Amphicheiral links with special properties, I

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Abstract

We provide necessary conditions for the Alexander polynomials of algebraically split component-preservingly amphicheiral links. We raise a conjecture that the Alexander polynomial of an algebraically split component-preservingly amphicheiral link with even components is zero. Our necessary conditions and some examples support the conjecture.

1 Introduction

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link in S^3 with $r \geq 1$. For an oriented knot K, we denote the orientation-reversed knot by $-K$. If φ is an orientation-reversing (orientation-preserving, respectively) homeomorphism of S^3 so that $\varphi(K_i) = \varepsilon_{\sigma(i)} K_{\sigma(i)}$ for all $i = 1, \ldots, r$ where $\varepsilon_i = +$ or $-$, and σ is a permutation of $\{1, 2, \ldots, r\}$, then L is said an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*amphicheiral link* (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*invertible link*, respectively). A term "amphicheiral link" is used as a general term for an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link. A link is said an *interchangeable link* if it is an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link such that σ is not the identity. An $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link is said an *invertible link* simply if there exists $1 \leq i \leq r$ such that $\varepsilon_i = -$. If σ is the identity, then an amphicheiral link is said a *component-preservingly amphicheiral link*, and σ may be omitted from the notation. We mainly deal with component-preservingly amphicheiral link in the present paper. If every $\varepsilon_i = \varepsilon$ is identical for all $i = 1, \ldots, r$ (including the case that σ is not the identity), then an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link, respectively) is said an (ε)-amphicheiral link (an (ε)-invertible link, respectively). We use the notations $+ = +1 = 1$ and $- = -1$. A link L with at least 2-component is said an *algebraically split link* if the linking number of every 2-component sublink of L is zero. We note that a component-preservingly (ε)-amphicheiral link is an algebraically split link.

Necessary conditions for the Alexander polynomials of amphicheiral knots are studied by R. Hartley [3], R. Hartley and A. Kawauchi [4], and A. Kawauchi [14]. In

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[14], non-invertibility of 8_{17} is firstly proved by the conditions. On the other hand, T. Sakai [20] proved that any one-variable Laurent polynomial $f(t)$ over $\mathbb{Z}$ such that $f(t) = f(t^{-1})$ and $f(1) = 1$ is realized by the Alexander polynomial of a strongly invertible knot in S^3. B. Jiang, X. Lin, Shicheng Wang and Y. Wu [6] showed that (1) a twisted Whitehead doubled knot is amphicheiral if and only if it is the unknot or the figure eight knot, and (2) a prime link with at least 2 components and up to 9 crossings is component-preservingly $(+)$-amphicheiral if and only if it is the Borromean rings. They used S. Kojima and M. Yamasaki's η-function [17]. Shida Wang [23] determined prime component-preservingly $(+)$-amphicheiral links with at least 2 components and up to 11 crossings by the same method as [6]. There are four such links. For geometric studies of symmetries of arborescent knots, see F. Bonahon and L. C. Siebenmann [2]. In [7], we determined symmetries such as invertibility, amphicheirality and interchangeability of 2-bridge links by the parameters such as Schubert's normal form, Conway's normal form and Conway's normal form whose entries are even integers.

In the present paper, we study necessary conditions for the Alexander polynomials of algebraically split component-preservingly amphicheiral links by computing the Reidemeister torsions of surgered manifolds along the link. The results and the techniques of the present paper have been already applied to some directions. The author and A. Kawauchi [10] obtained a necessary condition by invariants deduced from the quadratic form of a link [13, 16], showed a partial affirmative answer for the conjecture stated below, and determined prime amphicheiral links with up to 9 crossings by combining with the present results. The author [8] developed methods to detect component-preservingly amphicheiral links by the results in [6] and [3, 4, 14], and determined prime amphicheiral links with up to 11 crossings by techniques including the present results. There are 27 prime amphicheiral links with up to 11 crossings. The techniques of the present paper are based on V. G. Turaev [22]. By the same techniques, the following two results on Dehn surgeries are shown: In [11], the author, N. Maruyama and M. Shimozawa determined all Dehn surgeries yieding lens spaces (i.e. lens surgeries). In [9], we will show that the λ-component Milnor link with $\lambda \geq 4$ does not have a lens surgery by the Reidemeister torsion and some geometric techniques.

Let $\Delta_L = \Delta_L(t_1, \dots, t_r)$ be the Alexander polynomial of L which is an element of an r-variable Laurent polynomial ring $\Lambda_r := \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ over $\mathbb{Z}$ where t_i $(i = 1, 2, \dots, r)$ is a variable corresponding to a meridian of K_i.

We raise a conjecture:

Conjecture 1.1 *For an even-component algebraically split component-preservingly amphicheiral link L, we have $\Delta_L = 0$.*

Our results stated below support the conjecture. In Section 6, we explain about other supporting results in [8, 10]. We remark that the similar statement for the odd-component case does not hold. For example, a link which is connected sums of copies of the Borromean rings is a component-preservingly amphicheiral link with odd components, and has the non-zero Alexander polynomial. For any odd number, there are such examples. For an algebraically split component-preservingly amphicheiral link L, if L' is obtained from L by taking untwisted parallels of some components, and the number of components of L' is strictly greater than that of L, then L' is also an algebraically split component-preservingly amphicheiral link, and we have $\varDelta_{L'} \equiv 0$. Therefore we cannot find a counterexample for the conjecture by the construction. We raise supporting examples in Section 6.

