EXCEPTIONAL SURGERIES ON COMPONENTS OF TWO-BRIDGE LINKS

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ABSTRACT. In this paper, we give a complete classification of exceptional Dehn surgeries on a component of a hyperbolic two-bridge link in the 3-sphere.

1. INTRODUCTION

A Dehn surgery on a link L in a 3-manifold M is defined as an operation as; take the exterior $E(L)$ of L, i.e., remove the interior of the tubular neighborhood $N(L)$ of L from M, and then, glue solid tori to $E(L)$.

One of the motivation to study Dehn surgery is given by the fact [10, Theorem 5.8.2] due to Thurston: On each component of a hyperbolic link, there are only finitely many Dehn surgeries In view of this, a Dehn surgery on a hyperbolic link giving a non-hyperbolic manifold is said to be an *exceptional surgery*.

In the study of exceptional surgery, one of the most important problems, related to Knot theory, is: Completely classify the exceptional surgeries on hyperbolic links in the 3-sphere S^3. This seems to be considerably challenging, and the problem much easier to tackle is to give a complete classification of the exceptional surgeries on some class of links. Along this line, we consider in this paper the hyperbolic *2-bridge links* in S^3.

A link in S^3 is called a *2-bridge link* if it admits a diagram with exactly two maxima and minima. See [5] for more details. We will follow the definition and notations about 2-bridge link from [4, 11]. In the following, we denote by $L_{p/q}$ the 2-bridge link associated to a rational number p/q.

In this paper, we give a complete classification of exceptional surgeries on a component of a hyperbolic two-bridge link in S^3.

To state our result, we set our notation as follows. For a knot K in S^3, by using a standard meridian-longitude system, we have a one-to-one correspondence between the set of slopes on the peripheral torus of K and the set of rational numbers, or equivalently irreducible fractions, with $1/0$. See [8] for example. Let L be a 2-bridge link. We denote $L(r)$ the manifold obtained by Dehn surgery on a component of L along the slope $r \in \mathbb{Q}$, i.e., the rational number r corresponds to the slope determined by the meridian of the attached solid torus.

We here recall the classification of exceptional surgery on a component of a hyperbolic link. A Dehn surgery on one component of a 2-component hyperbolic link is exceptional, i.e., it yields a non-hyperbolic 3-manifold with torus boundary, if and only if the obtained manifold contains an essential disk, annulus, 2-sphere, or torus. See [10] as the original reference.

Now we give our classification theorem as follows.

Theorem. *Let L be a hyperbolic 2-bridge link in S^3 and $L(r)$ denote the 3-manifold obtained by Dehn surgery on a component of L along the slope r. Then the following hold.*

(1) *$L(r)$ contains neither essential disks nor essential 2-spheres.*
(2) *$L(r)$ contains an essential torus if and only if L is equivalent to $L_{[2w,v,2u]}$ and $r = -w - u$ with*
  (a) *$w = 1, u = -1, |v| \geq 2$,*
  (b) *$w \geq 2, |u| \geq 2, |v| = 1$.*
  (c) *$w \geq 2, |u| \geq 2, |v| \geq 2$.*
  In all the cases, $L(r)$ is never Seifert fibered, and $L(r)$ gives a graph manifold if and only if the parameters u, v, w satisfies the first and the second conditions.
(3) *$L(r)$ contains an essential annulus, but contains no essential tori, equivalently $L(r)$ is a small Seifert fibered space if and only if L is equivalent to*
  (a) *$L_{[3,2u+1]}$ and $r = u$,*
  (b) *$L_{[2w+1,3]}$ and $r = -w - 1$,*
  (c) *$L_{[3,-3]}$ and $r = -1$, or,*
  (d) *$L_{[2w+1,2u+1]}$ and $r = -w + u$*
  with $w \geq 1, u \neq 0, -1$.

This theorem will be proved in the last section. As a preliminary, we will give a key lemma in the next section.

We here recall the known results on exceptional surgeries on hyperbolic 2-bridge links. These are the motivation of our study, and actually our proof of the theorem heavily due to the following.

On hyperbolic 2-bridge knots, Brittenham and Wu gave in [1] a complete classification of exceptional surgeries. For example, they showed that only 2-bridge knots $K_{[b_1,b_2]}$ admits exceptional surgeries. Here, by $[a_1, a_2, \cdots, a_n]$, we mean a continued fraction expansion following [4].

