Amphicheiral links with special properties, II

Teruhisa KADOKAMI

July 11, 2011

Abstract

We determine prime amphicheiral links with at least 2 components and up to 11 crossings. There are 27 such links. We check also special amphicheiralities. Most of prime links with up to 11 crossings are detected not to be amphicheiral by a condition on the Jones polynomial. For the rest links, we applied conditions from the Alexander polynomial. We added new necessary conditions for a special case.

1 Introduction

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link in S^3 with $r \geq 1$. For an oriented knot K, we denote the orientation-reversed knot by $-K$. If φ is an orientation-reversing (orientation-preserving, respectively) homeomorphism of S^3 so that $\varphi(K_i) = \varepsilon_{\sigma(i)} K_{\sigma(i)}$ for all $i = 1, \ldots, r$ where $\varepsilon_i = +$ or $-$, and σ is a permutation of $\{1, 2, \ldots, r\}$, then L is said an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*amphicheiral link* (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*invertible link*, respectively). A term "amphicheiral link" is used as a general term for an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link. A link is said an *interchangeable link* if it is an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link such that σ is not the identity. An $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link is said an *invertible link* simply if there exists $1 \leq i \leq r$ such that $\varepsilon_i = -$. If σ is the identity, then an amphicheiral link is said a *component-preservingly amphicheiral link*, and σ may be omitted from the notation. If every $\varepsilon_i = \varepsilon$ is identical for all $i = 1, \ldots, r$ (including the case that σ is not the identity), then an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link, respectively) is said an (ε)-amphicheiral link (an (ε)-invertible link, respectively). We use the notations $+ = +1 = 1$ and $- = -1$. A link L with at least 2-component is said an *algebraically split link* if the linking number of every 2-component sublink of L is zero. We note that a component-preservingly (ε)-amphicheiral link is an algebraically split link.

Necessary conditions for the Alexander polynomials of amphicheiral knots are studied by R. Hartley [4], R. Hartley and A. Kawauchi [6], and A. Kawauchi [13] (cf. Lemma 2.2). In [13], non-invertibility of 8_{17} is firstly proved by the conditions. On the other

2010 *Mathematics Subject Classification*: 57M25, 57M27.
Keywords: amphicheiral link; Jones polynomial; HOMFLY polynomial; Alexander polynomial.

hand, T. Sakai [21] proved that any one-variable Laurent polynomial $f(t)$ over $\mathbb{Z}$ such that $f(t) = f(t^{-1})$ and $f(1) = 1$ is realized by the Alexander polynomial of a strongly invertible knot in S^3. B. Jiang, X. Lin, Shicheng Wang and Y. Wu [8] showed that (1) a twisted Whitehead doubled knot is amphicheiral if and only if it is the unknot or the figure eight knot, and (2) a prime link with at least 2 components and up to 9 crossings is component-preservingly (+)-amphicheiral if and only if it is the Borromean rings. They used S. Kojima and M. Yamasaki's η-function [15]. Shida Wang [26] determined prime component-preservingly (+)-amphicheiral links with at least 2 components and with up to 11 crossings by the same method as [8]. There are four such links (cf. Theorem 1.3 (3)). For geometric studies of symmetries of arborescent knots, see F. Bonahon and L. C. Siebenmann [2]. The author [10] studied necessary conditions for the Alexander polynomials of algebraically split component-preservingly amphicheiral links by computing the Reidemeister torsions of surgered manifolds along the link (cf. Lemma 3.5). The author and A. Kawauchi [11] obtained necessary conditions by invariants deduced from the quadratic forms of a link [12, 14], and by using the conditions they showed that the Alexander polynomial of an algebraically split component-preservingly (ε)-amphicheiral link with even components is zero (cf. Conjecture 4.1) and determined amphicheiral links with up to 9 crossings (cf. Lemma 3.6).

We determine prime amphicheiral links with at least 2 components and up to 11 crossings. For a link with the crossing number up to 9, we use the notation of D. Rolfsen's book [20], and for a link with the crossing number 10 or 11, we use a slightly modified notation from M. Thistlethwaite's table on D. Bar-Natan and S. Morrison's website [1] (see Section 3). In the present paper, we used information of the Jones polynomial and the multi-variable Alexander polynomial for the class on the website [1]. For prime links with up to 10 and 11 crossings, firstly we checked a condition on the Jones polynomial (cf. Lemma 3.1). Most of them are ruled out by the condition. For the rest links, we applied the same conditions on the Alexander polynomial as in [11] (cf. Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 3.6). 10^2_{a51} and 11^2_{n127} could not be detected not to be amphicheiral by the conditions and the HOMFLY polynomial (see Figure 4). Thus we made the following condition for a special case (cf. Lemma 1.1) which is inspired by the method in [8], and the result of [6, 13] on the Alexander polynomials of amphicheiral knots. The conditions would be useful in determining the *link-symmetric group* of a link (cf. [7, 9, 27]).

For an r-component link $L = K_1 \cup \ldots \cup K_r$, let $\mathit{\Delta}_L(t_1, \ldots, t_r)$ be the r-variable Alexander polynomial of L which is an element of the r-variable Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ over $\mathbb{Z}$ where t_i $(i = 1, \ldots, r)$ is a variable corresponding to a meridian of K_i. For two elements A and B in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ $((\mathbb{Z}/2\mathbb{Z})[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$, respectively), we denote by $A \doteq B$ ($A \doteq_2 B$, respectively) if they are equal up to multiplications of trivial units. A one variable Laurent polynomial $r(t) \in \mathbb{Z}[t^{\pm 1}]$ is *of type X* if there are integers $n \geq 0$ and $\lambda \geq 3$ with λ odd, and $f_i(t) \in \mathbb{Z}[t, t^{-1}]$ $(i = 0, 1, \ldots, n)$ such that $f_i(t) \doteq f_i(t^{-1})$, $|f_i(1)| = 1$, and for $i > 0$, $f_i(t) \doteq_2 f_0(t)^{2^i} p_\lambda(t)^{2^{i-1}}$ where

$p_\lambda(t) = (t^\lambda - 1)/(t-1)$, and

$$r(t) \doteq \begin{cases} f_0(t)^2 & (n=0), \\ f_0(t)^2 f_1(t)\cdots f_n(t) & (n \geq 1). \end{cases}$$

Lemma 1.1 *Let $L = K_1 \cup K_2$ be an oriented 2-component link in S^3 such that the linking number ℓ of L is not zero, and K_2 is the trivial knot. Let $\tilde{K}$ be the lifted knot of K_1 in the p-fold branched covering over K_2 where $p \geq 2$ is coprime to ℓ.*

(1) If L is component-preservingly amphicheiral, then $\tilde{K}$ is an amphicheiral knot.

