ON L^p INEQUALITY FOR DIFFERENTIAL FORMS AND L^p COHOMOLOGY OF A SEMIALGEBRAIC SET FOR $p >> 1$

LEONID SHARTSER

Abstract. We study Poincaré type L^p inequality on a compact semialgebraic subset of $\mathbb{R}^n$ for $p >> 1$. First we derive a local inequality by using a Lipschitz deformation retraction with estimates on its derivatives. Then, we extend the local inequality to a global inequality by employing double complex technique. As a consequence we obtain an isomorphism between L^p cohomology and singular cohomology of a normal compact semialgebraic pseudomanifold.

Contents

1. Introduction

Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set and ω a smooth k-form on X_{reg}, the regular part of X. We say that ω is L^p bounded when

$$\|\omega\|_{L^p} := \left(\int_{X_{reg}} |\omega(x)|^p dVol(x) \right)^{1/p} < \infty .$$

Assume ω is a closed L^p bounded smooth k-form on X_{reg}. We prove that if all of the integrals of ω on the cycles in X vanish (see Section 5 for the precise definition

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of an integral of an L^p bounded form on a cycle in X) then there exists a smooth $(k-1)$-form ξ such that $\omega = d\xi$ and, moreover,

$$\|\xi\|_{L^p(X)} \leq C\|\omega\|_{L^p(X)} \tag{1.1}$$

holds for $p >> 1$, where C depends only on the set X and p. Of course for a contractible semialgebraic set X, there are no cycles in X. Consequently there is a ξ such that $\omega = d\xi$ and inequality (1.1) holds on X for $p >> 1$.

In [S1] we proved a generalization of inequality (1.1), but on compact manifolds. Namely, we constructed for every smooth exact k-form ω on a compact Riemannian manifold M, $\dim M = n$, a smooth $(k-1)$-form ξ on M such that $\omega = d\xi$ and inequality

$$\|\xi\|_{L^p(M)} \leq C\|d\omega\|_{L^q(M)} \tag{1.2}$$

holds for p and q in the standard range (i.e. $p < q$ or $p \geq q$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{n}$) with a positive constant C depending only on p, q, k and the manifold M. We proved (1.2) first on a convex set following the arguments of Lemma 3.11 of [BoMi] and then derived the global version by means of a method (suggested to us by P. Milman) based on the Weil's double complex. The local version of (1.2) appeared in [IwLu]. In this article we will make use of the 'globalization' method of the proof of (1.2) in [S1].

The main difficulty in extending the proof of (1.2) to a neighborhood of a point in a set with singularities is that we can no longer connect any two points by a straight line that lies entirely in the set.

To overcome this difficulty we make use of a Lipschitz deformation retraction r to a single point with estimates on its derivatives, namely:

Theorem 6.12 *(Lipschitz deformation retraction theorem) Let Σ_0 be a stratification of $\mathbb{R}^n$, $X = \cup X_j$, $X_j \in \Sigma_0$, $0 \in \overline{X}_j \cap X$, $j = 1, \ldots, m$. There exist a stratified neighborhood (U, Σ_U) of 0 in $\mathbb{R}^n$ with Σ_U a cell subdivision such that $\Sigma_U \prec \Sigma \cap U$ and a Lipschitz semialgebraic deformation retraction $r : U \times I \to U$, such that*

(1) $r_0(x) = 0$, $r_1(x) = x$,
(2) $r|_{S\times(0,1]}$ is smooth ,
(3) $|\det Dr_t| \gtrsim t^\mu$, for some $\mu \geq 0$,
(4) $\|Dr_t\| \lesssim t^\lambda$ for some $\lambda > 0$,

where $r_t(x) := r(x,t)$, Dr_t is the tangent map of r_t and $\|Dr_t\|$ denotes the operator max-norm of the tangent map.

By means of the latter we define a homotopy operator R such that for a closed form ω we have $\omega = dR\omega$ and our estimates on the derivatives of the deformation retraction r allow us to prove that for $p >> 1$ our homotopy operator R is an L^p bounded operator. Consequently we conclude that $\xi := R\omega$ is the solution to inequality (1.1) on a neighborhood of a point in X.

Fortunately, 'globalization' of the local L^p inequality (1.1) to a semialgebraic set can be carried out essentially just like for a smooth manifold in [S1]. The basic two facts that are needed for proving this global version is the validity of the local version of inequality (1.1) and the existence of a partition of unity (with locally bounded differentials), which semialgebraic sets admit.

One of the most important applications of the local version of inequality (1.1) is in the theory of L^p cohomology on semialgebraic sets. To define L^p cohomology we consider a differential complex consisting of the L^p bounded forms with L^p bounded weak exterior derivatives on the regular part of the set in question. L^p cohomology is defined as the factor space of closed L^p bounded forms by the exact L^p bounded forms. Of course for compact semialgebraic sets, L^p cohomology is an invariant of the induced metric. But the question of finiteness of the latter in general (for any $p \leq \infty$) was open. The L^p cohomology theory is addressed in several special cases by various authors (see e.g. [Ch],[Y],[HP],[GKS],[GKS2],[GKS3],[Gr]).

In this article we show that as a consequence of inequality (1.1) L^p Poincaré lemma is valid for $p >> 1$ and hence L^p cohomology coincides with the singular cohomology of a compact (normal) semialgebraic set.

1.1. **Organization of the article.** In Section 3 we prove a local inequality of the form of inequality (1.1) for smooth L^p bounded forms in a neighborhood of a point in a semialgebraic set.

In Section 4 we give an application of the local L^p inequality to L^p cohomology of a compact normal semialgebraic set X. We show that for $p >> 1$ the L^p cohomology of such sets coincides with the singular cohomology by means of a sheaf theoretic argument.

In Section 5 we extend the local inequality to a global inequality on compact semialgebraic sets. We show that under certain conditions, which we express combinatorially, closed L^p bounded forms satisfy (1.1) for $p >> 1$.

Section 6 is introduction to the construction of the Lipschitz deformation retraction with estimates on derivatives.

In Section 7 we introduce some technical material needed for our construction of Lipschitz deformation retraction.

Finally, in Section 8 we prove the Lipschitz deformation retraction Theorem 6.12.

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2. Notations and basic definitions

Let $X \subset \mathbb{R}^n$ be a semialgebraic set. Denote by X_{reg} the subset of X consisting of points where X is a smooth manifold and set $X_{sing} := X - X_{reg}$. Denote by $\overline{X}$ the closure of X and by bd X the topological boundary of X.

- $(\Omega^\bullet(X_{reg}), d)$ denotes the complex of smooth k-forms on X_{reg} and with exterior derivative $d : \Omega^k(X_{reg}) \to \Omega^{k+1}(X_{reg})$.

- For a form $\omega \in \Omega^k(X_{reg})$ define the L^p norm by

$$\|\omega\|_{L^p} := \left(\int_{X_{reg}} |\omega(x)|^p dVol(x) \right)^{1/p} < \infty,$$

where $|\omega(x)|$ is the pointwise norm of ω at the point $x \in X_{reg}$ defined by

$$\sup_{v \in \wedge^k(X_{reg})} \frac{|\omega(x;v)|}{|v|}.$$

Suppose that X_{reg} is of dimension n and $\omega \in \Omega^k_{L^p}(X_{reg})$. A form $\gamma \in \Omega^{k+1}_{L^p}(X_{reg})$ is said to be **the weak exterior derivative** of ω if for every point $p \in X_{reg}$ there exists a neighborhood U such that for every smooth $(n-k-1)$-form ϕ supported in U we have

$$\int_U \omega \wedge d\phi = (-1)^{k+1} \int_U \gamma \wedge \phi.$$

The weak exterior derivative of ω is denoted by $\overline{d}\omega$.

3. Local L^p inequality on a semialgebraic set

Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set with $a \in X$. Denote by $(\Omega^\bullet_{L^p}(X_{reg}), \overline{d})$ the complex of L^p bounded forms with L^p bounded weak exterior derivatives, i.e., forms ω with

$$\|\omega\|_{L^{p,1}} := \|\omega\|_{L^p} + \|\overline{d}\omega\|_{L^p} < \infty.$$

We say that X admits a **local L^p estimate** near a if there is a neighborhood U of a in X such that for every closed smooth L^p bounded k-form ω, $k \geq 1$, defined in U there is a smooth form ξ, defined in U, such that

$$\begin{cases} \omega = d\xi \ \text{ in } U, \\ \|\xi\|_{L^p(U)} \leq C\|\omega\|_{L^p(U)} \end{cases} \tag{3.3}$$

where $C > 0$ is independent of ω.

We prove in this section that X admits local L^p estimate for $p >> 1$. The main technical tool is our Lipschitz deformation retraction Theorem 6.12.

3.1. **Homotopy Opertator.** Let (U, Σ) be a stratified neighborhood and $r : U \times I \to U$, $I := [0,1]$, be the Lipschitz semialgebraic deformation retraction obtained by applying Theorem 6.12 to the set X and any stratification of $\mathbb{R}^n$ that is compatible with X. Let $\varepsilon > 0$. We associate a homotopy operator R_ε with the deformation retraction r as follows:
Let α be an L^p bounded smooth k-form on X_{reg}. The pull back $r^*\alpha$ is a form on $U \times I$ and can be represented as $\alpha_0 + dt \wedge \alpha_1$ where t is the coordinate in I. Define an operator

$$P : \Omega^k_{L^p}(U) \to \Omega^{k-1}_{L^p}(U \times I), \quad P\alpha := \alpha_1.$$

Set

$$R_\varepsilon \alpha := \int_\varepsilon^1 \alpha_1(x,t) dt.$$

Observe that $R_\varepsilon\alpha$ is defined almost everywhere on every stratum of Σ that is contained in U. Next we show that R_ε is an L^p bounded operator (for p large enough) and therefore $R_\varepsilon\alpha$ defines an element in L^p. We will need the following lemma.

Lemma 3.1. *Suppose that $S \subset \mathbb{R}^n$ is a locally closed oriented submanifold of dimension k and $\phi : D \to S$ is a bi-Lipschitz diffeomorphism from an open and bounded domain $D \subset \mathbb{R}^k$. Then,*

$$\int_S f(x)dVol(x) \sim \int_D f(\phi(x))dx_1 \dots dx_k \text{ for any } f : S \to \mathbb{R} \ ,$$

where $dVol(x)$ is the volume form on S and $x_1, \dots, x_k$ are coordinates in D.

Proof. In coordinates $x = (x_1, \dots, x_k)$ on D and $y = (y_1, \dots, y_n)$ on $\mathbb{R}^n \supset S$ the description of ϕ is $y_1 = \phi_1(x), \dots, y_n = \phi_n(x)$. Also

$$\frac{\partial}{\partial x_i} := \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} \in T_yS \subset \mathbb{R}^n, \ 1 \le i \le k,$$

is a basis of tangent vectors to S at the point $y = \phi(x)$. Thus the volume form $dVol(x)$ in the induced from $\mathbb{R}^n$ Riemannian metric on $S \subset \mathbb{R}^n$ can be written as

$$dVol(x) = \sqrt{\det\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle_{i,j}} dx_1 \dots dx_k \ ,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Since ϕ is bi-Lipschitz, the 'volume density' function $\sqrt{\det\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle_{i,j}}$ is bounded and is non vanishing. Therefore this function is 'equivalent' to a constant. □

Theorem 3.2. *(Local L^p inequality Theorem) Suppose that ω is a smooth L^p bounded k-form, $k \in \mathbb{N}$, defined on X_{reg} near $0 \in X$. Then there is a neighborhood U of $0 \in X$ such that*

(i) $\|R_\varepsilon\omega\|_{L^p(U)} \le C\|\omega\|_{L^p(U)}$,
(ii) $R_\varepsilon\omega \to R_0\omega$ *in* L^p,
(iii) $\|r_\varepsilon^*\omega\|_{L^p(U)} \to 0$ *as* $\varepsilon \to 0$.

where $p >> 1$ and $C > 0$ depend only on the set X.

Proof. Let (U, Σ) be a stratified neighborhood of $0 \in X$ and $r : U \times I \to U$ be a Lipschitz deformation retraction given by Theorem 6.12. Clearly it is enough to prove Theorem 3.2 for every $S \in \Sigma$ of dimension $\dim X$ which is a stratum contained in U. So let S be such a stratum. Let $\omega_1 := P\omega$. Then

$$\|R_\varepsilon\omega\|_{L^p(S)} \quad = \quad \| \int_\varepsilon^1 \omega_1(x,t)dt\|_{L^p(S)} \ .$$

Note that $\omega_1(x, t; \cdot) = r^*\omega(x, t; \frac{\partial}{\partial t}, \cdot)$ or equivalently, for every $v \in \wedge^{k-1}(\mathbb{R}^n)$

$$\omega_1((x,t);v) = \omega(r(x,t); r_* \frac{\partial}{\partial t} \wedge r_* v) \ ,$$

holds, where r_* denotes the push forward map of the deformation retraction r.

According to Theorem 6.12 there is $\lambda > 0$ such that an upper bound $\|Dr_t\| \lesssim t^\lambda$ holds. It follows that

$$\begin{aligned} |\omega_1(x,t)| &= \sup_{|v|=1} \left| \omega(r(x,t); r_* \frac{\partial}{\partial t} \wedge r_* v) \right| \\ &\leq |\omega(r(x,t)| \sup_{|v|=1} \left| r_* \frac{\partial}{\partial t} \wedge r_* v) \right| \\ &\leq C\, |\omega(r(x,t)|\, \|Dr_t\|^{k-1} \\ &\leq C\, |\omega(r(x,t)|\, t^{(k-1)\lambda} \ . \end{aligned}$$

Consequently

$$\begin{aligned} \| \int_\varepsilon^1 \omega_1(x,t) dt \|_{L^p(S,dx)} &\lesssim \| \int_\varepsilon^1 t^{(k-1)\lambda} |\omega(r(x,t))| dt \|_{L^p(S,dx)} \\ &\leq \int_\varepsilon^1 \| |\omega(r(x,t))| \|_{L^p(S,dx)} t^{(k-1)\lambda} dt \\ &= \int_\varepsilon^1 \| |\omega(z)| \left| \frac{\partial x}{\partial z} \right|^{\frac{1}{p}} \|_{L^p(S,dz)} t^{(k-1)\lambda} dt \end{aligned}$$

and since $\left|\frac{\partial z}{\partial x}\right| \gtrsim t^\mu$ for $z := r(x,t)$, the upper bound on the latter is

$$\begin{aligned} &\leq \int_\varepsilon^1 t^{-\frac{\mu}{p}+(k-1)\lambda} \| |\omega(z)| \|_{L^p(S,dz)} dt \\ &= \frac{p}{p\,(1+(k-1)\lambda)-\mu} \left(1 - \varepsilon^{-\frac{\mu}{p}+(k-1)\lambda+1} \right) \|\omega\|_{L^p(S)} \ . \end{aligned}$$

Therefore, for any $p > \frac{\mu}{1+(k-1)\lambda}$ and any $\varepsilon \geq 0$ the homotopy operator R_ε is a bounded operator between the L^p spaces of differential forms.

