$\liminf \frac{\ln \Delta_n}{\ln n} \geq 1/\gamma, \qquad \mathbb{P}\text{-}a.s..$$

Proof. Firstly, using (9.6) and the law of large numbers, we see that for c small enough

$$\mathbb{P}[\tau_{cn} \leq \Delta_n] \to 1.$$

Using (9.2) and a reasoning similar to the proof of Lemma 9.1

$$\min_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[\tau_1 \geq n \mid D=\infty] \geq c\mathbf{E}[(1-C/c_*([e_1,2e_1]))^n]$$
$$\geq cP_*[c_* \geq n^{1/(1+\varepsilon_1/2)}] \geq cn^{-(\gamma(1+\varepsilon_1))},$$

for any $\varepsilon_1 > 0$.

Notice that if $\sum_{i=1}^{cn-1}(\tau_{i+1}-\tau_i) \leq n^{\frac{1}{\gamma}-\varepsilon}$ then $\tau_{i+1}-\tau_i \leq n^{\frac{1}{\gamma}-\varepsilon}$ for any $i \leq cn-1$. The previous two equations imply for any $\varepsilon > 0$ we have

$$\mathbb{P}[\Delta_n \leq n^{\frac{1}{\gamma}-\varepsilon}] \leq \mathbb{P}\Big[\sum_{i=1}^{cn-1}(\tau_{i+1}-\tau_i) \leq n^{\frac{1}{\gamma}-\varepsilon}\Big] + o(1)$$
$$\leq o(1) + C\prod_{i=1}^{cn-1} \max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}_0^a[\tau_1 < cn^{\frac{1}{\gamma}-\varepsilon} \mid D=\infty]$$
$$\leq o(1) + C(1-cn^{-(\frac{1}{\gamma}-\varepsilon)\gamma(1+\varepsilon_1)})^{cn-1},$$

by Theorem 7.4 and Lemma 7.5, for any $\varepsilon_1 > 0$.

Then taking $\varepsilon_1 > 0$ small enough such that $(\frac{1}{\gamma}-\varepsilon)\gamma(1+\varepsilon_1) < 1$, we see that

$$\mathbb{P}[\Delta_n \leq n^{\frac{1}{\gamma}-\varepsilon}] \to 1.$$

This being true for all $\varepsilon > 0$, we have the lemma. □

9.3. **Upper-bound on the fluctuations of the random walk.** We can follow the ideas of Lemma 8.5 to prove the following lemma.

Lemma 9.3. *Assume that* $-\lim \frac{\ln P_*[c_*>n]}{\ln n} = \gamma < 1$. *For any* $\varepsilon > 0$, *there exists* K_0 *such that, for any* $K \geq K_0$

$$\max_{a\in[1/K,K]^{\mathcal{E}}} \mathbb{P}^a[\tau_1^{(K)} > n \mid D=\infty] \leq C(K)n^{-(\gamma-\varepsilon)}$$

Proof. To simplify the notations, we will do the proof for $\mathbb{P}$. In a similar fashion, we could do it for any $\mathbb{P}^a$ for $a \in [1/K,K]^{\mathcal{E}}$.

In this proof, we will point out the K dependence of constants. Fix $\varepsilon > 0$. Denote χ the smallest integer so that $\{X_i,\ i \in [0, \tau_1]\} \subset B(\chi, \chi^\alpha)$. By (8.8), on $\{0-\text{regen}\}$

$$\tau_1 = \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{\tau_1} \mathbf{1}\{X_i = x\}$$
$$\leq \sum_{x \in \text{GOOD}(\omega)} \sum_{i=1}^{\tau_1} \mathbf{1}\{X_i = x\} + \sum_{x \in \partial\text{BAD}(\omega)} \sum_{i=1}^{\tau_1} \mathbf{1}\{X_i = x\} T^+_{\text{GOOD}(\omega)} \circ \theta_i.$$

Recalling the definition of χ at the beginning of the proof of Lemma 8.6. On $\{0-\text{regen}\}$

$$E^\omega[\mathbf{1}\{\chi \leq n^\varepsilon\}\tau_1]$$
$$\leq E^\omega\big[\mathbf{1}\{\chi \leq n^\varepsilon\} \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{\tau_1} \mathbf{1}\{X_i = x\}\big]$$
$$\leq \sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} E^\omega\big[\sum_{i=1}^{\infty} \mathbf{1}\{X_i = x\}\big]$$
$$\leq \sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} \big[\mathbf{1}\{x \in \text{GOOD}(\omega)\} E^\omega\big[\sum_{i=1}^{\infty} \mathbf{1}\{X_i = x\}\big]$$
$$+ \mathbf{1}\{x \in \partial\text{BAD}(\omega)\} E^\omega\big[\sum_{i=1}^{\infty} \mathbf{1}\{X_i = x\} T^+_{\text{GOOD}(\omega)} \circ \theta_i\big]\big].$$

