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Hyperbolicity and identification of Berge knots of types VII and VIII

Teruhisa KADOKAMI

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Abstract

T. Saito and M. Teragaito asked in their paper whether Berge knots of type VII, which can be situated on the fiber surface of the lefthand trefoil, are hyperbolic, and showed that some infinite sequences of the knots are all hyperbolic. We show that Berge knots of types VII and VIII are hyperbolic except the known sequence of torus knots. We used the Reidemeister torsions. Consequently, the Alexander polynomials of them have already shown their hyperbolicities. We also show that the standard parameters identify Berge knots of types VII and VIII up to orientations and mirror images, and study what kind of information identify them.

1 Introduction

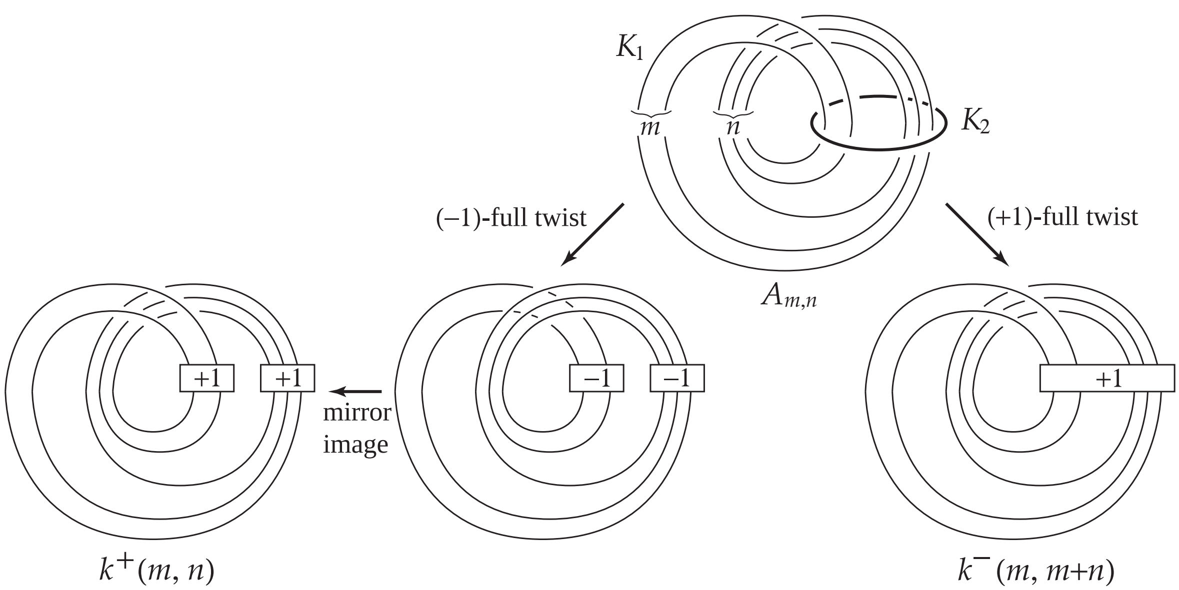

Figure 1: $A_{m,n}$, and Berge knots of type VII ($k^+(m,n)$) and VIII ($k^-(m,m+n)$)

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For two positive coprime integers m and n, let $A_{m,n} = K_1 \cup K_2$ be a 2-component link in the upper side of Figure 1, where K_1 is an (m,n)-torus knot $T(m,n)$ and K_2 is the unknot. By operating 1-surgery (i.e. (-1)-full twist) along K_2 and taking the mirror image, we obtain a knot $k^+(m,n)$ in the lefthand side of Figure 1. By operating (-1)-surgery (i.e. $(+1)$-full twist) along K_2, we obtain a knot $k^-(m,m+n)$ in the righthand side of Figure 1. Then $k^+(m,n)$ and $k^-(m,m+n)$ are Berge knots of types VII and VIII respectively [Ber], and they can be situated on the fiber surfaces of the lefthand trefoil and the figure eight knot respectively. We say that the *standard parameters* of $k^+(m,n)$ and $k^-(m,m+n)$ are $(+,m,n)$ and $(-,m,n)$, respectively. We adopt the notations in [Yam1] (see also [KY, ST, Yam2, Yam3]). Since $A_{m,n} = A_{n,m}$ as ordered oriented links, we have $k^+(m,n) = k^+(n,m)$ and $k^-(m,m+n) = k^-(n,m+n)$ as oriented knots, and we may assume $1 \le m \le n$ and $\gcd(m,n) = 1$. As trivial cases, $k^+(1,n)$ and $k^-(1,n+1)$ are an $(n+1,n)$-torus knot and an $(n+1,n+2)$-torus knot respectively, and we have $k^+(1,n) = k^-(1,n) = T(n,n+1)$. Hence we study hyperbolicity and identification problem of $k^+(m,n)$ and $k^-(m,m+n)$ for $2 \le m < n$. K. Baker [Ba] showed that families of Berge knots of types VII and VIII include hyperbolic knots with arbitrarily large volume.

T. Saito and M. Teragaito [ST] asked hyperbolicity of $k^+(m,n)$, and showed that some infinite sequences of the knots are all hyperbolic by using the necessary and sufficient condition for hyperbolicity of a Berge knot due to Saito [Sai] (Recently J. Greene [Gr] showed that every doubly primitive knot is a Berge knot), and by elementary number theoretical results on Fibonacci series. In the present paper, we show that both $k^+(m,n)$ and $k^-(m,m+n)$ are hyperbolic for $2 \le m < n$ (Theorem 4.6). We used the Reidemeister torsions. Consequently from the proof of Theorem 4.6 and the author's previous works [Kad1, Kad2], the Alexander polynomials of them have already shown their hyperbolicities (Theorem 4.7).

In [ST, Theorem 1.1], it is shown that there are infinite pairs of two distinct knots in S^3 whose same integer surgeries yield homeomorphic lens spaces (Theorem 5.1). We point out that the resulting lens space up to orientations by an odd integer surgery along a hyperbolic Berge knot of type VII or VIII identifies the standard parameter and the knot itself up to orientations and mirror images (Lemma 5.2, Corollary 5.3). We show that a pair of the odd surgery coefficient and the Seifert genus identifies a Berge knot of type VII or VIII up to orientations and mirror images (Theorem 5.4, Corollary 5.5). Y. Yamada [Yam3] defined $k^+(m,n)$ and $k^-(m,m+n)$ for every coprime pair (m,n), and discussed equivalence among them up to orientations and mirror images. For every element of the extended class, there is one representative with $1 \le m \le n$ (see Lemma 5.8 (3)). Since a double torus knot is strongly invertible, a non-trivial torus knot is non-amphicheiral, and a non-torus knot yielding a lens space is non-amphicheiral by the Cyclic Surgery Theorem [CGLS], the extended class can be also classified completely (see also Remark 5.6 (2)).

We ask whether one of the odd surgery coefficient p and the Seifert genus g of a Berge knot of type VII or VIII determines uniquely the standard parameter or not. For the case of p, we give the complete answer which has been already described in [Ber] by a theory of quadratic fields (Lemma 5.9). We make a complete table of p, $(\pm,m,n)$ and g for $p \le 500$. By the table, we can give examples for the case of g.

In Section 2, we give a definition of the Reidemeister torsion, and prepare the surgery formulae for the Reidemeister torsions and the Franz lemma. In Section 3, we compute the

Alexander polynomials of $k^+(m,n)$ and $k^-(m,m+n)$ from that of $A_{m,n}$, determine the genera of $k^+(m,n)$ and $k^-(m,m+n)$ by the Alexander polynomials, and compute the Reidemeister torsions of lens spaces obtained from $k^+(m,n)$ and $k^-(m,m+n)$. In Section 4, we show that both $k^+(m,n)$ and $k^-(m,m+n)$ are hyperbolic for $2 \leq m < n$ (Theorem 4.6). In Section 5, we give the complete answer for identification problem of $k^+(m,n)$ and $k^-(m,m+n)$ (Theorem 5.4), and make a table of the standard parameters. In Section 6, we give a remark about a relation between our method and K. Ichihara, T. Saito and M. Teragaito [IST].

Throughout the paper, for coprime integers p and q, a lens space of type (p,q) is the result of $-p/q$-surgery along the unknot, and it is denoted by $L(p,q)$.

2 Surgery formulae for the Reidemeister torsions, and Franz lemma

For a precise definition of the Reidemeister torsion, the reader refer to V. Turaev [Tur]. In this section, we give surgery formulae of the Reidemeister torsions, and the Franz lemma. Throughout the paper, ζ_d denotes a primitive d-th root of unity, and $(\mathbb{Z}/d\mathbb{Z})^\times$ is the multiplicative group of a ring $\mathbb{Z}/d\mathbb{Z}$.