For 2-bridge links, it follows from the result obtained by Wu in [11]: If a 3-manifold obtained by a Dehn surgery on a component of a 2-bridge link L contains an essential disk, annulus, or 2-sphere, then L is equivalent to $L_{[b_1,b_2]}$. Recall that an embedded disk, annulus, 2-sphere in a 3-manifold is called *essential* if it is incompressible and not boundary-parallel. We remark that Dehn surgery on a hyperbolic link yielding 3-manifolds with essential disk, annulus, or 2-sphere, is a typical example of exceptional surgery.

Further, in [4], Goda, Hayashi and Song obtained a complete classification (resp. a necessary condition) of 2-bridge links on a component of which a Dehn surgery yields a non-trivial, non-core torus knot exterior or a cable knot exterior (resp. a prime satellite knot exterior) in a lens space.

2. Surfaces in 2-bridge link exterior

To prove our theorem, a key investigation is to study essential surfaces embedded in 2-bridge link exteriors of genus at most one. Most parts of such studies have been achieved in [4], which is based on the machinery of [2]. In this section, we give a lemma which concerns the remaining cases of [4].

Lemma. *If a hyperbolic 2-bridge link exterior contains a meridionally incompressible essential planer surface F with at most two meridional boundaries on a component of the link and non-empty boundary on the other component if and only if the link is equivalent to $L_{[2,n,-2]}$ with $|n| \geq 2$ and F is an essential two punctured disk with two meridional punctures on a component on the link and a single longitudinal boundary on the other component.*

Here a surface F in $E(L)$ is called *meridionally incompressible* if, for any disk $D \subset S^3$ with $D \cap F = \partial D$ and D meeting L transversely in one point in the interior of D, there is a disk $D' \subset F \cup L$ with $\partial D' = \partial D$, D' also meeting L transversely in one interior point.

Actually, in [2], Floyd and Hatcher studied meridionally incompressible essential surfaces in 2-bridge link exteriors, and gave a complete

description of such surfaces. See [2] and [4] for details. In the following, we assume that the readers are familiar to a certain extent.

Proof of Lemma. Let $L = K_1 \cup K_2$ be a hyperbolic 2-bridge link in S^3, and $E(L)$ its exterior.

Suppose that there exists a meridionally incompressible essential planer surface F in $E(L)$ with at most two meridional boundaries on a component of the link, say K_2, and non-empty boundary on the other component K_1. Then, by [2, Theorem 3.1 (a)], the surface F is carried by a branched surface Σ_γ for some minimal edge-path γ in the diagram D_t in [2]. See also [4].

Since F has meridional boundaries only on ∂K_2, we see that the minimal edge-path γ is in D_∞. Moreover, observing the sub-branched surfaces corresponding to the edges in the diagram depicted in [2, Figure 3.1] and [4, Figure 4], the edge-path γ must consist of edges labeled by B or D only.

Note that the edge-path γ connects $1/0$ to p/q, where q must be even since L is a 2-bridge link, and an edge labeled by D can connect the two vertices with even denominators. Thus, if γ contains edges labeled by B, the edges labeled by B appears in pairs. However, by observing the shape of the sub-branched surface corresponding to the edge labeled by B, if γ contains edges labeled by B in pairs, then F would have positive genus, contradicting the assumption that F is planer. It concludes that the edge-path γ consists of only edges labeled by D.

Moreover, by observing the shape of the sub-branched surface corresponding to the edge labeled by D, the number of meridional boundary components are at least the number of the edges labeled by D in γ. Since we are assuming that F has at most two meridional boundaries on $\partial N(K_2)$, it follows that the length of γ is at most two.

If γ is of length one, then L must be equivalent to $L_{1/2}$ which is non-hyperbolic, contradicting the assumption that L is hyperbolic.

If γ is of length two, then the slopes p/q and $1/2$ has distance two, and so L must be equivalent to $L_{[2,n,-2]}$ with some non-zero integer $|n| \geq 2$.

