Relativistic Quantum and Classical Mechanics in the Hilbert Space

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Abstract

We present a formulation of relativistic classical and quantum mechanics in the Hilbert space with a linkage through the Ehrenfest quantization. We start with the covariant form of the Lorentz force and consistently derive the Dirac equation along its classical counterpart as a classical spinorial equation. This equation fills a missing link between relativistic quantum and classical mechanics and takes the role of the Koopmann-von Neumann equation in the relativistic regime.

1 introduction

One of the first attempts to describe classical mechanics in terms of the Hilbert space was done by Koopman and von Neumann [1, 2]. This approach was further explored by Deoto and Mauro [3, 4, 5], and more recently enriched with the help of the Ehrenfest theorem in [6]. A different approach based on the formulation of classical/quantum mechanics in terms of the Algebra of Physical Space (APS), as a geometric algebra variety, provided new insights on the appearance of the spin in a classical relativistic framework [7, 8, 9, 10, 11]. This work carries further the Ehrenfest quantization method described in [6] by developing a covariant relativistic formulation of quantum and classical mechanics. Our formulation starts from the covariant form of the Lorentz force, which allows us to deduce the Dirac equation as well as the corresponding equation that appears in the classical limit.

This classical equation, which constitutes the most important novelty introduced in this paper, is a relativistic classical spinor equation in the time-extended phase-space that shows the remarkable presence of the interaction of the spin with the external electromagnetic field and a spin-orbit interaction.

2 Background: The Lorentz force

The Lagrangian of the relativistic particle in an electromagnetic field is

$$\mathcal{L} = \frac{m}{2} u^\mu u_\mu + eA^\mu u_\mu + \frac{m}{2}, \tag{1}$$

where $u^\mu = \frac{dx^\mu}{ds}$ is the four velocity also known as proper velocity and the Minkowski metric is taken with diagonal elements $\{1, -1, -1, -1\}$. More precisely, this is the time-extended form of the Lagrangian of a particle with electromagnetic interaction. In this formalism the condition

$$u^\mu u_\mu = 1 \tag{2}$$

is not enforced as a constraint but recovered as an integration condition in the moment to find physical solutions.

The canonical momentum is denoted with covariant indices as

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial u^\mu} = m u_\mu + eA_\mu \tag{3}$$

Let us remember that the physical momentum and four-vector potential are denoted with contravariant indices $p^\mu = (E, \mathbf{p})$ and $A^\mu = (\phi, \mathbf{A})$, where bold symbols stand for vectors in the standard Euclidean space.

The time-extended Hamiltonian is

$$\mathcal{H} \equiv p_\mu u^\mu - \mathcal{L} = \frac{(p^\mu - eA^\mu)(p_\mu - eA_\mu)}{2m} - \frac{m}{2}. \tag{4}$$

This Hamiltonian does not depend on the parameter s explicitly, so it is a dynamic invariant integral. The integration that leads to physical solutions occurs when

$$\mathcal{H} = 0, \tag{5}$$

which implies (2).

The equation of motion can be obtained accordingly with the application of the time-extended Poisson brackets defined as

$$\{F,G\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial G}{\partial x^\mu}\frac{\partial F}{\partial p_\mu} \tag{6}$$

such that

$$\frac{dF}{ds} = \{F,\mathcal{H}\} \tag{7}$$

The four-velocity is recovered as

$$\begin{aligned} \frac{dx^\mu}{ds} &= \frac{\partial \mathcal{H}}{\partial p_\mu} \\ \frac{dx^\mu}{ds} &= \frac{p^\mu - eA^\mu}{m} \end{aligned} \tag{8}$$

and the canonical four-force equation is

$$\begin{aligned} \frac{dp_\mu}{ds} &= -\frac{\partial \mathcal{H}}{\partial x^\mu} = \frac{e}{m}(\partial_\mu A_\nu)(p^\nu - eA^\nu) \\ \frac{dp_\mu}{ds} &= e(\partial_\mu A_\nu)u^\nu \end{aligned} \tag{9}$$