Let X be a finite CW complex and $\pi : \tilde{X} \to X$ its maximal abelian covering. Then $\tilde{X}$ has a CW structure induced by that of X and π, and the cell chain complex $\mathbf{C}_*$ of $\tilde{X}$ has a $\mathbb{Z}[H]$-module structure, where $H = H_1(X;\mathbb{Z})$ is the first homology of X. For an integral domain R and a ring homomorphism $\psi : \mathbb{Z}[H] \to R$, $\mathbf{C}^\psi_* = \mathbf{C}_* \otimes_{\mathbb{Z}[H]} Q(R)$ is the chain complex of $\tilde{X}$ related with ψ, where $Q(R)$ is the quotient field of R. Note that there is a natural inclusion $R \hookrightarrow Q(R)$. Then the Reidemeister torsion of X related with ψ, denoted by $\tau^\psi(X)$, is obtained from $\mathbf{C}^\psi_*$, and is an element of $Q(R)$ up to multiplication of $\pm\psi(h)$ $(h \in H)$. If $R = \mathbb{Z}[H]$ and ψ is the identity map, then we denote $\tau^\psi(X)$ by $\tau(X)$. We note that $\tau^\psi(X)$ is not zero if and only if $\mathbf{C}^\psi_*$ is acyclic. If A and B are elements of $Q(R)$, and there exists an element $h \in H$ such that $A = \pm\psi(h)B$, then we denote the equality by $A \doteq B$.

Let H be a finitely generated abelian group with a direct sum $H \cong B \oplus T$, where B is a free abelian group of rank r and T is a finite abelian group. Then, by the Chinese Remainder Theorem or the Pontrjagin duality, $\mathbb{Q}[H]$ is a direct sum, as commutative rings, of $\mathbb{Q}(\zeta_{d_i}) \otimes F \cong F(\zeta_{d_i})$, where $F = \mathbb{Q}(B)$ is a rational function field with r variables over $\mathbb{Q}$. In particular, if $H = \langle t \mid t^p = 1\rangle \cong \mathbb{Z}/p\mathbb{Z}$, then we have the canonical isomorphism:

$$\mathbb{Q}[t,t^{-1}]/(t^p-1) \cong \bigoplus_{d|p,d\geq 1} \mathbb{Q}[t,t^{-1}]/(\mathbf{\Phi}_d(t)), \tag{2.1}$$

where $\mathbf{\Phi}_d(t)$ is the d-th cyclotomic polynomial, and $t \pmod{t^p-1}$ in the lefthand side corresponds to $(t \pmod{\mathbf{\Phi}_d(t)})_{d|p,d\geq 1}$ in the righthand side. (2.1) deduces an isomorphism:

$$\mathbb{Q}[H] \cong \bigoplus_{d|p,d\geq 1} \mathbb{Q}(\zeta_d). \tag{2.2}$$

However isomorphisms

$$\mathbb{Q}[t,t^{-1}]/(t^p-1) \cong \mathbb{Q}[H] \quad \text{and} \quad \mathbb{Q}[t,t^{-1}]/(\mathbf{\Phi}_d(t)) \cong \mathbb{Q}(\zeta_d) \tag{2.3}$$

are not uniquely determined. In the former case, t^i can be a generator of H for any $i \in (\mathbb{Z}/p\mathbb{Z})^\times$. In the latter case, t can correspond to ζ_d^i for any $i \in (\mathbb{Z}/d\mathbb{Z})^\times$. In discussing the Reidemeister torsions, only the field components in the direct sum of $\mathbb{Q}[H]$ are essential.

Let $\psi_d : \mathbb{Z}[t, t^{-1}]/(t^p - 1) \to \mathbb{Q}(\zeta_d)$ be a ring homomorphism defined by $\psi_d(t) = \zeta_d$. The map ψ_d can be regarded as a projection to a field component in (2.2). Since a primitive d-th root of unity is not unique for $d \geq 3$, the map ψ_d has some possibilities. Its ambiguity is the Galois group of $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$, which is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^\times$.

Let E be a compact 3-manifold whose boundary ∂E consists of tori, $M = E \cup V$ a 3-manifold obtained by attaching a solid torus V to E by an attaching map $f : \partial V \to \partial E$, and $\iota : E \hookrightarrow M$ the natural inclusion. Let l' be the core of V, and $[l']$ its homology class in $H_1(M)$.

Lemma 2.1 (Surgery formula I) *Let F be a field and $\psi : \mathbb{Z}[H_1(M)] \to F$ a ring homomorphism. If $\psi([l']) \neq 1$, then*

$$\tau^\psi(M) \doteq \tau^{\psi'}(E) \cdot (\psi([l']) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_*$ (ι_* is a ring homomorphism induced by ι).

For an oriented μ-component link $L = K_1 \cup \ldots \cup K_\mu$ in S^3, let E_L be the complement of L, and $\Delta_L(t_1, \ldots, t_\mu)$ the Alexander polynomial of L where t_i is represented by a meridian of K_i. The Reidemeister torsion is closely related with the Alexander polynomial.

Lemma 2.2 (Milnor [Mil]) *We have*

$$\tau(E_L) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & (\mu = 1), \\ \Delta_L(t_1, \ldots, t_\mu) & (\mu \geq 2). \end{cases}$$

For an oriented μ-component link $L = K_1 \cup \ldots \cup K_\mu$ in S^3, let m_i and l_i be a meridian and a longitude of the i-th component K_i, $[m_i]$ and $[l_i]$ their homology classes, and $(L; p_1/q_1, \ldots, p_\mu/q_\mu)$ the result of $(p_1/q_1, \ldots, p_\mu/q_\mu)$-surgery along L where p_i and q_i are coprime integers or $p_i/q_i = \emptyset$. We take integers r_i and s_i satisfying $p_i s_i - q_i r_i = -1$. For coprime integers p and q, the inverse of q in $(\mathbb{Z}/p\mathbb{Z})^\times$ is denoted by $\overline{q} \pmod p$, and also $\overline{q} \pmod d$ for any divisor d (≥ 2) of p.

Lemma 2.3 (Surgery formula II; Sakai [Sak], Turaev [Tur])

(1) For $M = (K; p/q)$ ($|p| \geq 2$) and a diviser $d \geq 2$ of p, we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}$$

where $q\overline{q} \equiv 1 \pmod p$.

(2) For $M = (L; p_1/q_1, \ldots, p_\mu/q_\mu)$ ($\mu \geq 2$), let F be a field and $\psi : \mathbb{Z}[H_1(M)] \to F$ a ring homomorphism. If $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ ($i = 1, \ldots, \mu$), then we have

$$\tau^\psi(M) \doteq \Delta_L(\psi([m_1]), \ldots, \psi([m_\mu])) \prod_{i=1}^{\mu} (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

Example 2.4 By Lemma 2.3 (1), for a divisor $d \geq 2$ of p, we have

$$\tau^{\psi_d}(L(p,q)) \doteq (\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod{p}$.

Remark 2.5 By the ambiguity in (2.3), the Reidemeister torsion for the result of Dehn surgery along a knot can include information of a meridian of the knot.

W. Franz [Fz] showed the following, and classified lens spaces by using it.

Lemma 2.6 (Franz [Fz]) *For a_i and $b_i \in (\mathbb{Z}/p\mathbb{Z})^\times$ $(i = 1, \ldots, n)$, suppose*

$$\prod_{i=1}^{n}(\zeta_p^{a_i} - 1) \doteq \prod_{i=1}^{n}(\zeta_p^{b_i} - 1).$$

Then there exists a permutation σ of $\{1, \ldots, n\}$ such that $a_i = \pm b_{\sigma(i)}$ for all $i = 1, \ldots, n$. In other words, $\{\pm a_i \pmod{p}\} = \{\pm b_i \pmod{p}\}$ as multiple sets.

3 The Alexander polynomials of $k^+(m,n)$ and $k^-(m, m+n)$

We compute the Alexander polynomials of $k^+(m,n)$ and $k^-(m,m+n)$ from that of $A_{m,n}$, determine the genera of $k^+(m,n)$ and $k^-(m,m+n)$ by the Alexander polynomials, and compute the Reidemeister torsions of lens spaces obtained from $k^+(m,n)$ and $k^-(m,m+n)$. Some results are from a work of the author with Y. Yamada [KY].

For a coprime positive pair (m,n), we define a set $\mathfrak{I}(m,n)$ by

$$\begin{aligned} \mathfrak{I}(m,n) &= (m\mathbb{Z} \cup n\mathbb{Z}) \cap \{k \in \mathbb{Z} \mid 0 \leq k \leq mn\} \\ &= \{0, m, 2m, \ldots, nm\} \cup \{0, n, 2n, \ldots, mn\} \end{aligned}$$

Note that the cardinality of $\mathfrak{I}(m,n)$ is $m+n$. We sort the all elements in $\mathfrak{I}(m,n)$ as

$$0 = k_0 < k_1 < k_2 < \cdots < k_{m+n-1} = mn \qquad (k_i \in \mathfrak{I}(m,n)).$$

Theorem 3.1 ([KY]) *The Alexander polynomial of the link $A_{m,n}$ is*

$$\Delta_{A_{m,n}}(t,x) \doteq \sum_{i=0}^{m+n-1} t^{k_i}x^i.$$

We define u_j $(j = 0, 1, 2, \ldots, n)$ and w_j $(j = 0, 1, 2, \ldots, m)$ by

$$k_{u_j} = jm \quad \text{and} \quad k_{w_j} = jn.$$

Then we have the following:

Lemma 3.2 ([KY])

(1)

$$u_j = \begin{cases} 0 & (j=0), \\ \left[\frac{jm}{n}\right] + j & (j=1,2,\ldots,n-1), \\ m+n-1 & (j=n), \end{cases}$$

and $u_j + u_{n-j} = m+n-1$*, where* $[\ \cdot\ ]$ *is the gaussian symbol.*

(2)

$$w_j = \begin{cases} 0 & (j=0), \\ \left[\frac{jn}{m}\right] + j & (j=1,2,\ldots,m-1), \\ m+n-1 & (j=m), \end{cases}$$

and $w_j + w_{m-j} = m+n-1$.

