arXiv:1107.0174v1 [math.AG] 1 Jul 2011

THE SYMPLECTIC TOPOLOGY OF PROJECTIVE MANIFOLDS WITH SMALL DUAL

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1. INTRODUCTION AND SUMMARY OF THE MAIN RESULTS

In this paper we study a special class of complex algebraic manifolds called *projective manifolds with small dual*. A projectively embedded algebraic manifold $X \subset \mathbb{C}P^N$ is said to have small dual if the dual variety $X^* \subset (\mathbb{C}P^N)^*$ has (complex) codimension ≥ 2. Recall that the dual variety X^* of a projectively embedded algebraic manifold $X \subset \mathbb{C}P^N$ is by definition the space of all hyperplanes $H \subset \mathbb{C}P^N$ that are not transverse to X, i.e.

$$X^* = \{H \in (\mathbb{C}P^N)^* \mid H \text{ is somewhere tangent to } X\}.$$

Let us mention that for "most" manifolds the codimension of X^* is 1, however in special situations the codimension might be larger. To measure to which extent X deviates from the typical case one defines the defect (see [Tev2]) of an algebraic manifold $X \subset \mathbb{C}P^N$ by

$$\mathrm{def}(X) = \mathrm{codim}_{\mathbb{C}}(X^*) - 1.$$

Thus we will call manifolds with small dual also *manifolds with positive defect*. Note that this is not an intrinsic property of X, but rather of a given projective embedding of X.

The class of algebraic manifolds with small dual was studied by many authors, for instance see [Ein2, Ein1, GH2, Kle2, Sno]. These studies show that these manifolds have very special geometry. In this paper we will show that manifolds with small dual also exhibit unique properties from the point of view of symplectic topology.

In symplectic topology one has the class of *subcritical Stein manifolds* which has been studied in many contexts [EG, Eli2, BC1, BC2]. We will first show that the class of projective manifolds with positive defect is intimately related to the class of subcritical Stein manifolds. Let Y be a Stein manifold. Define

$$\mathrm{ind}(Y) = \min\left\{\mathrm{ind}_{\max}(\varphi) \mid \varphi : Y \to \mathbb{R} \text{ exhausting p.s.h Morse function}\right\},$$

where $\mathrm{ind}_{\max}(\varphi) = \max\{\mathrm{ind}_z(\varphi) \mid z \in \mathrm{Crit}(\varphi)\}$ is the maximal Morse index of φ. By exhausting we mean that the function is proper and bounded from below, and p.s.h. stands for *plurisubharmonic*. We refer to the number $\mathrm{ind}(Y)$ as the subcriticality index

Date: July 4, 2011.

of Y. It is well known [EG, Eli2] that $\mathrm{ind}(Y) \leq \dim_{\mathbb{C}}(Y)$. A Stein manifold Y is called subcritical if $\mathrm{ind}(Y) < \dim_{\mathbb{C}}(Y)$, i.e. it admits an exhausting Morse plurisubharmonic function $\varphi : Y \to \mathbb{R}$ with $\mathrm{ind}_z(\varphi) < \dim_{\mathbb{C}}(Y)$ for every $z \in \mathrm{Crit}(\varphi)$. (We refer the reader to [EG, Eli2] for the symplectic theory of Stein manifolds.) Our first result is:

Theorem A. *Let $X \subset \mathbb{C}P^N$ be a projective manifold with small dual and let $\Sigma \subset X$ be a smooth hyperplane section of X. Then the Stein manifold $X \setminus \Sigma$ is subcritical. In fact:*

$$\mathrm{ind}(X \setminus \Sigma) \leq dim_{\mathbb{C}}(X) - \mathrm{def}(X).$$

See Theorem 8.D for a partial converse to this theorem.

Our next results are concerned with geometric properties of a smooth hyperplane section $\Sigma \subset X$ of a manifold $X \subset \mathbb{C}P^N$ with small dual, under the additional assumption that $b_2(X) = 1$. (Here and in what follows we denote by $b_j(X) = \dim H^j(X;\mathbb{R})$ the j'th Betti-number of X.) By a well known result of Ein [Ein2] the assumption $b_2(X) = 1$ implies that both X and Σ are Fano manifolds.