The left side of this equation can be expressed as

$$\frac{dp_\mu}{ds} = m\frac{du_\mu}{ds} + e\frac{dA_\mu}{ds} = m\frac{du_\mu}{ds} + e\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{ds} = m\frac{du_\mu}{ds} + e\frac{\partial A_\mu}{\partial x^\nu}u^\nu \tag{10}$$

and introduced in (9) to obtain the covariant form of the Lorentz force

$$\begin{aligned} m\frac{du_\mu}{ds} &= e(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu \\ m\frac{du_\mu}{ds} &= eF_{\mu\nu}u^\nu, \end{aligned} \tag{11}$$

where the Faraday electromagnetic tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

3 The Derivation of the Dirac Equation

According to (9) and (8), the dynamical equations in consideration are

$$\frac{dp_\mu}{ds} = \frac{e}{m}(\partial_\mu A_\nu)(p^\nu - eA^\nu), \tag{12}$$

$$\frac{dx^\mu}{ds} = \frac{p^\mu - eA^\mu}{m}. \tag{13}$$

The average of these equations over an assemble of particles written in the Hilbert space formalism (see, e.g., [6]) takes the form of the Ehrenfest theorems

$$\frac{d\langle\Psi(s)|\hat{p}_\mu|\Psi(s)\rangle}{ds} = \langle\Psi(s)|\frac{e}{m}(\partial_\mu\hat{A}_\nu)(\hat{p}^\nu - e\hat{A}^\nu)|\Psi(s)\rangle, \quad (14)$$

$$\frac{d\langle\Psi(s)|\hat{x}^\mu|\Psi(s)\rangle}{ds} = \langle\Psi(s)|\frac{\hat{p}^\mu - e\hat{A}^\mu}{m}|\Psi(s)\rangle. \quad (15)$$

Stones's theorem gives

$$\hbar\frac{d|\Psi(s)\rangle}{ds} = iG|\Psi(s)\rangle, \quad (16)$$

which can be viewed as the definition of the generator of motion G. Following the non-relativistic case [6], we transform the Ehrenfest theorems (12) and (13) into the following general commutator equations

$$\frac{i}{\hbar}[\hat{p}_\mu, G] = \frac{e}{m}(\partial_\mu A_\nu)(\hat{p}^\nu - e\hat{A}^\nu), \quad (17)$$

$$\frac{i}{\hbar}[\hat{x}^\mu, G] = \frac{\hat{p}^\mu - e\hat{A}^\mu}{m}. \quad (18)$$

Assuming that $G = D(\hat{x}^\mu, \hat{p}_\mu)$ and the following commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = -i\hbar\delta^\mu{}_\nu, \quad (19)$$

we get the differential equations for the generator

$$\frac{\partial D}{\partial\hat{x}^\mu} = -\frac{e}{m}(\partial_\mu\hat{A}_\nu)(\hat{p}^\nu - e\hat{A}^\nu), \quad (20)$$

$$\frac{\partial D}{\partial\hat{p}_\mu} = \frac{\hat{p}^\mu - e\hat{A}^\mu}{m}. \quad (21)$$

The solution for the generator D has the same form as the classical time-extended Hamiltonian (4), which reads

$$D = \frac{1}{2m}(\hat{p}^\mu - e\hat{A}^\mu)(\hat{p}_\mu - e\hat{A}_\mu) - \frac{m}{2}. \quad (22)$$

Assuming that the underlying Hilbert space is four-component, we can upgrade the operator D to the product of operators in a Clifford algebra with basis γ^μ obeying

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1} \quad (23)$$

such that the generator takes the form

$$D \to \mathbf{D} = \frac{1}{2m}\Big(\gamma^\mu(\hat{p}_\mu - e\hat{A}_\mu) + m\Big)\Big(\gamma^\nu(\hat{p}_\nu - e\hat{A}_\nu) - m\Big). \quad (24)$$

This step looks more natural observing that, for commuting operators or classical variables, one would have $\mathbf{D} = D \times \mathbf{1}$, or in other words $D = \frac{1}{4}Tr(\mathbf{D})$.