If L is an r-component algebraically split link with $r \geq 2$, then the Alexander polynomial of L is of the form:

$$\varDelta_L = \varDelta_L(t_1, \dots, t_r) \doteq (t_1 - 1) \cdots (t_r - 1) f(t_1, \dots, t_r)$$

where we can take $f = f(t_1, \dots, t_r) \in \varLambda_r$ satisfying $f(t_1, \dots, t_r) = f(t_1^{-1}, \dots, t_r^{-1})$. Note that $f(t_1, \dots, t_r)$ is uniquely determined up to multiplication of ± 1. We set $I_r = \{1, 2, \dots, r\}$. If we take $I = \{i_1, i_2, \dots, i_s\} \subset I_r$, then we denote $L_I = K_{i_1} \cup \cdots \cup K_{i_s}$, $|I| = s$, and $\varLambda_I = \mathbb{Z}[t_{i_1}^{\pm 1}, \dots, t_{i_s}^{\pm 1}] \cong \varLambda_s$. If $s \geq 2$, then

$$\varDelta_{L_I} = \varDelta_{L_I}(t_{i_1}, \dots, t_{i_s}) \doteq \prod_{i \in I} (t_i - 1) f_I(t_{i_1}, \dots, t_{i_s})$$

where we can take $f_I = f_I(t_{i_1}, \dots, t_{i_s}) \in \varLambda_I$ satisfying $f_I(t_{i_1}, \dots, t_{i_s}) = f_I(t_{i_1}^{-1}, \dots, t_{i_s}^{-1})$, and the sign of f_I is uniquely determined by (4.2) and Lemma 4.1. In particular, $f = f_{I_r}$. We set $u(I) = (u_i)_{i \in I_r \setminus I}$ where $u_i \in \{1, -1\}$. For $J = \{j_1, \dots, j_k\} \supset I$, we use the following notations:

$$\begin{aligned}
k_J(u(I)) \quad & : \text{the number of 1 in } u_i \ (i \in J \setminus I), \\
\eta_J(u(I)) \quad & = (-1)^{k_J(u(I))}, \\
F_J(I) \quad & : \text{the polynomial obtained by substituting } t_i = 1 \ (i \in J \setminus I) \text{ to } f_J, \\
F(I) \quad & = F_{I_r}(I).
\end{aligned}$$

We set

$$S_{u(I)}^{\text{even}} = \sum_{\substack{J \supset I, |J \setminus I| : \text{even} \\ 2 \leq |J| \leq r}} \eta_J(u(I)) F_J(I),$$

and

$$S_{u(I)}^{\text{odd}} = \sum_{\substack{J \supset I, |J \setminus I| : \text{odd} \\ 2 \leq |J| \leq r}} \eta_J(u(I)) F_J(I).$$

The following is our first main theorem:

Theorem 1.2 *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component algebraically split component-preservingly amphicheiral link where $r \geq 2$, and $I \subset I_r$. Then for any $u(I)$, we have the following:*

(1) If $|I| = 1$, then $S_{u(I)}^{\text{odd}} = 0$.

(2) If $2 \leq |I| \leq r-1$, then $S^{\mathrm{even}}_{u(I)} = 0$ or $S^{\mathrm{odd}}_{u(I)} = 0$.

To prove the theorem, we compute the Reidemeister torsions of the manifolds surgered along L with associated coefficients to $u(I)$. We can deduce some corollaries.

Corollary 1.3 *Under the same assumption as in Theorem 1.2, we have the following:*

(1) If r is even and $|I| = 1$, then $F(I) = 0$.

(2) If $I = I_r \setminus \{i\}$ (i.e. $|I| = r-1$) and $\Delta_{L_I} \neq 0$, then f is divisible by $t_i - 1$.

(3) If $I = I_r \setminus \{i\}$ (i.e. $|I| = r-1$) and $F(I) \neq 0$, then $\Delta_{L_I} = 0$.

In particular, if $r = 2$, then we have the following:

Corollary 1.4 *If $L = K_1 \cup K_2$ is an algebraically split component-preservingly amphicheiral link, then Δ_L is divisible by $(t_1-1)^2(t_2-1)^2$.*

The following is our second main theorem:

Theorem 1.5 *If $L = K_1 \cup \cdots \cup K_r$ is an r-component algebraically split component-preservingly (ε)-amphicheiral link with r even, and $\varepsilon = +$ or $-$, then the Alexander polynomial of L satisfies $\Delta_L(t^{\eta_1}, \ldots, t^{\eta_r}) = 0$ where $\eta_i \in \{1, -1\}$ $(i = 1, \ldots, r)$.*

To prove the theorem, we span a Seifert surface on L. This method is a slightly extended argument in [3, Theorem 2.1]. In particular, if $r = 2$, then we have the following:

Corollary 1.6 *If $L = K_1 \cup K_2$ is an algebraically split $(\varepsilon, \varepsilon)$-amphicheiral link where $\varepsilon = +$ or $-$, then Δ_L is divisible by $(t_1-1)^2(t_2-1)^2(t_1t_2-1)(t_1-t_2)$.*

We remark that after proving the results above, the author and A. Kawauchi [10] showed that Conjecture 1.1 is afffirmative for even-component algebraically split component-preservingly (ε)-amphicheiral links by invariants deduced from the quadratic form of a link [13, 16].

In Section 2, we prepare facts on the Alexander polynomials. In Section 3, we discuss about basic properties on amphicheiral links and invertible links. We give an almost purely algebraic proof for a lemma due to Hartley [3]. In Section 4, we prove Theorem 1.2 and its corollaries. In Section 5, we prove Theorem 1.5. In Section 6, we raise some examples which support Conjecture 1.1.

2 Alexander polynomials as Reidemeister torsions

Let X be a finite CW complex, and $\psi : \mathbb{Z}[H_1(X)] \to R$ is a ring homomorphism where R is an integral domain. Then we denote the Reidemeister torsion of X related to ψ by $\tau^\psi(X) \in Q(R)$ where $Q(R)$ is the quotient field of R (see [22]). The value $\tau^\psi(X)$ is determined up to multiplication of $\pm\psi(h)$ ($h \in H_1(X)$). An equation between two values A and $B \in Q(R)$ is denoted by $A \doteq B$ if $A = \pm\psi(h)B$ for some $h \in H_1(X)$. When ψ is the identity, we denote $\tau^\psi(X)$ by $\tau(X)$.

The Alexander polynomial is a kind of the Reidemeister torsion.

Lemma 2.1 ([19, 22]) *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component link, and E_L the complement of L. Then we have*

$$\tau(E_L) \doteq \begin{cases} \dfrac{\Delta_L(t_1)}{t_1 - 1} & (r = 1), \\ \Delta_L(t_1, \ldots, t_r) & (r \geq 2). \end{cases}$$

We will use the surgery formula to show Theorem 1.1.