By Lemma 3.2 (1) and (2), we have

$$\sum_{i=0}^{m+n-1} t^{k_i} x^i = 1 + \sum_{i=1}^{n-1} t^{im} x^{u_i} + \sum_{j=1}^{m-1} t^{jn} x^{w_j} + t^{mn} x^{m+n-1}.$$

Here we denote $k^+(m,n)$ and $k^-(m,m+n)$ by K^+ and K^-, respectively. Let g be the Seifert genus of $K^\pm$. Since every knot yielding a lens space is fibered Y. Ni [Ni], the degree of the Alexander polynomial of a Berge knot including $K^\pm$ equals to $2g$.

Proposition 3.3 (Saito and Teragaito [ST, Lemma 5.1]) *The Alexander polynomial of* $K^\pm$ *is*

$$\Delta_{K^\pm}(t) \doteq \frac{t-1}{t^{m+n}-1} \cdot \sum_{i=0}^{m+n-1} t^{k_i \mp i(m+n)},$$

and the genus of $K^\pm$ *is*

$$g = \frac{(m+n-1)^2 \mp mn}{2}.$$

Proof For $A_{m,n} = K_1 \cup K_2$ and $M = (A_{m,n}; \emptyset, \pm 1)$, let E be the complement of $A_{m,n}$, V an attaching solid torus to $\partial N(K_2) \subset \partial E$ where $N(K_2)$ is a regular neighborhood of K_2 in S^3, and $M = E \cup V$. Let m_i and l_i $(i=1,2)$ be a meridian and a longitude of K_i respectively, and m' and l' be a meridian and a core of V respectively. We note that M is homeomorphic to the complement of $K^\pm$. Then we study homological conditions.

In $H_1(E)$,

$$[l_1] = [m_2]^{m+n}, \quad [l_2] = [m_1]^{m+n},$$

and

$$\begin{aligned} H_1(E) &\cong \langle [m_1],[l_1],[m_2],[l_2] \mid [l_1] = [m_2]^{m+n}, [l_2] = [m_1]^{m+n} \rangle \\ &\cong \langle [m_1],[m_2] \mid - \rangle \cong \mathbb{Z}^2. \end{aligned}$$

In $H_1(M)$,

$$[m'] = [m_2]^{\pm 1}[l_2] = 1, \quad [l'] = [m_2],$$

and

$$\begin{aligned} H_1(M) &\cong \langle [m_1],[m_2] \mid [m_1]^{m+n}[m_2]^{\pm 1} = 1\rangle \\ &\cong \langle t \mid - \rangle \cong \mathbb{Z}, \end{aligned}$$

where $t = [m_1]$. Then

$$\begin{aligned} [m_2] &= t^{\mp(m+n)}, \\ [l'] &= [m_2] = t^{\mp(m+n)}. \end{aligned} \tag{3.1}$$

By Theorem 3.1, (3.1) and the surgery formula I (Lemma 2.1), we have the Alexander polynomial of $K^{\pm}$ as

$$\Delta_{K^{\pm}}(t) \doteq \frac{(t-1)\Delta_{A_{m,n}}(t, t^{\mp(m+n)})}{t^{m+n}-1} \doteq \frac{t-1}{t^{m+n}-1} \cdot \sum_{i=0}^{m+n-1} t^{k_i \mp i(m+n)}$$

(i) The K^+ case

If $k_i = jm$ ($j = 0,1,2,\ldots,n-1$) (i.e. $i = u_j$), then by Lemma 3.2 (1), we have

$$k_i - i(m+n) = jm - \left(j + \left[\frac{jm}{n}\right]\right)(m+n) = -\left\{jn + \left[\frac{jm}{n}\right](m+n)\right\},$$

and it is monotonely decreasing about j. Moreover

$$\begin{aligned} & k_{u_{n-1}} - (m+n-2)(m+n) = (n-1)m - (m+n-2)(m+n) \\ > \ & k_{u_n} - (m+n-1)(m+n) = -\{(m+n-1)(m+n) - mn\}. \end{aligned}$$

If $k_i = jn$ ($j = 0,1,2,\ldots,m-1$) (i.e. $i = w_j$), then by Lemma 3.2 (2), we have

$$k_i - i(m+n) = jn - \left(j + \left[\frac{jn}{m}\right]\right)(m+n) = -\left\{jm + \left[\frac{jn}{m}\right](m+n)\right\},$$

and it is monotonely decreasing about j. Moreover

$$\begin{aligned} & k_{w_{m-1}} - \left(m+n-2-\left[\frac{n}{m}\right]\right)(m+n) = (m-1)n - \left(m+n-2-\left[\frac{n}{m}\right]\right)(m+n) \\ > \ & k_{w_m} - (m+n-1)(m+n) = -\{(m+n-1)(m+n) - mn\}. \end{aligned}$$

Hence the degree of $\Delta_{K^+}(t)$ is

$$2g = \{(m+n-1)(m+n) - mn\} + 1 - (m+n) = (m+n-1)^2 - mn.$$

(ii) The K^- case

If $k_i = jm$ ($j = 0,1,2,\ldots,n-1$) (i.e. $i = u_j$), then by Lemma 3.2 (1), we have

$$k_i + i(m+n) = jm + \left(j + \left[\frac{jm}{n}\right]\right)(m+n)$$

is monotonely increasing about j. Moreover

$$\begin{aligned} & k_{u_{n-1}} + (m+n-2)(m+n) = (n-1)m + (m+n-2)(m+n) \\ < \ & k_{u_n} + (m+n-1)(m+n) = mn + (m+n-1)(m+n). \end{aligned}$$

If $k_i = jn$ $(j = 0, 1, 2, \ldots, m-1)$ (i.e. $i = w_j$), then by Lemma 3.2 (2), we have

$$k_i + i(m+n) = jn + \left(j + \left[\frac{jn}{m}\right]\right)(m+n)$$

is monotonely increasing about j. Moreover

$$\begin{aligned} & k_{w_{m-1}} + \left(m+n-2-\left[\frac{n}{m}\right]\right)(m+n) = (m-1)n + \left(m+n-2-\left[\frac{n}{m}\right]\right)(m+n) \\ < \ & k_{w_m} + (m+n-1)(m+n) = mn + (m+n-1)(m+n). \end{aligned}$$

Hence the degree of $\Delta_{K^-}(t)$ is

$$2g = \{(m+n-1)(m+n) + mn\} + 1 - (m+n) = (m+n-1)^2 + mn.$$

□

The following lemma is obtained from Kirby moves [Kir] and Rolfsen moves [Ro].

Lemma 3.4 ([KY, Yam3]) *$M = (A_{m,n}; mn, r)$ is a lens space for every integer $r \in \mathbb{Z}$. Then $M \cong L(rmn - (m+n)^2, m\overline{n})$. In particular, we have*

$$\mp(A_{m,n}; mn, \pm 1) \cong (K^{\pm}; (m+n)^2 \mp mn) \cong L((m+n)^2 \mp mn, m\overline{n}).$$

For $A_{m,n} = K_1 \cup K_2$ and $M = (A_{m,n}; mn, r)$ with $r \in \mathbb{Z}$, let E be the complement of $A_{m,n}$, V_i $(i = 1, 2)$ an attaching solid torus to $\partial N(K_i) \subset \partial E$ where $N(K_i)$ is a regular neighborhood of K_i in S^3, and $M_1 = E \cup V_1$ (then $M = M_1 \cup V_2$). Let m_i and l_i $(i = 1, 2)$ be a meridian and a longitude of K_i respectively, and m_i' and l_i' $(i = 1, 2)$ be a meridian and a core of V_i respectively. We set $p = (m+n)^2 - rmn$. Then we study homological conditions.

In $H_1(E)$,

$$[l_1] = [m_2]^{m+n}, \quad [l_2] = [m_1]^{m+n},$$

and

$$\begin{aligned} H_1(E) &\cong \langle [m_1], [l_1], [m_2], [l_2] \mid [l_1] = [m_2]^{m+n}, [l_2] = [m_1]^{m+n} \rangle \\ &\cong \langle [m_1], [m_2] \mid - \rangle \cong \mathbb{Z}^2. \end{aligned}$$

In $H_1(M_1)$,

$$[m_1'] = [m_1]^{mn}[l_1] = 1, \quad [l_1'] = [m_1],$$

and

$$\begin{aligned} H_1(M_1) &\cong \langle [m_1], [m_2] \mid [m_1]^{mn}[m_2]^{m+n} = 1 \rangle \\ &\cong \langle T \mid - \rangle \cong \mathbb{Z}, \end{aligned}$$

where $T = [m_1]^u [m_2]^v$ for integers u and v such that $(m+n)u - mnv = 1$. Then

$$\begin{aligned} [m_1] &= [m_1]^{(m+n)u - mnv} = ([m_1]^u[m_2]^v)^{m+n}([m_1]^{mn}[m_2]^{m+n})^{-v} = T^{m+n}, \\ [m_2] &= [m_2]^{(m+n)u - mnv} = ([m_1]^u[m_2]^v)^{-mn}([m_1]^{mn}[m_2]^{m+n})^{u} = T^{-mn}, \\ [l_1'] &= [m_1] = T^{m+n}. \end{aligned} \tag{3.2}$$