Conversely, if L is equivalent to $L_{[2,n,-2]}$ with $|n| \geq 2$, then we can find a two-punctured disk naturally spanned by K_1. It is incompressible or boundary-incompressible, otherwise, after compression or boundary-compression, we can find a meridionally incompressible essential planer surface in $E(L)$ with single boundary on ∂K_2, contradicting that L is hyperbolic in the same way as above.

This completes the proof.

$\square$

3. Proof

In this section, we give a proof of our theorem.

Proof of Theorem. Let $L = K_1 \cup K_2$ be a hyperbolic 2-bridge link in S^3 and $L(r)$ denote the 3-manifold obtained by Dehn surgery on $K_1 \subset L$ along the slope r. Note that, since the component K_2 remains unfilled, $L_{(}r)$ has a torus boundary. Also note that it is known by [6] that L is hyperbolic if and only if L is not equivalent to $L_{1/n}$ for some integer n.

Now suppose that $L(r)$ is non-hyperbolic. Then, as remarked before, $L(r)$ contains an essential disk, sphere, annulus or torus.

In the following, we give our proof of the theorem divided into four claims.

Claim 1. *There are no essential sphere in $L(r)$.*

Proof. Suppose for a contrary that there exists an essential sphere in $L(r)$. Then, by the standard argument, the link exterior $E(L)$ contains a connected, orientable, essential (i.e., incompressible and ∂-incompressible), properly embedded planer surface F. The surface F has non-empty boundary components on $\partial N(K_1)$ with boundary slope r and no boundary components on $\partial N(K_2)$.

First suppose that F is meridionally incompressible. Then, again by [2, Theorem 3.1 (a)], the surface F is carried by a branched surface Σ_γ for some minimal edge-path γ in the diagram D_t in [2]. See also [4]. In this case, we can apply the argument given in [4, Lemma 12.1]. Then we see that the minimal edge-path γ is in D_∞ and is composed of only two edges with label B with endpoints $1/0$ and p/q, where $L_{p/q}$ is equivalent to L. However, as seen in [2, Figure 1.1] or [4, Figure 2], it implies that $L_{p/q}$ is equivalent to $L_{\pm 1/m}$ for some m, contradicting L is hyperbolic.

Next suppose that F is meridionally compressible. Perform meridional compressions as possible. It can be checked by the standard argument that meridional compressions preserve essentiality of surfaces. Then, since any boundary curve of a meridionally compressing disk is separating on F, there must exist some component which is meridionally incompressible essential planer surface with single meridional boundary on $\partial N(K_2)$ and with non-empty boundaries on $\partial N(K_1)$. However, by Lemma in Section 2, such a surface must have exactly two meridional boundaries on $\partial N(K_2)$. A contradiction occurs.   □

Claim 2. *There are no essential disk in $L(r)$.*

Proof. Suppose for a contrary that there exists an essential disk in $L(r)$. It follows that there is a compressible disk for $\partial L(r)$ in $L(r)$. By

compression, $L(r)$ must be a solid torus. Otherwise we would have an essential sphere in $L(r)$ contradicting Claim 1.

Then, considering the exterior of K_2, we can regard K_1 as a knot in a handlebody. Since the surgery on K_1 yields a solid torus again, by the result given in [3], K_1 is either a 0 or 1-bridge braid in the solid torus $E(K_2)$. Then, together with the result of [7, Proposition 3.2], K_1 must be knotted in S^3. This contradicts that L is a 2-bridge link. □

Claim 3. *There exists an essential torus in $L(r)$ if and only if L is equivalent to $L_{[2w,v,2u]}$ and $r = -w - u$ with*
  (1) $w = 1, u = -1, |v| \geq 2$,
  (2) $w \geq 2, |u| \geq 2, |v| = 1$.
  (3) $w \geq 2, |u| \geq 2, |v| \geq 2$.
In all the cases, $L(r)$ is never Seifert fibered, and $L(r)$ gives a graph manifold if and only if the parameters u, v, w satisfies the first and the second conditions.

Proof. Suppose that there exists an essential torus in $L(r)$.

As seen in the proof of Claim 1, the link exterior $E(L)$ contains a connected, orientable, essential properly embedded surface F of genus one with non-empty boundaries on $\partial N(K_1)$ with boundary slope r and no boundary components on $\partial N(K_2)$.