The Dirac generator can also be written as

$$\mathbf{D} = \frac{1}{2m}(\not{p} - e\not{A} + m)(\not{p} - e\not{A} - m). \quad (25)$$

The wave function equation that is consistent with the condition (5) leads to the following expression known as the quadratic form of the Dirac equation

$$(\not{p} - e\not{A} + m)(\not{p} - e\not{A} - m)\Psi = 0, \quad (26)$$

which is the composition of the more fundamental first order Dirac equation for positive and negative mass, respectively

$$(\not{p} - e\not{A} - m)\Psi = 0, \quad (27)$$

$$(\not{p} - e\not{A} + m)\Psi = 0. \quad (28)$$

4 The Derivation of the Relativistic Koopman-von Neumann Equation

The relativistic form of the Koopman-von Neumann equation can be obtained starting from the general commutator equations

$$i[\hat{P}_\mu, G] = \frac{e}{m}(\partial_\mu \hat{A}_\nu)(\hat{p}^\nu - e\hat{A}^\nu), \quad (29)$$

$$i[\hat{X}^\mu, G] = \frac{\hat{P}^\mu - e\hat{A}^\mu}{m}, \quad (30)$$

supplied with the commutation relationship for the position and momentum operators

$$[\hat{X}^\mu, \hat{P}_\nu] = 0. \quad (31)$$

Assume the dependence of the classical generator $G = K(\hat{X}^\mu, \hat{P}_\mu, \hat{\Lambda}_\mu, \hat{\Theta}^\mu)$ on two additional operators $\hat{\Lambda}_\mu$ and $\hat{\Theta}^\mu$ that satisfy the following commutation relations

$$[\hat{X}^\mu, \hat{\Lambda}_\nu] = -i\delta^\mu{}_\nu, \quad (32)$$

$$[\hat{P}_\mu, \hat{\Theta}^\nu] = -i\delta^\nu{}_\mu, \quad (33)$$

$$[\hat{\Lambda}_\mu, \hat{\Theta}^\nu] = 0. \quad (34)$$

Then, the commutator equations reduce to the system of differential equations

$$\frac{\partial K}{\partial \Theta^\mu} = \frac{e}{m}(\partial_\mu \hat{A}_\nu)(\hat{P}^\nu - e\hat{A}^\nu), \tag{35}$$

$$\frac{\partial K}{\partial \Lambda_\mu} = \frac{\hat{P}^\mu - e\hat{A}^\mu}{m}, \tag{36}$$

whose solution is

$$K = \frac{1}{m}(\hat{P}^\mu - e\hat{A}^\mu)(\hat{\Lambda}_\mu + e\partial_\nu \hat{A}_\mu \hat{\Theta}^\nu) + f(\hat{X}, \hat{P}), \tag{37}$$

with f as an arbitrary function of the classical operators $\hat{X}^\mu$ and $\hat{P}_\mu$.

Finally, the relativistic Koopman-von Neumann equation can be readily stated as

$$K\psi = 0, \tag{38}$$

with the substitution of $\hat{\Lambda}_\nu$ and $\hat{\Theta}^\nu$ by the corresponding differential operators that preserve the commutation relations (32)–(34).

5 The Derivation of the Vlasov Equation

The classical dynamics of a relativistic particle can be expressed in terms of coordinates and velocities instead of coordinates and momenta, which is in fact more common, such that

$$m\frac{du_\mu}{ds} = eF_{\mu\nu}u^\nu \tag{39}$$

$$\frac{dx^\mu}{ds} = u^\mu. \tag{40}$$

The application of the Ehrenfest quantization leads to the following commutation equations

$$i[\hat{u}_\mu, G] = eF_{\mu\nu}\hat{u}^\nu \tag{41}$$

$$i[\hat{x}^\mu, G] = \hat{u}^\mu. \tag{42}$$

The rest of the derivation of the generator resembles the case treated in the previous section with a generator $G = \mathcal{V}(\hat{x}^\mu, \hat{u}_\mu, \hat{\Lambda}_\mu, \hat{\Theta}^\mu)$, such that

$$\frac{\partial \mathcal{V}}{\partial \hat{\Theta}^\mu} = eF_{\mu\nu}\hat{u}^\nu, \quad (43)$$
$$\frac{\partial \mathcal{V}}{\partial \hat{\Lambda}_\mu} = \hat{u}^\mu. \quad (44)$$