In $H_1(M)$,

$$[m_2'] = [m_2]^r[l_2] = 1, \quad [l_2'] = [m_2],$$

and

$$\begin{aligned} H_1(M) &\cong \langle [m_1],[m_2] \mid [m_1]^{mn}[m_2]^{m+n} = 1, [m_1]^{m+n}[m_2]^r = 1\rangle \\ &\cong \langle T \mid T^p = 1\rangle \cong \mathbb{Z}/p\mathbb{Z}, \end{aligned}$$

where T is the induced one from T in (3.2). Then

$$[l_2'] = [m_2] = T^{mn}. \tag{3.3}$$

Lemma 3.5 ([KY]) *Let T be an indeterminancy.*

$$\Delta_{A_{m,n}}(T^{m+n}, T^{-mn}) \doteq \frac{(T^{mn}-1)(T^{m+n}-1)}{(T^m-1)(T^n-1)}.$$

Lemma 3.6 *Let $\varphi_p : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_p)$ be a ring homomorphism such that $\varphi_p(T) = \zeta_p$, where T is the same one in (3.3). Then we have*

$$\tau^{\varphi_p}(M) \doteq (\zeta_p^m - 1)^{-1}(\zeta_p^n - 1)^{-1}.$$

Proof By (3.2) and (3.3), we have $\varphi_p([l_1']) \neq 1$ and $\varphi_p([l_2']) \neq 1$. By the surgery formula II (Lemma 2.3 (2)), Theorem 3.1, Lemma 3.5, (3.2) and (3.3), we have the result. □

4 Hyperbolicity of $k^+(m,n)$ and $k^-(m,m+n)$

A hyperbolic knot in S^3 is characterized as a prime, non-torus and non-satellite knot. Since the tunnel number of every Berge knot is one, every Berge knot is prime by F. H. Norwood [No]. Hence we show that both $k^+(m,n)$ and $k^-(m,m+n)$ are non-torus and non-satellite. This problem is origined by T. Saito and M. Teragaito [ST] for the case $k^+(m,n)$.

To show the main theorem, we need some known results on torus knots and satellite knots.

Theorem 4.1 (Moser [Mos]) *For an (r,s)-torus knot, the surgery coefficient p/q yielding a lens space satisfies $|p-qrs| = 1$, and the resulting lens space is $L(p, -qr^2)$.*

Theorem 4.2 (Gordon [Go]; Wu [Wu]) *A satellite knot yielding a lens space is a $(2, 2rs \pm 1)$-cable of an (r,s)-torus knot. Then the surgery coefficient is $4rs \pm 1$, and the resulting lens space is $L(4rs \pm 1, \mp 4r^2)$.*

Theorem 4.3 (Seifert [Se]) *Let $K_{r,s}$ be an (r,s)-cable of a knot K. Then the Alexander polynomial of $K_{r,s}$ is:*

$$\Delta_{K_{r,s}}(t) \doteq \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)}\Delta_K(t^r).$$

Lemma 4.4 *(1) For an (r,s)-torus knot $T(r,s)$ with $r>0$ and $s>0$, the genus is $(r-1)(s-1)/2$, and the Reidemeister torsion of $M = (T(r,s);p)$ with $p = rs \pm 1$ is*

$$\tau^{\psi_p}(M) \doteq (\zeta_p^r - 1)^{-1}(\zeta_p^s - 1)^{-1}$$

where a ring homomorphism $\psi_p : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_p)$ maps the meridian of $T(r,s)$ to ζ_p.

(2) For a $(2, 2rs \pm 1)$-cable of an (r, s)-torus knot, denoted by K, the Alexander polynomial of K is

$$\Delta_K(t) \doteq \frac{(t^{2(2rs\pm1)} - 1)(t^{2rs} - 1)(t - 1)}{(t^{2rs\pm1} - 1)(t^{2r} - 1)(t^{2s} - 1)},$$

and the Reidemeister torsion of $M = (K; p)$ with $p = 4rs \pm 1$ is

$$\tau^{\psi_p}(M) \doteq (\zeta_p^{2r} - 1)^{-1}(\zeta_p^{2s} - 1)^{-1}$$

where a ring homomorphism $\psi_p : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_p)$ maps the meridian of K to ζ_p.

Proof (1) By Theorem 4.3, we have

$$\Delta_{T(r,s)}(t) \doteq \frac{(t^{rs} - 1)(t - 1)}{(t^r - 1)(t^s - 1)}.$$

Since $T(r, s)$ is fibered, the genus is $(r - 1)(s - 1)/2$. By the surgery formula II (Lemma 2.3 (1)), we have $\tau^{\psi_p}(M) \doteq (\zeta_p^r - 1)^{-1}(\zeta_p^s - 1)^{-1}$.

(2) By Theorem 4.3, we have the Alexander polynomial. By the surgery formula II (Lemma 2.3 (1)), we have $\tau^{\psi_p}(M) \doteq (\zeta_p^{2r} - 1)^{-1}(\zeta_p^{2s} - 1)^{-1}$. □

The following is an easy but much important observation.

Lemma 4.5 *For $K^\pm = k^+(m, n)$ and $k^-(m, m + n)$ respectively, we set $p = (m + n)^2 \mp mn$ and $2g = (m + n - 1)^2 \mp mn$. Then p is a lens surgery coefficient and g is the Seifert genus of $K^\pm$, and we have*

$$p - 2g = 2(m + n) - 1$$

and p is odd.

Proof By Proposition 3.3, Lemma 3.4 and a direct calculation, we have the result. □

The following is our main theorem.

Theorem 4.6 *Both $k^+(m, n)$ and $k^-(m, m + n)$ are hyperbolic for $2 \le m < n$.*

Proof Firstly we show that both $K^+ = k^+(m, n)$ and $K^- = k^-(m, m+n)$ are non-torus. We set $p = rs \pm 1$, $g = (r - 1)(s - 1)/2$ and $K = K^+$ or K^-. Suppose that K is an (r, s)-torus knot with $2 \le r < s$. Then $p = (m+n)^2 - mn$ or $(m+n)^2 + mn$ by Lemma 3.4 and Theorem 4.1, and the Reidemeister torsion of $M = (K; p)$ is

$$\tau^{\varphi_p}(M) \doteq (\zeta_p^m - 1)^{-1}(\zeta_p^n - 1)^{-1} \tag{4.1}$$

by Lemma 3.6, where a ring homomorphism $\varphi_p : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_p)$ maps the meridian of K to ζ_p^{m+n} by (3.2). From Lemma 4.4 (2) and (4.1), we have

$$(\zeta_p^r - 1)(\zeta_p^s - 1) \doteq (\zeta_p^{m\cdot\overline{m+n}} - 1)(\zeta_p^{n\cdot\overline{m+n}} - 1). \tag{4.2}$$

From $(m + n)^2 \equiv \pm mn \pmod{p}$, we have

$$m \cdot \overline{m+n} \equiv \pm(m\overline{n} + 1) \quad \text{and} \quad n \cdot \overline{m+n} \equiv \pm(\overline{m}n + 1) \pmod{p}. \tag{4.3}$$

By applying the Franz lemma (Lemma 2.6) to (4.2), we may assume

$$r \equiv \pm(m\overline{n}+1) \quad \text{and} \quad s \equiv \pm(\overline{m}n+1) \pmod p.$$

Thus we have the following four cases:

(i) $(r-1)(s-1) \equiv 1 \pmod p$.

(ii) $(r-1)(s+1) \equiv -1 \pmod p$.

(iii) $(r+1)(s-1) \equiv -1 \pmod p$.

(iv) $(r+1)(s+1) \equiv 1 \pmod p$.

We set the lefthand side of the equations above as U. Then we have

$$2p-1-U \geq 2(rs-1)-1-(r+1)(s+1) = (r-1)(s-1)-5,$$

and

$$1 < (r-1)(s-1) \leq U < 2p-1$$

except the cases $(r,s) = (2,3)$ and $(2,5)$ in (iv) and $p = rs-1$.

Case 1 $(r,s) \neq (2,3)$ and $(2,5)$.

We have $U = p+1$ for (i) and (iv), and $U = p-1$ for (ii) and (iii).

(i) $(r-1)(s-1) = p+1$.

$$p+1-(r-1)(s-1) \geq (rs-1)+1-(r-1)(s-1) = r+s-1 > 0.$$

It is a contradiction.

(ii) $(r-1)(s+1) = p-1$.

$$p-1-(r-1)(s+1) \geq (rs-1)-1-(r-1)(s+1) = -r+s-1 > 0$$

except $s = r+1$ and $p = rs-1$.

(iii) $(r+1)(s-1) = p-1$.

$$p-1-(r+1)(s-1) \leq (rs+1)-1-(r+1)(s-1) = r-s+1 < 0$$

except $s = r+1$ and $p = rs+1$.

(iv) $(r+1)(s+1) = p+1$.

$$p+1-(r+1)(s+1) \leq (rs+1)+1-(r+1)(s+1) = -r-s+1 < 0.$$

It is a contradiction.

We show the rest cases:

(a) $s = r+1$ and $p = rs-1$ in (ii).

(b) $s = r+1$ and $p = rs+1$ in (iii).

conclude contradiction.

(a) By Proposition 3.3, Lemma 4.4 (1) and Lemma 4.5, we have

$$p = r(r+1) - 1 = (m+n)^2 \mp mn \quad \text{and} \quad p - 2g = 2r - 1 = 2m + 2n - 1.$$

Thus we have

$$m + n = r \quad \text{and} \quad mn = \pm(r-1).$$

Then there is no solution for $2 \leq m < n$.