Using Markov's property and Lemma 8.1 we obtain

$$E^\omega[\mathbf{1}\{\chi \leq n^\varepsilon\}\tau_1]$$
$$\leq C(K) \sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} [\mathbf{1}\{x \in \text{GOOD}(\omega)\} + \mathbf{1}\{x \in \partial\text{BAD}(\omega)\} E^\omega_x[T^+_{\text{GOOD}(\omega)}]],$$

and now, since $\gamma < 1$, we have

$$E^\omega[\mathbf{1}\{\chi \leq n^\varepsilon\}\tau_1^{\gamma-\varepsilon}] \leq E^\omega[\mathbf{1}\{\chi \leq n^\varepsilon\}\tau_1]^{\gamma-\varepsilon}$$
$$\leq C(K) \Big[\sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} [\mathbf{1}\{x \in \text{GOOD}(\omega)\} + \mathbf{1}\{x \in \partial\text{BAD}(\omega)\} E^\omega_x[T^+_{\text{GOOD}(\omega)}]]\Big]^{\gamma-\varepsilon}$$

We may now apply Lemma 8.3

$$E^\omega[\mathbf{1}\{\chi \leq n^\varepsilon\}\tau_1^{\gamma-\varepsilon}]$$
$$\leq C(K) \Big[\sum_{x \in B(n^\varepsilon, n^{C\varepsilon})} \mathbf{1}\{x \in \text{GOOD}(\omega)\} + \mathbf{1}\{x \in \partial\text{BAD}(\omega)\} E^\omega_x[T^+_{\text{GOOD}(\omega)}]\Big]^{\gamma-\varepsilon}$$

$$
\begin{aligned}
\leq & C(K)\Big[\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x\in \mathrm{GOOD}(\omega)\} \\
&\quad + \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x\in \partial\mathrm{BAD}(\omega)\}\exp(3\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|)\Big(1+\sum_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))} c_*(e)\Big)\Big]^{\gamma-\varepsilon} \\
\leq & C(K)n^{C\varepsilon}\max_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x\in \partial\mathrm{BAD}(\omega)\}\,|E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))|\exp(3\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|) \\
&\qquad \times\Big(1+\max_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))} c_*(e)^{\gamma-\varepsilon}\Big) \\
\leq & C(K)n^{C\varepsilon}\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x\in \partial\mathrm{BAD}(\omega)\}\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|) \\
&\qquad \times\Big(1+\sum_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))} c_*(e)^{\gamma-\varepsilon}\Big).
\end{aligned}
$$

Now

$$
\begin{aligned}
& \mathbb{E}[\mathbf{1}\{\chi\leq n^{\varepsilon}\}\tau_1^{\gamma-\varepsilon}\mid D=\infty] \\
\leq & C(K)\mathbf{E}\Big[\mathbf{1}\{0\in \mathrm{GOOD}(\omega)\}n^{C\varepsilon}\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{1}\{x\in \partial\mathrm{BAD}(\omega)\}\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|) \\
&\qquad \times\Big(1+\sum_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))} c_*(e)^{\gamma-\varepsilon}\Big)\Big],
\end{aligned}
$$

so using a reasoning similar to (8.12) and (8.13) with Lemma 8.4 yields

$$
\begin{aligned}
& \mathbb{E}[\mathbf{1}\{\chi\leq n^{\varepsilon}\}\tau_1^{\gamma-\varepsilon}\mid D=\infty]/(C(K)n^{C\varepsilon}) \\
\leq & \sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{E}\Big[\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|)\Big(1+\sum_{e\in E(\mathrm{BAD}^{\mathrm{s}}_x(\omega))} c_*(e)^{\gamma-\varepsilon}\Big)\Big] \\
\leq & C(K)(1+E_*[c_*^{\gamma-\varepsilon}])\sum_{x\in B(n^{\varepsilon},n^{C\varepsilon})} \mathbf{E}[|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|^d\exp(3\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|)].
\end{aligned}
$$

Now, we may see that Lemma 8.2 implies that

$$
\mathbf{E}[\mathbf{1}\{x\in \partial\mathrm{BAD}(\omega)\}\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|^d\exp(4\lambda\,|\partial\mathrm{BAD}^{\mathrm{s}}_x(\omega)|)]<C(K)<\infty.
$$

which means that for any $\varepsilon>0$

$$
\mathbb{E}[\mathbf{1}\{\chi\leq n^{\varepsilon}\}\tau_1^{\gamma-\varepsilon}\mid D=\infty]\leq C(K)n^{C\varepsilon}.
$$

From this, using Chebyshev's inequality, we get

$$
\begin{aligned}
\mathbb{P}[\chi\leq n^{\varepsilon},\tau_1>n\mid D=\infty] &= \mathbb{P}[\mathbf{1}\{\chi\leq n^{\varepsilon}\}\tau_1>n\mid D=\infty] \\
&\leq n^{-(\gamma-\varepsilon)}\mathbb{E}[\mathbf{1}\{\chi\leq n^{\varepsilon}\}\tau_1^{\gamma-\varepsilon}\mid D=\infty] \\
&\leq C(K)n^{-(\gamma-\varepsilon)}n^{C_1\varepsilon}.
\end{aligned}
$$