This differential equation can be readily integrated, obtaining the following generator

$$\mathcal{V} = \hat{u}^\mu \hat{\Lambda}_\mu + eF_{\mu\nu}\hat{u}^\nu \hat{\Theta}^\mu + f(x,u). \quad (45)$$

This equation can be upgraded into a different context for the description of a plasma without collisions and with long range interactions. In this case, $F_{\mu\nu}$ takes into account both the external field and the effective field created by the particles in a self-consistent fashion. Choosing $f(x,p) = 0$, the Vlasov equation [12] for the density ρ in the context of the description of plasma reads

$$u^\mu \partial_\mu \rho + eF_{\mu\nu}u^\nu \frac{\partial \rho}{\partial u_\mu} = 0. \quad (46)$$

6 The Derivation of the Koopmann-von Neumann-Dirac Equation

The commutator relations from the Ehrenfest quantization are

$$\frac{i}{\hbar}[\hat{p}_\mu, G] = \frac{e}{m}(\partial_\mu \hat{A}_\nu)(\hat{p}^\nu - e\hat{A}^\nu), \quad (47)$$
$$\frac{i}{\hbar}[\hat{x}^\mu, G] = \frac{\hat{p}^\mu - e\hat{A}^\mu}{m}. \quad (48)$$

Assuming the dependence of the generator $G = W(\hat{x}^\mu, \hat{p}_\mu, \hat{\lambda}_\mu, \hat{\theta}^\mu)$ on the quantum operators $\hat{x}^\mu$ and $\hat{p}_\mu$ as well as on the auxiliary operators $\hat{\lambda}_\mu$ and $\hat{\theta}^\mu$ obeying the commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = -i\hbar\kappa\delta^\mu{}_\nu, \quad (49)$$
$$[\hat{x}^\mu, \hat{\lambda}_\nu] = -i\delta^\mu{}_\nu, \quad (50)$$
$$[\hat{p}_\mu, \hat{\theta}^\nu] = -i\delta^\nu{}_\mu, \quad (51)$$
$$[\hat{\lambda}_\mu, \hat{\theta}^\nu] = 0, \quad (52)$$

one reaches the following differential equations

$$\frac{i}{\hbar}\left(i\hbar\kappa\frac{\partial W}{\partial x^\mu} - i\frac{\partial W}{\partial \theta^\mu}\right) = \frac{e}{m}(\partial_\mu \hat{A}_\nu)(\hat{p}^\nu - e\hat{A}^\nu), \quad (53)$$

$$\frac{i}{\hbar}\left(-i\hbar\kappa\frac{\partial W}{\partial p_\mu} - i\frac{\partial W}{\partial \lambda_\mu}\right) = \frac{\hat{p}^\mu - e\hat{A}^\mu}{m}. \quad (54)$$

The solution of these equations reads

$$W = \frac{1}{2m\kappa}(\hat{p}^\mu - e\hat{A}^\mu)(\hat{p}_\mu - e\hat{A}_\mu) - \frac{m}{2} - f(x^\mu + \hbar\kappa\theta^\mu, p_\mu - \hbar\kappa\lambda_\mu), \quad (55)$$

with f as an arbitrary function. Retracing the steps when obtaining the Dirac equation, we upgrade the generator W to the Clifford algebra spanned by the γ^μ matrices

$$W \to \mathbf{W} = \frac{1}{2m\kappa}\gamma^\mu(\hat{p}_\mu - e\hat{A}_\mu)\gamma^\nu(\hat{p}_\nu - e\hat{A}_\nu) - \frac{m}{2} - f(x^\mu + \hbar\kappa\theta^\mu, p_\mu - \hbar\kappa\lambda_\mu). \quad (56)$$