(b) By Proposition 3.3, Lemma 4.4 (1) and Lemma 4.5, we have

$$p = s(s-1) + 1 = (m+n)^2 \mp mn \quad \text{and} \quad p - 2g = 2s - 1 = 2m + 2n - 1.$$

Thus we have

$$m + n = s \quad \text{and} \quad mn = \pm(s-1).$$

Then there is no solution for $2 \leq m < n$.

Case 2 $(r, s) = (2, 3)$ or $(2, 5)$.

This case corresponds to (iv). By the assumption $2 \leq m < n$, we have $m + n \geq 5$. By Proposition 3.3, Lemma 4.4 and Lemma 4.5, we have

$$p - 2g = 2m + 2n - 1 = r + s \quad \text{or} \quad r + s - 2.$$

If $(r, s) = (2, 3)$ or $(2, 5)$, then we have $m + n \leq (r + s + 1)/2 \leq 4$. It is a contradiction.

Secondly we show that both $K^+ = k^+(m, n)$ and $K^- = k^-(m, m+n)$ are non-satellite. We set $p = 4rs \pm 1$, and $K = K^+$ or K^-. Suppose that K is a $(2, 2rs \pm 1)$-cable of an (r, s)-torus knot with $2 \leq r < s$. Then $p = (m+n)^2 - mn$ or $(m+n)^2 + mn$ by Lemma 3.4 and Theorem 4.2. In the similar way as the first case, by Lemma 4.4 (2) and (4.1), we have

$$(\zeta_p^{2r} - 1)(\zeta_p^{2s} - 1) \doteq (\zeta_p^{m \cdot \overline{m+n}} - 1)(\zeta_p^{n \cdot \overline{m+n}} - 1). \tag{4.4}$$

By applying the Franz lemma (Lemma 2.6) to (4.4), and (4.3), we may assume

$$2r \equiv \pm(m\overline{n} + 1) \quad \text{and} \quad 2s \equiv \pm(\overline{m}n + 1) \pmod{p}.$$

Thus we have the following four cases:

(i) $(2r-1)(2s-1) \equiv 1 \pmod{p}$.

(ii) $(2r-1)(2s+1) \equiv -1 \pmod{p}$.

(iii) $(2r+1)(2s-1) \equiv -1 \pmod{p}$.

(iv) $(2r+1)(2s+1) \equiv 1 \pmod{p}$.

We set the lefthand side of the equations above as U. Then we have

$$2p - 1 - U \geq 2(4rs - 1) - 1 - (2r + 1)(2s + 1) = (2r - 1)(2s - 1) - 5 \geq 10,$$

$$1 < (2r - 1)(2s - 1) \leq U < 2p - 1,$$

$U = p + 1$ for (i) and (iv), and $U = p - 1$ for (ii) and (iii). However U is odd, and $p \pm 1$ is even. This is a contradiction. $\square$

Consequently from the proof of Theorem 4.6, we have the following:

Theorem 4.7 *The Alexander polynomials of $k^+(m, n)$ and $k^-(m, m + n)$ for $2 \leq m < n$ are not those of torus knots and satellite knots (i.e. The Alexander polynomials of $k^+(m, n)$ and $k^-(m, m + n)$ for $2 \leq m < n$ have already shown their hyperbolicities).*

We note that there may be a hyperbolic knot yielding a lens space whose Alexander polynomial is the same as that of a torus knot or a satellite knot. We also note that though the Seifert genera of $k^+(m, n)$ and $k^-(m, m + n)$ are uniquely determined by their Alexander polynomials from their fiberedness due to [Ni], lens surgery coefficients of them are not always uniquely determined by their Alexander polynomials.

To prove Theorem 4.7, we need some results.

Lemma 4.8 ([Kad1]) *Let K be a knot whose Alexander polynomial of an (r, s)-torus knot with $2 \leq r < s$. Suppose that $M = (K; p)$ with $p \geq 2$ has the same Reidemeister torsions as those of a lens space. Then we have*

$$r \equiv \pm 1 \quad \textit{or} \quad s \equiv \pm 1 \quad \textit{or} \quad rs \equiv \pm 1 \pmod p.$$

Lemma 4.9 ([Kad2]) *Let K be a knot whose Alexander polynomial of a $(2, 2rs \pm 1)$-cable of an (r, s)-torus knot with $2 \leq r < s$. Suppose that $M = (K; p)$ with $p \geq 2$ has the same Reidemeister torsions as those of a lens space. Then we have*

$$r \equiv \pm 1 \quad \textit{or} \quad s \equiv \pm 1 \quad \textit{or} \quad 2rs \equiv \pm 1 \quad \textit{or} \quad 4rs \pm 1 \equiv 0 \pmod p.$$

Proof of Theorem 4.7 Suppose that $K^{\pm}$ has the same Alexander polynomial as that of an (r, s)-torus knot with $2 \leq r < s$. Let p be a lens surgery coefficient, and $g = (r - 1)(s - 1)/2$ the Seifert genus of $K^{\pm}$. By Lemma 4.8, $p = rs \pm 1$ or

$$p \leq \frac{rs + 1}{2}, \tag{4.5}$$

or

$$r = 2 \quad \text{and} \quad p = s + 1. \tag{4.6}$$

The case $p = rs \pm 1$ does not happen by the proof of Theorem 4.6. Suppose (4.5). Then we have

$$2(p - 2g) \leq rs + 1 - 2(r - 1)(s - 1) = -(r - 2)(s - 2) + 3 \leq 3.$$

Suppose (4.6). Then we have

$$p - 2g = 2.$$

By Lemma 4.5, we have

$$p - 2g = 2(m+n) - 1 \geq 9.$$

It is a contradiction.

Suppose that $K^{\pm}$ has the same Alexander polynomial as that of a $(2, 2rs \pm 1)$-cable of an (r, s)-torus knot with $2 \leq r < s$. Let p be a lens surgery coefficient, and $g = (r-1)(s-1) + (2rs - 1 \pm 1)/2$ the Seifert genus of $K^{\pm}$. By Lemma 4.9, $p = 4rs \pm 1$ or

$$p \leq \max\left(\frac{4rs+1}{2}, 2rs+1\right) = 2rs + 1. \tag{4.7}$$

The case $p = 4rs \pm 1$ does not happen by the proof of Theorem 4.6. We suppose (4.7). Then we have

$$p - 2g \leq 2rs + 1 - 2(r-1)(s-1) - (2rs - 2) = -2(r-1)(s-1) + 3 \leq -1.$$

By Lemma 4.5, we have a contradiction. □

5 Identification of $k^+(m, n)$ and $k^-(m, m+n)$

We ask what kind of information identify a Berge knot of type VII or VIII. We consider the cases that (i) the resulting lens space up to orientations by an odd integer surgery along the knot, (ii) a pair of an odd integer surgery p and the Seifert genus g of the knot, and (iii) one of p and g.

A background of the case (i) is the following fact:

Theorem 5.1 (Saito and Teragaito [ST, Theorem 1.1]) *There are infinite pairs of two distinct knots in S^3 whose same integer surgeries yield homeomorphic lens spaces. Then we can take the two knots from the classes of torus knots, satellite knots, and hyperbolic knots arbitrarily except the case that both knots are satellite knots.*

The Cyclic Surgery Theorem [CGLS] says that the number of lens surgery coefficients of a hyperbolic knot is at most 2, and they are successing integers if it is 2. Hence the odd lens surgery coefficient of a hyperbolic knot is uniquely determined if it exists.

Lemma 5.2 *The resulting lens space up to orientations by an odd integer surgery along a hyperbolic Berge knot of type VII or VIII identifies uniquely the standard parameter.*

Proof For $k^+(m, n)$ or $k^-(m, m+n)$ with $2 \leq m < n$ (i.e. a hyperbolic Berge knot of type VII or VIII), $p = (m+n)^2 \mp mn$ is the odd lens surgery coefficient. Then the result of p-surgery is $L(p, m\overline{n})$ by Lemma 3.4. We set its standard parameter as (ε, m, n). Suppose that there is a hyperbolic Berge knot of type VII or VIII yielding a homeomorphic lens space with its standard parameter (ε', m', n'). Then we have

$$m\overline{n} \equiv \pm m'\overline{n'} \quad \text{or} \quad \pm\overline{m'}n' \pmod{p},$$

and

$$mn' + m'n \quad \text{or} \quad mn' - m'n \quad \text{or} \quad mm' + nn' \quad \text{or} \quad mm' - nn' \equiv 0 \pmod{p}.$$

By the Cauchy-Schwartz inequation, we have

$$\begin{aligned} 0 &\leq \min(mn' + m'n, |mn' - m'n|, mm' + nn', |mm' - nn'|) \\ &< \max(mn' + m'n, |mn' - m'n|, mm' + nn', |mm' - nn'|) \\ &\leq \sqrt{m^2 + n^2} \cdot \sqrt{(m')^2 + (n')^2} \\ &\leq \max(m^2 + n^2, (m')^2 + (n')^2) < p, \end{aligned}$$

and hence only the case $|mn' - m'n| = 0$ is possible. This is equivalent to $(m, n) = (m', n')$. It deduces $\varepsilon = \varepsilon'$. □

By Lemma 5.2, we have the following.