First suppose that F is meridionally incompressible. Then, by [2, Theorem 3.1 (a)], the surface F is carried by a branched surface Σ_γ for some minimal edge-path γ in the diagram D_t in [2]. See also [4]. Again we can apply the argument given in [4, Lemma 12.1]. Then, in this case, γ has length 4 in D_∞ with endpoints $1/0$ and p/q, where $L_{p/q}$ is equivalent to L. As claimed in the proof of [4, Theorem 1.5], $L_{p/q}$ must be equivalent to $L_{[2w,v,2u]}$ with $w \geq 2, |v| \geq 1, |u| \geq 2$.

It remains to show that $L_{[2w,v,2u]}$ actually contains essential torus for $w \geq 2, |v| \geq 1, |u| \geq 2$. We here imitate the arguments used in the proofs of [11, Theorem 5.1] and [4, Theorem 11.1]. By performing a band sum of K_2 and the curve parallel to the one on $\partial N(K_1)$ with slope $r = -w - u$, equivalently, using a Kirby move on the framed knot (K_1, r), it can be checked directly from the illustration that the surgered manifold $L_{[2w,v,2u]}(r)$ is homeomorphic to the exterior of a satellite knot with a torus knot as a companion in a lens space. See Figure 1.

Moreover, in the case where $|v| \neq 1$ (resp. $|v| = 1$), we can see that the companion knot is a torus knot and the pattern knot is a hyperbolic knot (resp. a cable knot). See also [4, Theorem 11.1] in the case where $|v| = 1$. Note that we have $L_{[2w,\pm1,2u]}(-w - u) \equiv L_{[2w'+1,2u'+1]}(-w' + u' \pm 1)$ for some w' and u'.
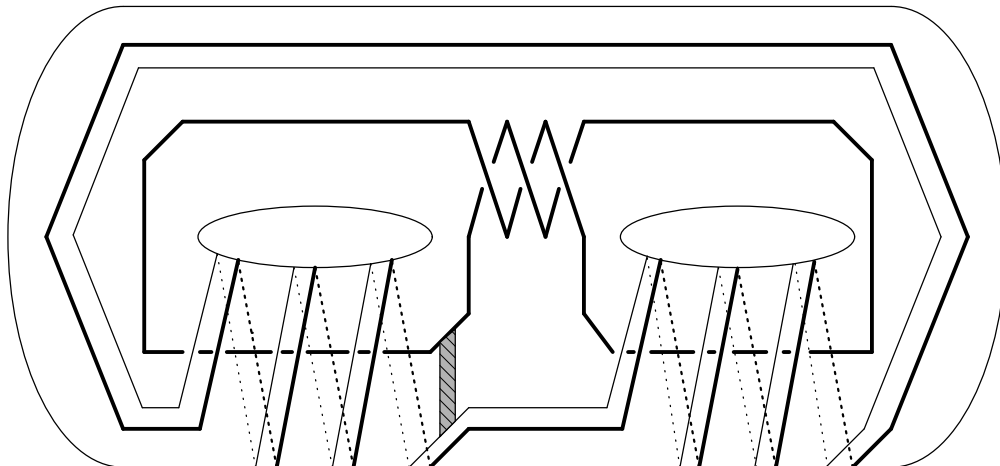
FIGURE 1. band sum for $L_{[6,3,6]}$

Next suppose that F is meridionally compressible. As in the proof of Claim 1, perform meridional compressions as possible. It can be checked by the standard argument that meridional compressions preserve essentiality of surfaces. If some boundary curve of a meridionally compressing disk on F is separating, then the same contradiction could occur as in Claim 1, and so, there must be single meridional compression for F along the non-separating curve on F. Then, by Lemma in Section 2, the link is equivalent to $L_{[2,n,-2]}$ with $|n| \geq 2$ and F is an essential two punctured disk with two meridional punctures on $\partial N(K_2)$ and a single longitudinal boundary on $\partial N(K_1)$. Actually, by tubing operation, we can find a once-punctured torus or klein bottle embedded in $E(L)$ coming from a spanning surface for K_1.