The classical generator is obtained in the limit $\kappa \to 0$ if f is chosen to be

$$f = \frac{1}{2m\kappa}\gamma^\mu(\hat{p}_\mu - \hbar\kappa\hat{\lambda}_\mu - e\hat{A}_\mu(\hat{x} + \hbar\kappa\hat{\theta}))\gamma^\nu(\hat{p}_\nu - \hbar\kappa\hat{\lambda}_\nu - e\hat{A}_\nu(\hat{x} + \hbar\kappa\hat{\theta})) - \frac{m}{2}, \quad (57)$$

along with the following transformation of operators from the current quantum operators to a new set of classical operators (34)

$$\hat{x}^\mu = \hat{X}^\mu - \frac{\hbar\kappa}{2}\hat{\Theta}^\mu, \quad (58)$$

$$\hat{p}_\mu = \hat{P}_\mu + \frac{\hbar\kappa}{2}\hat{\Lambda}_\mu, \quad (59)$$

$$\hat{\lambda}_\mu = \hat{\Lambda}_\mu, \quad (60)$$

$$\hat{\theta}^\mu = \hat{\Theta}^\mu, \quad (61)$$

which have the remarkable property of preserving the commutation relations of the original quantum operators. The connection between quantum and classical operators are schematically represented in Figure 1

The quantum generator, written in terms of the classical operators, is

$$\mathbf{W} = \frac{\hbar}{m}\gamma^\mu\hat{P}_\mu\gamma^\nu\hat{\Lambda}_\nu + \frac{1}{m\kappa}\gamma^\mu\hat{P}_\mu\gamma^\nu(\hat{A}^+_\nu - \hat{A}^-_\nu) - \frac{\hbar}{2m}\gamma^\mu\hat{\Lambda}_\mu\gamma^\nu(\hat{A}^+_\nu + \hat{A}^-_\nu) + \frac{e^2\gamma^\mu\gamma^\nu}{2m\kappa}(\hat{A}^-_\mu\hat{A}^-_\nu - \hat{A}^+_\mu\hat{A}^+_\nu), \quad (62)$$

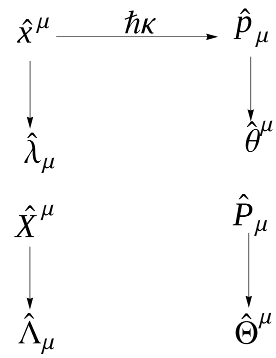

Figure 1: Schematic layout of the quantum and classical operators.

with $\hat{A}^+_\nu = \hat{A}^+_\nu(\hat{X}^\mu + \frac{\hbar\kappa}{2}\hat{\Theta}^\mu)$ and $\hat{A}^-_\nu = \hat{A}^-_\nu(\hat{X}^\mu - \frac{\hbar\kappa}{2}\hat{\Theta}^\mu)$.

From (62), we get

$$\hbar\mathbf{K} = \lim_{\kappa\to 0}\mathbf{W}, \tag{63}$$

leading to

$$\begin{aligned}\hbar\mathbf{K} &= \frac{\hbar}{m}\gamma^\mu\hat{P}_\mu\gamma^\nu\hat{\Lambda}_\nu + \frac{\hbar}{m}\gamma^\mu\hat{P}_\mu\gamma^\nu(\partial_\alpha\hat{A}_\nu\hat{\Theta}^\alpha) &(64)\\ &\quad -\frac{\hbar}{m}\gamma^\mu\hat{\Lambda}_\mu\gamma^\nu\hat{A}_\nu - \frac{\hbar e^2}{2m}\gamma^\mu\hat{A}_\mu\partial_\alpha\gamma^\nu\hat{A}_\nu\hat{\Theta}^\alpha, &(65)\end{aligned}$$

which can be arranged to derive the classical limit of the Dirac generator

$$\mathbf{K} = \frac{1}{m}\gamma^\mu(\hat{P}_\mu - \hat{A}_\mu)\gamma^\nu(\hat{\Lambda}_\mu + \partial_\alpha\hat{A}_\nu\hat{\Theta}^\alpha) - \frac{e^2}{2m}[\gamma^\mu\hat{\Lambda}_\mu, \gamma^\nu\hat{A}_\nu]. \tag{66}$$

The commutator can be manipulated as

$$[\gamma^\mu\hat{\Lambda}_\mu, \gamma^\nu\hat{A}_\nu] = i[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + [\gamma^\mu, \gamma^\nu]\hat{A}_\nu\hat{\Lambda}_\mu, \tag{67}$$

revealing the interaction of the classical spin with the external electromagnetic field on the left and the classical spin-orbit coupling on the right.