For a space Y we will denote from now on by

$$H^*(Y) := H^*(Y;\mathbb{Z})/\mathrm{torsion}$$

the torsion-free part of the integral cohomology $H^*(Y;\mathbb{Z})$. (We will sometime denote this also by $H^*(Y;\mathbb{Z})_{fr}$.) Denote by $QH^*(\Sigma;\Lambda) = (H^\bullet(\Sigma) \otimes \Lambda)^*$ the quantum cohomology ring of Σ with coefficients in the Novikov ring $\Lambda = \mathbb{Z}[q, q^{-1}]$ (see below for our grading conventions), and endowed with the quantum product $*$. We prove:

Theorem B. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual, $b_2(X) = 1$ and $\dim_{\mathbb{C}}(X) \geq 2$. Let Σ be a smooth hyperplane section of X. Let ω be the restriction of the Fubini-Study Kähler form of $\mathbb{C}P^N$ to Σ. Then*

$$[\omega] \in QH^2(\Sigma;\Lambda)$$

is an invertible element with respect to the quantum product.

We will actually prove a slightly stronger result in §5 (see Theorem 5.B and the discussion after it). In Theorem 9.A in §9 we will establish a much more general, though less precise, version of this theorem.

A classical result of Lanteri and Struppa [LS] on the topology of projective manifolds with positive defect states that if $X \subset \mathbb{C}P^N$ is a projective manifold with $dim_{\mathbb{C}}X = n$ and $\mathrm{def}(X) = k > 0$ then:

$$b_j(X) = b_{j+2}(X) \ \ \forall\, n - (k-1) \leq j \leq n + k - 1.$$

(In §3 we will reprove this fact using Morse theory). As we will see in Corollary C below, Theorem B implies stronger topological restrictions in the case $b_2(X) = 1$.

As mentioned above, under the assumption $b_2(X) = 1$ the manifold Σ is Fano. The quantum cohomology $QH^*(\Sigma;\Lambda) = (H^\bullet(\Sigma)\otimes\Lambda)^*$ admits a grading induced from both factors $H^\bullet(\Sigma)$ and Λ. Here we grade Λ by taking $deg(q) = 2C_\Sigma$, where

$$C_\Sigma = \min\left\{c_1^\Sigma(A) > 0 \mid A \in \text{image}\,(\pi_2(\Sigma)\to H_2(\Sigma;\mathbb{Z}))\right\} \in \mathbb{N}$$

is the minimal Chern number of Σ. Here we have denoted by $c_1^\Sigma \in H^2(\Sigma;\mathbb{Z})$ the first Chern class of the tangent bundle $T\Sigma$ of Σ. Theorem B implies that the map

$$*[\omega] : QH^*(\Sigma;\Lambda) \longrightarrow QH^{*+2}(\Sigma;\Lambda), \quad a \longmapsto a * [\omega],$$

is an isomorphism. In our case, a computation of Ein [Ein2] gives:

$$2C_X = n+k+2, \quad 2C_\Sigma = n+k.$$

(It is well known, by a result of Landman, that n and k must have the same parity. See §2.) Define now the cohomology of X graded cyclically as follows:

$$\widetilde{H}^i(X) = \bigoplus_{l\in\mathbb{Z}} H^{i+2C_X l}(X), \quad \widetilde{b}_i(X) = \text{rank}\,\widetilde{H}^i(X). \tag{1}$$

Define $\widetilde{H}^i(\Sigma)$ and $\widetilde{b}_i(\Sigma)$ in a similar way (note that in the definition of $\widetilde{H}^i(\Sigma)$ one has to replace also C_X by C_Σ). Theorem B together with a simple application of the Lefschetz hyperplane section theorem give the following result:

Corollary C. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual and $b_2(X) = 1$. Then $\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X)$, $\forall\, j \in \mathbb{Z}$. Moreover, if $\Sigma \subset X$ is a smooth hyperplane section then similarly to X we have $\widetilde{b}_j(\Sigma) = \widetilde{b}_{j+2}(\Sigma)$, $\forall\, j \in \mathbb{Z}$.*

A similar result (for subcritical manifolds) has been previously obtained by He [He] using methods of contact homology.