Corollary 5.3 *The standard parameter with $2 \leq m < n$ identifies uniquely a hyperbolic Berge knot of type VII or VIII up to mirror images.*

For the case (ii), we set

$$\begin{aligned} \mathcal{S} &= \{(\varepsilon, m, n) \in \{+, -\} \times \mathbb{N} \times \mathbb{N} \mid 1 \leq m \leq n,\ \gcd(m, n) = 1\}, \\ \mathcal{S}_2 &= \{(\varepsilon, m, n) \in \{+, -\} \times \mathbb{N} \times \mathbb{N} \mid 2 \leq m < n,\ \gcd(m, n) = 1\}, \\ \mathcal{B}_{+,1} &= \{k^+(1, n) \mid 1 \leq n\} = \{T(n, n+1) \mid 1 \leq n\}, \\ \mathcal{B}_{-,1} &= \{k^-(1, n+1) \mid 1 \leq n\} = \{T(n+1, n+2) \mid 1 \leq n\}, \\ \mathcal{B}_2 &= \{k^+(m, n) \mid 2 \leq m < n,\ \gcd(m, n) = 1\} \\ &\quad \cup \{k^-(m, m+n) \mid 2 \leq m < n,\ \gcd(m, n) = 1\}, \\ \mathcal{B} &= \mathcal{B}_{+,1} \cup \mathcal{B}_{-,1} \cup \mathcal{B}_2, \end{aligned}$$

where $\mathcal{S}$ is the set of the standard parameters, and knots are considered up to orientations and mirror images. We define maps

$$F : \mathcal{S} \to \mathbb{N} \times \mathbb{N} \quad \text{and} \quad G : \mathcal{S} \to \mathcal{B}$$

by

$$\begin{aligned} F(\varepsilon, m, n) &= (p, 2g) = ((m+n)^2 - \varepsilon mn, (m+n-1)^2 - \varepsilon mn), \\ G(\varepsilon, m, n) &= k^+(m, n) \text{ for } \varepsilon = + \text{ and } k^-(m, m+n) \text{ for } \varepsilon = -. \end{aligned}$$

Theorem 5.4 *(1) The map F is injective.*

(2) $\mathcal{B}_{+,1} \supset \mathcal{B}_{-,1}$ and $\mathcal{B}_{+,1} \setminus \mathcal{B}_{-,1} = \{k^+(1, 1)\}$.

(3) The map G is surjective, and for any $K \in \mathcal{B}$, $\sharp\{G^{-1}(K)\} = 1$ or 2 where $\sharp\{\cdot\}$ implies the cardinality. $\sharp\{G^{-1}(K)\} = 2$ if and only if $K \in \mathcal{B}_{-,1}$ (i.e. For some $n \geq 2$, $K = T(n, n+1)$ and $G^{-1}(K) = \{(+, 1, n), (-, 1, n-1)\}$).

Proof (1) By Lemma 4.5, we have

$$p - 2g = 2(m+n) - 1 \quad \text{and} \quad p - (m+n)^2 = -\varepsilon mn.$$

Since $m+n$ and mn are uniquely determined from ε and $(p, 2g)$, m and n are also uniquely determined from ε and $(p, 2g)$.

Suppose that $F(+, m, n) = F(-, m', n') = (p, 2g)$. By the relations above, we have $m+n = m'+n'$ and

$$p - (m+n)^2 = -mn = m'n'.$$

It is a contradiction. Therefore F is injective.

(2) Since $k^+(1, n) = k^-(1, n) = T(n, n+1)$, we have the result.

(3) Surjectivity of G is clear. By Theorem 4.6 and the Cyclic Surgery Theorem [CGLS], the odd lens surgery coefficient of an element in $\mathcal{B}_2$ is uniquely determined as p. By (1) and (2), we have the result. □

By Theorem 5.4, we have the following.

Corollary 5.5 *The map $G \circ F^{-1} : F(\mathcal{S}_2) \to \mathcal{B}$ is injective, and its image is $\mathcal{B}_2$. That is, a pair $(p, 2g)$ with $2 \leq m < n$ identifies uniquely a hyperbolic Berge knot of type VII or VIII up to mirror images.*

Remark 5.6 (1) Lemma 5.2 and Corollary 5.3 say that p and $\pm m\overline{n} \pmod p$ with $2 \leq m < n$ determine (ε, m, n) and a hyperbolic Berge knot of type VII or VIII uniquely. Theorem 5.4 and Corollary 5.5 say that p and g with $2 \leq m < n$ determine (ε, m, n) and a hyperbolic Berge knot of type VII or VIII uniquely.

(2) Y. Yamada [Yam2] defines $k^+(m, n)$ and $k^-(m, m+n)$ for any coprime pair (m, n), and their mirror images $l^+(m, n)$ and $l^-(m, m+n)$, respectively. They have unique reprsentatives in $\mathcal{B}$ up to orientations and mirror images (see Lemma 5.8 (3)). Since a double torus knot is strongly invertible, a non-trivial torus knot is non-amphicheiral, and a non-torus knot yielding a lens space is non-amphicheiral by the Cyclic Surgery Theorem [CGLS], the extended class can be also classified completely.

For the case (iii), we define maps

$$\mathrm{proj}_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \quad \text{and} \quad \mathrm{proj}_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

by

$$\mathrm{proj}_1(x, y) = x \quad \text{and} \quad \mathrm{proj}_2(x, y) = y,$$

respectively. We set $H_{\varepsilon,i} = \mathrm{proj}_i \circ F_\varepsilon$ and $H_i = \mathrm{proj}_i \circ F$ ($\varepsilon = +, -;\ i = 1, 2$). Then we study $\mathrm{Im}\,(H_{\varepsilon,i})$, $\mathrm{Im}\,(H_{+,i}) \cap \mathrm{Im}\,(H_{-,i})$ and $(H_2)^{-1}(2g)$.

We need number theoretical results on $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$. Let O_+ and O_- be the integer rings of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$ respectively, and $U_+ = O_+^\times$ and $U_- = O_-^\times$ the unit groups of O_+ and O_- respectively. We set

$$\omega_+ = \frac{1+\sqrt{-3}}{2}, \quad \overline{\omega_+} = \frac{1-\sqrt{-3}}{2}, \quad \omega_- = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\omega_-} = \frac{1-\sqrt{5}}{2}.$$

Then $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega_+)$ and $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\omega_-)$, and their Galois groups are

$$\mathrm{Gal}\,(\mathbb{Q}(\omega_+)/\mathbb{Q}) = \langle \sigma_+ \mid \sigma_+^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{Gal}\,(\mathbb{Q}(\omega_-)/\mathbb{Q}) = \langle \sigma_- \mid \sigma_-^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}$$

where $\sigma_\pm(\omega_\pm) = \overline{\omega_\pm}$. For $x \in \mathbb{Q}(\omega_\pm)$, we set $\overline{x} = \sigma_\pm(x)$. It is easy to see that

$$\omega_+ + \overline{\omega_+} = 1,\ \omega_+\overline{\omega_+} = 1,\ (\omega_+)^2 = -\overline{\omega_+},\ (\omega_+)^3 = -1,\ (\omega_+)^6 = 1,$$

and

$$\omega_- + \overline{\omega_-} = 1,\ \omega_-\overline{\omega_-} = -1,\ (\omega_-)^2 = 1 + \omega_- = \frac{3+\sqrt{5}}{2}.$$

Let $\{a_k\}_{k\in\mathbb{Z}}$ be the *Fibonacci series* defined by

$$a_{k+2} = a_{k+1} + a_k \quad \text{and} \quad a_1 = a_2 = 1.$$

The following is elementary results on a theory of quadratic fields except (3).

Proposition 5.7 *(1) $O_+ = \mathbb{Z}[\omega_+]$ and $O_- = \mathbb{Z}[\omega_-]$ as rings, and*

$$O_+ \cong \mathbb{Z} \oplus \mathbb{Z}\cdot\omega_+ \quad \textit{and} \quad O_- \cong \mathbb{Z} \oplus \mathbb{Z}\cdot\omega_- \cong \mathbb{Z} \oplus \mathbb{Z}\cdot(\omega_-)^2$$

as abelian groups. Both O_+ and O_- have the class number one (i.e. principal ideal domains).

(2) $U_+ = \langle \omega_+ \mid (\omega_+)^6 = 1\rangle \cong \mathbb{Z}/6\mathbb{Z}$ and $U_- = \langle \omega_-, -1 \mid (-1)^2 = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(3) (a) For $k \geq 1$, a_k is a positive integer.

(b) For $k \in \mathbb{Z}$,

$$a_k = \frac{(\omega_-)^k - (\overline{\omega_-})^k}{\omega_- - \overline{\omega_-}}, \quad a_{-k} = (-1)^{k+1}a_k,$$

and

$$(\omega_-)^k = a_{k-1} + a_k\omega_-.$$

(4) Let ℓ be a positive prime of $\mathbb{Z}$.

(a) ℓ is not a prime in O_+ if and only if $\ell = 3$ or $\equiv 1 \pmod 3$. Their prime factorizations in O_+ are

$$3 = -(\sqrt{-3})^2 \quad \textit{or} \quad \ell = \mathfrak{l}\cdot\bar{\mathfrak{l}} \quad (\mathfrak{l} \in O_+, \mathfrak{l} \neq \bar{\mathfrak{l}}),$$

respectively.

(b) ℓ is not a prime in O_- if and only if $\ell = 5$ or $\equiv \pm 1 \pmod 5$. Their prime factorizations in O_- are

$$5 = (\sqrt{5})^2 \quad \textit{or} \quad \ell = \mathfrak{l}\cdot\bar{\mathfrak{l}} \quad (\mathfrak{l} \in O_-, \mathfrak{l} \neq \bar{\mathfrak{l}}),$$

respectively.