For part (ii), we have to show that if $\varepsilon, \varepsilon'$ are small then $\|R_\varepsilon \omega - R_{\varepsilon'} \omega\|_{L^p(S)}$ is small. The same type of computation as in the previous paragraph (replacing integration from ε to 1 by integration from ε' to ε) implies part (ii).

To prove part (iii) we will make use of

$$\begin{aligned} \|r_\varepsilon^* \omega\|_{L^p(S)}^p &= \int_S |r_\varepsilon^* \omega|^p dx \\ &\leq \int_S |\omega(r_\varepsilon(x))|^p \|Dr_\varepsilon\|^{pk} dx \\ &\leq \int_S |\omega(r_\varepsilon(x))|^p \varepsilon^{pk\lambda} dx \end{aligned}$$

and with $z = r_\varepsilon(x)$ the upper bound for the latter is

$$\begin{aligned} &\leq \int_S |\omega(z)|^p \left|\frac{\partial x}{\partial z}\right| \varepsilon^{pk\lambda} dz \\ &\lesssim \int_S |\omega(z)|^p \varepsilon^{-\mu} \varepsilon^{pk\lambda} dz \end{aligned}$$

Therefore $p > \frac{\mu}{k\lambda}$ implies $\|r_\varepsilon^* \omega\|_{L^p} \to 0$ as $\varepsilon \to 0$, as required.

□

Next, we show that R_ε satisfies the classical homotopy identity.

Proposition 3.3. *Suppose that V is a stratified neighborhood of $0 \in X$ provided by Theorem 6.12 and consequently the operator R_ε is defined on V. The homotopy operator R_ε satisfies the following homotopy identity:*

$$\overline{d} R_\varepsilon \alpha + R_\varepsilon d\alpha = \alpha - r_\varepsilon^* \alpha$$

for any smooth L^p bounded form α defined on $U := X_{reg} \cap V$.

Proof. In order to check this identity let ϕ be a smooth $(n-k)$-form with a compact support in U. We have to show that

$$\int_U (\alpha - r_\varepsilon^* \alpha - R_\varepsilon d\alpha) \wedge \phi = (-1)^k \int_U R_\varepsilon \alpha \wedge d\phi. \tag{3.4}$$

Note that since r is Lipschitz and α is smooth on X_{reg} the pullback $r^*\alpha$ of the form α is an L^∞ form on X in the sense of [SV] on

$$\{x \in S : d(x, X_{sing} \cap S) \geq \varepsilon\} \times (0, 1]$$

for every stratum $S \in \Sigma$, $S \subset X_{reg}$ and any $\varepsilon > 0$. As a result, the forms $\alpha_1 \wedge \phi$ and $\alpha_1 \wedge d\phi$, where $\alpha_1 := P\alpha$ are also L^∞ forms in the sense of [SV]. To be precise we mean that there is a stratification $\mathcal{C}$ of $U \times [\varepsilon, 1]$ such that the forms $\alpha_1 \wedge \phi$ and $\alpha_1 \wedge d\phi$ are stratified and bounded with stratified and bounded exterior derivatives. In the computation below exterior derivative of a form is calculated on each stratum separately.

Below we denote by d_x the exterior derivative with respect to x (ignoring the variable t). We begin by analyzing the right hand side of (3.4):

$$\begin{aligned}
\int_U R_\varepsilon \alpha \wedge d\phi &= \int_U \int_\varepsilon^1 \alpha_1(x,t) dt \wedge d\phi \\
&= \int_{U\times[\varepsilon,1]} \alpha_1(x,t) dt \wedge d\phi \\
&= (-1)^k \int_{U\times[\varepsilon,1]} d(\alpha_1(x,t)dt\wedge\phi) - d_x\alpha_1 dt \wedge \phi \\
&= (-1)^k \int_{U\times[\varepsilon,1]} d(\alpha_1(x,t)dt\wedge\phi) - (-1)^k \int_{U\times[\varepsilon,1]} d_x\alpha_1 dt\wedge\phi \\
&= (-1)^{k+1} \int_{U\times[\varepsilon,1]} d_x\alpha_1 dt \wedge \phi.
\end{aligned} \tag{3.5}$$

The latter equality makes use of the Stokes' formula for L^∞ forms. Indeed,

$$\begin{aligned}
\int_{U\times[\varepsilon,1]} d(\alpha_1(x,t)dt\wedge\phi) &= \int_{\partial(U\times[\varepsilon,1])} \alpha_1(x,t)dt\wedge\phi \\
&= \int_{\partial U\times[\varepsilon,1]} \alpha_1(x,t)dt\wedge\phi + (-1)^n \int_{U\times\partial[\varepsilon,1]} \alpha_1(x,t)dt\wedge\phi
\end{aligned}$$

The first summand in the latter equation equals to zero since ϕ is compactly supported in U and the second summand vanishes since on $U\times\partial[\varepsilon,1]$ the variable t is locally constant and therefore $dt=0$.

Next we simplify the left hand side of the equation (3.4):

$$\int_U (\alpha - r_\varepsilon^*\alpha - R_\varepsilon d\alpha)\wedge\phi = \int_U (\alpha - r_\varepsilon^*\alpha)\wedge\phi - \int_{U\times[\varepsilon,1]} \left(\frac{\partial\alpha_0}{\partial t} - d_x\alpha_1\right) dt\wedge\phi \ ,$$

where $\alpha_0 := r^*\alpha - P\alpha$. Formula (3.4) follows from

$$\int_U (\alpha - r_\varepsilon^*\alpha)\wedge\phi = \int_{U\times[\varepsilon,1]} \frac{\partial\alpha_0}{\partial t} dt\wedge\phi,$$

which we prove below.

By refining the stratification $\mathcal{C}$ we may assume that each stratum $S\in\mathcal{C}$ is a cell in $\mathbb{R}^n\times[\varepsilon,1]$. In particular, it means that the projection of each cell $S\in\mathcal{C}$ to the first n coordinates is a cell in U. Hence, we may assume that $\mathcal{C}=\{S_{i,j}\}$, where $i,j\in\mathbb{N}$, and

$$S_{i,j} := \{(x,t)\in U\times[\varepsilon,1] : \eta_{i,j}(x)\le t\le\eta_{i,j+1}(x), x\in S_i'\} \ ,$$

where S_i' is the projection to the first n coordinates of the set $S_{i,j}$ (for any j) and $\eta_{i,j}$ are smooth semialgebraic functions defined over S_i'.

Next, since $r^*\alpha$ is stratified it follows that $r^*\alpha|_{t=\eta_{i,j}(x)}$ is well defined for $x \in S_i'$ and, in particular, $\alpha_0(x, \eta_{i,j}(x))$ is well defined. Therefore,

$$\begin{aligned}
\int_{U\times[\varepsilon,1]} \frac{\partial\alpha_0}{\partial t} dt \wedge \phi &= \int_U \left(\int_\varepsilon^1 \frac{\partial \alpha_0}{\partial t} dt \right) \wedge \phi \\
&= \sum_{i,j} \int_{S_i'} \left(\int_{\eta_{i,j}(x)}^{\eta_{i,j+1}(x)} \frac{\partial\alpha_0}{\partial t} dt \right) \wedge \phi \\
&= \sum_i \int_{S_i'} (\alpha_0(x,1) - \alpha_0(x,\varepsilon)) \wedge \phi \\
&= \int_U (\alpha - r_\varepsilon^*\alpha) \wedge \phi \ ,
\end{aligned}$$

as required.

□

3.2. **Finding a smooth solution to problem (3.3).** As a corollary of the results of the previous section the following holds

Corollary 3.4. *In the setting of Proposition 3.3 assume ω is a smooth L^p bounded closed form defined on U. Then there exists a smooth L^p bounded form ξ solving problem (3.3).*

To prove this corollary we will need a theorem from [Y].

Theorem 3.5. *(Theorem 2.7.1 [Y]) Let M be Riemannian manifold. Suppose that ω in an L^p bounded form on X with $\overline{d}\omega$ a smooth L^p bounded form. Then for any $\varepsilon > 0$ there exits a form ψ_ε such that $\|\psi_\varepsilon\|_{L^p} + \|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$ and $\omega + \overline{d}\psi_\varepsilon$ is smooth.*

Proof of Corollary 3.4. Denote $R_\varepsilon\omega$ by ξ_ε'. According to (ii) of Theorem 3.2 $\xi' := \lim_{\varepsilon\to 0} \xi_\varepsilon'$ is L^p bounded. Proposition 3.3 implies

$$\overline{d}\xi_\varepsilon' = \omega - r_\varepsilon^*\omega.$$

Hence (iii) of Theorem 3.2 and passing to limit as $\varepsilon \to 0$ imply $\overline{d}\xi' = \omega$. Moreover, (i) of Theorem 3.2 implies $\|\xi'\|_{L^p} \le C\|\omega\|_{L^p}$. According to Theorem 3.5 there is a form ψ such that $\|\psi\|_{L^p} + \|\overline{d}\psi\|_{L^p} < \|\omega\|_{L^p}$ and $\xi := \xi' + \overline{d}\psi$ is smooth. Therefore $d\xi = \overline{d}\xi'$ and

$$\|\xi\|_{L^p} = \|\xi' + \overline{d}\psi\|_{L^p} \le (C+1)\|\omega\|_{L^p},$$

as required. □

4. L^p-cohomology

In this section we consider an L^p cohomology theory of a normal compact semialgebraic set X.

The L^p cohomology is the cohomology of the complex $(\Omega^\bullet_{L^p}(X), \overline{d})$ commonly defined by

$$H^k_{L^p}(X) := \frac{\text{Ker } (\overline{d} : \Omega^k_{L^p}(X_{reg}) \to \Omega^{k+1}_{L^p}(X_{reg}))}{\text{Im } (\overline{d} : \Omega^{k-1}_{L^p}(X_{reg}) \to \Omega^k_{L^p}(X_{reg}))} \ .$$

Let
$$\Lambda^k_{L^p}(X) := \Omega^k(X_{reg}) \cap \Omega^k_{L^p}(X_{reg}),$$
and denote the k^{th} cohomology group of $(\Lambda^\bullet_{L^p}, d)$ by $H^k(\Lambda^\bullet_{L^p}(X))$.

Then Theorem 3.5 implies that the cohomology of $\Lambda^\bullet_{L^p}(X)$ is isomorphic to the L^p cohomology:

Proposition 4.1. $H^k(\Lambda^\bullet_{L^p}(X)) = H^k_{L^p}(X)$.

Proof. We define a homomorphism
$$i : H^k_{L^p}(X) \to H^k(\Lambda^\bullet_{L^p}(X))$$
as follows. Assume $\omega \in \Omega^k_{L^p}(X_{reg})$ is a closed form. Denote by $[\omega]$ the L^p cohomology class of ω. According to Theorem 3.5 there is a form ψ with $\|\psi\|_{L^{p,1}} < \infty$ such that $\omega + \overline{d}\psi$ is smooth.

Set $i[\omega]$ to be the Λ_{L^p} cohomology class of $\omega + \overline{d}\psi$. First note that i is well defined. Indeed, since if ψ' is another form such that $\|\psi'\|_{L^{p,1}} < \infty$ and $\omega + \overline{d}\psi'$ is smooth then
$$\overline{d}(\psi - \psi') = \omega + \overline{d}\psi - (\omega + \overline{d}\psi')$$
is a smooth form. Therefore, applying Theorem 3.5 once more, we obtain a form ξ, $\|\xi\|_{L^{p,1}} < \infty$ such that $\psi - \psi' + \overline{d}\xi$ is smooth. Finally,
$$(\omega + \overline{d}\psi) - (\omega + \overline{d}\psi') = d(\psi - \psi' + \overline{d}\xi)\ ,$$
as we claimed. The proof of surjectivity of i is straightforward. The homomorphism i is also injective. Indeed, if $i[\omega] = 0$ then there is a form ψ, $\|\psi\|_{L^{p,1}} < \infty$ such that $\omega + \overline{d}\psi = d\gamma$ for a smooth form γ with $\|\gamma\|_{L^{p,1}} < \infty$ and injectivity follows. □

Definition 4.2. We will refer to a k dimensional subset $X \subset \mathbb{R}^n$ as **normal** if for any $x \in X$, there exists $\varepsilon > 0$ such that $S^{n-1}(x, \varepsilon) \cap X_{reg}$ is connected, where $S^{n-1}(x, \varepsilon)$ is an $(n-1)$-sphere in $\mathbb{R}^n$ centered at x with radius ε.

Since we work with compact sets, L^p boundedness is a local property and hence germs of L^p bounded k-forms define a sheaf on X. Namely, for every open set $U \subset X$ we associate the set $\Omega^k_{L^p}(U \cap X_{reg})$ or $\Lambda^k_{L^p}(U \cap X_{reg})$. We denote the sheaf of L^p bounded k-forms by $\Omega^k_{L^p}$ and the sheaf of smooth L^p bounded forms by $\Lambda^k_{L^p}$. The sheaves $\Omega^k_{L^p}$ and $\Lambda^k_{L^p}$ are fine on compact semialgebraic set $X \subset \mathbb{R}^n$. Indeed, every open cover of X can be extended to an open cover of a neighborhood of X in $\mathbb{R}^n$, on which existence of a partition of unity is evident.

In Section 3 we proved that, locally, smooth closed k-forms on a semialgebraic set X are exact for $p >> 1$ and $k > 0$. If X is normal then closed 0-forms are locally constant functions. It follows from here that the sheaf complex $\Lambda^\bullet_{L^p}$ on a normal set X comprises a fine resolution of the constant sheaf $\mathbb{R}$ on X and therefore, a standard argument from sheaf theory implies that the singular cohomology of X naturally coincides with the cohomology of $\Omega^\bullet_{L^p}(X)$. That is we have proven

Theorem 4.3. *Let $X \subset \mathbb{R}^n$ be a normal compact semialgebraic set. There exists $p >> 1$ such that the cohomology of the L^p complex $\Omega^\bullet_{L^p}(X)$ is isomorphic to the singular cohomology of X.*

We remark that for small p the isomorphism of Theorem 4.3 need not hold.

Example. Let X be a semialgebraic set such that (X_{reg}, g) is a smooth Riemannian manifold diffeomorphic to $M \times (0,1]$, where M is a smooth compact m-dimensional manifold with a Riemannian metric g_M. Suppose that $g = dr^2 + r^{2\alpha} g_M$, where r is the coordinate in $(0,1]$ and $\alpha \geq 1$. Topologically, X is a cone over the manifold M. Let ω be a smooth, closed, non exact and radially constant k-form on X_{reg}, i.e. ω does not contain any terms of the form $dr \wedge \dots$ and the coefficients of ω are independent of r. The volume form on X_{reg} is given by $dV = r^{\alpha m} dr \wedge dV_M$, where dV_M stands for the volume form on M. The pointwise norm of ω is given by

$$|\omega(x,r)| = r^{-k\alpha} |\omega(x,r)|_M, \quad (x,r) \in M \times (0,1] ,$$

where $|\cdot|_M$ is the pointwise norm on M. Note that since ω is closed and radially constant, it is independent of r and, in abuse of notation, we will write $\omega(x)$ instead of $\omega(x,r)$.