If $dim_{\mathbb{C}}(X) = n$ and $\text{def}(X) = k$, Theorem C implies the following relations among the Betti numbers of X:

$$b_j(X) + b_{j+n+k+2}(X) = b_{j+2}(X) + b_{j+n+k+4}(X), \ \ \forall\, 0 \le j \le n+k-1,$$
$$b_{n+k}(X) = b_{n+k+2}(X) + 1, \quad b_{n+k+1}(X) = b_{n+k+3}(X),$$

and the following ones for those of Σ:

$$b_j(\Sigma) + b_{j+n+k}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma), \ \ \forall\, 0 \le j \le n+k-3,$$
$$b_{n+k-2}(\Sigma) = b_{n+k}(\Sigma) + 1, \quad b_{n+k-1}(\Sigma) = b_{n+k+1}(\Sigma).$$

We will prove a slightly stronger result in §5, see Corollary 5.D.

Example. Consider the complex Grassmannian $X = Gr(5,2) \subset \mathbb{C}P^9$ of 2-dimensional subspaces in $\mathbb{C}^5$ embedded in projective space by the Plücker embedding. It is known that $\mathrm{def}(X) = 2$, see [Tev2]. We have $\dim_{\mathbb{C}}(X) = 6$ and $2C_X = 10$. The table of Betti numbers of X is given as follows:

q	0	1	2	3	4	5	6	7	8	9	10	11	12
$b_q(X)$	1	0	1	0	2	0	2	0	2	0	1	0	1

Further implications of Theorem B are obtained by studying the algebraic properties of the inverse $[\omega]^{-1}$. First note that due to degree reasons the inverse element should be of the form

$$[\omega]^{-1} = \alpha_{n+k-2} \otimes q^{-1} \in QH^{-2}(\Sigma;\Lambda)$$

where $\alpha_{n+k-2} \in H^{n+k-2}(\Sigma)$ is a nontrivial element. Moreover, this element needs to satisfy the following conditions:

$$[\omega] \cup \alpha_{n+k-2} = 0, \quad ([\omega] * \alpha_{n+k-2})_1 = 1,$$

where $([\omega] * \alpha_{n+k-2})_1 \in H^0(\Sigma)$ is determined by the condition that

$$\langle ([\omega] * \alpha_{n+k-2})_1, - \rangle = GW_1^{\Sigma}(PD[\omega], PD(\alpha_{n+k-2}), -).$$

Here PD stands for Poincaré duality, and for $a \in QH^l(\Sigma;\Lambda)$ and $i \in \mathbb{Z}$ we denote by $(a)_i \in H^{l-2iC_\Sigma}(\Sigma)$ the coefficient of q^i in a. The notation $GW_j^{\Sigma}(A,B,C)$ stands for the Gromov-Witten invariant counting the number of rational curves $u : \mathbb{C}P^1 \to \Sigma$ passing through three cycles representing the homology classes A, B, C with $c_1(u_*[\mathbb{C}P^1]) = jC_\Sigma$.

So in our case, the fact that $([\omega] * \alpha_{n+k-2})_1 \neq 0$ implies that Σ is uniruled. The uniruldness of Σ (as well as that of X) was previously known and the variety of rational curves on it was studied by Ein in [Ein2]. Finally, note that the uniruldness of X follows also from the results of He [He] in combination with Theorem A above.

The method of proof of Theorem B is an application of the theory of Hamiltonian fibrations and, in particular, their Seidel elements, see [Sei]. In [Sei] Seidel constructed a representation of $\pi_1(Ham(\Sigma,\omega))$ on $QH(\Sigma;\Lambda)$ given by a group homomorphism

$$S : \pi_1(Ham(\Sigma,\omega)) \longrightarrow QH(\Sigma;\Lambda)^{\times},$$

where $QH(\Sigma;\Lambda)^{\times}$ is the group of invertible elements of the quantum cohomology algebra.

Theorem B follows from:

Theorem D. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual and $b_2(X) = 1$. Let $\Sigma \subset X$ be a smooth hyperplane section of X and denote by ω the symplectic structure induced on Σ from $\mathbb{C}P^N$. There exists a* nontrivial *element $1 \neq \lambda \in \pi_1(Ham(\Sigma,\omega))$ whose Seidel element is given by*

$$S(\lambda) = [\omega] \in QH^2(\Sigma;\Lambda).$$

See Theorems 5.B and 9.A for more general statements.

Before we turn to examples, let us mention that by results of [BFS], based on Mori theory, the classification of manifolds with small dual is reduced to the case $b_2(X) = 1$. Here is a list of examples of manifolds with small dual and $b_2(X) = 1$ (see [Tev2] for more details):

Examples. (1) $X = \mathbb{C}P^n \subset \mathbb{C}P^{n+1}$ has $\mathrm{def}(X) = n$.