(2) If L is component-preservingly $(-,+)$-amphicheiral, then $\tilde{K}$ is a $(-)$-amphicheiral knot, and there is an element $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $|f(1)| = 1$, $f(t^{-1}) \doteq f(-t)$, and

$$\Delta_{\tilde{K}}(t^2) \doteq \prod_{i=1}^{p-1} \Delta_L(t^2, \zeta_p^i) \doteq f(t)f(t^{-1})$$

where ζ_p is a primitive p-th root of unity.

(3) If L is component-preservingly $(+,-)$-amphicheiral, then $\tilde{K}$ is a $(+)$-amphicheiral knot, and there are $r_j(t) \in \mathbb{Z}[t, t^{-1}]$ of type X and an odd number α_j $(j = 1, \ldots, m)$ such that

$$\Delta_{\tilde{K}}(t) \doteq \prod_{i=1}^{p-1} \Delta_L(t, \zeta_p^i) \doteq \prod_{j=1}^{m} r_j(t^{\alpha_j})$$

where ζ_p is a primitive p-th root of unity. In particular, if $\tilde{K}$ is hyperbolic, then we can take $m = 1$ and $\alpha_1 = 1$.

Corollary 1.2 *Under the same setting as Lemma 1.1, if L is $(-,+)$-amphicheiral, then in the prime factorization of $|\Delta_{\tilde{K}}(-1)|$, the power of a prime which is congruent to 3 modulo 4 is even. In particular, if ℓ is odd, then $|\Delta_L(-1,-1)|$ satisfies the condition.*

Let $\mathcal{A}_n$ ($\mathcal{C}_n$, respectively) be the set of prime amphicheiral links (component-preservingly amphicheiral links, respectively) with at least 2 components and up to n crossings, and $\mathcal{A}_n^\varepsilon$ the subset of $\mathcal{A}_n$ consisting of (ε)-amphicheiral links ($\mathcal{C}_n^\varepsilon$ the subset of $\mathcal{C}_n$ consisting of component-preservingly (ε)-amphicheiral links, respectively) where $\varepsilon = +$ or $-$. It is clear that $\mathcal{A}_n \supset \mathcal{C}_n$, $\mathcal{A}_n^\pm \supset \mathcal{C}_n^\pm$, $\mathcal{A}_n \supset \mathcal{A}_n^\pm$ and $\mathcal{C}_n \supset \mathcal{C}_n^\pm$.

Theorem 1.3 *Under the setting above, we have the following:*

(1) $\mathcal{C}_{11} = \{2_1^2, 6_2^2, 6_2^3, 8_8^2, 8_6^3, 8_3^4,$

$$10^2_{a56}, 10^2_{a81}, 10^2_{a83}, 10^2_{a86}, 10^2_{a116}, 10^2_{a120}, 10^2_{a121}, 10^3_{a136}, 10^3_{a140}, 10^4_{a169},$$
$$10^2_{n36}, 10^2_{n46}, 10^4_{n107}\}.$$

(2) $\mathcal{A}_{11} \setminus \mathcal{C}_{11} = \{8^3_4, 9^2_{61}, 10^3_{a151}, 10^3_{a156}, 10^3_{a158}, 10^2_{n59}, 10^4_{n105}, 11^2_{n247}\}$.

(3) $\mathcal{C}^+_{11} = \{6^3_2, 10^3_{a140}, 10^2_{n36}, 10^4_{n107}\}$, $\mathcal{C}^-_{11} = \{10^2_{n36}\}$.

(4) $\mathcal{A}^+_{11} \setminus \mathcal{C}^+_{11} = \{8^3_4, 8^3_6, 8^4_3, 10^3_{a151}, 10^2_{n59}, 11^2_{n247}\}$,
$\mathcal{A}^-_{11} \setminus \mathcal{C}^-_{11} = \{6^3_2, 8^3_4, 8^3_6, 8^4_3, 10^3_{a140}, 10^3_{a151}, 10^2_{n59}, 10^4_{n107}, 11^2_{n247}\}$.

We remark that Theorem 1.3 (3) corresponds to the theorem of Shida Wang [26].

The following would also be useful if we determine prime amphicheiral links with the crossing number greater than 11 in the future (see also Conjecture 4.2 in Section 4).

Lemma 1.4 *The minimal crossing number of an alternating amphicheiral link is even.*

In Section 2, we prove Lemma 1.1, Corollary 1.2 and Lemma 1.4. In Section 3, we prove Theorem 1.3. In Section 4, we give some remarks related to our previous results [10].

2 Proof of Lemma 1.1, Corollary 1.2 and Lemma 1.4

To show Lemma 1.1, we need two results.

K. Murasugi [18] gave a formula of the Alexander polynomial of a periodic knot in S^3, M. Sakuma [22] extended it to the case of periodic links, and V. G. Turaev [25] extended them for more general settings. In the present paper, we use only the case of periodic knots in S^3.

Lemma 2.1 (Murasugi [18]; Sakuma [22]; Turaev [25]) *Let $L = K_1 \cup \ldots \cup K_r \cup K_{r+1}$ be an $(r+1)$-component link in S^3 such that $r \geq 1$ and K_{r+1} is the trivial knot. Let $\tilde{L} = \tilde{L}_1 \cup \ldots \cup \tilde{L}_r$ be the lifted link in the p-fold branched covering over K_{r+1} where $p \geq 2$ and $\tilde{L}_i$ is the lifted link of K_i $(i = 1, \ldots, r)$. Then $\tilde{L}$ is a p-periodic link in S^3, and we have*

$$\Delta_{\tilde{L}}(t_1, \ldots, t_r) \doteq \prod_{i=1}^{p-1} \Delta_L(t_1, \ldots, t_r, \zeta_p^i)$$

where ζ_p is a primitive p-th root of unity.