We say that an element $b + c\omega_\pm \in O_\pm$ $(b, c \in \mathbb{Z})$ is *primitive* if $\gcd(b, c) = 1$.

Lemma 5.8 *(1) For $x = b + c\omega_\pm \in O_\pm$ $(b, c \in \mathbb{Z})$, we set $x_k = x(\omega_\pm)^{k-1}$ $(k \in \mathbb{Z})$.*

(a) For $x = b + c\omega_+ \in O_+$ $(b, c \in \mathbb{Z})$, we have

$$\begin{array}{rclrcl} x = x_1 &=& b + c\omega_+, & x_2 &=& -c + (b+c)\omega_+, \\ x_3 &=& -(b+c) + b\omega_+, & x_4 &=& -b - c\omega_+, \\ x_5 &=& c - (b+c)\omega_+, & x_6 &=& (b+c) - b\omega_+, \\ \overline{x} = \overline{x_1} &=& (b+c) - c\omega_+, & \overline{x_2} &=& b - (b+c)\omega_+, \\ \overline{x_3} &=& -c - b\omega_+, & \overline{x_4} &=& -(b+c) + c\omega_+, \\ \overline{x_5} &=& -b + (b+c)\omega_+, & \overline{x_6} &=& c + b\omega_+, \end{array}$$

and $x_{k+6} = x_k$.

(b) For $x = b + c\omega_- \in O_-$ $(b, c \in \mathbb{Z})$, we set $x_k = b_k + b_{k+1}\omega_-$ $(b_k \in \mathbb{Z})$. Then $\{b_k\}_{k \in \mathbb{Z}}$ is the Fibonacci series such that

$$b_{k+2} = b_{k+1} + b_k, \quad b_1 = b \quad \text{and} \quad b_2 = c,$$

and we have $b_k = ba_{k-2} + ca_{k-1}$ where $\{a_k\}_{k \in \mathbb{Z}}$ is the same as in Proposition 5.7 (3), and $\overline{x_k} = b_{k+2} - b_{k+1}\omega_-$.

(2) For $x = b + c\omega_\pm \in O_\pm$ $(b, c \in \mathbb{Z})$, we set

$$\mathrm{Orb}\,(x) = \{xu,\ \overline{x}u \mid u \in U_\pm\}.$$

Then one element of Orb (x) *is primitive if and only if every element of* Orb (x) *is primitive.*

(3) Let $x \in O_\pm$ be a primitive element. Then we have the following.

(a) There is a unique element $b + c\omega_+ \in \mathrm{Orb}\,(x)$ $(b, c \in \mathbb{Z})$ such that $1 \leq b \leq c$ (and $\gcd(b, c) = 1$).

(b) There is a unique element $b + c(\omega_-)^2 = (b+c) + c\omega_- \in \mathrm{Orb}\,(x)$ $(b, c \in \mathbb{Z})$ such that $1 \leq b \leq c$ (and $\gcd(b, c) = 1$).

(4) Let $x, y \in O_\pm$ be two primitive elements. Then xy is also primitive if and only if for every prime factor $\mathfrak{l}$ of x, $\overline{\mathfrak{l}}$ is not a prime factor of y.

Proof (1) We can show by straight calculations.

(2) By (1) and Proposition 5.7 (2), we have the result.

(3) By (1) and (2), they can be checked without difficulty.

(4) By Proposition 5.7 (4), we have the result. □

We can factorize

$$\begin{aligned} m^2 + mn + n^2 &= (m + n\omega_+)(m + n\overline{\omega_+}), \\ m^2 + 3mn + n^2 &= \{m + n(\omega_-)^2\}\{m + n(\overline{\omega_-})^2\} \\ &= \{(m+n) + n\omega_-\}\{(m+n) + n\overline{\omega_-}\} \end{aligned} \tag{5.1}$$

in O_+ and O_-, respectively. Note that every factor in the righthand side of (5.1) is primitive, and satisfies the conditions of Lemma 5.8 (3).

Lemma 5.9 *(1)* (Berge [Ber, Theorem 4]) *Let p be a positive integer with a prime factorization $p = 3^e \ell_1^{e_1} \cdots \ell_r^{e_r}$ where $\ell_1, \dots, \ell_r$ are distinct primes other than 3, $e \geq 0$ and $e_i \geq 1$ $(i = 1, \dots, r)$. Then $p \in \mathrm{Im}\,(H_{+,1})$ if and only if $e = 0$ or 1, and every $\ell_i \equiv 1 \pmod 3$. Moreover the number of elements of the set $(H_{+,1})^{-1}(p)$ is 2^{r-1}.*

(2) (Berge [Ber, Theorem 5]) *Let p be a positive integer with a prime factorization $p = 5^e \ell_1^{e_1} \cdots \ell_r^{e_r}$ where $\ell_1, \dots, \ell_r$ are distinct primes other than 5, $e \geq 0$ and $e_i \geq 1$ $(i = 1, \dots, r)$. Then $p \in \mathrm{Im}\,(H_{-,1})$ if and only if $e = 0$ or 1, and every $\ell_i \equiv 1$ or $-1 \pmod 5$. Moreover the number of elements of the set $(H_{-,1})^{-1}(p)$ is 2^{r-1}.*

(3) Let p be a positive integer with a prime factorization $p = \ell_1^{e_1} \cdots \ell_r^{e_r}$ where $\ell_1, \dots, \ell_r$ are distinct primes and $e_i \geq 1$ $(i = 1, \dots, r)$. Then $p \in \mathrm{Im}\,(H_{+,1}) \cap \mathrm{Im}\,(H_{-,1})$ if and only if every $\ell_i \equiv 1$ or $4 \pmod{15}$.

Proof (1) By Proposition 5.7 (4) (a), Lemma 5.8 (4), and (5.1), we have the result.

(2) By Proposition 5.7 (4) (b), Lemma 5.8 (4), and (5.1), we have the result.

(3) By (1) and (2), we have the result. □

By Proposition 3.3, Lemma 3.4 and Lemma 5.9, we make a complete table of p, (ε, m, n) and g for $p \leq 500$.

p	(ε, m, n)	g
7	$(+, 1, 2)$	1
11	$(-, 1, 2)$	3
13	$(+, 1, 3)$	3
19	$(+, 2, 3)$	5
	$(-, 1, 3)$	6
21	$(+, 1, 4)$	6
29	$(-, 1, 4)$	10
31	$(+, 1, 5)$	10
	$(-, 2, 3)$	11
37	$(+, 3, 4)$	12
39	$(+, 2, 5)$	13
41	$(-, 1, 5)$	15
43	$(+, 1, 6)$	15
49	$(+, 3, 5)$	17
55	$(-, 1, 6)$	21
57	$(+, 1, 7)$	21
59	$(-, 2, 5)$	23
61	$(+, 4, 5)$	22
	$(-, 3, 4)$	24
67	$(+, 2, 7)$	25
71	$(-, 1, 7)$	28
73	$(+, 1, 8)$	27
79	$(+, 3, 7)$	30
	$(-, 3, 5)$	32
89	$(-, 1, 8)$	36
91	$(+, 1, 9)$	36
	$(+, 5, 6)$	35
93	$(+, 4, 7)$	36
95	$(-, 2, 7)$	39
97	$(+, 3, 8)$	38
101	$(-, 4, 5)$	42
103	$(+, 2, 9)$	41
109	$(+, 5, 7)$	43
	$(-, 1, 9)$	45
111	$(+, 1, 10)$	45
121	$(-, 3, 7)$	51
127	$(+, 6, 7)$	51
129	$(+, 5, 8)$	52
131	$(-, 1, 10)$	55
133	$(+, 1, 11)$	55

p	(ε, m, n)	g
	$(+, 4, 9)$	54
139	$(+, 3, 10)$	57
	$(-, 2, 9)$	59
145	$(-, 3, 8)$	62
147	$(+, 2, 11)$	61
149	$(-, 4, 7)$	64
151	$(+, 5, 9)$	62
	$(-, 5, 6)$	65
155	$(-, 1, 11)$	66
157	$(+, 1, 12)$	66
163	$(+, 3, 11)$	68
169	$(+, 7, 8)$	70
179	$(-, 5, 7)$	78
181	$(+, 4, 11)$	76
	$(-, 1, 12)$	78
183	$(+, 1, 13)$	78
191	$(-, 2, 11)$	83
193	$(+, 7, 9)$	81
199	$(+, 2, 13)$	85
	$(-, 3, 10)$	87
201	$(+, 5, 11)$	85
205	$(-, 4, 9)$	90
209	$(-, 1, 13)$	91
	$(-, 5, 8)$	92
211	$(+, 1, 14)$	91
	$(-, 6, 7)$	93
217	$(+, 3, 13)$	93
	$(+, 8, 9)$	92
219	$(+, 7, 10)$	93
223	$(+, 6, 11)$	95
229	$(+, 5, 12)$	98
	$(-, 3, 11)$	101
237	$(+, 4, 13)$	102
239	$(-, 1, 14)$	105
241	$(+, 1, 15)$	105
	$(-, 5, 9)$	107
247	$(+, 3, 14)$	107
	$(+, 7, 11)$	106
251	$(-, 2, 13)$	111
259	$(+, 2, 15)$	113