For any $\varepsilon_1 > 0$, we may apply the previous equality for a small ε (which depends only on γ and C_1) to obtain

$$\mathbb{P}[\chi \le n^{\varepsilon}, \tau_1 > n \mid D = \infty] \le C(K) n^{-(\gamma - \varepsilon_1)}.$$

Hence (8.16) and the previous equation imply that for any $\varepsilon_1 > 0$ there exists K large enough, we obtain

$$\begin{aligned}\mathbb{P}[\tau_1^{(K)} > n \mid D = \infty] &\le \mathbb{P}[\chi > n^{\varepsilon} \mid D = \infty] + \mathbb{P}[\chi \le n^{\varepsilon}, \tau_1^{(K)} > n \mid D = \infty] \\ &\le C(K) n^{-(\gamma-\varepsilon_1)},\end{aligned}$$

which proves the lemma. □

Hence, we have

Proposition 9.2. *If* $\lim \frac{\ln P_*[c_* > n]}{\ln n} = -\gamma$ *with* $\gamma < 1$ *then*

$$\limsup \frac{\ln \Delta_n}{\ln n} \le \frac{1}{\gamma}, \qquad \mathbb{P}\text{-}a.s..$$

Proof. Fix $M \ge 1$. We know, by Lemma 9.3 and Theorem 7.4, that there exists K large enough such that for $i \le M + 1$

$$\begin{aligned}\mathbb{E}[\mathrm{card}\{j \le n,\ \tau_j^{(K)} - \tau_{j-1}^{(K)} \ge n^{i/(M\gamma)}\}] =& \mathbb{E}\Big[\sum_{j \le n} 1\{\tau_j^{(K)} - \tau_{j-1}^{(K)} \ge n^{i/(M\gamma)}\}\Big] \\ \le& C n n^{-(i/M)(1-1/M)} = C n^{(1-i/M)+2/M},\end{aligned}$$

where we used Lemma 7.5. Hence by Markov's inequality for any L

$$\begin{aligned}&\mathbb{P}\Big[\mathrm{card}\{j \le n,\ \tau_j^{(K)} - \tau_{j-1}^{(K)} \ge n^{i/(M\gamma)}\} \ge \frac{1}{2M} n^{1/\gamma + L/M - (i+1)/(M\gamma)}\Big] \\ \le& C(M) n^{(1-1/\gamma)(1-i/M) + (1/\gamma + 2 - L)/M}.\end{aligned}$$

Fix $L \ge 1/\gamma + 3$ (which does not depend on M), we denote the event

$$\begin{aligned}&B(n, M) \\ =&\Big\{\mathrm{card}\{j \le n,\ \tau_j^{(K)} - \tau_{j-1}^{(K)} \in (n^{i/(M\gamma)}, n^{(i+1)/(M\gamma)}]\} \ge \frac{1}{2M} n^{1/\gamma + L/M - (i+1)/(M\gamma)}\Big\},\end{aligned}$$

and we get that for any fixed M and $i \le M$

$$\mathbb{P}[B(n, i, M)] \le C n^{-1/M} = o(1), \tag{9.7}$$

since $\gamma \le 1$.

In the same way, by Lemma 9.3 and Theorem 7.4, we get that

$$B(n, M+1, M) = \{\mathrm{card}\{j \le n,\ \tau_j^{(K)} - \tau_{j-1}^{(K)} \ge n^{(M+1)/(M\gamma)}\} \ge 1\},$$

verifies

$$\mathbb{P}[B(n, M+1, M)] \le n^{-\varepsilon} = o(1).$$

Also denoting $B(n,M)=\cup_{j=0}^{M+1}B(n,i,M)$ we have

$$\mathbb{P}[B(n,M)]=o(1).$$

Now, on $B(n,M)^c$, we can give an upper bound for $\tau_n^{(K)}$ by

$$\tau_n^{(K)}\le\sum_{i=0}^{M}n^{(i+1)/(M\gamma)}\Big(\frac{1}{2M}n^{1/\gamma+L/M-(i+1)/(M\gamma)}\Big)=\frac{M+1}{2M}n^{1/\gamma+L/M}.$$

Since by (7.3), we necessarily have $X_{\tau_n^{(K)}}\cdot e_1\ge n$, it follows that $\Delta_n\le\tau_n^{(K)}\le n^{1/\gamma+L/M}$ on $B(n,M)^c$. Hence,

$$\text{on } B(n,M)^c,\qquad \frac{\ln\Delta_n}{\ln n}\le 1/\gamma+L/M.$$

Hence we have proved that for any $M\ge 1$, by (9.7)

$$\limsup\frac{\ln\Delta_n}{\ln n}\le 1/\gamma+L/M,\qquad \mathbb{P}\text{-a.s.},$$

and letting M go to infinity we get the result (we recall that L does not depend on M). □

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Courant Institute of Mathematical Sciences, 251 Mercer Street, New York University, New York, 10012-1185, U.S.A.

E-mail address: `friberg h@cims.nyu.edu`