Lemma 2.2 (surgery formula) *Let M_0 be a compact 3-manifold whose boundary consists of tori, V a solid torus whose core is l', and $M = M_0 \cup_f V$ is the result of Dehn filling where $f : \partial V \to \partial M_0$ is an attaching map. Let $\psi : \mathbb{Z}[H_1(M)] \to R$ be a ring homomorphism where R is an integral domain, and $\psi_0 : \mathbb{Z}[H_1(M_0)] \to R$ the induced map from ψ. If $\psi([l']) \neq 1$, then we have*

$$\tau^\psi(M) \doteq \tau^{\psi_0}(M_0)(\psi([l']) - 1)^{-1}.$$

We raise some properties on the Alexander polynomials.

Lemma 2.3 (duality [15, 19, 21, 22]) *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component link. Then we have*

$$\Delta_L(t_1) = t_1^a \Delta_L(t_1^{-1}) \quad (r = 1)$$

where a is even, and

$$\Delta_L(t_1, \ldots, t_r) = (-1)^r t_1^{a_1} \cdots t_r^{a_r} \Delta_L(t_1^{-1}, \ldots, t_r^{-1}) \quad (r \geq 2)$$

where $a_i \equiv 1 + \sum_{j \neq i} \mathrm{lk}\,(K_i, K_j) \pmod 2$ $(i = 1, \ldots, r)$.

The Torres condition is a special case of the surgery formula.

Lemma 2.4 (Torres condition [15, 21, 22]) *Let $L = K_1 \cup \cdots \cup K_r \cup K_{r+1}$ be an oriented $(r+1)$-compnent link, and $L' = K_1 \cup \cdots \cup K_r$ an r-component sublink. Then we have*

$$\Delta_L(t_1, 1) \doteq \frac{t_1^\ell - 1}{t_1 - 1} \Delta_{L'}(t_1) \quad (r = 1)$$

where ℓ is the linking number of K_1 and K_2, and

$$\Delta_L(t_1, \ldots, t_r, 1) \doteq (t_1^{\ell_1} \cdots t_r^{\ell_r} - 1) \Delta_{L'}(t_1, \ldots, t_r) \quad (r \geq 2)$$

where ℓ_i is the linking number of K_i and K_{r+1} $(i = 1, \ldots, r)$.

One necessary condition for the Alexander polynomial of an amphicheiral or invertible link is the following:

Lemma 2.5 *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component $(\varepsilon_1, \ldots, \varepsilon_r)$-amphicheiral or invertible link where $\varepsilon_i = +$ or $-$ $(i = 1, \ldots, r)$. Then we have*

$$\Delta_L(t_1, \ldots, t_r) \doteq \Delta_L(t_1^{\varepsilon_1}, \ldots, t_r^{\varepsilon_r}).$$

3 Amphicheiral link and invertible link

We raise basic properties of amphicheiral links and invertible links.

Lemma 3.1 *Let $L = K_1 \cup \cdots \cup K_r$ be an r-component link.*

(1) If L is an $(\varepsilon_1, \ldots, \varepsilon_r)$-amphicheiral link, then a sublink $L' = K_{i_1} \cup \cdots \cup K_{i_s}$ $(1 \leq i_1 < \cdots < i_s \leq r)$ is an $(\varepsilon_{i_1}, \ldots, \varepsilon_{i_s})$-amphicheiral link.

(2) If $L' = K_{i_1} \cup \cdots \cup K_{i_s}$ $(1 \leq i_1 < \cdots < i_s \leq r)$ is an s-component sublink of L such that $s \geq 3$ is odd, and $\ell_{1,2} \cdot \ell_{2,3} \cdots \ell_{s-1,s} \cdot \ell_{s,1} \neq 0$ where $\ell_{p,q}$ is the linking number of K_{i_p} and K_{i_q}, then L is not component-preservingly amphicheiral.

(3) If L is an $(\varepsilon_1, \ldots, \varepsilon_r)$-invertible link, then a sublink $L' = K_{i_1} \cup \cdots \cup K_{i_s}$ $(1 \leq i_1 < \cdots < i_s \leq r)$ is an $(\varepsilon_{i_1}, \ldots, \varepsilon_{i_s})$-invertible link.

(4) Let $L = K_1 \cup K_2$ be a 2-component link with non-zero linking number. If L is an invertible link, then L is a $(-,-)$-invertible link.

The linking numbers are the first information to detect both amphicheirality and invertibility as in Lemma 3.1 (2) and (4). By Lemma 3.1 (1) and (3), to study sublinks is also important for the problems. Hartley [3] showed the following by the JSJ (Jaco-Shalen-Johanson) decomposition. The result is also about relation between amphicheirality and the linking number. We reprove it by another way.

Lemma 3.2 (Hartley [3]) *Let $L = K_1 \cup K_2$ be a 2-component link with non-zero even linking number. Then L is not component-preservingly amphicheiral.*

Proof Suppose that L is component-preservingly amphicheiral, and the linking number of K_1 and K_2 is non-zero and even. By Lemma 2.3, we may assume

$$\Delta_L(t_1, t_2) = t_1 t_2 \Delta_L(t_1^{-1}, t_2^{-1}) \tag{3.1}$$

By Lemma 2.5 and Lemma 3.1 (4), we may assume

$$\Delta_L(t_1, t_2) = \eta t_1^{b_1} t_2^{b_2} \Delta_L(t_1^{-1}, t_2)$$

where $\eta = +$ or $-$, and $b_1, b_2 \in \mathbb{Z}$. By substituting $t_2 = 1$ to (3.1), we have $\eta = +$ and $b_1 = 1$. By substituting $t_1 = 1$ to (3.1), we have $b_2 = 0$, and hence we have

$$\Delta_L(t_1, t_2) = t_1 \Delta_L(t_1^{-1}, t_2)$$

By substituting $t_1 = -1$ to the equation above, we have $\Delta_L(-1, t_2) = 0$. In the similar way, we have $\Delta_L(t_1, -1) = 0$, and hence $\Delta_L(t_1, t_2)$ is divisible by $(t_1 + 1)(t_2 + 1)$. We set $\Delta_L(t_1, t_2) = (t_1 + 1)(t_2 + 1)g(t_1, t_2)$ where $g(t_1, t_2) \in \Lambda_2$.