p	(ε, m, n)	g
	$(+, 5, 13)$	112
269	$(-, 4, 11)$	120
271	$(+, 9, 10)$	117
	$(-, 1, 15)$	120
273	$(+, 1, 16)$	120
	$(+, 8, 11)$	118
277	$(+, 7, 12)$	120
281	$(-, 7, 8)$	126
283	$(+, 6, 13)$	123
291	$(+, 5, 14)$	127
295	$(-, 3, 13)$	132
301	$(+, 4, 15)$	132
	$(+, 9, 11)$	131
305	$(-, 1, 16)$	136
307	$(+, 1, 17)$	136
309	$(+, 7, 13)$	135
311	$(-, 5, 11)$	140
313	$(+, 3, 16)$	138
319	$(-, 2, 15)$	143
	$(-, 7, 9)$	144
327	$(+, 2, 17)$	145
331	$(+, 10, 11)$	145
	$(-, 3, 14)$	149
337	$(+, 8, 13)$	148
341	$(-, 1, 17)$	153
	$(-, 4, 13)$	154
343	$(+, 1, 18)$	153
349	$(+, 3, 17)$	155
	$(-, 5, 12)$	158
355	$(-, 6, 11)$	161
359	$(-, 2, 15)$	163
361	$(+, 5, 16)$	160
	$(-, 8, 9)$	164
367	$(+, 9, 13)$	162
373	$(+, 4, 17)$	166
379	$(+, 7, 15)$	168
	$(-, 1, 18)$	171
381	$(+, 1, 19)$	171
389	$(-, 5, 13)$	177
395	$(-, 2, 17)$	179

p	(ε, m, n)	g
397	$(+, 11, 12)$	176
399	$(+, 5, 17)$	178
	$(+, 10, 13)$	177
401	$(-, 7, 11)$	183
403	$(+, 2, 19)$	181
	$(+, 9, 14)$	179
409	$(+, 8, 15)$	182
	$(-, 3, 16)$	186
417	$(+, 7, 16)$	186
419	$(-, 1, 19)$	190
421	$(+, 1, 20)$	190
	$(-, 4, 15)$	192
427	$(+, 3, 19)$	192
	$(+, 6, 17)$	191
431	$(-, 5, 14)$	197
433	$(+, 11, 13)$	193
439	$(+, 5, 18)$	197
	$(-, 6, 13)$	201
449	$(-, 8, 11)$	206
451	$(-, 3, 17)$	206
	$(-, 9, 10)$	207
453	$(+, 4, 19)$	204
455	$(-, 7, 12)$	204
457	$(+, 7, 17)$	205
461	$(-, 1, 20)$	210
463	$(+, 1, 21)$	210
469	$(+, 3, 20)$	212
	$(+, 12, 13)$	210
471	$(+, 11, 14)$	211
479	$(-, 2, 19)$	219
481	$(+, 5, 19)$	217
	$(+, 9, 16)$	216
487	$(+, 2, 21)$	221
489	$(+, 8, 17)$	220
491	$(-, 7, 13)$	226
499	$(+, 7, 18)$	225
	$(-, 9, 11)$	230

From the table, we raise examples of g such that $\sharp\{(H_2)^{-1}(2g)\} \geq 2$, where $\sharp\{\cdot\}$ implies the cardinality, except the trivial case $(H_2)^{-1}(n(n-1)) = \{(+,1,n),(-,1,n-1)\}$ (Then $n \geq 2$, $H_1(+,1,n) = n^2+n+1$ and $H_1(-,1,n-1) = n^2+n-1$).

p	(ε, m, n)	g
89	$(-,1,8)$	36
91	$(+,1,9)$	36
93	$(+,4,7)$	36
121	$(-,3,7)$	51
127	$(+,6,7)$	51
145	$(-,3,8)$	62
151	$(+,5,9)$	62
179	$(-,5,7)$	78
181	$(-,1,12)$	78
183	$(+,1,13)$	78
199	$(+,2,13)$	85
201	$(+,5,11)$	85
209	$(-,5,8)$	92
217	$(+,8,9)$	92
211	$(-,6,7)$	93
217	$(+,3,13)$	93
219	$(+,7,10)$	93
241	$(-,5,9)$	107
247	$(+,3,14)$	107
269	$(-,4,11)$	120
271	$(-,1,15)$	120
273	$(+,1,16)$	120
277	$(+,7,12)$	120

p	(ε, m, n)	g
295	$(-,3,13)$	132
301	$(+,4,15)$	132
327	$(+,2,17)$	145
331	$(+,10,11)$	145
389	$(-,5,13)$	177
399	$(+,10,13)$	177
395	$(-,2,17)$	179
403	$(+,9,14)$	179
409	$(-,3,16)$	186
417	$(+,7,16)$	186
421	$(-,4,15)$	192
427	$(+,3,19)$	192
431	$(-,5,14)$	197
439	$(+,5,18)$	197
449	$(-,8,11)$	206
451	$(-,3,17)$	206
453	$(+,4,19)$	204
455	$(-,7,12)$	204
461	$(-,1,20)$	210
463	$(+,1,21)$	210
469	$(+,12,13)$	210

As stated in Theorem 4.7, the Alexander polynomials of hyperbolic Berge knots of types VII and VIII are not those of torus knots and satellite knots. However we do not know the following:

Question 5.10 *Do the Alexander polynomials of hyperbolic Berge knots of types VII and VIII identify the knots ?*

From the table above, the degrees of the Alexander polynomials do not identify hyperbolic Berge knots of types VII and VIII completely.

6 Final Remark

We give a remark about a relation between our method and K. Ichihara, T. Saito and M. Teragaito [IST] They show a formula of the Alexander polynomial of a doubly primitive knot in S^3. Let K be a doubly primitive knot in S^3. Suppose that $(K;p)$ with $p \geq 2$ is a lens space

$L(p,q)$. Then the dual knot of K in $L(p,q)$ can be expressed by using a genus two Heegaard splitting of $L(p,q)$ and a certain parameter k (*Saito parameter*) with $1 \leq k \leq p-1$. We denote the knot by $K(L(p,q),k)$.

For $i \in \mathbb{Z}$, let $\Psi(i)$ be the integral lift of $i\overline{q} \pmod p$ such that $1 \leq \Psi(i) \leq p$, and

$$\Phi(i) = \sharp\{j \mid \Psi(j) < \Psi(i) \quad \text{and} \quad 1 \leq j \leq k-1\},$$

where $\sharp\{\cdot\}$ implies the cardinality.

Theorem 6.1 (Ichihara, Saito and Teragaito [IST, Theorem 1.1]) *The Alexander polynomial of a doubly primitive knot K, whose dual knot is $K(L(p,q),k)$, is*

$$\Delta_K(t) \doteq \frac{t-1}{t^k-1} \cdot \sum_{i=0}^{k-1} t^{\Phi(i)p-\Psi(i)k}.$$

Lemma 6.2 (K. Ichihara, T. Saito and M. Teragaito [IST, Lemma 2.1]) $\gcd(k,p)=1$.

Proof We give an alternative proof here. By Theorem 6.1, we have

$$\left(\sum_{i=0}^{k-1} t^i\right) \cdot \Delta_K(t) \doteq \sum_{i=0}^{k-1} t^{\Phi(i)p-\Psi(i)k}.$$

Suppose that $d=\gcd(k,p) \geq 2$. Substituting $t=\zeta_d$ to the equation above, we have

$$0 = k \neq 0.$$

This is a contradiction. □

By the surgery formula II (Lemma 2.3 (1)), the Reidemeister torsion of $M=(K;p)$ is

$$\begin{aligned}
\tau^{\psi_d}(M) &\doteq \frac{\zeta_d-1}{\zeta_d^k-1} \cdot \frac{\zeta_d^{k^2\overline{q}}-1}{\zeta_d^{k\overline{q}}-1} \cdot \frac{1}{(\zeta_d-1)^2} \\
&\doteq \frac{\zeta_d^{k^2\overline{q}}-1}{(\zeta_d-1)(\zeta_d^k-1)(\zeta_d^{k\overline{q}}-1)},
\end{aligned}$$

where $d \geq 2$ is a divisor of p.

Lemma 6.3 *We have*

$$q \equiv \pm k^2 \pmod p$$

and

$$\tau^{\psi_d}(M) \doteq (\zeta_d^k-1)^{-1}(\zeta_d^{\overline{k}}-1)^{-1}.$$

Proof By the Franz lemma (Lemma 2.6), we have

$$k^2\overline{q} \equiv \pm 1 \quad \text{or} \quad \pm k \quad \text{or} \quad \pm k\overline{q} \pmod p.$$

From one of the latter two cases, we have

$$\tau^{\psi_d}(M) \doteq (\zeta_d - 1)^{-2},$$

which is the Reidemeister torsion of $L(p, \pm 1)$. By results due to P. Ozsváth and Z. Szabó [OS] and M. Tange [Ta], K is the unknot or the trefoil. The knots satisfy the conditions. From the first case, we have the result. □

By Lemma 6.3, $\{\pm k, \pm \overline{k} \pmod p\}$ has the same information as the Reidemeister torsions. In [ST], $k \equiv -n \cdot \overline{m+n} \pmod p$ has been obtained for the case $K = k^+(m, n)$. The relations (4.2) and (4.4) can also be obtained from the facts. M. Tange gave a comment to the author that since the Saito parameter k is determined for every Berge knot, to determine hyperbolicity of every Berge knot is not difficult by the same way as the present paper.

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Teruhisa KADOKAMI
Department of Mathematics, East China Normal University,
Dongchuan-lu 500, Shanghai, 200241, China
mshj@math.ecnu.edu.cn, kadokami2007@yahoo.co.jp