7 General features of the Koopmann-von Neumann-Dirac equation

The Koopmann-von Neumann-Dirac equation (KND) is defined in terms of the application of the generator (66) on a spinor field

$$\mathbf{K}\Psi = 0. \tag{68}$$

In particular, the KND equation without spin interaction can be written in terms of the APS representation of the geometric algebra within the phase space as

$$\mathbf{K} = \frac{1}{m}(P - A)\left(\partial + \partial_\alpha A \frac{\partial}{\partial P_\alpha}\right), \tag{69}$$

where the spinor Ψ is represented by a 2×2 complex matrix with the current and density in this representation as

$$j = \Psi\Psi^\dagger, \tag{70}$$

$$\rho = \frac{1}{4}Tr(\Psi\Psi^\dagger). \tag{71}$$

Not every solution of the KND equation represents a consistent classical system. A *necessary condition* for a truly classical solution is that the current needs to satisfy the KND equation

$$\mathbf{K}\Psi\Psi^\dagger = 0 \Rightarrow \Psi \in \text{classical world}. \tag{72}$$

Hence, the KND equation can be reformulated as the equation for the current

$$\mathbf{K}j = 0, \tag{73}$$

which constrains the form of the spinor Ψ. According to the singular value decomposition, the spinor can always be written as the product of a Hermitian part B and a unitary part U such that

$$\Psi = BU. \tag{74}$$

Applying the KND operator on the corresponding current leads to

$$\mathbf{K}j = \mathbf{K}B^2, \tag{75}$$

which implies that if a solution Ψ of the KND equation is purely Hermitian, then the corresponding current satisfies the KND equation as well. In other words, the Hermiticity of the spinor is a sufficient condition to be a classically valid solution

$$\Psi^\dagger = \Psi \Rightarrow \mathbf{K}j = 0 \Rightarrow \Psi \in \text{Classical solution.} \tag{76}$$

A less restrictive sufficient condition for the current to satisfy the KND equation reads

$$[\Psi, \Psi^\dagger] = 0 \wedge \mathbf{K}\Psi^\dagger = 0 \Rightarrow \Psi \in \text{Classical solution} \tag{77}$$

The KND equation must be invariant under gauge transformations $A \to A + \partial\chi$. This fact can be translated into the possibility to be able to act on any spinor Ψ with the following operator

$$\mathcal{R} = \exp\left(e\partial_\mu\chi\frac{\partial}{\partial p_\mu}\right), \tag{78}$$

and still maintaining $\mathcal{R}\Psi$ as a valid solution of the KND equation.

The KND equation is in general very difficult to solve but some particular analytical solutions can be found for cases involving constant electromagnetic fields. The case of a constant electric field along the x^3 direction demands the following equation to be solve

$$\left(\partial + e\partial_3\phi(x^3)\frac{\partial}{\partial p_3}\right)\Psi = 0, \tag{79}$$

with $\partial_3\phi(x^3) = -E_0$. A specific spinor solution can be pursued in the form

$$\Psi = f(x,p)\sqrt{u}, \tag{80}$$

with $f(x,p)$ as a real scalar function and a constant Hermitian spinor $\sqrt{u}$. Introducing (80) into the KND equation, the differential equation for $f(x,p)$ is

$$\partial_0 f - eE_0\frac{\partial}{\partial p_3}f = 0, \tag{81}$$

which has the following solution

$$f = f(p_0, p_1, p_2, p_3 + E_0x^0 - \mathfrak{p}_3), \tag{82}$$

with $\mathfrak{p}_3$ as a constant and the shell condition for physically allowed solutions

$$(p_0 - \phi(z))^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = m^2 \tag{83}$$

A particular solution in the configuration space where all the particles have an initial four-momentum $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3$ at time $x^0 = 0$ can be found by selecting

$$u = \frac{\mathfrak{p}}{m}, \tag{84}$$

where the square root can be calculated with the help of formula (112) such that the final solution is the following Hermitian expression

$$\Psi_{\mathfrak{p}} = f(\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, p_3 + eE_0 x^0 - \mathfrak{p}_3)\sqrt{\frac{\mathfrak{p}}{m}} \tag{85}$$