By substituting $t_2 = 1$ to $\Delta_L(t_1, t_2)$, and Lemma 2.4, we have

$$\Delta_L(t_1, 1) = 2(t_1 - 1)g(t_1, 1) \doteq \frac{t_1^\ell - 1}{t_1 - 1}\Delta_{K_1}(t_1).$$

Since the righthand side is not divisible by 2, we have a contradiction. ∎

Both Lemma 3.1 and Lemma 3.2 motivate us to study amphicheirality of algebraically split links. We remark that our proof of Lemma 3.2 works for the case that L is in an integral homology sphere with an orientation-reversing autohomeomorphism.

4 Proof of Theorem 1.2, Corollary 1.3 and Corollary 1.4

To prove Theorem 1.2, we study the form of the Alexander polynomial of an algebraically split link, and compute the Reidemeister torsions of surgered manifolds along the link.

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component algebraically split link where $r \geq 2$. We add one component K_{r+1} to L such that $L_i = K_i \cup K_{r+1}$ $(i = 1, \ldots, r)$ is the connected sum of K_i and the Hopf link, where the linking number of K_i and K_{r+1} is 1. Then we have

$$\Delta_{L_i}(t_i, t_{r+1}) \doteq \Delta_{K_i}(t_i) \tag{4.1}$$

We set $\overline{L} = L \cup K_{r+1}$. By Lemma 2.4, we have

$$\begin{aligned} \Delta_{\overline{L}}(t_1, \ldots, t_{r+1}) &= (t_1 \cdots t_r - 1)(t_1 - 1) \cdots (t_r - 1) f(t_1, \ldots, t_r) \\ &\quad + (t_{r+1} - 1) g(t_1, \ldots, t_{r+1}) \end{aligned} \tag{4.2}$$

where $f(t_1, \ldots, t_r) \in \Lambda_r$ and $g(t_1, \ldots, t_{r+1}) \in \Lambda_{r+1}$. By Lemma 2.3, we may assume that

$$f(t_1, \ldots, t_r) = f(t_1^{-1}, \ldots, t_r^{-1}) \tag{4.3}$$

and

$$\Delta_{\overline{L}}(t_1, \ldots, t_{r+1}) = (-1)^{r+1} t_1^2 \cdots t_r^2 t_{r+1}^a \Delta_{\overline{L}}(t_1^{-1}, \ldots, t_{r+1}^{-1}) \tag{4.4}$$

where $a \equiv r+1 \pmod 2$. For $I = \{i_1, i_2, \ldots, i_\mu\} \subset I_r$, we set $L_I = K_{i_1} \cup \cdots \cup K_{i_s}$, $\overline{L}_I = L_I \cup K_{r+1}$, and $g_I \in \Lambda_{\overline{I}}$ is obtained by substituting $t_j = 1$ for all $j \in I_r \setminus I$ to $g(t_1, \ldots, t_{r+1})$.

By (4.1) and (4.4), if $I = \{i\}$ $(s = 1)$, then we may take

$$\Delta_{\overline{L}_I}(t_i, t_{r+1}) = \Delta_{K_i}(t_i) \tag{4.5}$$

where $\Delta_{K_i}(t_i) = t_i^2 \Delta_{K_i}(t_i^{-1})$. If $2 \leq s \leq r$ $(r \geq 2)$, then we may take

$$\begin{aligned} \Delta_{\overline{L}_I}(t_{i_1}, \ldots, t_{i_s}, t_{r+1}) \;=\; & \left(\prod_{i \in I} t_i - 1\right) \prod_{i \in I}(t_i - 1) f_I(t_{i_1}, \ldots, t_{i_s}) \\ & +(t_{r+1} - 1) g'_I(t_{i_1}, \ldots, t_{i_s}, t_{r+1}) \end{aligned} \tag{4.6}$$

where

$$\Delta_{L_I}(t_{i_1}, \ldots, t_{i_s}) = \prod_{i \in I}(t_i - 1) f_I(t_{i_1}, \ldots, t_{i_s}),$$

$$f_I(t_{i_1}, \ldots, t_{i_s}) = f_I(t_{i_1}^{-1}, \ldots, t_{i_s}^{-1}) \in \Lambda_I$$

and

$$g'_I(t_{i_1}, \ldots, t_{i_s}, t_{r+1}) \in \Lambda_{\overline{I}}.$$

We set $f_I = f_I(t_{i_1}, \ldots, t_{i_s})$ and $g'_I = g'_I(t_{i_1}, \ldots, t_{i_s}, t_{r+1})$. We remark that $f_{I_r} = f(t_1, \ldots, t_r)$ and $g'_{I_r} = g(t_1, \ldots, t_r)$.

Lemma 4.1 *Under the situation above, for $1 \leq s \leq r-1$, we have*

$$g_I = (t_{r+1} - 1)^{r-s-1} \Delta_{\overline{L}_I}(t_{i_1}, \ldots, t_{i_s}, t_{r+1}).$$

Proof By applying Lemma 2.4 repeatedly, we have the result. ∎

We exapand $g(t_1, \ldots, t_{r+1})$-part in (4.2) as follows:

Lemma 4.2 *If $r \geq 2$, then we have*

$$\begin{aligned} g(t_1, \ldots, t_{r+1}) \;=\; & (t_{r+1} - 1)^{r-2} \prod_{i=1}^{r} \Delta_{K_i}(t_i) \\ & + \sum_{\substack{I \subset I_r \\ 2 \leq s = |I| \leq r-1}} \prod_{i \in I}(t_i - 1)(t_{r+1} - 1)^{r-s-1} \\ & \cdot \left\{ \left(\prod_{i \in I} t_i - 1\right) f_I + (t_{r+1} - 1) h_I \right\} \\ & + \prod_{i=1}^{r}(t_i - 1) h \end{aligned}$$

where $f_I \in \Lambda_I$, $h_I \in \Lambda_{\overline{I}}$, and $h \in \Lambda_{r+1}$.