Clearly $\|\omega\|_{L^p(X)} < \infty$ if and only if $p < \frac{\alpha m+1}{k\alpha}$:

$$\begin{aligned} \int_X |\omega(x)|^p dV &= \int_0^1 \int_M |\omega|^p r^{\alpha m} dr dV_M \\ &\approx \int_0^1 r^{-\alpha k p + \alpha m} dr < \infty. \end{aligned}$$

In fact, an L^p bounded closed (not exact) radially constant form defines a nontrivial cohomology class in $H^k_{L^p}$. Indeed, otherwise assume $\omega = d\xi$, $\|\xi\|_{L^p} < \infty$. Since ω is radially constant ξ must be radially constant as well. But then $\omega|_M = d\xi|_M$ which contradicts the assumption of ω not being exact. However, for $p >> 1$ it follows from Theorem 4.3 that $H^k_{L^p}(X) = 0$ for $k > 0$. In our example, the minimal p for which $H^k_{L^p}(X) = 0$ equals $\frac{\alpha m+1}{k\alpha}$.

5. Global L^p inequality on a semialgebraic set

For a closed form considered in Section 3 the problem of finding an antiderivative with the bound (3.3) can be generalized to a global problem on a compact semialgebraic set X. In the case that X is a compact smooth manifold, such problem is treated in [S1]. In general, due to the existence of topological obstructions, of course there are no antiderivatives for some closed forms.

To overcome this obstacle we derive a combinatorial condition under which closed forms are exact. In the case that X is a compact smooth manifold, a closed form on X is exact if and only if its integrals vanish on every cycle in X. We generalize this condition to closed L^p bounded forms by extending the notion of integration over cycles in X to the case of closed L^p bounded forms. In particular, our generalized condition for closed L^p bounded forms to be exact is the usual one (mentioned above) whenever X is a (nonsingular) manifold. Our definition of an integral of a closed L^p bounded form over a cycle in X is placed in the forthcoming subsection and is of a combinatorial nature. In [S1] we derive a combinatorial formula for an integral of a closed form over a cycle in a manifold. It is constructed iteratively by means of a process of an application of the L^p inequality for forms on various contractible subsets. This process can be carried out for any class of forms that satisfy the L^p inequality for forms on a 'good' (or even 'weakly good') covering by contractible subsets. We prove in the forthcoming subsection that closed L^p bounded forms satisfy the L^p inequality for forms on contractible sets, which would allow us to extend the notion of an integral over cycles to the L^p bounded forms.

Remark 5.1. In a paper by Gol'dshtein, Kuz'minov and Shvedov [GKS], the authors defined an integral of forms in $W^k_{p,q}$ over any k-dimensional manifold parametrized by a Lipschitz map. However, for our purposes, it suffices to define integrals of closed L^p bounded forms just over cycles.

5.1. **Definition of an integral of a closed L^p bounded form.** To define an integral of an L^p bounded form over a cycle we consider the Čech-De Rham double complex. We refer the reader to [S1] for the related definitions and generalities.

Assume $X \subset \mathbb{R}^n$ is a compact semialgebraic set and let $\mathcal{U} = \{U_i\}_{i=1,\dots,N}$ be a finite open cover of X. We associate a differential Čech complex with values in the L^p bounded k-forms to the cover $\mathcal{U}$.

Definition 5.2. The Čech complex with values in $\Omega^k_{L^p}$ we denote by $(K^{k,\bullet}(\mathcal{U}, \Omega^k_{L^p}), \delta)$, where

$$\delta : K^{k,j}(\mathcal{U}, \Omega^k_{L^p}) \to K^{k,j+1}(\mathcal{U}, \Omega^k_{L^p})$$

is defined by

$$(\delta\varphi)_{i_0,\dots,i_{j+1}} := \sum_k (-1)^k \varphi_{i_0,\dots\hat{i_k},\dots,i_{j+1}}.$$

The 'combined' double complex Čech and the complex of L^p bounded forms is defined by $(K^\bullet(\mathcal{U}, \Omega^\bullet_{L^p}), D)$, where

$$K^j(\mathcal{U}) := \bigoplus_{l+k=j} C^l(\mathcal{U}, \Omega^k_{L^p})$$

and $D : K^j(\mathcal{U}) \to K^{j+1}(\mathcal{U})$ is defined by $D := d + (-1)^l\delta$ on $C^l(\mathcal{U}, \Omega^k_{L^p})$.

Denote by $H^j(K^\bullet(\mathcal{U}))$ the cohomology of the complex $K^\bullet$ and denote by $H^j(C^\bullet(\mathcal{U}, \Omega^r_{L^p}))$ the cohomology of the complex $C^\bullet(\mathcal{U}, \Omega^r_{L^p})$. In [S1] a good cover is defined as a cover consisting of convex sets. In this article we will work with slightly weaker condition on covers namely:

Definition 5.3. If $\mathcal{U} = \{U_i\}$ is a cover of X, we say that $\mathcal{U}$ is a **weakly good cover** if each finite intersection of U_i's is contractible.

The **nerve complex** of a cover $\mathcal{U}$ is a simplicial complex $(C(\mathcal{U})_\bullet, \partial)$ with simplex $[I]$ associated with every non empty intersection U_I. The boundary operator $\partial : C_l(\mathcal{U}) \to C(\mathcal{U})_{l-1}$ is defined as usually

$$\partial[I] := \sum_{j>0} (-1)^j [i_0, \dots, \hat{i_j}, \dots, i_l], \quad I = [i_0, \dots, i_l].$$

It is a well known fact that if $\mathcal{U}$ is a weakly good cover of X then the homology of the nerve complex $C_\bullet(\mathcal{U})$ coincides with the singular homology of X (see e.g. [H] Corollary 4G.3).

Remark 5.4. Every triangulable set X has a weakly good cover. Indeed, if T is a triangulation of X with $V := \{1, \dots, |T|\}$ being the set of vertices of T, then let $\mathcal{U} := \{U_i\}_{i \in V}$ be a cover of X, where U_i is the star of vertex i. We claim that $\{U_i\}_{i\in V}$ is a weakly good cover. Let $I := (i_0, \dots, i_l)$ and assume that $U_I := U_{i_0} \cap \dots \cap U_{i_l} \neq \emptyset$. Every simplex of dimension $\dim X$ in the closure of U_I contains the vertex i_0. Therefore, it is possible to deformation retract the closure of U_I to i_0. Consequently, every finite intersection U_I is contractible.

Lemma 5.5. *Assume that X is a compact contractible semialgebraic set. There exists $p >> 1$ such that for every closed k-form ω in $\Lambda^k_{L^p}(X_{reg})$, $k \geq 1$ there exists a form $\xi \in \Omega^{k-1}_{L^p}(X)$ such that*

$$\begin{cases} \omega = d\xi \quad \text{on } X_{reg}, \\ \|\xi\|_{L^p(X)} \leq C\|\omega\|_{L^p(X)} \end{cases} \tag{5.6}$$

Proof. The proof is by induction on k. When $k = 1$ Corollary 3.4 implies that there is a cover $\{U_i\}_{i=1}^N$, $X = \cup_{i=1}^N \overline{U}_i$ such that (5.6) holds on U_i with ξ being $\tilde{\xi}_i$, for some form $\tilde{\xi}_i$ on U_i. Let $\{B_i\}$ be a weakly good cover of X that refines $\{U_i\}$. For any pair i, j with $B_i \subset U_j$ we, in abuse of notation, denote within the proof of this lemma $(k-1)$-form $\tilde{\xi}_j$ by ξ_i. In this case we have

$$\|\xi_i\|_{L^p(B_i)} = \|\tilde{\xi}_j\|_{L^p(B_i)} \leq \|\tilde{\xi}_j\|_{L^p(U_j)} \lesssim \|\omega\|_{L^p(X)}.$$

Note that $\xi_{i,j} := \xi_i - \xi_j$ is a closed 0-form on $B_i \cap B_j$ and therefore is a constant on $B_i \cap B_j$. Define

$$\xi := \xi_i + c_i \quad \text{on } B_i \ ,$$

where c_i are constants such that $c_i - c_j = \xi_j - \xi_i$ on $B_i \cap B_j$. Existence of such constants follows from the fact that U is contractible. Indeed, consider the 'nerve' complex $N_\bullet := C_\bullet(\{B_i\})$ and let $f : N_1 \to \mathbb{R}$ be 1-cochain defined by $f([ij]) := (\delta\xi)_{i,j} := (\xi_j - \xi_i)|_{B_i \cap B_j}$. Clearly, f is closed and since X is contractible f is exact. Therefore $f = \delta g$ where g is 0-cochain of K. Hence, $(\xi_j - \xi_i)|_{B_i\cap B_j} = g(i) - g(j)$. Denote $C_i := g(i)$. These constants C_i solve a system of linear equations

$$(\delta c)_{i,j} = (\xi_j - \xi_i)|_{B_i \cap B_j} \ .$$

Therefore,

$$C_i = \sum A_{i,j}(\xi_j - \xi_i)|_{B_i\cap B_j} \ ,$$

where $A_{i,j} \in \mathbb{R}$ are constants that depend only on the combinatorics of the cover $\{B_i\}$. Below we estimate the L^p norms of the constants C_i:

$$\begin{aligned}
\|C_i\|_{L^p(B_i)} &\leq \sum |A_{i,j}| \|(\xi_j - \xi_i)|_{B_i \cap B_j}\|_{L^p(B_i)} \\
&= \sum |A_{i,j}| \|(\xi_j - \xi_i)|_{B_i \cap B_j}\|_{L^p(B_i \cap B_j)} \left(\frac{Vol(B_i)}{Vol(B_i \cap B_j)} \right)^{1/p} \\
&\leq \sum |A_{i,j}| \left\{ \|\xi_j\|_{L^p(B_i)} + \|\xi_i\|_{L^p(B_j)} \right\} \left(\frac{Vol(B_i)}{Vol(B_i \cap B_j)} \right)^{1/p} \\
&\lesssim \|\omega\|_{L^p(X)} \ .
\end{aligned}$$

It follows

$$\|\xi\|_{L^p(X)} \leq \sum \|\xi_i + C_i\|_{L^p(B_i)} \leq \sum \|\xi\|_{L^p(B_i)} + \sum \|C_i\|_{L^p(B_i)} \lesssim \|\omega\|_{L^p(X)} \ ,$$

which completes the proof of in the case that $k = 1$.

When $k > 1$, our proof is similar to that of the case when $k = 1$. Assume that $X = \cup_{i=1}^N \overline{U_i}$, where $\{U_i\}$ is a cover such that (5.6) holds for $U := U_j$ with $\xi := \tilde{\xi}_j$ for a form $\tilde{\xi}_j$. Similarly to the case of $k = 1$ let $\{B_i\}$ be a weakly good cover that refines $\{U_i\}$ and for any pair i, j with $B_i \subset U_j$ we once again denote by ξ_i the form $\tilde{\xi}_j$.

Once more, note that $\xi_{i,j} := \xi_i - \xi_j$ is a closed $(k-1)$-form on $B_i \cap B_j$. Therefore, by the induction hypothesis, $\xi_{i,j} = d\xi^1_{i,j}$ and estimate (5.6) holds on $B_i \cap B_j$. From here, we can run the 'globalization' process as described in [S1] to obtain solutions ξ_I^{l+1} to the equations $(\delta\xi^l)_I = d\xi_I^{l+1}$ on B_I (see Section 3.1, Def. 3.8 and Example 3.7 illustrating all of the important features of the 'globalization' construction). In the final step we have a collection of 0-forms ξ_I^{k-1} with $(\delta\xi^{k-1})_I$ being constants. By an argument similar to the one in the case that $k = 1$, with f being a closed (and hence exact) cochain $f : N_k \to \mathbb{R}$, defined by $f([I]) := (\delta\xi^{k-1})_I$ there are constants C_I such that

$$(\delta\xi^{k-1} + C)_J = 0$$

and, moreover,

$$\|C_I\|_{L^p(B_I)} \lesssim \|\omega\|_{L^p(X)}.$$

As is described in [S1] Section 3.2, we may find a collection of $(k-1)$-forms x_I^{k-1} defined on B_I such that $(\delta x^{k-1})_J = \xi_J^{k-1} - C_J$ and $\delta x^{k-t} = \xi^{k-t} - dx^{k-t+1}$ for $t > 1$. As in the proof of Proposition 3.12 (replacing '(p, q)-Poincaré inequality for forms' by the local Poincaré L^p inequality) it follows that the forms x_I^s admit the following estimates:

$$\|x_I^s\|_{L^p(B_I)} \lesssim \|\omega\|_{L^p(X)},$$

and

$$\|dx_J^s\|_{L^p(B_J)} \lesssim \|\omega\|_{L^p(X)}.$$

It is then straightforward to show that $\xi := x^0$ is a global solution to problem (5.6). □

Proposition 5.6. *Assume X is normal and $\mathcal{U}$ is a weakly good cover of X. Then there are isomorphisms*

$$h_1 : H^j(K^\bullet(\mathcal{U})) \to H^j(K^{k,\bullet}(\mathcal{U}, \Omega^\bullet_{L^p}))$$

and

$$h_2 : H^j(K^\bullet(\mathcal{U})) \to H^j(\Omega^\bullet_{L^p}(X)))$$

induced by the homomorphisms of the respective differential complexes.

Proof. When X is a smooth manifold, constructions of h_1 and h_2 can be found in [BT] (Theorem 8.1, Proposition 8.8 and Theorem 8.9). To adapt these constructions in our setting one has to substitute the classical Poincaré lemma by Lemma 5.5, cf. the proof of our Proposition 3.14 in [S1] in which this construction is carried out in complete details. ☐

Denote by

$$Int : H^j(\Omega^\bullet_{L^p}(X))) \to H^j(C^\bullet(\mathcal{U}, \Omega^k_{L^p}))$$

the isomorphism $h_1 \circ h_2^{-1}$.

Remark 5.7. It is proved in [S1] that if X is a compact manifold and ω is a closed smooth k-form on M then $(Int\ \omega)c = \int_c \omega$ for every cycle c in X. Moreover, if $\mathcal{U}$ is a good cover and c is a cycle given by $\sum_I a_I[I]$, then

$$(Int\ \omega)c = (-1)^{\lfloor \frac{k}{2} \rfloor} \sum a_I \delta \xi_I^{k-1},$$

where ξ_I^{k-1} are the forms on U_I constructed for every form ω by the inductive relation:

$$d\xi_I^{s+1} = (\delta \xi^s)_I \quad \text{on } U_I,$$

where $\xi^0_{i_0}$ is a solution to $d\xi^0_{i_0} = \omega|_{U_{i_0}}$ satisfying $\|\xi^0_{i_0}\|_{L^p(U_{i_0})} \lesssim \|\omega\|_{L^p(U_{i_0})}$ given by Lemma 5.5.