R. Hartley [4], R. Hartley and A. Kawauchi [6], and A. Kawauchi [13] gave necessary conditions on the Alexander polynomials of amphicheiral knots.

Lemma 2.2 (Hartley [4]; Hartley and Kawauchi [6]; Kawauchi [13])

(1) *Let K be a $(-)$-amphicheiral knot. Then there is an element $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $|f(1)| = 1$, $f(t^{-1}) \doteq f(-t)$, and*

$$\Delta_K(t^2) \doteq f(t)f(t^{-1}).$$

(2) *Let K be a $(+)$-amphicheiral knot. Then there are $r_j(t) \in \mathbb{Z}[t, t^{-1}]$ of type X (cf. Section 1) and an odd number α_j $(j = 1, \ldots, m)$ such that*

$$\Delta_K(t) \doteq \prod_{j=1}^{m} r_j(t^{\alpha_j}).$$

In particular, if K is hyperbolic, then we can take $m = 1$ and $\alpha_1 = 1$.

Proof of Lemma 1.1 Let $L = K_1 \cup K_2$ be an oriented 2-component component-preservingly amphicheiral link in S^3 with the linking number $\ell \neq 0$. Since the exterior of L is orienation-preserving homeomorphic to that of the mirror image L^* of L with preserving boundary components (which may not preserve orientations), $\tilde{K}$ is amphicheiral for every p which shows (1). Suppose that L is $(-,+)$-amphicheiral. Then it is easy to see that $\tilde{K}$ is a $(-)$-amphicheiral knot, and the condition of the Alexander polynomial is obtained from Lemma 2.1 and Lemma 2.2 (1). Suppose that L is $(+,-)$-amphicheiral. Then it is easy to see that $\tilde{K}$ is a $(+)$-amphicheiral knot, and the condition of the Alexander polynomial is obtained from Lemma 2.1 and Lemma 2.2 (2). ∎

Proof of Corollary 1.2 By substituting $t = \sqrt{-1}$ in the equation in Lemma 1.1 (2) (or Lemma 2.2 (1)), and a result from elementary number theory on primes in $\mathbb{Z}[\sqrt{-1}]$, we have the result. ∎

To show Lemma 1.4, we need two results.

Let $D = D_1 \cup \ldots \cup D_r$ be an oriented link diagram for an r-component link $L = K_1 \cup \ldots \cup K_r$, $w(D)$ the *writhe* of D which is the sum of the signs of the crossings, and $c(D)$ the the *crossing number* of D.

Lemma 2.3 *In the situation above, let D' be an oriented diagram obtained from D by reversing the orientation of the i-th component D_i. Then we have*

$$w(D') = w(D) - 4 \sum_{\substack{1 \leq j \leq r \\ j \neq i}} \mathrm{lk}\,(K_i, K_j)$$

where $\mathrm{lk}\,(K_i, K_j)$ is the linking number of K_i and K_j, and $w(D)$ $(\mathrm{mod}\ 4)$ does not depend on the orientation of D.

W. Menasco and M. Thistlethwaite [16] gave the affirmative answer for Tait's flyping conjecture. For an alternating link L, a *reduced diagram* of L is a diagram of L which is an alternating diagram without nugatory crossings. A *flyping* is an operation on a link diagram.

Lemma 2.4 (Menasco and Thistlethwaite [16]) *Let L be an oriented prime alternating link. Let D and D' be two reduced diagrams of L. Then D and D' are related by a finite sequence of flypings. As consequences, we have $w(D) = w(D')$, and $c(D) = c(D')$ which is the minimal crossing number of L.*

Proof of Lemma 1.4 Let L be an oriented prime alternating link, and D a reduced diagram of L. By Lemma 2.4, $w(D)$ is an invariant of L. By Lemma 2.3, $w(D) \pmod 4$ is an invariant of L as an unoriented link. Let D^* be the mirror image diagram of D. Suppose that L is amphicheiral. Then we have $w(D^*) = -w(D) \equiv w(D) \pmod 4$, and hence $2w(D) \equiv 0 \pmod 4$. It implies that the crossing number of L is even. Since a reduced diagram for a non-prime alternating link is realized by connected sums of reduced diagrams of the prime factors (see [19]), we have the result. ∎

3 Amphicheiral links with up to 11 crossings

In this section, we determine prime amphicheiral links with at least 2 components and up to 11 crossings. For a link with the crossing number up to 9, we use the notation of D. Rolfsen's book [20], and for a link with the crossing number 10 or 11, we use a slightly modified notation from M. Thistlethwaite's table on D. Bar-Natan and S. Morrison's website [1] below. In Rolfsen's table [20], an r-component link such that $r \geq 2$ and the crossing number c is denoted by c^r_k where k is the ordering of the link in the table. In Thistlethwaite's table [1], an r-component link such that $r \geq 2$ and the crossing number c is denoted by Lcak or Lcnk where 'a' implies that the link is alternating, 'n' implies that the link is non-alternating, and k is the ordering of the link in the table. We modify the notations Lcak and Lcnk into c^r_{ak} and c^r_{nk}, respectively.

We raise some conditions on the Jones polynomial and the Alexander polynomial of an amphicheiral link without proofs. For an oriented r-component link $L = K_1 \cup \ldots \cup K_r$, let $V_L(t)$ be the Jones polynomial of L with one variable t, and $P_L(m, l)$ the HOMFLY polynomial of L with two variables m and l. Then $V_L(t)$ is an element of $\mathbb{Z}[t^{\pm\frac{1}{2}}]$, and $P_L(m, l)$ is an element of $\mathbb{Z}[m^{\pm 1}, l^{\pm 1}]$. Let $L_{\varepsilon_1,\ldots,\varepsilon_r} = \varepsilon_1 K_1 \cup \ldots \cup \varepsilon_r K_r$ be an oriented link obtained from L by changing the oriented i-th component K_i into $\varepsilon_i K_i$ ($i = 1, \ldots, r$) where $\varepsilon_i = +$ or $-$, and $-L = L_{-,\ldots,-}$. Let $L^* = K_1^* \cup \ldots \cup K_r^*$ be the mirror image of L with the induced orientation.