Proof We show by induction on r.

(i) The case $r=2$.

By Lemma 4.1 and (4.5), $g(t_1,t_2,t_3)-\Delta_{K_1}(t_1)\Delta_{K_2}(t_2)$ is divisible by $(t_1-1)(t_2-1)$. Hence we have the result.

(ii) The case $r\geq 3$.

Suppose the case $r-1$. By (4.6), Lemma 4.1 and the assumption, we have

$$\begin{aligned} g(t_1,\ldots,t_{r+1}) &= \Delta_{\overline{L}_I}(t_1,\ldots,t_{r-1},t_{r+1})+(t_r-1)H_I \\ &= \left(\prod_{i\in I} t_i-1\right)\prod_{i\in I}(t_i-1)f_I+(t_{r+1}-1)g'_I+(t_r-1)H_I \\ &= \left(\prod_{i\in I} t_i-1\right)\prod_{i\in I}(t_i-1)f_I+(t_{r+1}-1)^{r-2}\prod_{i\in I}\Delta_{K_i}(t_i) \\ &\quad +\sum_{\substack{J\subset I \\ 2\leq k=|J|\leq r-2}}\prod_{i\in J}(t_i-1)(t_{r+1}-1)^{r-k-1} \\ &\quad \cdot\left\{\left(\prod_{i\in J} t_i-1\right)f_J+(t_{r+1}-1)h_J\right\} \\ &\quad +\prod_{i\in I}(t_i-1)h_I+(t_r-1)H_I \end{aligned}$$

where $I=I_r\setminus\{r\}$, $f_I\in\Lambda_I$, $g'_I\in\Lambda_{\overline{I}}$, $h_J\in\Lambda_{\overline{J}}$, and $H_I\in\Lambda_{r+1}$.

We set

$$\begin{aligned} G(t_1,\ldots,t_{r+1}) &= g(t_1,\ldots,t_{r+1})-(t_{r+1}-1)^{r-2}\prod_{i=1}^{r}\Delta_{K_i}(t_i) \\ &\quad -\sum_{\substack{I\subset I_r \\ 2\leq s=|I|\leq r-1}}\prod_{i\in I}(t_i-1)(t_{r+1}-1)^{r-s-1} \\ &\quad \cdot\left\{\left(\prod_{i\in I} t_i-1\right)f_I+(t_{r+1}-1)h_I\right\}. \end{aligned}$$

Then we have $G(t_1,\ldots,t_{r-1},1,t_{r+1})=0$. In the similar way, if we substitute $t_i=1$ for any $1\leq i\leq r$ to $G(t_1,\ldots,t_{r+1})$, then we have 0, and hence $G(t_1,\ldots,t_{r+1})$ is divisible by $\prod_{i=1}^{r}(t_i-1)$. ∎

Let E be the complement of $\overline{L}$, and $M=(\overline{L};u_1,\ldots,u_{r+1})$ the result of $(u_1,\ldots,u_{r+1})$-surgery along $\overline{L}$ where $u_i\in\mathbb{Q}\cup\{\infty,\emptyset\}$. For $I=\{i_1,i_2,\ldots,i_s\}\subset I_r$, we suppose that $u_i=\emptyset$ if $i\in I$, $u_i\in\mathbb{Q}\cup\{\infty\}$ if $i\in I_r\setminus I$, and $u_{r+1}\in\{\infty,\emptyset\}$. We set $u(I)=(u_i)_{i\in I_r\setminus I}$

and $u(\overline{I}) = u(I) \cup \{u_{r+1}\}$. If $u_{r+1} = \emptyset$, then we set $M = M_{u(\overline{I})}$, and if $u_{r+1} = \infty$, then we set $M = M_{u(I)}$. We set the natural inclusion $\iota : M_{u(\overline{I})} \hookrightarrow M_{u(I)}$. From now on, we consider only the case $u_i \in \{1, -1\}$ $(i \in I_r \setminus I)$.

Let m_i and l_i $(i = 1, \ldots, r+1)$ be a meridian and a longitude of K_i. We denote the homology class of a loop γ by $[\gamma]$. In $H_1(E)$, we have

$$\begin{aligned} [l_i] &= [m_{r+1}] \qquad\qquad (i = 1, \ldots, r) \\ [l_{r+1}] &= [m_1] \cdots [m_r] \end{aligned} \tag{4.7}$$

In $H_1(M_{u(\overline{I})})$, we have

$$[m'_i] = [m_i]^{r_i}[l_i] = [m_i]^{r_i}[m_{r+1}] = 1, \quad [l'_i] = [m_i] \qquad (i \in I_r \setminus I) \tag{4.8}$$

where m'_i and l'_i are a meridian and a longitude of the attaching solid torus for K_i $(i \in I_r \setminus I)$. In $H_1(M_{u(I)})$, we have

$$[m'_{r+1}] = [m_{r+1}] = 1, \quad [l'_{r+1}] = [l_{r+1}] \tag{4.9}$$

We note that

$$H_1(E) = \langle t_1, \ldots, t_r, t_{r+1} \rangle \cong \mathbb{Z}^{r+1},$$
$$H_1(M_{u(\overline{I})}) = \langle t_{i_1}, \ldots, t_{i_s}, t_{r+1} \rangle \cong \mathbb{Z}^{s+1},$$

and

$$H_1(M_{u(I)}) = \langle t_{i_1}, \ldots, t_{i_s} \rangle \cong \mathbb{Z}^{s},$$

where we set $t_i = [m_i]$ $(i = 1, \ldots, r+1)$.