As a consequence of Lemma 5.5 and Remark 5.7 one may extend the definition of an integral over the cycles in X to all closed L^p bounded forms as follows

$$\int_c \omega := (Int\ \omega)c = (-1)^{\lfloor \frac{k}{2} \rfloor} \sum a_I \delta \xi_I^{k-1},$$

where $c = \sum_I a_I[I]$ is a cycle in X.

5.2. **Global L^p inequality.** A global analog of problem (3.3) can be formulated as follows. Say that X satisfies the global L^p inequality for forms if there exists a constant $C > 0$ such that for every closed form $\omega \in \Lambda^k_{L^p}(X)$ with zero integrals over every cycle in X, there is a form $\xi \in \Lambda^{k-1}_{L^p}(X)$ such that

$$\text{(5.7)} \qquad \begin{cases} \omega = d\xi \quad \text{on } X_{reg}, \\ \|\xi\|_{L^p(X)} \le C\|\omega\|_{L^p(X)} \ . \end{cases}$$

Proposition 5.8. *For sufficiently large $p >> 1$ if ω is a closed k-form in $\Lambda^k_{L^p}(X)$ and $\int_c \omega = 0$ for every $c \in H_k(X)$ then ω is exact and (5.7) holds for ω.*

Proof. The proof of this proposition follows the same argument as the proof of Theorem 3.1 in [S1]. We construct the form ξ satisfying (5.7) following faithfully the structure of the construction in [S1] Sections 3.1 and 3.2 for a finite weakly good cover $\mathcal{U}$ of X, but replacing the '(p, q) Poincaré inequality' for forms by Lemma 5.5 . □

6. Introduction to Lipschitz Retraction Theorem

Deformation retractions play an important role in De Rham theory. For instance, a standard proof of classical Poincaré lemma on a star shaped domain $U \subset \mathbb{R}^n$ uses a smooth deformation retraction $r : U \times I \to U$ to construct a primitive of a closed form ω in the following way. Let us assume for simplicity that U is star shaped (from $0 \in \mathbb{R}^n$). Let $r_t(x) = r(x, t) := tx$. Assume that ω is a closed form, then we have the following (unique) decomposition of the pull back of ω by r:

$$r^*\omega = \omega_0 + dt \wedge \omega_1,$$

where the differential forms ω_0 and ω_1 do not contain any terms involving dt. Set

$$\gamma(x) := \int_0^1 \omega_1(x, t)dt.$$

Now $d\gamma = \omega$. Indeed, since $d\omega = 0$ and d commutes with r^* we have

$$0 = dr^*\omega = d_x\omega_0 + dt \wedge (\frac{\partial \omega_0}{\partial t} - d_x\omega_1),$$

where d_x represents the exterior derivative with respect to x. Therefore, $\frac{\partial \omega_0}{\partial t} = d_x\omega_1$ and hence

$$d\gamma(x) = \int_0^1 d_x\omega_1 dt = \int_0^1 \frac{\partial \omega_0}{\partial t}dt = \omega_0(x, t)|_{t=0}^{t=1} = r_1^*\omega(x) - r_0^*\omega(x) = \omega(x).$$

However, in this article we deal with semialgebraic sets X, which need not have star shaped neighborhoods of every point. Therefore, to extend the Poincaré lemma to our setting we will have to construct Lipschitz semialgebraic deformation retractions with controlled growth of their derivatives. The main techniques of our construction are based on Lipschitz semialgebraic geometry theory developed in [V1].

In what follows, we represent points $q \in \mathbb{R}^{n+1}$ by pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}$.

Definition 6.1. A **cell** in $\mathbb{R}^n$ is defined by induction on n. For $n = 1$, a cell is a point or an open interval. For $n > 1$ a cell is either a graph of a semialgebraic function or a band delimited by two semialgebraic functions over a cell in $\mathbb{R}^{n-1}$.

A cell is called **Lipschitz cell** if all the graphs and bands involved in its construction are defined by means of Lipschitz semialgebraic functions.

A **cell subdivision** of $\mathbb{R}^n$ is a subdivision of $\mathbb{R}^n$ into a disjoint collection of cells. A cell subdivision of $\mathbb{R}^n$ is said to be **compatible** with a set A if A can be represented as a union of the cells of this subdivision.

For every collection of semialgebraic sets in $\mathbb{R}^n$ there exists a cell subdivision compatible with them.

Theorem 6.2. *Let $A_1, \dots, A_m \subset \mathbb{R}^n$ be semialgebraic sets. There exists a cell subdivision of $\mathbb{R}^n$ compatible with A_i for $i = 1, \dots, m$.*

In general, providing a cell subdivision is often not sufficient for a study of a semialgebraic set since cell subdivision does not include information on how the cells come in contact with the neighboring cells. Let $\mathcal{A}$ be a collection of cells in $\mathbb{R}^n$. We say that $\mathcal{A}$ satisfies the **frontier condition** if the boundary of each cell in $\mathcal{A}$ is a union of cells in $\mathcal{A}$. Next we introduce a concept of stratification.

Definition 6.3. A **stratification** of a set X is a collection Σ of smooth manifolds called **strata** such that their union is the set X and the boundary of each stratum is union of the strata of lower dimension.

If S and S' are two strata in Σ such that $S' \subset \partial S$ then we write $S' \leq S$.

Denote by Σ^k the collection of all strata in Σ of dimension k, by $\Sigma^{(k)}$ the collection of all strata up to (and including) dimension k and by $|\Sigma|$ the union of all strata in Σ. A **refinement** of Σ is a stratification Σ' such that each stratum of Σ is a union of strata of Σ'. We then write $\Sigma' \prec \Sigma$. If $f : X \to Y$ is a map and Σ is a stratification of X then we write $f(\Sigma)$ to denote the collection of sets $\{f(S) : S \in \Sigma\}$.

We say that Σ is a **Whitney A** stratification if for every two strata $S' \leq S$ and a sequence of points $p_n \in S$ converging to $p \in S'$ we have $\lim_{n\to\infty} T_{p_n} S \supset T_p S'$ whenever the limit on the left hand side exists.

Every semialgebraic set admits a Whitney A stratification, moreover

Theorem 6.4. *Suppose that $X \subset \mathbb{R}^n$ is a semialgebraic set. There exists a Whitney A stratification of $\mathbb{R}^n$ compatible with X.*

For a proof see e.g., [BCR].

Remark 6.5. A Lipschitz cell C of dimension k in $\mathbb{R}^{n+1}$ is bi-Lipschitz equivalent to a k-dimensional Lipschitz cell $D \subset \mathbb{R}^k$. A bi-Lipschitz homeomorphism $\phi : C \to D$ can be constructed by induction as follows. The cell C is either a graph of a Lipschitz semialgebraic function or a band bounded by two Lipschitz semialgebraic functions over a cell $C' \subset \mathbb{R}^n$. Assume $\phi' : C' \to D'$ is a bi-Lipschitz homeomorphism, where D' is a $(k-1)$-dimensional Lipschitz cell in $\mathbb{R}^{k-1}$. Now,

if C is a graph of $\theta : \mathcal{C}' \to \mathbb{R}$ then set

$$\phi(x, \theta(x)) := (\phi'(x), \theta(x)),$$

if C is a band bounded by $\theta_i : \mathcal{C}' \to \mathbb{R}$, $i = 1, 2$, $\theta_1 < \theta_2$ define

$$\phi(x, y) := (\phi'(x), y)\ .$$

Since θ_1, θ_2 and θ are Lipschitz, the map ϕ is bi-Lipschitz. Note that the map ϕ^{-1} defines coordinates $u = (u_1, \dots, u_k)$ on C. Also, observe that a function $f : C \to \mathbb{R}$ is Lipschitz if and only if $f \circ \phi^{-1}$ is Lipschitz.

Definition 6.6. Suppose that C is a cell of $\mathbb{R}^{n+1}$ given by a graph of a function θ or by a band bounded by graphs of functions θ_1 and θ_2 over a cell $C' \subset \mathbb{R}^n$. Moreover, assume that we have a deformation retraction $r' : C' \times I \to C'$. The **standard lift** of r' is a deformation retraction $r : C \times I \to C$ defined by

$$r_t(q) := r(q,t) := (r'(x,t), (1-\tau(q))\theta_1(r'(x,t)) + \tau(q)\theta_2(r'(x,t))) \ , \quad q = (x,y) \ ,$$

where $\tau(q) := \frac{y-\theta_1(x)}{\theta_2(x)-\theta_1(x)}$ in the case that $C = \{q : \theta_1(x) < y < \theta_2(x), \ x \in C'\}$ and by

$$r_t(q) := r(q,t) := (r'(x,t), \theta(r'(x,t))) \ ,$$

in the case that $C = \{q : y = \theta(x), \ x \in C'\}$. Note that $\tau(r(q,t)) = \tau(q)$.

In the reminder of this section we give an intuitive derivation of our main Lipschitz deformation retraction Theorem 6.12. In what follows we describe a rough idea of our construction of a Lipschitz semialgebraic deformation retraction r on a neighborhood of a point in a semialgebraic set. We remark that the Lipschitz semialgebraic deformation retraction of Theorem 6.12 has additional estimates on its derivatives, but for the sake of simplicity we will only deal with the Lipschitz property of r for now (i.e. in this section).

Assume that C is a cell in $\mathbb{R}^{n+1}$ bounded by two Lipschitz semialgebraic functions $\theta_1 < \theta_2$ defined over a cell $C' \subset \mathbb{R}^n$ and that $r' : C' \times I \to C'$ is a Lipschitz semialgebraic deformation retraction. It is not always true that the standard lift of r' to C is Lipschitz as the following example shows.

Example 6.7. Let $\xi(x) : \mathbb{R}^2 \to \mathbb{R}$, $\xi(x) = |x_1^2 - x_2|$, $r'_t(x) := tx$. Assume that C is a cell in $\mathbb{R}^3$ defined by $\{(x,y) \in \mathbb{R}^2 \times \mathbb{R} : 0 \le y \le \xi(x)\}$. Let r_t be the standard lift of r' from $\mathbb{R}^2$ to $\mathbb{R}^3$. Clearly ξ is Lipschitz. Let us show that r_t is not Lipschitz. Note that

$$r_t(x,y) = (tx, y\frac{\xi(tx)}{\xi(x)}).$$

Observe that r_t is continuous and differentiable almost everywhere, so r_t is Lipschitz if and only if all partial derivatives of its components are bounded. In particular, if r_t is Lipschitz then $\frac{\xi(tx)}{\xi(x)}$, being the derivative of the last component of r_t with respect to y, has to be bounded. We will show that $\frac{\xi(tx)}{\xi(x)}$ is not bounded. Indeed, set

$$x_1 = t, \qquad x_2 = t^2 + t^5$$

and observe that

$$\frac{\xi(tx)}{\xi(x)} = \frac{|t^4 - t^3 - t^6|}{|t^5|} \to \infty \text{ as } t \to 0.$$

It is possible to redefine the deformation retraction r' on $\mathbb{R}^2$ in such a way that its standard lift would be Lipschitz. Indeed, let $r'_t(x) := (tx_1, t^2x_2)$ then

$$\frac{\xi(r'_t(x))}{\xi(x)} = t^2,$$

and hence the standard lift r_t is Lipschitz.

This example leads us to formulate a condition for a standard lift of a Lipschitz semialgebraic deformation retraction to be Lipschitz.

Proposition 6.8. *Assume that $C \subset \mathbb{R}^{n+1}$ is a cell which is a graph of a Lipschitz semialgebraic function θ_1 or a band bounded by Lipschitz semialgebraic functions θ_2 and θ_3 over a cell C' of $\mathbb{R}^n$. Let $r' : C' \times I \to C'$ be a Lipschitz semialgebraic deformation retraction and r be its standard lift. The standard lift r is Lipschitz in the case that C is a graph . When C is a band, the standard lift r is Lipschitz if and only if*

$$|\theta_2(r'_t(x)) - \theta_3(r'_t(x))| \lesssim |\theta_2(x) - \theta_3(x)|, \tag{6.8}$$

where $r'_t(x) := r'(x, t)$.

Proof. Since semialgebraic functions are generically smooth we only have to check that partial derivatives of r are bounded. By Remark 6.5 there exist bi-Lipschitz maps $\phi' : C' \to D'$ and $\phi : C \to D$ such that the diagram

$$\begin{array}{ccc} C & \stackrel{\phi}{\to} & D \\ \downarrow\pi_{n+1} & & \downarrow\pi_k \\ C' & \stackrel{\phi'}{\to} & D' \end{array}$$

is commutative, where $\pi_j : \mathbb{R}^j \to \mathbb{R}^{j-1}$ is the standard projection to the first $j-1$ coordinates. Therefore, we may assume (by replacing C with D and C' with D') that C' is a cell of dimension n in $\mathbb{R}^n$ and C is either a graph or a band over C'.

The map r_t can be written as $r_t(x, y) = (r'_t(x), r_{n,t}(x, y))$. Set $D_j := \frac{\partial}{\partial x_j}$ and $D_t := \frac{\partial}{\partial t}$. By our assumption r' is Lipschitz. Therefore, we only have to check that $|D_j r_{n,t}|$ and $\left|\frac{\partial r_{n,t}}{\partial t}\right|$ are bounded.

In the case that C is a graph of θ_1 we have

$$|D_j r_{n,t}(x, y)| = |D_j(\theta_1(r'(x, t)))| = |\sum_i D_i\theta_1(r'(x, t))D_j r'_{i,t}(x)|,$$

which is bounded since θ_1 and r'_t are Lipschitz. For the same reason

$$|D_t r_{n,t}(x, y)| = |\sum_j D_j\theta_1(r'(x, t))D_t r'_{j,t}(x)|$$

is bounded.