(2) $X = Gr(2l+1, 2)$ embedded via the Plücker embedding has $\mathrm{def}(X) = 2$.

(3) $X = \mathbb{S}_5 \subset \mathbb{C}P^{15}$ the 10–dimensional spinor variety has $\mathrm{def}(X) = 4$.

(4) In any of the examples (1)–(3) one can take iterated hyperplane sections and still get manifolds with def > 0 and $b_2 = 1$, provided that the number of iterations does not exceed the defect-1. (See §2.)

The manifolds in (1)–(3) together with the corresponding hyperplane sections (4) are the only known examples of projective manifolds with small dual and $b_2(X) = 1$, see [BS, Sno]. On the basis of these examples, it is conjectured in [BS] that all non-linear algebraic manifolds with $b_2(X) = 1$ have $\mathrm{def}(X) \leq 4$.

Organization of the paper. The rest of the paper is organized as follows: In §2 we review basic facts on projective manifolds with small dual. In §3 we prove Theorem A. In §4 we review relevant results from the theory of Hamiltonian fibrations and the Seidel representation. In §5 we explain the relation between manifolds with small dual and Hamiltonian fibrations. In §6 we prove Theorems B and D. Corollary C is proved in §7. In §8 we present more applications of our methods to questions on the symplectic topology and algebraic geometry of manifolds with small dual. We also outline an alternative proof of Corollary C based on Lagrangian Floer theory. In §9 we explain how to generalize Theorem B to the case $b_2(X) > 1$ (or more generally to non-monotone manifolds). In the same section we also work out explicitly such an example. Finally, in §10 we discuss some open questions and further possible directions of study.

2. Basic results on projective manifolds with small dual

Let $X \subset \mathbb{C}P^N$ be an algebraic manifold of $dim_{\mathbb{C}} X = n$. Denote by $(\mathbb{C}P^N)^*$ the dual projective space parametrizing hyperplanes $H \subset \mathbb{C}P^N$. To X one associates the dual variety $X^* \subset (\mathbb{C}P^N)^*$, which (in the case X is smooth) is defined as

$$X^* = \{H \mid H \text{ is somewhere tangent to } X\}.$$

We refer the reader to [Tev2] for a detailed account on the subject of projective duality. In this section we will review basic properties of projective manifolds with positive defect.

Define the defect of X to be

$$\mathrm{def}(X) = \mathrm{codim}_{\mathbb{C}} X^* - 1.$$

Note that when X^* is a hypersurface the defect of X is zero. An important feature of the defect is the following: if $\mathrm{def}(X) = k$ then for a smooth point of the dual variety, $H \in X^*_{\mathrm{sm}}$, the singular part $\mathrm{sing}(X \cap H)$ of $X \cap H$ is a projective space of dimension k linearly embedded in $\mathbb{C}P^N$. Thus, X is covered by projective spaces of dimension k, and in particular there is a projective line through every point of X (see [Kle1]).

Next, the defect of X and that of a hyperplane section $\Sigma \subset X$ of X are related as follows (see [Ein1]):

$$\mathrm{def}(\Sigma) = \max\{\mathrm{def}(X) - 1, 0\}. \tag{2}$$

A well known (unpublished) result of Landman states that for manifolds X with small dual we have the following congruence $\dim_{\mathbb{C}}(X) \equiv \mathrm{def}(X) \pmod 2$ (see [Ein2, Tev2] for a proof of this).

Later, Ein proved in [Ein2] the following. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $\dim_{\mathbb{C}}(X) = n$ and $\mathrm{def}(X) = k > 0$. Denote by c_1^X the first Chern class of X. Then through every point in X there exists a projective line S with*

$$c_1^X(S) = \frac{n+k}{2} + 1.$$

Of special importance is the case $b_2(X) = 1$, which was extensively studied by Ein in [Ein2]. In this case we have:

$$c_1^X = \left(\frac{n+k}{2} + 1\right) \cdot h, \tag{3}$$

where $h \in H^2(X) \cong \mathbb{Z}$ is the positive generator, which is also the class of the restriction (to X) of the Kähler form of $\mathbb{C}P^N$. In particular, in this case both X and Σ are Fano manifolds.