The particle current is calculated as

$$j_{\mathfrak{p}} = \Psi\Psi^\dagger = f(\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, p_3 + eE_0 x^0 - \mathfrak{p}_3)^2 \frac{\mathfrak{p}}{m}, \tag{86}$$

which is also a solution of the KND equation. In this case, the density of particles is identified as

$$\rho = \frac{1}{4} Tr(\Psi\Psi^\dagger) = f(\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, p_3 + eE_0 x^0 - \mathfrak{p}_3)^2 \frac{\mathfrak{p}_0}{m}, \tag{87}$$

where $\mathfrak{p}_0$ is obtained from the shell condition. If f represents a Dirac delta function, the solution corresponds exactly with the single-particle solution because the particle is constrained to be located at

$$-p^3 + eE_0 x^0 - \mathfrak{p}_3 = 0. \tag{88}$$

For example, if the particle starts at the rest, this condition can be written as

$$\gamma \frac{dx^3}{dx^0} = \frac{eE_0}{m} x^0, \tag{89}$$

with $\gamma = \sqrt{1 - (\frac{dx^3}{dx^0})^2}$. Further arrangement of this equation leads to

$$\frac{dx^3}{dx^0} = \frac{x^0 \frac{eE}{m}}{\sqrt{1 + (x^0 \frac{eE}{m})^2}}, \tag{90}$$

which can be integrated to obtain a well known result

$$x^3 = \sqrt{\left(\frac{m}{eE}\right)^2 + (x^0)^2}, \tag{91}$$

for the initial condition $x^3(0) = |\frac{m}{eE}|$.

8 Conclusions

We developed a unified formulation of relativistic quantum and classical mechanics in the Hilbert space. We applied the Ehrenfest quantization based on the classical covariant Lorentz force to obtain the Dirac equation as well as the Koopman-von Neumann-Dirac (KND) equation in the classical limit as a relativistic classical spinor equation. The existence of this equation has important consequences concerning the nature of the spin and its full consistency within the framework of classical mechanics.

A Canonical commutation relations

The commutation relation

$$[\hat{x}^\mu, \hat{p}_\nu] = -i\delta^\mu{}_\nu \hbar, \tag{92}$$

are satisfied with the following substitution

$$\hat{x}^\mu \quad \rightarrow \quad x^\mu \tag{93}$$

$$\hat{p}_\mu \quad \rightarrow \quad i\hbar\frac{\partial}{\partial x^\mu} \tag{94}$$

which is in agreement with the standard representation

$$E \quad \rightarrow \quad i\hbar\frac{\partial}{\partial t} \tag{95}$$

$$\mathbf{p} \quad \rightarrow \quad -i\hbar\nabla \tag{96}$$

Similarly, the following commutator relation

$$[\hat{x}^\mu, \hat{\lambda}_\nu] = -i\delta^\mu{}_\nu, \tag{97}$$

is satisfied with

$$\hat{x}^\mu \quad \rightarrow \quad x^\mu \tag{98}$$

$$\hat{\lambda}_\mu \quad \rightarrow \quad i\frac{\partial}{\partial x^\mu} \tag{99}$$

B Standard relativistic Lagrangian

The action on both forms must the the same

$$\int \mathcal{L} ds = \int L dt \tag{100}$$

$$\int \mathcal{L}\frac{ds}{dt} dt = \int L dt \tag{101}$$

so, up to exact differentials one has

$$L = \mathcal{L}\frac{ds}{dt}, \tag{102}$$

which leads to the more familiar form of the relativistic Lagrangian, where $u^\mu u_\mu = 1$ is enforced as a constraint.