In the case that C is a band bounded by $\theta_2 < \theta_3$, set $\theta(x) := \theta_3(x) - \theta_2(x)$ and let $D_y := \frac{\partial}{\partial y}$. Let $\tau(q) = \frac{y-\theta_2(x)}{\theta(x)}$ be as in Definition 6.6.

$$\begin{aligned} |D_j r_{n,t}(x, y)| &= |D_j\left(\theta_2(r'(x, t)) + \tau(q)\theta(r'(x, t))\right)| \\ &\le |D_j\theta_2(r'(x, t))| + |D_j\left(\tau(q)\theta(r'(x, t)\right)| \\ &\le C_1 + |(D_j\tau(q))\theta(r'(x, t)) + \tau(q)D_j\theta(r'(x, t))|. \end{aligned}$$

Note that

$$D_j\tau(q) = \frac{-D_j\theta_2(x)\theta(x) - (y - \theta_2(x))D_j\theta(x)}{\theta^2(x)}$$

Thus, by (6.8) and the fact that $0 < y - \theta_2(x) < \theta(x)$ we have

$$|D_j\tau(q)\theta(r'(x,t))| \le |D_j\theta_2(x)| + |D_j\theta(x)| \le C.$$

The estimate of $|D_y r_{n,t}(x,y)|$ is obtained as follows.

$$\begin{aligned}|D_y r_{n,t}(x,y)| &= |D_y\left(\theta_2(r'(x,t)) + \tau(q)\theta(r'(x,t))\right)| \\ &= |D_y\left(\tau(q)\theta(r'(x,t)\right)| \\ &= |(D_y\tau(q))\theta(r'(x,t)) + \tau(q)D_y\theta(r'(x,t))| \\ &= |\frac{1}{\theta(x)}\theta(r'(x,t))| \le C.\end{aligned}$$

We omit the proof of the boundedness of the partial derivative in t of the standard lift r because it is nearly identical to the proof of the boundedness of $D_j r$ above.

For the inverse implication in the case that C is a band, assume that $r_t(x,y)$ is Lipschitz. It follows that $|D_y r_{n,t}| = |\frac{1}{\theta(x)}\theta(r'(x,t))|$ is bounded, as required. □

We will make use of the basic construction of a Lipschitz semialgebraic deformation retraction on a Lipschitz cell as a standard lift of a Lipschitz semialgebraic deformation from lower dimensional cell. Our goal is to obtain a local Lipschitz semialgebraic deformation retraction to any point of X as a step in an inductive process. Criterion (6.8) derived in Proposition 6.8 results in the following Lipschitz deformation retraction theorem.

Theorem 6.9. *Let $X := \cup_{j=1}^m X_j \subset \mathbb{R}^n$ be a closed semialgebraic set, $0 \in \overline{X}_j \cap X, j = 1,\dots,m$, and $\xi_1,\dots,\xi_s : \mathbb{R}^n \to \mathbb{R}$ some continuous semialgebraic functions. Then there exists a neighborhood U of 0 in $\mathbb{R}^n$ and a Lipschitz deformation retraction $r : U \times I \to U$ that preserves X_j , $j = 1,\dots,m$ and satisfies*

$$\xi_j(r_t(x)) \lesssim \xi_j(x). \tag{6.9}$$

The latter theorem is too weak for our applications and is included here only as an introduction to the topic of the Lipschitz deformation retraction. Therefore we only sketch its proof. A 'stronger' version of this theorem is Theorem 8.4 which is proven in Section 8 in complete details.

Sketch of the proof of Theorem 6.9. The proof is by induction on n. The case of $n = 1$ is easy so we skip it and go directly to proving the inductive step. First we use a preparation theorem for functions ξ_j in combination with a bi-Lipschitz transformation to bring the semialgebraic sets X_i into a 'good position' and 'prepare' the functions ξ_j. More precisely, after a bi-Lipschitz transformation the topological boundaries of the sets X_j will belong to the union of graphs of globally defined Lipschitz semialgebraic functions $\eta_1 < \cdots < \eta_b$ defined over $\mathbb{R}^n$ and the functions ξ_j will be of the following form

$$\xi_j(q)\mid_{C_i} \sim |y - \eta_{l_{i,j}}|^{w_{i,j}} a_{i,j}(x) ,$$

where $\{C_i\}$ is a cell subdivision of $\mathbb{R}^{n+1}$ consisting of graphs and bands of functions η_j over the cells in $\mathbb{R}^n$. Next we apply the inductive hypothesis to the cells in $\mathbb{R}^n$ and the following collection of functions

$$\eta_j,\ |\eta_i-\eta_j|,\ \min\{|\eta_i-\eta_j|^{w_{i,j}}a_{i,j}(x),1\}\ \text{ for all } i,j\ .$$

As an output of the inductive step we obtain a deformation retraction r' on $\mathbb{R}^n$. Set r to be the standard lift of r' (see Definition 6.6). Applying the criterion of Proposition 6.8 the deformation retraction r is Lipschitz. To complete the inductive step we prove that condition (6.9) holds, namely:

Lemma 6.10. *Suppose that $C\subset\mathbb{R}^{n+1}$ is a cell bounded by graphs of Lipschitz semialgebraic functions $\theta_1<\theta_2$ over a cell $C'\subset\mathbb{R}^n$. Let r'_t be a Lipschitz semialgebraic deformation retraction on C' and r be its standard lift. Assume that $\zeta(q)=|y-\xi(x)|^w a(x)$, $w\in\mathbb{Q}$, is a bounded function such that $\xi\le\theta_1$. Set*

$$\theta(x):=|\theta_2(x)-\theta_1(x)|\ \text{and}\ \eta(x):=|\xi(x)-\theta_1(x)|\ .$$

If

$$\begin{aligned}\min(a(z_1)\eta(z_1)^w,1)\ &\lesssim\ \min(a(x)\eta(x)^w,1),\\ \min(a(z_1)\theta(z_1)^w,1)\ &\lesssim\ \min(a(x)\theta(x)^w,1).\end{aligned}\tag{6.10}$$

then

$$\zeta(r_t(q))\lesssim\zeta(q).$$

Proof. Note that

$$|y-\xi(x)|=|y-\theta_1(x)|+|\xi(x)-\theta_1(x)|\ .$$

Observe that

$$\zeta(q)\sim a(x)\left\{\begin{array}{ll}\min(|y-\theta_1(x)|^w,\eta(x)^w) & w<0\\ \max(|y-\theta_1(x)|^w,\eta(x)^w) & w>0\ .\end{array}\right.\tag{6.11}$$

Let $z:=z(t)=(z_1(t),z_2(t))$ be the components of the deformation retraction $r=r(q,t)$ where $(z_1(t),z_2(t))\in\mathbb{R}^n\times\mathbb{R}$, $q=(x,y)$. To simplify the notation we will write (z_1,z_2) instead of $(z_1(t),z_2(t))$. From Definition 6.6 and (6.11) it follows that:

$$\zeta(z)=a(z_1)\left\{\begin{array}{ll}\min\{|\tau(z)\theta(z_1)|^w\},\eta(z_1)^w\} & w<0\\ \max\{|\tau(z)\theta(z_1),\eta(z_1)^w\} & w>0\ .\end{array}\right.\tag{6.12}$$

Note that $\tau(q)=\tau(z)$.
If $w<0$ then, since ζ is bounded, it follows

$$\zeta(z)\sim\min\{\min(a_k(z_1)|\tau(z)\theta(z_1))|^w,1),\min(a(z_1)\eta(z_1)^w,1)\}\ .\tag{6.13}$$

Note that if condition (6.10) holds for f_1 and f_2 then it also holds for $\min\{f_1,f_2\}$. Also if f is a non-negative and bounded function then $f\sim\min(f,1)$. Therefore, it suffices to prove that

$$\min(a(z_1)|\tau(z)\theta(z_1))|^w,1)\lesssim\min(a(x)|\tau(q)\theta(x)|^w,1),$$

and

$$\min(a(z_1)\eta(z_1)^w,1) \lesssim \min(a(x)\eta(x)^w,1) \ ,$$

The latter inequality is a straightforward consequence of our assumption. For the former inequality, we note that

$$\min(a(z_1)\theta(z_1)^w,1) \lesssim \min(a(x)\theta(x)^w,1),$$

Therefore,

$$\begin{aligned}\min(a(z_1)|\tau(z)\theta(z_1))|^w,1) &= \min\{\tau(z)^w a(z_1)\theta(z_1)^w, \tau(z)^w, 1\} \\ &= \min\{\tau(z)^w \min(a(z_1)\theta(z_1)^w,1),1\} \\ &\lesssim \min\{\tau(q)^w a(x)\theta(x)^w, 1\} \ .\end{aligned}$$

Assume now that $w > 0$. It follows from the fact that ζ is bounded, formula (6.12) and the our assumption

$$a(z_1)|\tau(z)\theta(z_1)|^w \lesssim a(x)|\tau(q)\theta(x)|^w \ . \tag{6.14}$$

Therefore,

$$\begin{aligned}\zeta(z) &\sim \max\{(a(z_1)|\tau(z)\theta(z_1)|^w, a(z_1)\eta(z_1)^w\} \\ &\lesssim \max(a(x)|\tau\theta(x)|^w, a(x)\eta(x)^w) \\ &\sim \zeta(q) \ .\end{aligned}$$

□

Remark 6.11. The main result of this section is a strengthening of Theorem 6.9, in which we construct a deformation retraction with various estimates on its derivatives in terms of the deformation parameter t. This topic is technically the most important part of our work.

Theorem 6.12. *(Lipschitz deformation retraction theorem) Let Σ_0 be a stratification of $\mathbb{R}^n$, $X = \cup X_j$, $X_j \in \Sigma_0$, $0 \in \overline{X}_j \cap X$, $j = 1, \dots, m$. There exist a stratified neighborhood (U, Σ_U) of 0 in $\mathbb{R}^n$ with Σ_U a cell subdivision such that $\Sigma_U \prec \Sigma \cap U$ and a Lipschitz semialgebraic deformation retraction $r : U \times I \to U$ such that*

(1) $r_0(x) = 0$, $r_1(x) = x$,
(2) $r|_{S\times(0,1]}$ is smooth ,
(3) $|\det Dr_t| \gtrsim t^\mu$, for some $\mu \geq 0$,
(4) $\|Dr_t\| \lesssim t^\lambda$ for some $\lambda > 0$,

where $\det Dr_t$ is taken with respect to the coordinates of the respective cell of Σ_U (see Remark 6.5) and $\|Dr_t\|$ denotes the operator max-norm of the tangent map.

We prove this theorem in Section 8.

Remark 6.13. In contrast, condition (6.9) and the functions ξ_j of Theorem 6.9 are absent in the statement of Theorem 6.12: condition (6.9) was included only to carry the inductive step of the proof of Theorem 6.9 and not for our applications. The proof of Theorem 6.12 is by induction similar to that of the proof of Theorem 6.9. It involves functions analogous to the functions ξ_j of Theorem 6.9 and inequalities strengthening condition (6.9).

7. Regular families of hypersurfaces and bi-Lipschitz homeomorphisms

This section contains preliminaries made use of in Section 8 in order to construct a Lipschitz semialgebraic deformation retraction on a semialgebraic set with control on the growth of the derivatives.

In our sketch of proof of Theorem 6.9 we did not include an explanation on how to construct a bi-Lipschitz transformation of the ambient $\mathbb{R}^{n+1}$ that maps a given set with empty interior to a subset of a union of a finite number of graphs of Lipschitz semialgebraic functions. Construction of such a bi-Lipschitz map was essentially obtained by G.Valette in [V1], but we include it for the sake of completeness.

In our main Theorem 7.6 of this section we start with a stratification Σ of $\mathbb{R}^{n+1}$ and construct a bi-Lipschitz transformation h of $\mathbb{R}^{n+1}$ that maps a given cone, to a perhaps larger cone, such that the restriction of h to a certain refinement $\mathcal{A} \prec \Sigma$ is a diffeomorphism. Moreover, the images of the strata in $\mathcal{A}$ are graphs of Lipschitz semialgebraic functions or bands over cells in $\mathbb{R}^n$.

In what follows we will use the following notations. Let e_i , $i = 1, ..., n$ to be the standard basis and S^{n-1} the unit sphere of $\mathbb{R}^n$. For $\lambda \in S^{n-1} \subset \mathbb{R}^n$, we denote by N_λ the orthogonal to λ subspace of $\mathbb{R}^n$ (shortly $N_\lambda := \lambda^\perp$) and by π_λ the projection onto N_λ along λ. Given $q \in \mathbb{R}^n$, we denote by $q \mapsto q_\lambda$ the standard scalar product (in $\mathbb{R}^n$) with λ. We say that a set $H \subset \mathbb{R}^{n+1}$ is a graph **relative to** λ if there exists a function $\xi : N_\lambda \to \mathbb{R}$ such that

$$H = \{q \in \mathbb{R}^{n+1} : q_\lambda = \xi(\pi_\lambda(q))\}.$$

Definition 7.1. A **Lipschitz cell decomposition of** $\mathbb{R}^n$ is a cylindrical cell decomposition $\mathcal{C}$ of $\mathbb{R}^n$ which is also a stratification and is such that for $n > 1$ each cell $C \in \mathcal{C}$ is either a graph of a Lipschitz semialgebraic function or a band bounded by two Lipschitz semialgebraic functions over some cell C' in $\mathbb{R}^{n-1}$. The vector e_n is said to be **regular** for $\mathcal{C}$ if for each cell $C \in \mathcal{C}$ the restriction to C of $\pi_n := \pi_{e_n}$ is a one-to-one map and, also, there exists a Lipschitz function $\xi : \pi_n(C) \to \mathbb{R}$ such that C is the graph of ξ over $\pi_n(C)$.

We will need the following results from [V1] for our construction of the bi-Lipschitz transformation of this section.

Definition 7.2. A **regular family of hypersurfaces** of $\mathbb{R}^{n+1}$ is a family $H = (H_k; \lambda_k)_{1\le k\le b}$, $b \in \mathbb{N}$, of hypersurfaces of $\mathbb{R}^{n+1}$ together with elements λ_k of S^n such that the following properties hold for each $k < b$:

(i) The consecutive pairs H_k and H_{k+1} are the graphs relative to λ_k of two global Lipschitz functions ξ_k and, respectively, ξ'_k such that $\xi_k \le \xi'_k$;
(ii) $E(H_{k+1}; \lambda_k) = E(H_{k+1}; \lambda_{k+1})$, where $E(H_k; \lambda_k) = \{q \in \mathbb{R}^{n+1} : q_\lambda \le \xi(\pi_\lambda(q))\}$.

Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$ of empty interior. We say that the family H is **compatible** with A, if $A \subseteq \bigcup_{k=1}^{b} H_k$. An **extension** of H is a regular family compatible with the set $\bigcup_{k=1}^{b} H_k$.

We will also make use of the following notation

Notation 7.3. Let $\lambda \in S^{n-1}$ and $M \in [0,1)$. We denote

$$C_n(\lambda, M) := \{q \in \mathbb{R}^n : \frac{q \cdot \lambda}{|q|} \geq M\} \subset \mathbb{R}^n$$

(which are cones centered at 0 with the axis being λ).

Given two functions $f, g : A \to \mathbb{R}$ we say that f is **equivalent** to g, $f \sim g$, if there exist $c_1 > 0$ and $c_2 > 0$ such that $c_1 f \leq g \leq c_2 f$. If $f \leq c_1 g$, we write $f \lesssim g$. We say that f is **comparable** with g if the difference $f - g$ has a constant sign.