3. Subcriticality and projective defect

Let $Y \subset \mathbb{C}^N$ be a Stein manifold. The study of Morse theory on Stein manifolds was initiated in the classical paper [AF] of Andreotti and Frankel. Further aspects of Morse theory on Stein manifolds where studied by various authors [EG, Eli1, Eli2, BC1]. A function $\varphi : Y \to \mathbb{R}$ is called plurisubharmonic (p.s.h in short) if the form $\Omega = -dd^{\mathbb{C}}\varphi$ is a Kähler form on Y. Here $d^{\mathbb{C}}\varphi = d\varphi \circ J$, where J is the complex structure of Y. A plurisubharmonic function $\varphi : Y \to \mathbb{R}$ is called exhausting if it is proper and bounded from below.

For a plurisubharmonic Morse function $\varphi : Y \to \mathbb{R}$ denote

$$\mathrm{ind}_{\max}(\varphi) = \max\left\{\mathrm{ind}_z(\varphi) \mid z \in \mathrm{Crit}(\varphi)\right\},$$

where $\mathrm{ind}_z(\varphi)$ is the Morse index of the critical point $z \in \mathrm{Crit}(\varphi)$. The subcriticality index $\mathrm{ind}(Y)$ of a Stein manifold Y is defined by

$$\mathrm{ind}(Y) := \min\left\{\mathrm{ind}_{\max}(\varphi) \mid \varphi : Y \to \mathbb{R} \text{ is a p.s.h exhausting Morse function}\right\}.$$

A fundamental property of the index is that $0 \leq \mathrm{ind}(Y) \leq \dim_{\mathbb{C}}(Y)$. (See [AF, EG, Eli1, Eli2] for various proofs of this statement.) Recall also that a Stein manifold Y is called subcritical if $\mathrm{ind}(Y) < \dim_{\mathbb{C}}(Y)$.

We now prove Theorem A.

Proof. Write $n = \dim_{\mathbb{C}}(X)$, $k = \mathrm{def}(X)$ and assume $k > 0$. Denote by $Y = X \setminus \Sigma \subset \mathbb{C}P^N \setminus H = \mathbb{C}^N$. Denote by $h(\cdot,\cdot)$ the standard Hermitian form of $\mathbb{C}^N$, by $(\cdot,\cdot) = \mathrm{Re}h(\cdot,\cdot)$ the standard scalar product and by $|\cdot|$ the standard Euclidean norm on $\mathbb{C}^N$. Fix a point $w_0 \in \mathbb{C}^N$ and define $\varphi_w : Y \to \mathbb{R}$ to be the function

$$\varphi_{w_0}(z) := |z - w_0|^2, \quad \forall z \in Y.$$

By standard arguments, for a generic point $w_0 \in \mathbb{C}^N$, φ_{w_0} is an exhausting plurisubharmonic Morse function. It is well known (see e.g. [Voi]) that $z_0 \in Y$ is a critical point of φ_{w_0} if and only if $\overrightarrow{w_0 z_0} \perp T_{z_0}Y$. In order to compute the Hessian of φ_{w_0} at a critical point z_0 we need the second fundamental form. We will follow here the conventions from [Voi]. Denote by $\gamma : Y \longrightarrow \mathrm{Gr}(n, N)$ the Gauss map, $\gamma(x) = T_xY$. Consider the differential of this map $D\gamma_x : T_xY \longrightarrow T_{T_xY}\mathbb{C}^N = \hom(T_xY, \mathbb{C}^N/T_xY)$. This map induces a symmetric bilinear form:

$$\Phi : S^2T_xY \longrightarrow \mathbb{C}^N/T_xY$$

which is called the second fundamental form.

As $h(v, \overrightarrow{w_0 z_0}) = 0$ for every $v \in T_{z_0}Y$ we can define a symmetric complex bilinear form:

$$G : S^2T_{z_0}Y \longrightarrow \mathbb{C}, \quad G(u,v) = h(\Phi(u,v), \overrightarrow{w_0 z_0}).$$

A standard computation (see e.g. [Voi]) shows that the Hessian of φ_{w_0} is given by:

$$\mathrm{Hess}_{z_0}\varphi_{w_0}(u,v) = 2((u,v) + \mathrm{Re}G(u,v)), \ \forall u, v \in T_{z_0}Y. \tag{4}$$

Next, by a result of Katz we have: $\mathrm{rank}_{\mathbb{C}}G \leq n - k$. (See [Tev2, Tev1] and the references therein, e.g. exposé XVII by N. Katz in [SGA]. See also [GH2].) It follows that $\dim_{\mathbb{R}} \ker(\mathrm{Re}G) \geq 2k$.