C APS representation

The APS algebra can be represented in terms of the Pauli matrices, such that a general expression can be faithfully represented as a 2x2 complex matrix. The four-gradient is slitted in temporal and spatial components

$$\bar{\partial} = \partial_0 + \nabla \tag{103}$$

$$\partial = \partial_0 - \nabla, \tag{104}$$

with

$$\nabla = \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3 \tag{105}$$

In similar way, the four-vector potential can be represented as

$$\bar{A} = \partial_0 - \mathbf{A} \tag{106}$$

$$A = \partial_0 + \mathbf{A}, \tag{107}$$

with

$$\mathbf{A} = \sigma_1 A^1 + \sigma_2 A^2 + \sigma_3 A^3. \tag{108}$$

For the sake of concreteness, the gradient can be fully expanded as

$$\bar{\partial} = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} \tag{109}$$

The bar conjugation, also known as Clifford conjugation has the following effect on any general expression

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \Rightarrow \overline{\Psi} = \begin{pmatrix} \psi_{11} & -\psi_{12} \\ -\psi_{21} & \psi_{22} \end{pmatrix} \tag{110}$$

The four-velocity also known as proper velocity is defined as the derivative respect with the proper time τ

$$u = \frac{dx}{d\tau} = \gamma(1 + \mathbf{v}), \tag{111}$$

where $\mathbf{v}$ is the spatial velocity. The square root of the proper velocity can be calculated as

$$\sqrt{u} = \frac{u+1}{\sqrt{2(1+\gamma)}} \tag{112}$$

Other important useful elements are the following complementary projectors

$$\mathcal{P}_3 = \frac{1}{2}(\mathbf{1} + \sigma_3) \tag{113}$$

$$\overline{\mathcal{P}}_3 = \frac{1}{2}(\mathbf{1} - \sigma_3) \tag{114}$$

The gamma matrices in the Weyl representation can be expressed in terms of the Kronecker product of Pauli matrices

$$\gamma^0 = \sigma_1 \otimes \mathbf{1} \tag{115}$$

$$\gamma^k = i\sigma_2 \otimes \sigma_k \tag{116}$$

The fundamental Dirac generator can be rewritten as

$$i\gamma^\mu(\partial_\mu + eA_\mu) - m = i\gamma^0\partial_0 + i\gamma^k\partial_k + \gamma^0 A_0 + i\gamma^k A_k - m \tag{117}$$

$$= \sigma_1 \otimes \mathbf{1}(i\partial_0 + A_0) + i\sigma_2 \otimes \sigma_k(i\partial_k + A_k) - m \tag{118}$$

$$= \begin{pmatrix} -m & i\overline{\partial} + \overline{A} \\ i\partial + A & -m \end{pmatrix} \tag{119}$$

The four-column spinor can be written as

$$\psi = \begin{pmatrix} \overline{\Psi}^\dagger \mathcal{P}_3 \\ \Psi \mathcal{P}_3 \end{pmatrix}, \tag{120}$$

such that the Dirac equation is equivalent to the following coupled equations

$$(i\overline{\partial} + \overline{A})\Psi\mathcal{P}_3 = m\overline{\Psi}^\dagger\mathcal{P}_3 \tag{121}$$

$$(i\partial + A)\overline{\Psi}^\dagger\mathcal{P}_3 = m\Psi\mathcal{P}_3. \tag{122}$$

Conjugating the latter equation and arranging the terms we obtain

$$(i\overline{\partial} + \overline{A})\Psi\mathcal{P}_3 = m\overline{\Psi}^\dagger\mathcal{P}_3 \tag{123}$$

$$(-i\overline{\partial} + \overline{A})\Psi\overline{\mathcal{P}}_3 = m\overline{\Psi}^\dagger\overline{\mathcal{P}}_3 \tag{124}$$

The negative sign on the second equation can be absorbed as

$$i\overline{\partial}\Psi\sigma_3\overline{\mathcal{P}}_3 + \overline{A}\Psi\overline{\mathcal{P}}_3 = m\overline{\Psi}^\dagger\overline{\mathcal{P}}_3 \tag{125}$$

$$i\overline{\partial}\Psi\sigma_3\mathcal{P}_3 + \overline{A}\Psi\mathcal{P}_3 = m\overline{\Psi}^\dagger\overline{\mathcal{P}}_3, \tag{126}$$

which allows to write the Dirac equation in the following compact form

$$i\overline{\partial}\Psi\sigma_3 + \overline{A}\Psi = m\overline{\Psi}^\dagger \tag{127}$$

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