Theorem 7.4. *(Proposition 3.10 [V1]) For each semialgebraic set $A \subset \mathbb{R}^{n+1}$ with empty interior and $\varepsilon > 0$ there exists a regular family $(H_k; \lambda_k)_{1 \leq k \leq b}$ of hypersurfaces of $\mathbb{R}^{n+1}$ compatible with A and such that $\lambda_k \cdot e_{n+1} > 1 - \varepsilon$, $1 \leq k \leq b$.*

Given a regular system of hypersurfaces $H := (H_k; \lambda_k)_{1 \leq k \leq b}$ in $\mathbb{R}^{n+1}$ we associate with it a bi-Lipschitz map $h_H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ (Proposition 3.13 [V1]). For completeness we give the construction of h_H below.

Proposition 7.5. *(Proposition 3.13 [V1]) Let $H := (H_k; \lambda_k)_{1 \leq k \leq b}$ be a regular system of hyperplanes in $\mathbb{R}^{n+1}$. There exists a bi-Lipschitz mapping $h_H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps each hypersurface H_k to a hypersurface F_k which is a graph of a Lipschitz semialgebraic function η_k for e_n.*

Proof. We define h_H over $E(H_k; \lambda_k)$ by induction on k in such a way that

$$h_H(E(H_k; \lambda_k)) = E(F_k; e_n) \ , \tag{7.15}$$

where F_k is the graph of a Lipschitz function η_k relative to e_n . Note that then $h_H(H_k) = F_k$.

For $k = 1$ choose an orthonormal basis in N_{λ_1} and set $h_H(q) = (x_{\lambda_1}; q_{\lambda_1})$, where x_{λ_1} are the coordinates of $\pi_{\lambda_1}(q)$ in this basis. Then, let $k \geq 1$ and assume that h_H has been already constructed on $E(H_k; \lambda_k)$. Property (i) of Definition 7.2 says that H_k and H_{k+1} are the graphs relative to the same λ_k of two Lipschitz functions ζ_k and ζ'_k. For $q \in E(H_{k+1}; \lambda_k) \setminus E(H_k; \lambda_k)$ set

$$h_H(q) := h_H(\pi_{\lambda_k}(q); \zeta_k \circ \pi_{\lambda_k}(q)) + (q_{\lambda_k} - \zeta_k \circ \pi_{\lambda_k}(q)) e_n \ .$$

Due to property (ii) of Definition 7.2 we have $E(H_{k+1}; \lambda_{k+1}) = E(H_{k+1}; \lambda_k)$, so that h_H turns out to be defined over $E(H_{k+1}; \lambda_{k+1})$. Since ζ_k is Lipschitz h_H a bi-Lipschitz homeomorphism. Note also that (7.15) holds with F_{k+1} a graph of the following Lipschitz function

$$\eta_{k+1}(q) = \eta_k \circ \pi_{e_n}(q) + (\zeta'_k - \zeta_k) \circ \pi_{\lambda_k} \circ h^{-1}(q; \eta_k \circ \pi_{e_n}(q)) \ .$$

We now constructed h_H over $E(H_b;\lambda_b)$. To extend h_H to the whole of $\mathbb{R}^n$ we follow the case of $k=1$ (use λ_b instead of λ_1). Now it is straightforward to verify that h_H is a bi-Lipschitz homeomorphism. □

The following theorem is the main theorem of this section (cf. [V2] Corollary 2.2.4).

Theorem 7.6. *Let Σ be a stratification of $\mathbb{R}^{n+1}$ compatible with a cone $C := C_{n+1}(e_1, M)$. There exists a refinement $\mathcal{A} \prec \Sigma$ and a bi-Lipschitz map $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that*

(1) $h|_A$ is diffeomorphism for all $A \in \mathcal{A}$,
(2) there exists $0 < M' < 1$ such that $h(C) \subset C_{n+1}(e_1, M')$,
(3) every stratum $h(A)$, $A \in \mathcal{A}$, is either a graph of a Lipschitz semialgebraic function or a band bounded by graphs of Lipschitz semialgebraic functions over a stratum $h(A')$ in $\mathbb{R}^n$, $A' \in \mathcal{A}$.

For our proof the following proposition is crucial.

Proposition 7.7. *Let $A \subset \mathbb{R}^{n+1}$ be a semialgebraic set and let $H = (H_k; \lambda_k)_{1\le k\le b}$ be a regular system of hyperplanes compatible with A such that $\lambda_k \cdot e_{n+1} > 1-\varepsilon$. If $0 \in A \subset C_{n+1}(e_1, M)$ then there exists M' such that $h_H(A) \subset C_{n+1}(e_1, M')$, where h_H is provided by Proposition 7.5.*

Our proof of the latter proposition will make use of the following 3 lemmas.

Definition 7.8. Assume that $S \subset \mathbb{R}^{n+1}$ is a graph of a function ξ relative to $\lambda \in S^n$. Let map $\pi_S : \mathbb{R}^{n+1} \to S$ to be defined by $\pi_S(q) := \pi_\lambda(q) + \xi(\pi_\lambda(q))\lambda$,

Lemma 7.9. *Let $A' \subset C_n(e_1, M)$ and $\xi : A' \to \mathbb{R}$, $\xi(0) = 0$ be a Lipschitz semialgebraic function with Lipschitz constant L. Then, $\Gamma_\xi(A') \subset C_{n+1}(e_1, M/(1+L))$.*

Proof. We have to show that for $x \in A'$
$$\frac{(x,\xi(x))\cdot e_1}{\sqrt{\sum x_i^2 + \xi(x)^2}} \ge M/(1+L)\ .$$
Since $\xi(0) = 0$ we have
$$|\xi(x)| = |\xi(x) - \xi(0)| \le L|x|\ .$$
Therefore
$$\sqrt{\sum x_i^2 + \xi(x)^2} \le \sqrt{\sum x_i^2} + |\xi(x)| \le (1+L)|x|\ .$$
Since $x \in C_n(e_1, M)$ it follows that
$$\frac{(x,\xi(x))\cdot e_1}{\sqrt{\sum x_i^2 + \xi(x)^2}} \ge \frac{x\cdot e_1}{(1+L)|x|} \ge \frac{M}{1+L}\ .$$
□

Definition 7.10. Assume that $\lambda \in S^n$, $\lambda\cdot e_{n+1} > 0$. Let $e_{j,\lambda}$ be the unique vector in N_λ such that $\pi_{n+1} e_{j,\lambda} = e_j$.

Lemma 7.11. *Let $v := e_{1,\lambda} \in N_\lambda \subset \mathbb{R}^{n+1}$ and $\varepsilon > 0$. If $\lambda \cdot e_{n+1} \geq 1 - \varepsilon$ then*

$$C_{n+1}(e_1, M) \subset C_{n+1}(v/|v|, M')$$

with $M' := \frac{1}{|v|}\left(M - \frac{\sqrt{2\varepsilon}}{1-\varepsilon}\right)$.

Proof. Since v projects to e_1 there exists a constant $A \in \mathbb{R}$ such that $v = e_1 + Ae_{n+1}$. Since $N_\lambda = \lambda^\perp$ it follows that $0 = \lambda \cdot v = e_1 \cdot \lambda + Ae_{n+1} \cdot \lambda$. Hence $A = \frac{-e_1 \cdot \lambda}{e_{n+1} \cdot \lambda}$. Of course $\lambda = \sqrt{1-\delta^2} e_{n+1} + \delta w$ with $w \perp e_{n+1}$, $|w| = 1$ and $\delta \in \mathbb{R}_+$. Consequently, $1 - \varepsilon \leq \lambda \cdot e_{n+1} = \sqrt{1 - \delta^2}$,

$$\begin{aligned} 1 + \varepsilon^2 - 2\varepsilon &\leq 1 - \delta^2 \\ \delta^2 &\leq 2\varepsilon - \varepsilon^2. \end{aligned}$$

Therefore, $\delta \leq \sqrt{2\varepsilon}$ and the estimate for $|A|$ follows:

$$|A| = \left| \frac{-e_1 \cdot \lambda}{e_{n+1} \cdot \lambda} \right| \leq \left| \frac{|\delta w \cdot e_1|}{1 - \varepsilon} \right| \leq \left| \frac{\sqrt{2\varepsilon}}{1 - \varepsilon} \right| .$$

Finally, let $q \in C_{n+1}(e_1, M)$. Then $q \in C_{n+1}(v, M')$ due to

$$\begin{aligned} \frac{q \cdot v}{|q||v|} &= \frac{q \cdot e_1}{|q||v|} + A\frac{q \cdot e_{n+1}}{|q||v|} \\ &\geq \frac{1}{|v|}\left(M - \frac{\sqrt{2\varepsilon}}{1 - \varepsilon}\right) , \end{aligned}$$

as required. □

Lemma 7.12. *Let $A \subset C_{n+1}(e_{1,\lambda}, M)$, $\lambda \cdot e_{n+1} > 1 - \varepsilon$ and let $H_1 \subset \mathbb{R}^{n+1}$ be a graph of a Lipschitz semialgebraic function ξ, $\xi(0) = 0$ for λ_1, $\lambda_1 \cdot e_{n+1} > 1 - \varepsilon$. Then, $\pi_{H_1}(A) \subset C_{n+1}(e_{1,\lambda_1}, M')$.*

Proof. It follows from Lemma 7.11 that $C_{n+1}(e_{1,\lambda}, M) \subset C_{n+1}(e_{1,\lambda_1}, M'')$ for some M''. Hence $\pi_{\lambda_1}(A) \subset C_{n+1}(e_{1,\lambda_1}, M'')$ and since ξ is Lipschitz with $\xi(0) = 0$ it follows that $\pi_{H_1}(A) \subset C_{n+1}(e_{1,\lambda_1}, M')$, where $M' := M''/(1 + L))$ and L is the Lipschitz constant of ξ. □

Proof of Proposition 7.7. Assume first that $A \subset H_k$ for some k and H_k is a graph of ξ_k relative to λ_k. Observe that

$$\pi_{n+1} \circ (h_H|_{H_k})(q) = \pi_{\lambda_1} \circ \pi_{H_1} \circ \cdots \circ \pi_{H_{k-1}}(q).$$

It follows by iterating Lemma 7.12, that $\pi_{n+1} \circ (h_H|_{H_k})(A) \subset C_{n+1}(e_1, M'')$ for some M''. Since $h_H(H_K) = F_k$ is a graph of a Lipschitz semialgebraic function η_k with $\eta_k(0) = 0$ we conclude using Lemma 7.9 that $h_H(A) \subset C_{n+1}(e_1, M''/(1 + L_k))$, where L_k is the Lipschitz constant of η_k. In the general case of $A \subset \cup H_k$ we apply the argument as above to $A \cap H_k$ and then conclude that $h_H(A) \subset \cup C_{n+1}(e_1, M''/(1 + L_k)) \subset C_{n+1}(e_1, M')$, as required. □

Nect we prove Theorem 7.6.

Proof of Theorem 7.6. Let $H = (H_k; \lambda_k)_{1\le k\le b}$ be a regular system of hyperplanes compatible with the topological boundaries of Σ. Let $h_H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the bi-Lipschitz map given by Proposition 7.5 and let $\eta_k : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz semialgebraic functions such that $F_k := h_H(H_k)$ is a graph of η_k.

By Proposition 7.7 we have $h_H(C) \subset C_1 := C_{n+1}(e_1, M')$ and hence conclusion (2) is proven.

Next, we construct a stratification $\mathcal{A}$ of $\mathbb{R}^{n+1}$ such that $h|_A$ is a diffeomorphism for every $A \in \mathcal{A}$ and moreover, $h(A)$ is either included in a graph of one of the η_i's or is a band bounded by the graphs of two consecutive η_i's.

By induction on k we define a family of stratifications $\mathcal{F}_k := \{\mathcal{A}_{1,k}, \dots, \mathcal{A}_{k,k}\}$ such that for each i, $\mathcal{A}_{i,k}$ is a stratification of H_i that refines $\mathcal{A}_{i,k-1}$.

For $k = 1$ stratification $\mathcal{A}_{1,1}$ is a stratification of H_1. Assuming that $\mathcal{F}_k$ is constructed we construct $\mathcal{F}_{k+1}$ as follows.

Define $\mathcal{A}_{k+1-j,k+1}$ by induction on j. For $j = 0$ set $\mathcal{A}_{k+1,k+1}$ to be a stratification of H_{k+1}. Assuming that $\mathcal{A}_{k+1-j,k+1}$ is constructed we construct $\mathcal{A}_{k-j,k+1}$. Since the hypersurface H_{k+1-j} is a graph of a Lipschitz semialgebraic function relative to λ_{k-j}, we set $\mathcal{A}_{k-j,k+1}$ to be a refinement of $\mathcal{A}_{k-j,k}$ compatible with all $\pi^{-1}_{\lambda_{k-j}}(A) \cap H_{k-j}$ for $A \in \mathcal{A}_{k-j+1,k+1}$.

Consequently, the final family $\mathcal{F}_b$ consists of a stratification of the hypersurfaces $(H_k; \lambda_k)_{1\le k\le b}$ which induces a stratification $\mathcal{A}$ of $\mathbb{R}^{n+1}$ with the strata of $\mathcal{A}$ being:

- The strata of each $\mathcal{A}_{j,b}$, $j = 1, \dots, b$
- The bands bounded by the graphs of ζ_k and ζ'_k relative to λ_k intersected with $\pi^{-1}_{\lambda_k}(\mathrm{A})$, where $A \in \mathcal{A}_{k,b}$ and $k = 1, \dots, b$.
- $\{q : q_{\lambda_b} > \zeta_b(q)\}$ and $\{q : q_{\lambda_1} < \zeta_1(q)\}$

Our construction of h clearly guarantees that $h|_A$ is smooth for all $A \in \mathcal{A}$ and that $\Sigma_1 := \{h(A)\}_{A\in\mathcal{A}}$ forms a stratification of $\mathbb{R}^{n+1}$. □

8. Proof of Lipschitz deformation retraction theorem

In this section we prove the main technical result of our work, Theorem 6.12. The proof follows the same structure as the sketch of the proof of Theorem 6.9.

Recall that inequality $\xi_j(r'_t(x)) \lesssim \xi_j(x)$ with ξ_j being a difference of two Lipschitz functions that bound a cell, is a criterion for the standard lift r_t of r'_t to be Lipschitz. In order to show that Dr_t admits estimates in terms of t, we will have to enlarge inequality (6.9) from Theorem 6.9 to a group of several inequalities:

(1) (a) $\xi_j(r_t(q)) \lesssim \xi_j(q)$
  (b) if $\xi_j(0) = 0$ then $\xi_j(r_t(q)) \lesssim t^{\lambda_j}\xi_j(q)$ for $q \in X$, $\lambda_j > 0$.

(2) $\xi_j(r_t(q)) \gtrsim t^{\mu_j}\xi_j(q)$, $\mu_j \ge 0$

Then, we derive the estimates of $|\det Dr_t|$ and $\|Dr_t\|$ in terms of t by making use of the latter inequalities in a way similar to our proof of r being Lipschitz.