Lemma 3.1 *Under the settings above, we have the following:*

(1) $V_L(t) \in t^{\frac{r+1}{2}} \cdot \mathbb{Z}[t^{\pm 1}]$, $V_{-L}(t) = V_L(t)$, *and* $V_{L^*}(t) = V_L(t^{-1})$.

(2) $V_{L_{\varepsilon_1,\ldots,\varepsilon_r}}(t) = t^a \cdot V_L(t)$ *where* $a = \dfrac{3}{2} \displaystyle\sum_{1 \leq i < j \leq r} (1 - \varepsilon_i \varepsilon_j) \mathrm{lk}\,(K_i, K_j)$ *if* $r \geq 2$, *and* $a = 0$ *if* $r = 1$.

(3) If L is $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral, then we have $V_L(t^{-1}) = t^a \cdot V_L(t)$ where a is the same as in (2).

(4) If L is amphicheiral, then $V_L(t^{-1})$ is equal to $V_L(t)$ up to multiplication of t^k ($k \in \mathbb{Z}$) (i.e. the coefficients of $V_L(t)$ are symmetric).

(5) $P_L(m,l) \in (ml)^{r+1} \cdot \mathbb{Z}[m^{\pm 2}, l^{\pm 2}]$, $P_{-L}(m,l) = P_L(m,l)$, and $P_{L^*}(m,l) = P_L(m,l^{-1})$.

(6) If L is $(\varepsilon_1,\ldots,\varepsilon_r;\sigma)$-amphicheiral, then we have $P_{L_{\varepsilon_1,\ldots,\varepsilon_r}}(m,l) = P_L(m,l^{-1})$.

(7) $V_L(t) = P_L\left(\sqrt{-1}\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right), \sqrt{-1}t\right)$.

Lemma 3.1 (1), (2), (5) and (7) are basic properties. It is easy to see that (3) and (4) are deduced by (1) and (2), and (6) is deduced by (5). Let L_+, L_- and L_0 be three oriented links such that they are identical except the local parts as in Figure 1.

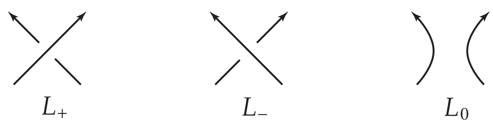


Figure 1: L_+, L_- and L_0

We computed the HOMFLY polynomial of a link by the following skein relation.

$$lP_{L_+}(m,l) + l^{-1}P_{L_-}(m,l) + mP_{L_0}(m,l) = 0, \quad P_U(m,l) = 1$$

where U implies the trivial knot.

Lemma 3.2 ([10, Lemma 2.5]) *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component $(\varepsilon_1,\ldots,\varepsilon_r;\sigma)$-amphicheiral link where $\varepsilon_i = +$ or $-$ ($i = 1,\ldots,r$), and σ is a permutation of $\{1,2,\ldots,r\}$. Then we have*

$$\Delta_L(t_1,\ldots,t_r) \doteq \Delta_L\left(t_{\sigma(1)}^{\varepsilon_{\sigma(1)}},\ldots,t_{\sigma(r)}^{\varepsilon_{\sigma(r)}}\right).$$

Lemma 3.3 ([10, Lemma 3.1], [11, Lemma 4.2]) *Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link.*

(1) If L is an $(\varepsilon_1,\ldots,\varepsilon_r;\sigma)$-amphicheiral link, then a sublink $L' = K_{i_1} \cup \cdots \cup K_{i_s}$ ($1 \le i_1 < \cdots < i_s \le r$) is an $(\varepsilon_{i_1},\ldots,\varepsilon_{i_s};\rho)$-amphicheiral link where σ is a permutation of $\{1,2,\ldots,r\}$, and ρ is a permutation of $\{i_1,i_2,\ldots,i_s\}$ induced by σ.

(2) If $r \ge 3$ is odd, and $\ell_{1,2} \cdot \ell_{2,3} \cdots \ell_{r-1,r} \cdot \ell_{r,1} \ne 0$ where $\ell_{p,q}$ is the linking number of K_p and K_q, then L is not component-preservingly amphicheiral.

(3) If $\ell_{1,2} \cdot \ell_{2,3} \cdot \ell_{3,1} \ne 0$ where $\ell_{p,q}$ is the linking number of K_p and K_q, then L is not amphicheiral.

Lemma 3.4 *Let $L = K_1 \cup K_2$ be a 2-component link with non-zero even linking number e. Then we have the following:*

(1) (Hartley [4], [10, Lemma 3.2]) *L is not component-preservingly amphicheiral.*

(2) ([11, Lemma 4.3]) *If $e \equiv 2 \pmod 4$, then L is not $(\pm, \mp; (1\ 2))$-amphicheiral where $(1\ 2)$ is the non-trivial permutation of $\{1, 2\}$.*

Lemma 3.5 ([10, Corollary 1.4]) *If $L = K_1 \cup K_2$ is an algebraically split component-preservingly amphicheiral link, then $\Delta_L(t_1, t_2)$ is divisible by $(t_1 - 1)^2(t_2 - 1)^2$.*

Lemma 3.6 ([11, Corollary 1.2]) *Let $L = K_1 \cup \ldots \cup K_r$ be an r-component amphicheiral link such that $r + \ell(L)$ is even where $\ell(L)$ is the total linking number. Then we have*

$$\Delta_L(-1, \ldots, -1) = 0.$$

In particular, if L is an (ε)-amphicheiral link where $\varepsilon = +$ or $-$, and r is even, then we have

$$\Delta_L(t, \ldots, t) = 0.$$

Prime links with up to 9 crossings have been determined in [11] (cf. Figure 2). From now on, we restrict the case that a link is prime with the crossing number 10 or 11.