For $J = \{j_1, \ldots, j_k\} \supset I$ and $p \in \Lambda_{\overline{J}}$, we use the following notations:

$$\begin{aligned}
k_J(u(I)) &\ : \text{the number of 1 in } u_i \ (i \in J \setminus I), \\
\rho_J(u(I)) &\ = (-1)^{k_J(u(I))} t_{r+1}^{-k_J(u(I))}, \\
\eta_J(u(I)) &\ = (-1)^{k_J(u(I))}, \\
\sigma_J(u(I)) &\ : \text{the sum of } u_i \ (i \in J \setminus I), \\
p(u(I)) &\ \ \text{the polynomial obtained by substituting } t_i = t_{r+1}^{-u_i} \ (i \in J \setminus I) \text{ to } p, \\
F_J(I) &\ : \text{the polynomial obtained by substituting } t_i = 1 \ (i \in J \setminus I) \text{ to } f_J, \\
F(I) &\ = F_{I_r}(I).
\end{aligned}$$

The Reidemeister torsions of $M_{u(\overline{I})}$ is the following:

Lemma 4.3 *Suppose that* $r \geq 2$ *and* $1 \leq |I| = s \leq r - 1$.

(1) If $I = \{x\}$ $(s = 1)$, then we have:

$$\begin{aligned}
\tau(M_{u(\overline{I})}) &\doteq \Delta_{K_x}(t_x) \prod_{i \in I_r \setminus \{x\}} \Delta_{K_i}(t_{r+1}^{-u_i}) \\
&\quad + (t_x - 1) \sum_{\substack{J \ni x \\ 2 \leq |J| \leq r}} \rho_J(u(I)) \left(t_x t_{r+1}^{-\sigma_J(u(I))} - 1 \right) f_J(u(I)) \\
&\quad + (t_{r+1} - 1) Q_I
\end{aligned}$$

where $Q_I \in \Lambda_{\overline{I}}$.

(2) If $2 \leq |I| \leq r - 1$, then we have:

$$\begin{aligned}
\tau(M_{u(\overline{I})}) &\doteq \prod_{i \in I} (t_i - 1) \sum_{\substack{J \supset I \\ 2 \leq |J| \leq r}} \rho_J(u(I)) \left(\prod_{i \in I} t_i t_{r+1}^{-\sigma_J(u(I))} - 1 \right) f_J(u(I)) \\
&\quad + (t_{r+1} - 1) Q_I
\end{aligned}$$

where $Q_I \in \Lambda_{\overline{I}}$.

Proof By Lemma 2.1, Lemma 2.2 and (4.8), we have

$$\begin{aligned}
\tau(M_{u(\overline{I})}) &\doteq \Delta_{\overline{L}}(u(I)) \prod_{i \in I_r \setminus I} (t_{r+1}^{-u_i} - 1)^{-1} \\
&\doteq \Delta_{\overline{L}}(u(I)) (t_{r+1} - 1)^{-(r-s)}
\end{aligned} \tag{4.10}$$

By (4.2), we have

$$\begin{aligned}
\Delta_{\overline{L}}(u(I)) &\doteq \left(\prod_{i \in I} t_i t_{r+1}^{-\sigma_J(u(I))} - 1 \right) \prod_{i \in I} (t_i - 1) \prod_{i \in I_r \setminus I} (t_{r+1}^{-u_i} - 1) f(u(I)) \\
&\quad + (t_{r+1} - 1) g(u(I)) \\
&= \rho_J(u(I)) \left(\prod_{i \in I} t_i t_{r+1}^{-\sigma_J(u(I))} - 1 \right) \prod_{i \in I} (t_i - 1)(t_{r+1} - 1)^{r-s} f(u(I)) \\
&\quad + (t_{r+1} - 1) g(u(I))
\end{aligned} \tag{4.11}$$

By Lemma 4.2, we have

$$\begin{aligned}
g(u(I)) &= (t_{r+1}-1)^{r-2}\prod_{i\in I}\Delta_{K_i}(t_i)\prod_{i\in I_r\setminus I}\Delta_{K_i}(t_{r+1}^{-u_i}) \\
&\quad+\sum_{\substack{J\subset I_r\\ 2\leq k=|J|\leq r-1}}\rho_J(u(I))\prod_{i\in I\cap J}(t_i-1)(t_{r+1}-1)^{r-|I\cap J|-1} \\
&\quad\cdot\left\{\left(\prod_{i\in I}t_it_{r+1}^{-\sigma_J(u(I))}-1\right)f_J(u(I))+(t_{r+1}-1)h_J(u(I))\right\} \\
&\quad+\rho_{I_r}(u(I))\prod_{i\in I}(t_i-1)(t_{r+1}-1)^{r-s}h(u(I))
\end{aligned}\tag{4.12}$$

where h_J and h are the same as in Lemma 4.2. In (4.12), we note that

$$0\leq|I\cap J|=|J|-|J\setminus I|\leq|I|=s,$$

and $|I\cap J|=s$ if and only if $J\supset I$. By the fact, and (4.10), (4.11) and (4.12), we have the result. ∎

The Reidemeister torsions of $M_{r(I)}$ are the following:

Lemma 4.4 *Suppose that $r\geq 2$ and $1\leq|I|\leq r-1$.*

(1) If $I=\{x\}$ ($s=1$), then we have:

$$\tau(M_{u(I)}) \doteq \left\{\Delta_{K_x}(t_x)+(t_x-1)^2\sum_{\substack{J\ni x\\ 2\leq|J|\leq r}}\eta_J(u(I))F_J(I)\right\}(t_x-1)^{-1}.$$

(2) If $2\leq|I|\leq r-1$, then we have:

$$\tau(M_{u(I)}) \doteq \prod_{i\in I}(t_i-1)\sum_{\substack{J\supset I\\ 2\leq|J|\leq r}}\eta_J(u(I))F_J(I).$$

Proof By (4.7) and (4.9), we have $[m_{r+1}]=t_{r+1}=1$ and $[l_{r+1}]=\prod_{i\in I}t_i$ in $H_1(M_{u(I)})$. Hence by Lemma 2.2, we have

$$\tau(M_{u(I)})\doteq\tau(M_{u(\overline{I})})|_{t_{r+1}=1}\left(\prod_{i\in I}t_i-1\right)^{-1}.$$

By combining with Lemma 4.3, we have the result. ∎

For $I \subset I_r$ with $1 \leq |I| \leq r$, and any $u(I) = (u_i)_{i \in I_r \setminus I}$, we set

$$S^{\mathrm{even}}_{u(I)} = \sum_{\substack{J \supset I, |J \setminus I|:\mathrm{even} \\ 2 \leq |J| \leq r}} \eta_J(u(I)) F_J(I),$$

$$S^{\mathrm{odd}}_{u(I)} = \sum_{\substack{J \supset I, |J \setminus I|:\mathrm{odd} \\ 2 \leq |J| \leq r}} \eta_J(u(I)) F_J(I),$$

and $-u(I)$ is obtained from $u(I)$ by replacing u_i into $-u_i$ for all $i \in I_r \setminus I$.