In the sketch of proof of Theorem 6.9 we remarked that the functions ξ_j of the theorem may be assumed (upon a bi-Lipschitz transformation) to be of the form $\xi_j(q) = |y - \eta_j(x)|^{w_j} a_j(x)$, where η_j are Lipschitz and a_j are continuous semialgebraic functions. We will make use of the following theorem and lemma (both from [V1]) to justify this assumption.

Theorem 8.1. *(Proposition 4.3 [V1]) Given a non negative semialgebraic function f on $\mathbb{R}^n$, there exist a finite number of semialgebraic subsets $W_1, \dots, W_s$, and a partition of $\mathbb{R}^n$ such that f is equivalent to a product of powers of distances to the W_j's on each element of the partition.*

Lemma 8.2. *Let $\eta_1, \dots, \eta_b : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz semialgebraic functions and $V \subset \mathbb{R}^{n+1}$ be a cell bounded by two consecutive η_i's over $V' := \pi_{n+1}(V)$. Suppose that $W_1, \dots, W_m \subset \cup\Gamma_{\eta_j}$ and $\xi : V \to \mathbb{R}_+$ is a bounded semialgebraic function equivalent to*

$$\prod d(\cdot, W_j)^{\alpha_j},\ \alpha_j \in \mathbb{Q}$$

Then, there exist Lipschitz semialgebraic functions $\theta_1 \le \dots \le \theta_{b'} : \mathbb{R}^n \to \mathbb{R}$, continuous semialgebraic functions $a_i : V \to \mathbb{R}$ and a subdivision $V = \cup V_i$ such that each V_i is a cell bounded by two consecutive θ_i's over V' and, moreover,

$$\xi(q)|_{V_i} \sim |y - \eta_{\nu_i}(x)|^{w_i} a_i(x)\ . \tag{8.16}$$

We prove this lemma following

Remark 8.3.

(1) If A is a union of graphs of semialgebraic functions $\theta_1, \dots, \theta_k$ over $\mathbb{R}^n$ then there is an ordered family of semialgebraic functions $\xi_1 \le \dots \le \xi_k$ such that A is a union of graphs of these functions.

(2) Given a family of Lipschitz semialgebraic functions $f_1, \dots, f_k$ defined over $\mathbb{R}^n$ there is a cell decomposition $\mathcal{C}'$ of $\mathbb{R}^n$ and some Lipschitz semialgebraic functions $\xi_1 \le \dots \le \xi_m$ on $\mathbb{R}^n$ such that over each cell $C = \{q = (x\ ;\ q_{n+1}) \in \mathbb{R}^{n+1} :\ x \in C',\ \xi_i(x) \le q_{n+1} \le \xi_{i+1}(x)\}$, where $C' \in \mathcal{C}'$, the semialgebraic functions $|q_{n+1} - f_i(x)|$ are comparable with each other and are comparable with the functions $f_i \circ \pi_n$. Indeed, it suffices to consider the graphs of functions f_i, $f_i + f_j$ and $\frac{f_i+f_j}{2}$ and then the family $\xi_1, \dots, \xi_m$ is provided by the first remark.

Proof of Lemma 8.2. Note that

$$d(q, W_j \cap \Gamma_{\eta_\nu}) \sim |y - \eta_\nu(x)| + d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))\ ,\quad q \in \mathbb{R}^{n+1}\ , \tag{8.17}$$

where $j \in J$ and $1 \le \nu \le b$. According to Remark 8.3, there is a collection of Lipschitz semialgebraic functions $\{\theta_1, \dots, \theta_{b'}\} \supset \{\eta_1, \dots, \eta_b\}$ on $\mathbb{R}^n$ such that there exists a cell decomposition $\mathcal{C}_0$ of $\mathbb{R}^{n+1}$ with the following properties:

(1) The cells of $\mathcal{C}_0$ consist of graphs and bands of θ_i's over the cells in $\mathbb{R}^n$.

(2) The semialgebraic functions $d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))$, η_ν, $|\eta_\nu - \eta_{\nu'}|$, and $|y - \eta_\nu|$, where $1 \le \nu, \nu' \le b$ and $j \in J$, are pairwise comparable over each cell in $\mathcal{C}_0$.

Let $\mathcal{C}$ be a stratification of $\mathbb{R}^{n+1}$ obtained from $\mathcal{C}_0$ by refining the cells in $\mathbb{R}^n$ in such a way that taking graphs and bands of the restrictions of θ_j's to those cells results in a Whitney A stratification of $\mathbb{R}^{n+1}$.

Let C be an open cell in $\mathbb{R}^{n+1}$ bounded by the graphs of θ_{j_0} and θ_{j_0+1} over a cell C' in $\mathbb{R}^n$. Due to the fact that the cell decomposition $\mathcal{C}$ is compatible with the graphs of the η_i's, we have either $\eta_i|_{C'} \geq \theta_{j_0+1}$ or $\eta_i|_{C'} \leq \theta_{j_0}$ for any $i \in \{1, \ldots, b\}$. Note that for any $j \in J$ and $q \in C$ we have

$$d(q, W_j) = \min_{\nu} d(q, W_j \cap \Gamma_{\eta_\nu}) \ .$$

Therefore,

$$\begin{aligned} \xi(q) &\sim \prod_{j \in J} (\min_{\nu} d(q, W_j \cap \Gamma_{\eta_\nu}))^{w_j} \\ &\sim \prod_{j \in J} (|y - \eta_\nu(x)| + d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu})))^{w_j} \ . \end{aligned}$$

Each expression of the form

$$(|y - \eta_\nu(x)| + d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu})))^{w_j}$$

is equivalent to

$$\min\left(|y - \eta_\nu(x)|^{w_j}, d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))^{w_j}\right) \quad \text{if } w_j < 0 \tag{8.18}$$

and is equivalent to

$$\max\left(|y - \eta_\nu(x)|^{w_j}, d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))^{w_j}\right) \quad \text{if } w_j > 0 \ . \tag{8.19}$$

Since over the cell C, functions from collection $\{|y - \eta_\nu(x)|, |\eta_\nu - \eta_{\nu'}|, d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))\}$ with $1 \leq \nu, \nu' \leq b$, $i \in I$ and $j \in J$, are pairwise comparable it follows that the expressions in (8.18) and (8.19) are equal to either $|y - \eta_\nu(x)|^{w_j}$ or $d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))^{w_j}$. Also, one of the following 3 properties holds

- $|y - \eta_\nu(x)| \sim |y - \eta_{\nu'}(x)|$
- $|y - \eta_\nu(x)| \sim |\eta_\nu(x) - \eta_{\nu'}(x)|$
- $|y - \eta_{\nu'}(x)| \sim |\eta_\nu(x) - \eta_{\nu'}(x)|$.

Consequently, there are constants ν_k, w, $w_{\nu,\nu'}$, and w'_j such that over the cell C (8.16) holds with

$$a(x) = \prod_{\nu,\nu'} |\eta_\nu - \eta_{\nu'}|^{w_{\nu,\nu'}} \prod_{j \in J} d(x, \pi_n(W_j \cap \Gamma_{\eta_\nu}))^{w'_j} \ .$$

$\square$

Next we prove Theorem 6.12 in a formulation convenient for a proof by induction on the dimension of the ambient $\mathbb{R}^n$.

Theorem 8.4. *Let Σ_0 be a stratification of $\mathbb{R}^n$ compatible with $X := \cup_{j \in J} X_j$, $X_j \in \Sigma_0$ (with J a finite index set) such that $0 \in \overline{X_j} \cap X$ for all $j \in J$ and $X \subset C_n(e_1, M)$. Let $\xi_1, \ldots, \xi_l : \mathbb{R}^n \to \mathbb{R}_+$ be bounded semialgebraic functions.*

Then, there are an open stratified neighborhood (U, Σ_U) *of* 0 *in* $\mathbb{R}^n$, $\Sigma_U \prec \Sigma_0$ *and a Lipschitz semialgebraic deformation retraction* $r : U \times [0,1] \to U$ *such that*

(1) $r|_{S\times(0,1]}$ *is smooth for all* $S \in \Sigma_U$ *;*
(2) $r : S \times (0,1] \to S$, $S \in \Sigma_U$, $r_0(q) = 0$, $r_1(q) = q$ *;*
(3) (a) $\xi_j(r_t(q)) \lesssim \xi_j(q)$ *;*
(b) if ξ_j *is continuous near* 0 *and* $\xi_j(0) = 0$ *then for some* $\lambda_j > 0$ *and all* $q \in X$ *inequality* $\xi_j(r_t(q)) \lesssim t^{\lambda_j}\xi_j(q)$ *holds;*
(4) $\xi_j(r_t(q)) \gtrsim t^{\mu_j}\xi_j(q)$ *for some* $\mu_j \geq 0$ *;*
(5) $|\det Dr_t| \gtrsim t^{\mu}$ *for some* $\mu \geq 0$ *;*
(6) $\|Dr_t|_{X_{reg}}\| \lesssim t^{\lambda}$ *for some* $\lambda > 0$ *.*

Proof. Assume $n = 1$. Let $r(x,t) := tx$. Each function ξ_j is a bounded semialgebraic function near 0 and therefore is continuous at 0. If $\xi_j(0) = a_j > 0$ then for x small enough we have $\xi_j \sim a_j$ and therefore estimates (3a), (4), (5) and (6) hold. If $\xi(0) = 0$ then expanding each ξ_j into a Puiseux series

$$\xi_j(x) = b_j x^{w_j} + R(x), \quad w_j \in \mathbb{Q}_+, \quad R(x) \in o(x^{w_j}),$$

it follows

$$\xi_j(x) \sim b_j x^{w_j}.$$

Consequently all estimates from (3) to (6) follow. Next we prove ($\mathbf{H_{n+1}}$) assuming ($\mathbf{H_n}$). Throughout the proof, we represent points of $\mathbb{R}^{n+1}$ by $q = (x,y) \in \mathbb{R}^n \times \mathbb{R}$. According to Theorem 8.1 there exists a finite partition $\{V_i\}_{i\in I}$ of $\mathbb{R}^{n+1}$ and a finite family of closed semialgebraic subsets $\{W_j\}_{j\in J}$ with empty interiors (otherwise, we would replace such W_j's by their topological boundaries), such that for every V_i with 0 in its closure holds:

$$\xi_k(q) \sim \prod_{j\in J} d(q, W_j)^{w_{ijk}} \ , \quad q \in V_i \ , \tag{8.20}$$

where $1 \leq k \leq l$ and $w_{ijk} \in \mathbb{Q}$. We may assume that $0 \in W_j$ for all $j \in J$ since the W_j that do not contain 0 are superfluous for the validity of (8.20). Consider a stratification $\tilde{\Sigma}$ that simultaneously refines stratification Σ_0 and both collections $\{V_i\}_{i\in I}$ and $\{W_j\}_{j\in J}$. Applying Theorem 7.6 with input $\tilde{\Sigma}$ results in a refining stratification $\mathcal{A}$ and a bi-Lipschitz map $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that all of the images of the topological boundaries of semialgebraic sets in $\Sigma_0 \cup \{V_i\}_{i\in I} \cup \{W_j\}_{j\in J}$ are included in the graphs of Lipschitz semialgebraic functions $\eta_1, \ldots, \eta_b : \mathbb{R}^n \to \mathbb{R}$ and $h(X) \subset C_{n+1}(e_1, M')$ for some $M' \in [0,1)$. Moreover, $h|_A$ is a diffeomorphism for all $A \in \mathcal{A}$ and each stratum from $\mathcal{A}$ is mapped to a stratum of the form of a graph over lower dimensional stratum of one of the η_j's or a band bounded by two consecutive η_j's.

Since the final conclusion of Theorem 8.4 is stable upon application of a semialgebraic bi-Lipschitz homeomorphism we may identity map h with the identify the map and rename M' by M.

Lemma 8.2 applied to the sets W_j and functions ξ_k results in a collection of Lipschitz semialgebraic functions $\theta_1 \leq \cdots \leq \theta_{b'}$ (including functions η_j) defined over $\mathbb{R}^n$ and a cell decomposition $\Sigma \prec \mathcal{A}$ of $\mathbb{R}^{n+1}$ such that the cells in $\mathbb{R}^{n+1}$ are given by graphs and bands of θ_i's over the cells in $\mathbb{R}^n$ and over each cell $C \in \Sigma$

$$\xi_k|_C(q) \sim |y - \eta_{k,C}(x)|^{w_{k,C}} a_{k,C}(x) \ . \tag{8.21}$$

We assume without loss of generality that X_j's are cells of Σ since otherwise, we may achieve this by refining the stratification Σ. Note that $X' \subset C_n(e_1, M')$. Apply the inductive hypothesis to the cells Σ' (cells of Σ in $\mathbb{R}^n$) with $X' := \cup\pi_n(X_j)$ and to a collection of semialgebraic functions $\mathcal{G}$ which we list following. The functions in $\mathcal{G}$ are the following:

- $d(x, 0)$
- $|\theta_j(x)|$ for $1 \le j \le b'$,
- $|\eta_j(x) - \eta_{j+1}(x)|$ for $1 \le j \le b-1$,
- $|\theta_j(x) - \theta_{j+1}(x)|$ for $1 \le j \le b'-1$,
- $\min\left(a_k(x)|\theta_j(x) - \theta_{j+1}(x)|^{w_k}, 1\right)$ for $1 \le j \le b'-1$, $1 \le k \le l$,
- $\min\left(a_k(x)|\theta_j(x) - \eta_{\nu_k}(x)|^{w_k}, 1\right)$ for $1 \le j \le b'$, $1 \le k \le l$,
- $\min\left(a_k(x)|\theta_{j+1}(x) - \eta_{\nu_k}(x)|^{w_k}, 1\right)$ for $1 \le j \le b'-1$, $1 \le k \le l$.

As a consequence we obtain a neighborhood U' of $0 \in \mathbb{R}^n$ and Lipschitz semialgebraic deformation retraction $r' : U' \times I \to U'$ with all the properties listed in the theorem. Let r be the standard lift of r' as defined in Definition 6.6. Note that r is continuous and smooth on $C \times (0, 1]$ for every cell $C \in \Sigma$. It is clear from the definition of r that conclusions (1) and (2) hold for r. We have to prove that r is Lipschitz and that the estimates from (3) to (6) hold. Note that it is enough to prove all these estimates on a cell $C \in \Sigma$ of X, so let C be a cell in X.

Proof that r is Lipschitz. If C is a graph of a Lipschitz semialgebraic function then by Proposition 6.8 the standard lift r of r' is Lipschitz. If C is a band bounded by $\theta_j \le \theta_{j+1}$ then let $\theta := |\theta_j - \theta_{j+1}|$ and note that $\theta \in \mathcal{G}$. Therefore we have $\theta(r'(x,t)) \lesssim \theta(x)$. According to the criterion of 6.8 the standard lift r is also Lipschitz.