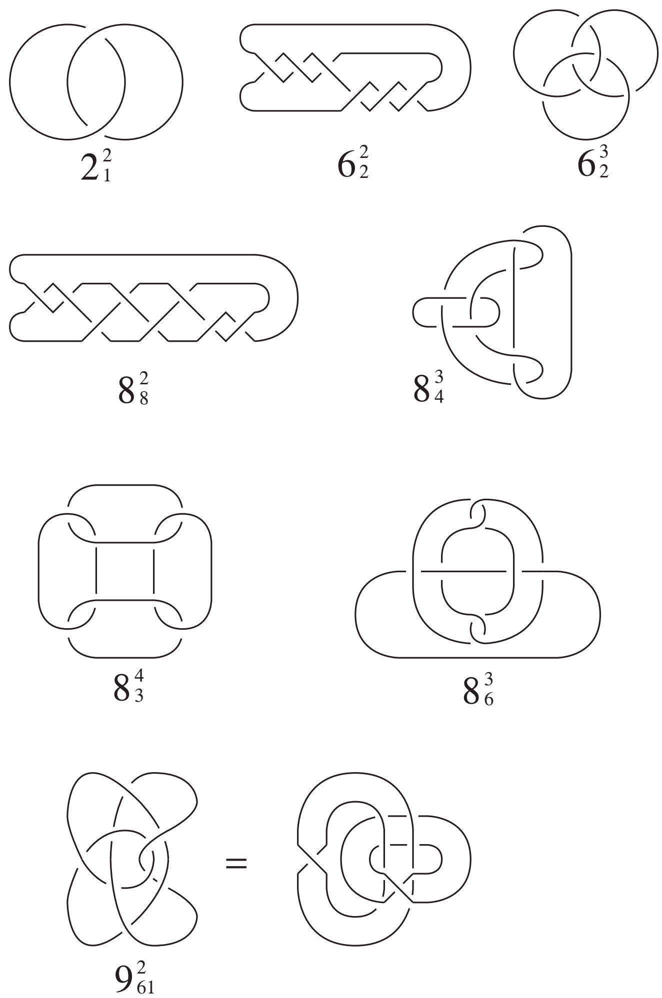


Figure 2: Prime amphicheiral links with up to 9 crossings

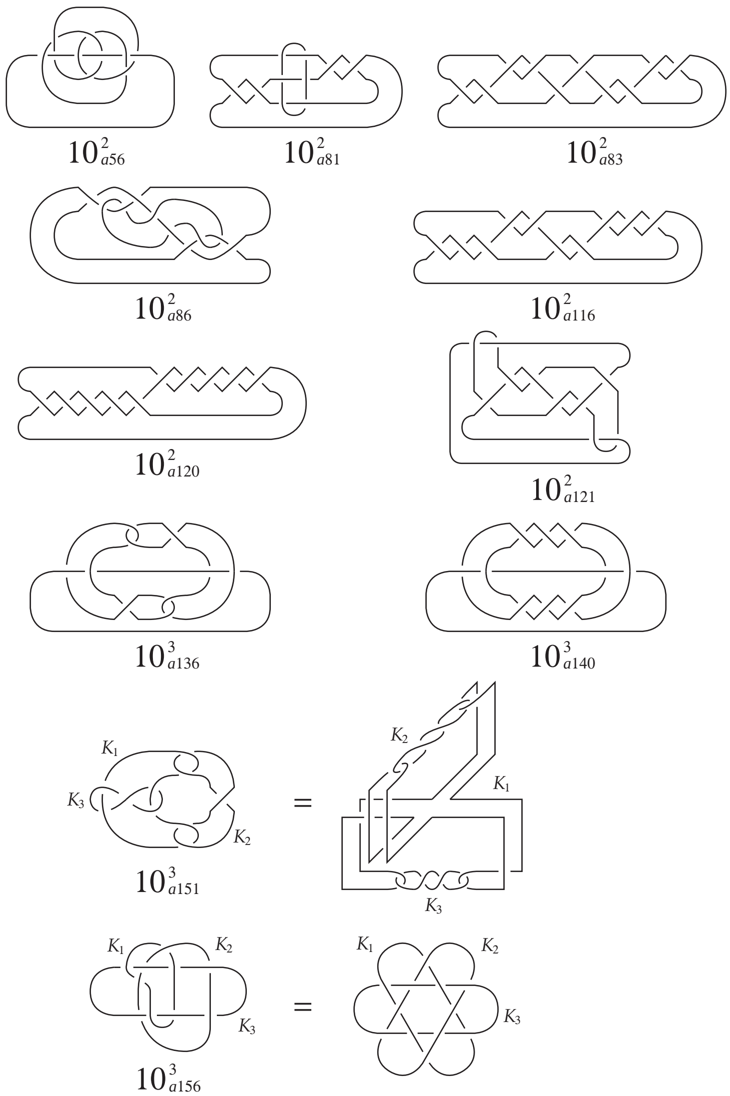
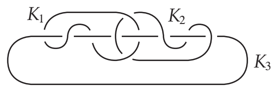
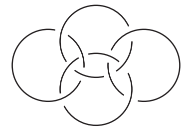


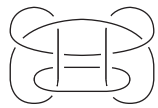
Figure 3: Prime amphicheiral links with 10 or 11 crossings 1



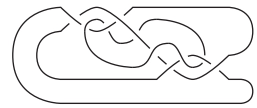
10^3_{a158}



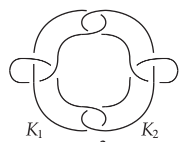
10^4_{a169}



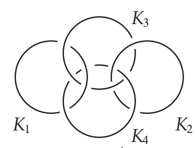
10^2_{n36}



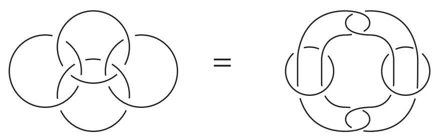
10^2_{n46}



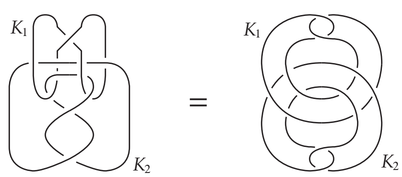
10^2_{n59}



10^4_{n105}



10^4_{n107}



11^2_{n247}

Figure 4: Prime amphicheiral links with 10 or 11 crossings 2

Firstly, we determine which links are amphicheiral, and whether they are in $\mathcal{C}_{11}$, $\mathcal{A}_{11} \setminus \mathcal{C}_{11}$, $\mathcal{C}_{11}^{\pm}$, or $\mathcal{A}_{11}^{\pm} \setminus \mathcal{C}_{11}^{\pm}$. By Figure 3 and Figure 4, 10^2_{a56}, 10^2_{a81}, 10^2_{a83}, 10^2_{a86}, 10^2_{a116}, 10^2_{a120}, 10^2_{a121}, 10^3_{a136}, 10^3_{a140}, 10^3_{a151}, 10^3_{a156}, 10^3_{a158}, 10^4_{a169}, 10^2_{n36}, 10^2_{n46}, 10^2_{n59}, 10^4_{n105}, 10^4_{n107} and 11^2_{n247} are amphicheiral. The reader can check by the figures whether they are in $\mathcal{C}_{11}$ or $\mathcal{C}_{11}^{\pm}$. Suppose that 10^3_{a140} is $(-,-,-)$-amphicheiral. Then Milnor's $\bar{\mu}$-invariant $\bar{\mu}(123)$ should be zero. However since