Lemma 4.5

$$S^{\mathrm{even}}_{-u(I)} = S^{\mathrm{even}}_{u(I)} \quad \textit{and} \quad S^{\mathrm{odd}}_{-u(I)} = -S^{\mathrm{odd}}_{u(I)}.$$

Proof For $J \supset I$, since $k_J(u(I)) + k_J(-u(I)) = |J \setminus I|$, we have

$$\eta_J(-u(I)) = (-1)^{|J \setminus I|} \eta_J(u(I)),$$

and the results. ∎

Proof of Theorem 1.2 Suppose that L is component-preservingly amphicheiral. Then $M_{u(I)}$ is homeomorphic to $M_{-u(I)}$.

(1) By Lemma 4.4 (1) and Lemma 4.5, we have

$$\tau(M_{u(I)}) \doteq \left\{ \Delta_{K_x}(t_x) + (t_x - 1)^2 \left(S^{\mathrm{even}}_{u(I)} + S^{\mathrm{odd}}_{u(I)} \right) \right\} (t_x - 1)^{-1}$$

and

$$\tau(M_{-u(I)}) \doteq \left\{ \Delta_{K_x}(t_x) + (t_x - 1)^2 \left(S^{\mathrm{even}}_{u(I)} - S^{\mathrm{odd}}_{u(I)} \right) \right\} (t_x - 1)^{-1}.$$

Since $\tau(M_{u(I)}) \doteq \tau(M_{-u(I)})$ and $\Delta_{K_x}(1) = 1 \neq 0$, we have $S^{\mathrm{odd}}_{u(I)} = 0$.

(2) By Lemma 4.4 (2) and Lemma 4.5, we have

$$\tau(M_{u(I)}) \doteq \prod_{i \in I} (t_i - 1) \left(S^{\mathrm{even}}_{u(I)} + S^{\mathrm{odd}}_{u(I)} \right)$$

and

$$\tau(M_{-u(I)}) \doteq \prod_{i \in I} (t_i - 1) \left(S^{\mathrm{even}}_{u(I)} - S^{\mathrm{odd}}_{u(I)} \right).$$

Since $\tau(M_{u(I)}) \doteq \tau(M_{-u(I)})$, we have $S^{\mathrm{even}}_{u(I)} = 0$ or $S^{\mathrm{odd}}_{u(I)} = 0$. ∎

Proof of Corollary 1.3 (1) We set $r = 2r'$ where $r' \in \mathbb{Z}$ and $r' \geq 1$. We prove by induction on r.

Suppose $r = 2$ ($r' = 1$). By Theorem 1.2 (1), we have $S^{\mathrm{odd}}_{u(I)} = \pm F(I) = 0$.

Suppose $r' \geq 2$. By the assumption of induction, we have $F_J(I) = 0$ for every J such that $|J|$ is even, and $2 \leq |J| \leq r - 2$. By Theorem 1.2 (1), we have $S^{\mathrm{odd}}_{u(I)} = \pm F(I) = 0$.

(2) Since $S^{\mathrm{even}}_{u(I)} = \pm \Delta_{L_I} \neq 0$, we have $S^{\mathrm{odd}}_{u(I)} = \pm F(I) = 0$ by Theorem 1.2 (2). The equation $F(I) = 0$ holds if and only if f is divisible by $t_i - 1$ from Lemma 2.3.

(3) Since $S^{\mathrm{odd}}_{u(I)} = \pm F(I) \neq 0$, we have $S^{\mathrm{even}}_{u(I)} = \pm \Delta_{L_I} = 0$ by Theorem 1.2 (2). ∎

Remark 4.6 By Corollary 1.3 (3) and Lemma 3.1 (1), for an algebraically split component-preservingly amphicheiral link L, if we can add one component K' to L satisfying that $L' = L \cup K'$ is also an algebraically split component-preservingly amphicheiral link such that $\Delta_{L'}$ is not divisible by $(t'-1)^2$ where t' corresponds to a meridian of K', then we have $\Delta_L = 0$. We hope that it is possible for the case that L is an algebraically split component-preservingly amphicheiral link with even components. If it is true, then Conjecture 1.1 is affirmative. However, if L is the Borromean rings (3-component link), then there does not exist a knot like K'.

Proof of Corollary 1.4 We take $I = \{2\}$. By Corollary 1.3 (2) (or Corollary 1.3 (1)), $f(t_1, t_2)$ is divisible by $t_1 - 1$. We can argue similarly for the case $I = \{1\}$. Therefore we have the result. ∎

5 Proof of Theorem 1.5 and Corollary 1.6

We prove Theorem 1.5 by a slightly generalized argument of Hartley [3, Theorem 2.1].

Proof of Theorem 1.5 Suppose that L is oriented. We span a Seifert surface F corresponding the orientation. We set a Seifert matrix from F as S. We can compute the one variable Alexander polynomial of L from S as:

$$(t-1)\Delta_L(t, \ldots, t) = \det(tS - S^T)$$

where S^T is the transposed matrix of S. Let φ be an orientation-reversing homeomorphism of S^3. Since L is $(\varepsilon, \ldots, \varepsilon)$-amphicheiral, $\varphi(F)$ is still a Seifert surface of L, and the corresponding Seifert matrix changed into $-S$. Since the S-equivalences do not change the one variable Alexander polynomial, we have

$$(t-1)\Delta_L(t, \ldots, t) = \det(-tS + S^T).$$

Since the size of S is odd, we have $\Delta_L(t, \ldots, t) = 0$. We can argue similarly if K_i is changed into $\eta_i K_i$. Therefore we have the result. ∎

Proof of Corollary 1.6 By Corollary 1.4 and Theorem 1.5, we have the result. ∎

6 Supporting examples

We raise examples which support Conjecture 1.1.