Proof of the estimates (3a) and (4). When C is a graph over a cell of Σ' condition (3a) is straight forward consequence of the induction hypothesis. Otherwise, the cell C is bounded by θ_j and θ_{j+1} and each ξ_k, $1 \le k \le s$, is of the form (8.21). Moreover, due to the construction preceding 8.21 also either $\eta_{\nu_k} \le \theta_j$ or $\eta_{\nu_k} \ge \theta_{j+1}$. We may assume, without loss of generality that the former case holds. Consequently

$$|y - \eta_{\nu_k}(x)| = |y - \theta_j(x)| + |\eta_{\nu_k}(x) - \theta_j(x)| \ .$$

Denote

$$\theta(x) := |\theta_{j+1}(x) - \theta_j(x)| \text{ and } \eta(x) := |\eta_{\nu_k}(x) - \theta_j(x)| \ .$$

Then

$$\xi_k(q) \sim a_k(x) \begin{cases} \min(|y - \theta_j(x)|^{w_k}, \eta(x)^{w_k}) & \text{if } \ w_k < 0 \\ \max(|y - \theta_j(x)|^{w_k}, \eta(x)^{w_k}) & \text{if } \ w_k > 0 \ . \end{cases} \tag{8.22}$$

Denote by $z := z(t) = (z_1(t), z_2(t))$ the components of the deformation retraction $r = r(q,t)$ with $(z_1(t), z_2(t)) \in \mathbb{R}^n \times \mathbb{R}$ and $q = (x, y)$. In abuse of notation we will write (z_1, z_2) instead of $(z_1(t), z_2(t))$. In terms of Definition 6.6 and due to (8.22) it follows that:

$$\xi_k(z) = a_k(z_1) \begin{cases} \min\{|\tau(z)\theta(z_1)|^{w_k}, \eta(z_1)^{w_k}\} & \text{if } w_k < 0 \\ \max\{|\tau(z)\theta(z_1)|^{w_k}, \eta(z_1)^{w_k}\} & \text{if } w_k > 0 \ . \end{cases} \tag{8.23}$$

Recall (Def.6.6) that $\tau(r(q,t)) = \tau(q)$ for all $t \in [0,1]$.
If $w_k < 0$ then boundedness of ξ_k implies

$$\xi_k(z) \sim \min\{\min(a_k(z_1)|\tau(z)\theta(z_1))|^{w_k}, 1), \min(a_k(z_1)\eta(z_1)^{w_k}, 1)\} \ . \tag{8.24}$$

Note that if conditions (3a) and (4) of the inductive hypothesis hold for f_1 and f_2 then they also hold for $\min\{f_1, f_2\}$ and that if f is a non-negative and bounded function then $f \sim \min(f, 1)$. Therefore, it suffices to prove that

$$\min(a_k(z_1)|\tau(z)\theta(z_1))|^{w_k}, 1) \lesssim \min(a_k(x)|\tau(q)\theta(x)|^{w_k}, 1),$$
$$\min(a_k(z_1)|\tau(z)\theta(z_1))|^{w_k}, 1) \gtrsim t^{\mu_k} \min(a_k(x)|\tau(q)\theta(x)|^{w_k}, 1)$$

and that

$$\min(a_k(z_1)\eta(z_1)^{w_k}, 1) \lesssim \min(a_k(x)\eta(x)^{w_k}, 1) \ ,$$
$$\min(a_k(z_1)\eta(z_1)^{w_k}, 1) \gtrsim t^{\mu_k} \min(a_k(x)\eta(x)^{w_k}, 1) \ ,$$

The latter two are immediate consequences of the inductive hypothesis. For the proof of the former two inequalities we note that the inductive hypothesis implies

$$\min(a_k(z_1)\theta(z_1)^{w_k}, 1) \lesssim \min(a_k(x)\theta(x)^{w_k}, 1) \ ,$$

and also that

$$\min(a_k(z_1)\theta(z_1)^{w_k}, 1) \gtrsim t^{\mu_k} \min(a_k(x)\theta(x)^{w_k}, 1) \ .$$

Therefore,

$$\begin{aligned} \min(a_k(z_1)|\tau(z)\theta(z_1))|^{w_k}, 1) &= \min\{\tau(z)^{w_k} a_k(z_1)\theta(z_1)^{w_k}, \tau(z)^{w_k}, 1\} \\ &= \min\{\tau(z)^{w_k} \min(a_k(z_1)\theta(z_1)^{w_k}, 1), 1\} \\ &\lesssim \min\{\tau(q)^{w_k} a_k(x)\theta(x)^{w_k}, 1\} \ . \end{aligned}$$

And similarly,

$$\begin{aligned} \min(a_k(z_1)|\tau(z)\theta(z_1))|^{w_k}, 1) &= \min\{\tau(z)^{w_k} a_k(z_1)\theta(z_1)^{w_k}, \tau(z)^{w_k}, 1\} \\ &= \min\{\tau(z)^{w_k} \min(a_k(z_1)\theta(z_1)^{w_k}, 1), 1\} \\ &\gtrsim \min\{\tau(q)^{w_k} t^{\mu_k} \min(a_k(x)\theta(x)^{w_k}, 1), 1\} \\ &\gtrsim t^{\mu_k} \min\{\tau(q)^{w_k} \min(a_k(x)\theta(x)^{w_k}, 1), 1\} \\ &= t^{\mu_k} \min\{\tau(q)^{w_k} a_k(x)\theta(x)^{w_k}, 1\} \ . \end{aligned}$$

Assume now that $w_k > 0$. Boundedness of ξ_k, formula (8.23) and the induction hypothesis imply that

$$a_k(z_1)|\tau(z)\theta(z_1)|^{w_k} \lesssim a_k(x)|\tau(q)\theta(x)|^{w_k} \ . \tag{8.25}$$
$$a_k(z_1)|\tau(z)\theta(z_1)|^{w_k} \gtrsim t^{\mu_k} a_k(x)|\tau(q)\theta(x)|^{w_k} \ .$$

Therefore,

$$\begin{aligned}\xi_k(z) &\sim \max\{(a_k(z_1)|\tau(z)\theta(z_1)|^{w_k}, a_k(z_1)\eta(z_1)^{w_k}\}\\ &\lesssim \max\{a_k(x)|\tau\theta(x)|^{w_k}, a_k(x)\eta(x)^{w_k}\}\\ &\sim \xi_k(q)\ .\end{aligned}$$

Similarly

$$\begin{aligned}\xi_k(z) &\sim \max\{(a_k(z_1)|\tau(z)\theta(z_1)|^{w_k}, a_k(z_1)\eta(z_1)^{w_k}\}\\ &\gtrsim t^{\mu_k}\max\{(a_k(x)|\tau(q)\theta(x)|^{w_k}, a_k(x)\eta(x)^{w_k}\}\\ &\sim t^{\mu_k}\xi_k(q)\ .\end{aligned}$$

Proof of the estimate (3b). Assume that ξ_j is continuous near 0 and $\xi_j(0)=0$. Note first that we may assume that $\xi_j(q) := d(q,0)$. Indeed, according to the Łojasiewicz inequality (Theorem 2.6.6 [BCR]) there is a continuous semialgebraic function $\tilde{\xi}_j$ and $L>0$ such that

$$\xi_j(q) = \tilde{\xi}_j(q)d(q,0)^L.$$

We may assume that $\tilde{\xi}_j$ is one of the functions ξ_i (by including it to begin with in the original collection) and then (3a) would imply $\tilde{\xi}_j(r_t(q)) \lesssim \tilde{\xi}_j(q)$. Therefore $d(r_t(q),0) \lesssim t^{\lambda_d}d(q,0)$ for some $\lambda_d>0$ would imply

$$\xi_j(r_t(q)) \lesssim t^{\lambda_d L}\xi_j(q).$$

Assume C is a cell in X.

Case 1: C is a graph of θ_j over C'. Then since $X \subset C_{n+1}(e_1,M)$ it follows that $\theta_j(0)=0$. Note that $d(q,0)\sim d(x,0)$. Indeed, assume $L_j>0$ is the Lipschitz constant of θ_j

$$d(q,0) \sim d(x,0) + |\theta_j(x)-\theta_j(0)| \le (1+L_j)d(x,0).$$

On the other hand, $d(q,0)\ge d(x,0)$. Therefore, $d(q,0)\sim d(x,0)$, as we claimed. Since the distance to zero function $x\mapsto d(x,0)$, is in $\mathcal{G}$ the induction hypothesis implies that $d(r'_t(x),0)\lesssim t^{\lambda_{d'}}d(x,0)$. Therefore,

$$d(r_t(q),0)\sim d(r'_t(x),0)\lesssim t^{\lambda_{d'}}d(x,0)\sim t^{\lambda_{d'}}d(q,0).$$

Case 2: Assume that C is a band bounded by the graphs of $\theta_j<\theta_{j+1}$ over C'. Note that $d(q,0)\le d(x,0)+|y|$. But $|y|\le Cd(x,0)$ since $q\in C_{n+1}(e_1,M)$ and hence $d(q,0)\lesssim d(x,0)$. On the other hand $d(q,0)\ge d(x,0)$ and therefore, $d(q,0)\sim d(x,0)$. Our proof of the remainder of the estimate is as in case 1 .

Proof of the estimate (5). Note that $\det Dr_t$ is well defined over each cell C. Indeed, every C is constructed iteratively as a graph or a band over a cell in a lower dimension. Assume $x_C := (x_{j_1},\dots,x_{j_k})$ are the coordinates on the band C. Then the remaining coordinates of $\mathbb{R}^{n+1}$ are Lipschitz semialgebraic functions of the coordinates x_C. Let $\pi: C\to D$ be the projection onto the x_C coordinate subspace on $\mathbb{R}^{n+1}$ and $\phi: D\to C$ be its inverse. Then $\det Dr_t := \det D\,(\pi\circ r_t\circ\phi)$.

Assume C is a graph of θ_j over a cell C'. Then $\det Dr_t = \det Dr'_t$ and therefore (5) holds by the inductive assumption.

Now assume C is a band bounded by the graphs of θ_j and θ_{j+1} over a cell C'. Then

$$r_{t,n+1}(q) = \theta_j(r'_t(x)) + \frac{(y-\theta_j(x))(\theta_{j+1}(r'_t(x)) - \theta_j(r'_t(x)))}{\theta_{j+1}(x) - \theta_j(x)} \ .$$

Note, that due to the iterative definition of r as the standard lifts from the lower dimensions the matrix Dr_t is lower triangular. In particular,

$$|\det Dr_t| = |\det Dr'_t| \left|\frac{\partial r_{t,n+1}}{\partial y}\right|.$$

Finally, the inductive hypothesis implies

$$\left|\frac{\partial r_{t,n+1}(q)}{\partial y}\right| = \left|\frac{\theta_{j+1}(r'_t(x)) - \theta_j(r'_t(x))}{\theta_{j+1}(x) - \theta_j(x)}\right| \gtrsim t^{\mu_{\theta_j}}$$

for some $\mu_{\theta_j} \geq 0$ and therefore,

$$|\det Dr_t| \gtrsim t^{\mu' + \mu_{\theta_j}} \ ,$$

as required.

Proof of the estimate (6). It suffices to show that

$$\left|\frac{\partial r_{t,n+1}}{\partial x_j}\right| \lesssim t^{\lambda}$$

and, also, that

$$\left|\frac{\partial r_{t,n+1}}{\partial y}\right| \lesssim t^{\lambda}$$

for some $\lambda > 0$. When C is a graph of θ_j over C' it follows due to the construction of the standard lift

$$r_{t,n+1}(q) = \theta_j(r'_t(x)) \ .$$

(and, in particular, does not depend on y). Therefore,

$$\left|\frac{\partial r_{t,n+1}}{\partial x_l}\right| = \left|\sum_i \frac{\partial \theta_j(r'_t(x))}{\partial x_i} \frac{\partial r'_{t,i(x)}}{\partial x_l}\right|.$$

By the induction hypothesis, $\left|\frac{r'_{t,i(x)}}{\partial x_l}\right| \lesssim t^{\lambda'}$ for some $\lambda' > 0$. Since θ_j is a Lipschitz semialgebraic function it follows $\left|\frac{\partial \theta_j(r'_t(x))}{\partial x_i}\right| \lesssim 1$ which completes the proof of (6) in the case of C being a graph over C'.

Now assume C is a band bounded by graphs of θ_j and θ_{j+1} over C'. Define $\theta(x) := \theta_{j+1}(x) - \theta_j(x)$. Then

$$r_{t,n+1}(q) = \theta_j(r'_t(x)) + (y - \theta_j(x)) \frac{\theta(r'_t(x))}{\theta(x)} \ .$$

The latter and the inductive assumption imply that

$$\left|\frac{\partial r_{t,n+1}}{\partial y}\right| = \left|\frac{\theta(r'_t(x))}{\theta(x)}\right| \lesssim t^{\lambda_\theta}$$

and that

$$\left|\frac{\partial r_{t,n+1}}{\partial x_l}\right| \leq \left|\sum_i \frac{\partial\theta_j(r'_t(x))}{\partial x_i}\frac{\partial r'_{t,i(x)}}{\partial x_l}\right| + \left|\frac{\partial\theta_j(x)}{\partial x_l}\frac{\theta(r'_t(x))}{\theta(x)}\right| + $$
$$+ \quad (y-\theta_j(x))\left|\frac{\theta(x)\sum_i \frac{\partial\theta(r'_t(x))}{\partial x_i}\frac{\partial r'_{t,i(x)}}{\partial x_l} - \theta(r'_t(x))\frac{\partial\theta(x)}{\partial x_l}}{\theta^2(x)}\right|.$$

Of course since $C \subset X$ is in a cone it follows $\theta_j(0) = \theta_{j+1}(0) = 0$ and $0 \leq y - \theta_j(x) \leq \theta(x)$ for $(x,y) \in C$. Finally, the induction hypothesis implies

$$(y-\theta_j(x))\left|\frac{\theta(x)\sum_i \frac{\partial\theta(r'_t(x))}{\partial x_i}\frac{\partial r'_{t,i(x)}}{\partial x_l} - \theta(r'_t(x))\frac{\partial\theta(x)}{\partial x_l}}{\theta^2(x)}\right| \leq \left|\sum_i \frac{\partial\theta(r'_t(x))}{\partial x_i}\frac{\partial r'_{t,i(x)}}{\partial x_l}\right|$$
$$+ \left|\frac{\theta(r'_t(x))\frac{\partial\theta(x)}{\partial x_l}}{\theta(x)}\right|$$
$$\lesssim t^{\lambda'} + t^{\lambda_\theta}$$

for some $\lambda_\theta > 0$ and (6) follows in the case of C being a band over C' as well, which completes the proof of Theorem 8.4. □

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Department of Mathematics, University of Toronto, 40 St. George st, Toronto, ON, Canada M5S 2E4

E-mail address: shartl@math.toronto.edu