$$\Delta_{10^3_{a140}}(t_1,t_2,t_3) \doteq (t_1-1)(t_2-1)(t_3-1)(t_2t_3+1)^2,$$

we have $\bar{\mu}(123) = \pm 2$ (cf. [3]), and hence $10^3_{a140} \notin \mathcal{C}_{11}^{-}$. Since 10^4_{n107} has a 3-component sublink which is equivalent to 6^3_2, it cannot be component-preservingly $(-)$-amphicheiral by Lemma 3.3 (1). However both 10^3_{a140} and 10^4_{n107} are in $\mathcal{A}_{11}^{-} \setminus \mathcal{C}_{11}^{-}$ by suitable orientations. The 2-component sublinks of 10^3_{a151} are the 2-component trivial link, the positive Whitehead link and the negative Whitehead link. Since the Whitehead link is not amphicheiral, $10^3_{a151} \in \mathcal{A}_{11} \setminus \mathcal{C}_{11}$, and it is in $\mathcal{A}_{11}^{+} \setminus \mathcal{C}_{11}^{+}$ and $\mathcal{A}_{11}^{-} \setminus \mathcal{C}_{11}^{-}$ by suitable orientations. Let $L = K_1 \cup K_2 \cup K_3$ be 10^3_{a156} or 10^3_{a158}. Then the 2-component sublinks of L are the 2-component trivial link $K_1 \cup K_2$, the positive Hopf link $K_1 \cup K_3$ and the negative Hopf link $K_2 \cup K_3$ by a suitable orientation. Suppose that L is component-preservingly amphicheiral. Then

$$\Delta_L(t_1,t_2,t_3) \doteq \Delta_L(t_1,t_2,t_3^{-1})$$

should be satisfied by Lemma 3.2. Since

$$\begin{aligned}
\Delta_{10^3_{a156}}(t_1,t_2,t_3) &\doteq (t_3-1)(t_1t_2t_3 - t_1t_2 + t_1 + t_2 - 1)(t_1t_2t_3 - t_1t_3 - t_2t_3 + t_3 - 1),\\
\Delta_{10^3_{a158}}(t_1,t_2,t_3) &\doteq (t_3-1)(t_1^2t_2^2 + t_1^2t_2t_3 - t_1^2t_2 - t_1^2t_3 + t_1t_2^2t_3 - t_1t_2^2 + t_1t_2t_3^2\\
&\quad -3t_1t_2t_3 + t_1t_2 - t_1t_3^2 + t_1t_3 - t_2^2t_3 - t_2t_3^2 + t_2t_3 + t_3^2),
\end{aligned}$$

10^3_{a156} and 10^3_{a158} are not component-preservingly amphicheiral. Suppose that 10^4_{n105} is component-preservingly amphicheiral. Then

$$\Delta_{10^4_{n105}}(t_1,t_2,t_3,t_4) \doteq \Delta_{10^4_{n105}}(t_1,t_2,t_3^{-1},t_4^{-1})$$

should be satisfied by Lemma 3.2. Since

$$\Delta_{10^4_{n105}}(t_1,t_2,t_3,t_4) \doteq t_1t_2t_3t_4 - t_1t_2t_3 + t_1t_3^2t_4 - t_1t_3t_4 - t_2t_3t_4 + t_2t_4 - t_3^2t_4 + t_3t_4,$$

10^4_{n105} is not component-preservingly amphicheiral. There is one essential torus T in the exterior of 11^2_{n247}. The torus T is trivial as a torus in S^3, and we denote the core of the separated solid torus by l_i $(i=1,2)$ where l_i is in the same connected component of K_i. Suppose that 11^2_{n247} is component-preservingly amphicheiral. Then two links $K_1 \cup l_2$ and $K_2 \cup l_1$ are amphicheiral. However since they are the positive Hopf link and the negative Hopf link, respectively, and they are not amphicheiral, 11^2_{n247} is not component-preservingly amphicheiral.

By applying Lemma 3.1 (4) for the rest links, we can see non-amphicheirality of them except 10^2_{a51}, 10^2_{a57}, 10^4_{a171}, 10^2_{n49}, 10^3_{n93}, 10^4_{n108}, 11^2_{n127}, 11^2_{n158}, 11^2_{n162}, 11^2_{n205}, 11^3_{n423},

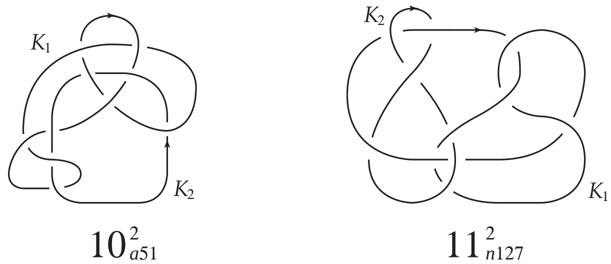


Figure 5: 10^2_{a51} and 11^2_{n127}

11^3_{n432} and 11^3_{n437}. By Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 3.6 (cf. [11, Example 4.5 and Example 4.6]), we can see non-amphicheirality of the links above except 10^2_{a51} and 11^2_{n127}.