Example 6.1 (1) In Figure 1, let M_λ be the λ-component Milnor link [18] where $\lambda \geq 3$. In particular, M_3 is the Borromean rings. The Alexander polynomial of M_λ is $\Delta_{M_3}(t_1, t_2, t_3) = (t_1-1)(t_2-1)(t_3-1)$ and $\Delta_{M_\lambda}(t_1, \ldots, t_\lambda) = 0$ $(\lambda \geq 4)$. The Borromean rings M_3 is $(+,+,+)$-amphicheiral, but it is not $(-,-,-)$-amphicheiral (cf. [6, 10]).

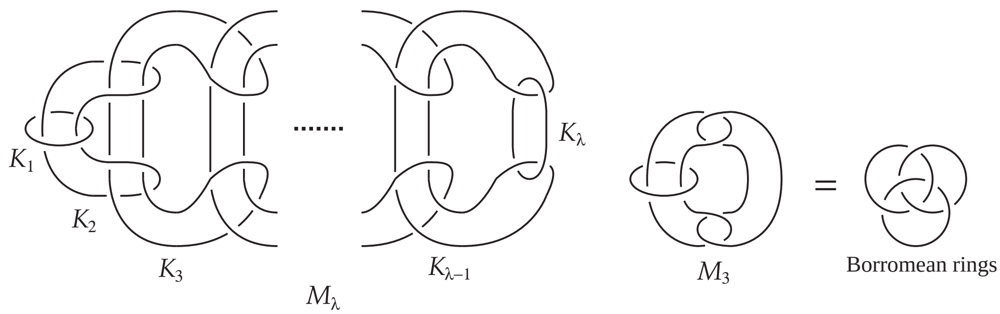

Figure 1: λ-component Milnor link M_λ

(2) In Figure 2, let $C(2a_1, 2b_1, \ldots, 2a_n)$ be a 2-component 2-bridge link where the number in a rectangle implies the number of half twists. A 2-component amphicheiral 2-bridge link is not algebraically split (see [7]), and the Alexander polynomial of every non-trivial 2-bridge link is not zero. As a special case, the Alexander polynomial of $L = C(2a, 2b, -2a)$ $(a \neq 0,\ b \neq 0)$ ($C(2, \pm 2, -2)$ is the Whitehead link) is

$$\Delta_L(t_1, t_2) = b(t_1 - 1)(t_2 - 1)\left\{\frac{(t_1t_2)^a - 1}{t_1t_2 - 1}\right\}^2$$

by Kanenobu's formula [12]. We can see that L is not amphicheiral by Corollary 1.4.

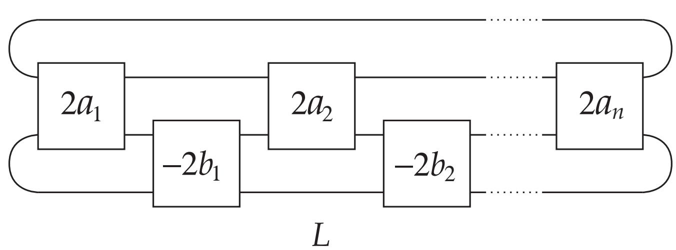

Figure 2: 2-bridge link $L = C(2a_1, 2b_1, \ldots, 2a_n)$

(3) For links with up to 11 crossings, we use slightly modified notations in a web site maintaied by D. Bar-Natan and S. Morrison [1]. In the class, only 10^2_{n36} and 10^4_{n107} are algebraically split component-preservingly amphicheiral links with even components. Moreover they are algebraically split component-preservingly (+)-amphicheiral links. We can confirm that the Alexander polynomials of them are 0 by direct computations or [10, Theorem 1.3]. We also remark that the condition "component-preservingly" is needed. 10^2_{n59} and 11^2_{n247} are algebraically split amphicheiral links with even components which are not component-preservingly amphicheiral (cf. [9]). The Alexander polynomials of them are

$$\begin{aligned} \Delta_{10^2_{n59}}(t_1, t_2) &\doteq (t_1 - 1)(t_2 - 1)(t_1 - t_2)(t_1t_2 - 1) \\ \Delta_{11^2_{n247}}(t_1, t_2) &= 0. \end{aligned}$$

We note that $\Delta_{10^2_{n59}}(t_1,t_2)$ satisfies the condition

$$\Delta_{10^2_{n59}}(t,t) = \Delta_{10^2_{n59}}(t,t^{-1}) = 0$$

in Theorem 1.5, and both 10^2_{n59} and 11^2_{n247} are $(\pm,\pm;(1\ 2))$-amphicheiral where $(1\ 2)$ is the nontrivial permutation of $\{1,2\}$.

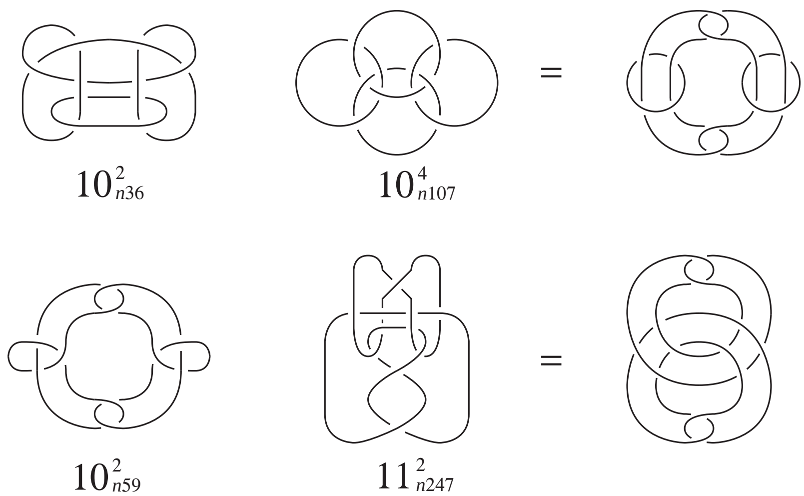

Figure 3: Examples of prime links with up to 11 crossings

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