Conversely, we can see that 0-surgery on $K_1 \subset L_{[2,n,-2]}$ with $|n| \geq 2$ gives the exterior of a knot K_2' in $S^2 \times S^1$. This K_2' intersects the level horizontal sphere in $S^2 \times S^1$ transversely twice. This implies that $E(K_2')$ contains a meridional annulus A. The annulus A is incompressible otherwise the meridian of K_2' bounds a disk in $E(K_2')$, contradicting Claim 2. Also A is not boundary parallel since it is non-separating. Thus we conclude that the surgered manifold $L_{[2,n,-2]}(0)$ contains an essential annulus A.

From the annulus, by tubing operation, we have a non-separating torus or klein bottle, which is incompressible by Claim 1 in the knot exterior.

It remains that $L_{[2,n,-2]}(0)$ is not a Seifert fibered space but a graph manifold. Then, since A is essential, A must be isotoped so that A is a union of Seifert fibers. Along this annulus A, we cut $L_{[2,n,-2]}(0)$ open

to get a compact manifold, say X_n, which is the exterior of a pair of properly embedded arcs $t \cup t'$ in $S^2 \times [0, 1]$. See Figure 2. Actually we can see that X_n is homeomorphic to the $(2, n)$-torus link exterior, and the copies of A appear as meridional annuli on the boundary ∂X_n of the knot exterior.
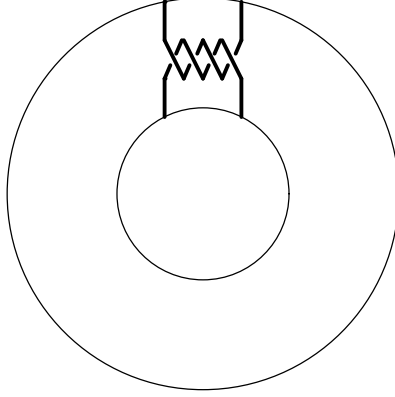

FIGURE 2. embedded arcs $t \cup t'$ in $S^2 \times [0, 1]$. $(n = 4)$

Thus we could actually verify that X_n is Seifert fibered, but, in the case where $|n| \geq 2$, such annuli cannot be a union of Seifert fibers in X_n. This means that the surgered manifold $L_{[2,n,-2]}(0)$ is not a Seifert fibered space but a graph manifold. $\square$

Claim 4. *There exists an essential annulus, but no essential torus in $L(r)$ if and only if $L(r)$ is a small Seifert fibered space and L is equivalent to*

(1) $L_{[3,2u+1]}$ *and* $r = u$,
(2) $L_{[2w+1,3]}$ *and* $r = -w - 1$,
(3) $L_{[3,-3]}$ *and* $r = -1$, *or,*
(4) $L_{[2w+1,2u+1]}$ *and* $r = -w + u$

with $w \geq 1, u \neq 0, -1$.

Proof. Suppose that there exists an essential annulus but no essential torus in $L(r)$. Then it is known that $L(r)$ must be a small Seifert fibered space.

Let r_2 be the slope on $\partial N(K_2)$ determined by the boundary of the essential annulus. Then it is shown that $r_2 \neq 1/0$ as follows. Suppose for a contrary that $r_2 = 1/0$, i.e., r_2 is meridional. Now we are assuming that $L(r)$ is a Seifert fibered space, and the essential annulus coming from the surface F must be vertical. This implies that the meridian of K_2 is a regular fiber of the Seifert fibration of $L(r)$. Then, as shown in

[9, Proof of Corollary 2.6], K_2 must be a core knot in the lens space. However it contradicts that $L(r)$ is not a solid torus as claimed before.

Thus we see that $r_2 \neq 1/0$. Then, as also shown in [9, Proof of Corollary 2.6], K_2 gives a non-trivial non-core torus knot in a lens space. In this case, if we perform suitable surgery on K_2, we have a reducible manifold, equivalently, a suitable surgery on the 2-bridge link L yields a reducible manifold. Then, as a consequence of [11, Theorem 5.1], L must be equivalent to a 2-bridge link corresponding to a continued irreducible fraction of length two.

Now we can apply [4, Theorem 11.1], which establishes a complete classification of such 2-bridge links and surgery slopes on which surgeries yield non-trivial non-core torus knots in lens spaces. This gives us the desired conclusions.

$\square$

By these claims, we have obtained our classification of exceptional Dehn surgeries on components of hyperbolic two-bridge links. $\square$

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