Let $L = K_1 \cup K_2$ be an oriented 10^2_{a51} or 11^2_{n127} as in Figure 5, and $L' = (-K_1) \cup K_2$. The HOMFLY polynomials of 10^2_{a51} and 11^2_{n127} are

$$\begin{aligned}
P_{10^2_{a51}}(m,l) &= P_{(10^2_{a51})'}(m,l^{-1}) \\
&= -m^7 l^{-1} + m^5(l + 4l^{-1} + 2l^{-3}) - m^3(2l + 6l^{-1} + 5l^{-3} + l^{-5}) \\
&\quad + m(l + 2l^{-1} + 3l^{-3} + l^{-5}) + m^{-1}(l^{-1} + l^{-3}), \\
P_{11^2_{n127}}(m,l) &= P_{(11^2_{n127})'}(m,l^{-1}) \\
&= m^5 l - m^3(2l^3 + 4l + l^{-1}) + m(l^5 + 5l^3 + 5l + 2l^{-1}) \\
&\quad - m^{-1}(l^5 + 2l^3 + 2l + l^{-1}).
\end{aligned}$$

By Lemma 3.1 (6), we cannot show they are not amphicheiral by the HOMFLY polynomials.

The components of 10^2_{a51} consist of two trivial knots, and the linking number of it is 1. Since the Alexander polynomial of it is

$$\varDelta_{10^2_{a51}}(t_1,t_2) \doteq (t_1t_2^2 - 2t_1t_2 + t_1 - t_2^2 + t_2 - 1)(t_1t_2^2 - t_1t_2 + t_1 - t_2^2 + 2t_2 - 1),$$

10^2_{a51} is component-preservingly amphicheiral by Lemma 3.2 if it is amphicheiral. Since we have

$$\varDelta_{10^2_{a51}}(t,-1) = (4t-3)(3t-4),$$

10^2_{a51} is not amphicheiral by Lemma 1.1 (2) and (3).

The components of 11^2_{n127} consist of one trivial knot and one figure eight knot, and the linking number of it is 1. It is component-preservingly amphicheiral by Lemma 3.3 (1) if it is amphicheiral. Since the Jones polynomial and the HOMFLY polynomial of $\tilde{K}$ which is the lifted knot of K_1 in the 2-fold branched covering over K_2 (cf. Figure 6) are

$$\begin{aligned}
V_{\tilde{K}}(t) &= -2t^6 + 4t^5 - 6t^4 + 9t^3 - 9t^2 + 9t - 7 + 5t^{-1} - 3t^{-2} + t^{-3}, \\
P_{\tilde{K}}(m,l) &= m^4(2l^2 - 1) - m^2(4l^4 + 3l^2 - l^{-2}) + (2l^6 + 4l^4 + 2l^2 + 1),
\end{aligned}$$

11^2_{n127} is not amphicheiral by Lemma 1.1 (1), Lemma 3.1 (3) and (6).

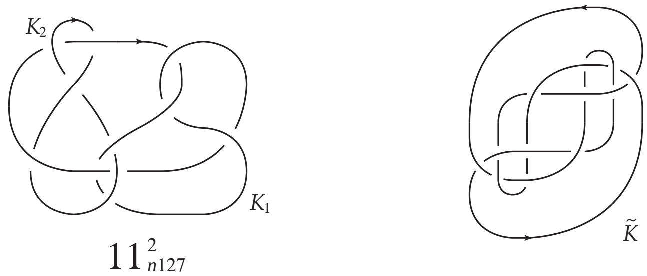


Figure 6: The lifted knot $\tilde{K}$ of K_1 in the 2-fold branched covering over K_2

We remark that we cannot complete the proof only by using the Alexander polynomial. Since the Alexander polynomial of 11^2_{n127} is

$$\begin{aligned}\Delta_{11^2_{n127}}(t_1, t_2) &\doteq t_1^2 t_2^2 - t_1^2 t_2 + t_1^2 - 2t_1 t_2^2 + t_1 t_2 - 2t_1 + t_2^2 - t_2 + 1 \\ &\doteq \left\{\left(t_1 + t_1^{-1}\right) - 2\right\}\left\{\left(t_2 + t_2^{-1}\right) - 1\right\} - 1\end{aligned} \tag{3.1}$$

we have

$$|\Delta_{11^2_{n127}}(-1, -1)| = 11 \not\equiv 1 \pmod 4,$$

and hence 11^2_{n127} is not $(-,+)$-amphicheiral by Corollary 1.2. Let ζ_p be a primitive p-th root of unity for $p \geq 2$. Then there exists $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta_{\tilde{K}}(t) \doteq \prod_{i=1}^{p-1} \Delta_{11^2_{n127}}(t, \zeta_p^i) \doteq \begin{cases} (f(t))^2(3t^2 - 5t + 3) & (p \text{ is even}), \\ (f(t))^2 & (p \text{ is odd}), \end{cases}$$

where

$$f(t) \doteq \prod_{1 \leq i < \frac{p}{2}} \Delta_{11^2_{n127}}(t, \zeta_p^i) \in \mathbb{Z}[t^{\pm 1}] \text{ (by (3.1))}.$$

Hence we cannot prove that 11^2_{n127} is not $(+,-)$-amphicheiral by Lemma 1.1 (2).

4 Further remarks

(1) In [10], the author raised a conjecture:

Conjecture 4.1 ([10, Conjecture 1.1]) *For an r-component algebraically split component-preservingly amphicheiral link L with r even, we have $\Delta_L(t_1, \ldots, t_r) = 0$.*

We gave a partial affirmative answer in [11, Theorem 1.3] for the case that L is an algebraically split component-preservingly (ε)-amphicheiral link with even components. In the prime links with up to 11 crossings, only 10^2_{n36} and 10^4_{n107} are algebraically split component-preservingly amphicheiral links with even components. They are also component-preservingly (+)-amphicheiral links (i.e. $10^2_{n36}, 10^4_{n107} \in \mathcal{C}^+_{11}$). We can confirm that the Alexander polynomials of them are 0. The condition "component-preservingly" is needed. 10^3_{a151}, 10^2_{n59} and 11^2_{n247} are algebraically split amphicheiral links in $\mathcal{A}_{11} \setminus \mathcal{C}_{11}$ and $\mathcal{A}^{\pm}_{11} \setminus \mathcal{C}^{\pm}_{11}$ whose Alexander polynomials are

$$\begin{aligned} \Delta_{10^3_{a151}}(t_1,t_2,t_3) &\doteq (t_1-1)(t_2-1)(t_3-1)(t_3^2-3t_3+1) \\ \Delta_{10^2_{n59}}(t_1,t_2) &\doteq (t_1-1)(t_2-1)(t_1-t_2)(t_1t_2-1) \\ \Delta_{11^2_{n247}}(t_1,t_2) &= 0. \end{aligned}$$

$\Delta_{10^2_{n59}}(t_1,t_2)$ satisfies the condition

$$\Delta_{10^2_{n59}}(t,t) = \Delta_{10^2_{n59}}(t,t^{-1}) = 0$$

in Lemma 3.6. We can find examples of λ-component algebraically split links in $\mathcal{C}_n \setminus \mathcal{C}^{\pm}_n$ with $\lambda \geq 4$ even in the Milnor links (see Figure 7). The Alexander polynomials of them are 0 (cf. [10, Example 6.1 (1)]).

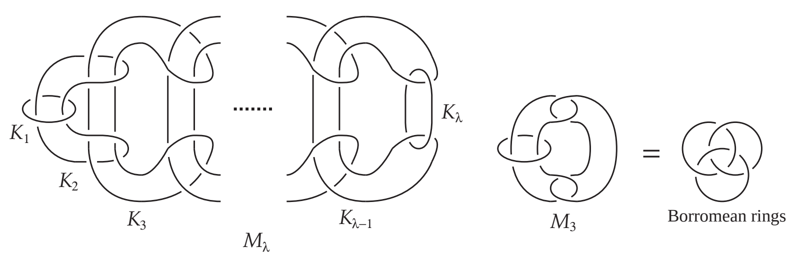


Figure 7: λ-component Milnor link M_λ

(2) In A. Stoimenow [23], the following conjecture is raised as Tait's conjecture IV.

Conjecture 4.2 ([23, Conjecture 2.4]) *The minimal crossing number of an amphicheiral knot is even.*

If there are counterexamples, then the knots are not alternating by Lemma 1.4. A. Stoimenow [24] found prime amphicheiral knots with the odd minimal crossing number c for every $c \geq 15$. He points out that if $c \leq 13$, then Conjecture 4.2 is affirmative. We have already found counterexamples 9^2_{61} and 11^2_{n247} for the case of links. However they are not component-preservingly amphicheiral. Recently Y. Kobatake found an example of a 2-component prime component-preservingly amphicheiral link with the minimal crossing number 21 and with the linking number 3, whose components consist of the unknot and an amphicheiral knot 15_{224980} in the table of [5].

Acknowledgements The author would like to express gratitude to Akio Kawauchi for giving him useful comments.

References

[1] D. Bar-Natan and S. Morrison, *Knot Atlas*, http://katlas.math.toronto.edu/wiki/Main_Page

[2] F. Bonahon and L. C. Siebenmann, *New geometric splittings of classical knots and the classification and symmetries of arborescent knots*, http://www-bcf.usc.edu/ fbonahon/Research/Preprints/BonSieb.pdf

[3] T. D. Cochran, *Concordance invariance of coefficients of Conway's link polynomial*, Invent. Math. **82** (1985), 527–541.

[4] R. Hartley, *Invertible amphicheiral knots*, Math. Ann., **252** (1980), 103–109.

[5] J. Hoste, M. Thistlethwaite and J. Weeks, *The first* $1,701,936$ *Knots*, Math. Intelligencer, **21** (1998), 33–48.

[6] R. Hartley and A. Kawauchi, *Polynomials of amphicheiral knots*, Math. Ann., **243** (1979), 63–70.

[7] J. Hillman, *Symmetries of knots and links, and invariants of abelian coverings (Part I)*, Kobe J. Math., **3** (1986), 7–27.

[8] B. Jiang, X. Lin, Shicheng Wang and Y. Wu, *Achirality of knots and links*, Top. Appl., **119** (2002), 185–208.

[9] T. Kadokami, *The link-symmetric groups of* 2*-bridge links*, to appear in Jounal of Knot Theory and its Ramifications.

[10] T. Kadokami, *Amphicheiral links with special properties, I*, to appear in Jounal of Knot Theory and its Ramifications, arXiv `math.GT/1107.0377`

[11] T. Kadokami and A. Kawauchi, *Amphicheirality of links and Alexander invariants*, to appear in SCIENCE CHINA Mathematics.

[12] A. Kawauchi, *On quadratic forms of* 3*-manifolds*, Invent. Math., **43** (1977), 177–198.

[13] A. Kawauchi, *The invertibility problem on amphicheiral excellent knots*, Proc. Japan Acad., Ser. A, Math. Sci., **55** (1979), 399–402.

[14] A. Kawauchi, *The quadratic form of a link*, in: Proc. of Low Dimensional Topology, Contemp. Math., **233** (1999), 97–116.

[15] S. Kojima and M. Yamasaki, *Some new invariants of links*, Invent. Math., **54** (1979), 213–228.

[16] W. Menasco and M. Thistlethwaite, *The classification of alternating links*, Ann. of Math., Second Series, **138** (1993), 113–171.

[17] J. W. Milnor, *Link groups*, Ann. of Math. (2) **59** (1954), 177–195.

[18] K. Murasugi, *On periodic knots*, Comment. Math. Helvetici, **46** (1971), 162–174.

[19] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology, **26** (1987), 187–194.

[20] D. Rolfsen, *Knots and Links*, Publish or Perish, Inc. (1976).

[21] T. Sakai, *Polynomials of invertible knots*, Math. Ann., **266** (1983), 229–232.

[22] M. Sakuma, *On the polynomials of periodic links*, Math. Ann., **257** (1981), 487–494.

[23] A. Stoimenow, *Tait's conjectures and odd crossing number of amphicheiral knots*, Bull. Amer. Math. Soc., **45** (2008), 285–291.

[24] A. Stoimenow, *Non-triviality of the Jones polynomial and the crossing number of amphicheiral knots*, arXiv `math.GT/0606255`

[25] V. G. Turaev, *Reidemeister torsion in knot theory*, Russian Math. Surveys, **41-1** (1986), 119–182.

[26] Shida Wang, *Strict achirality of prime links up to* 11*-crossings*, Acta Math. Sinica, **24** (2008), 997–1004.

[27] W. Whitten, *Symmetries of links*, Trans. Amer. Math. Soc., **135** (1969), 213–222.

Teruhisa KADOKAMI
Department of Mathematics,
East China Normal University,
Dongchuan-lu 500, Shanghai, 200241, China
mshj@math.ecnu.edu.cn
kadokami2007@yahoo.co.jp