Denote the non-zero eigenvalues of $\mathrm{Re}G$ (in some orthonormal basis) by λ_i, $i = 1, \ldots, r$, with $r \leq 2n - 2k$. It is well known that for real symmetric bilinear forms that appear as

the real part of complex ones (e.g. ReG) the following holds: λ is an eigenvalue if and only if $-\lambda$ is an eigenvalue (see [Voi]), moreover the multiplicities of λ and $-\lambda$ are the same. (See e.g. [Voi] for a proof.) It follows that the number of negative λ_i's can be at most $n-k$.

Coming back to (4), the eigenvalues of $\mathrm{Hess}_{z_0}\varphi_{w_0}$ are of the form $1+\lambda$ with λ being an eigenvalue of ReG. It follows that the number of negative eigenvalues of $\mathrm{Hess}_{z_0}\varphi_{w_0}$ is at most $n-k$. This shows that $\mathrm{ind}_{z_0}(\varphi_{w_0}) \leq n-k$ for every $z_0 \in \mathrm{Crit}(\varphi_{w_0})$. In particular, $\mathrm{ind}(Y) \leq n-k$. □

In particular, using standard arguments one gets from Theorem A the following version of the Lefschetz hyperplane theorem for manifolds with small dual, which was previously known and proved by other methods in [LS]:

Corollary 3.A. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $dim_{\mathbb{C}}X = n$ and $\mathrm{def}(X) = k$ and let $\Sigma \subset X$ be a smooth hyperplane section. Denote by $i : \Sigma \longrightarrow X$ the inclusion. The induced maps $i_* : H_j(\Sigma;\mathbb{Z}) \longrightarrow H_j(X;\mathbb{Z})$ and $i_* : \pi_j(\Sigma,*) \longrightarrow \pi(X,*)$ are:*

(1) Isomorphisms for $j < n+k-1$.
(2) Surjective for $j = n+k-1$.

Similarly, the restriction map $i^* : H^j(X;\mathbb{Z}) \longrightarrow H^j(\Sigma;\mathbb{Z})$ is an isomorphism for every $j < n+k-1$ and injective for $j = n+k-1$.

Another consequence is the following refinement of the hard Lefschetz theorem.

Corollary 3.B. *Let $X \subset \mathbb{C}P^N$ be as in Corollary 3.A. Denote by ω the Kähler form on X induced from the standard Kähler form of $\mathbb{C}P^N$. Then the map*

$$L : H^j(X;\mathbb{R}) \longrightarrow H^{j+2}(X;\mathbb{R}), \quad L(a) = a \cup [\omega],$$

is an isomorphism for every $n-k-1 \leq j \leq n+k-1$.

Proof. This follows from Corollary 3.A together with the Hard Lefschetz theorem applied both to Σ and X. □

4. Hamiltonian fibrations

In what follows we will use the theory of symplectic and Hamiltonian fibrations and their invariants. We refer the reader to [GLS, MS1, MS2] for the foundations.

Let $\pi : \widetilde{X} \to B$ be a smooth locally trivial fibration with fiber Σ and base B which are both closed manifolds. We will assume in addition that B is a *simply connected* manifold. Further, let $\widetilde{\Omega}$ be a closed 2-form on $\widetilde{X}$ such that the restriction $\Omega_b = \widetilde{\Omega}|_{\Sigma_b}$ to each fiber $\Sigma_b = \pi^{-1}(b)$, $b \in B$, is a symplectic form. Fix $b_0 \in B$, and let ω_Σ be a symplectic form

on Σ such that (Σ, ω_Σ) is symplectomorphic to $(\Sigma_{b_0}, \Omega_{b_0})$. This structure is a special case of a so called Hamiltonian fibration. It is well known that under these assumptions all fibers (Σ_b, Ω_b) are symplectomorphic and in fact the structure group of π can be reduced to $\mathrm{Ham}(\Sigma, \omega_\Sigma)$.

We will assume from now on that $B = S^2$. We identify $S^2 \cong \mathbb{C}P^1$ in a standard way and view S^2 as a Riemann surface whose complex structure is denoted by j.

4.1. **Holomorphic curves in Hamiltonian fibrations.** Let $\pi : (\widetilde{X}, \widetilde{\Omega}) \to S^2$ be a Hamiltonian fibration as above. Denote by $T^v\widetilde{X} = \ker(D\pi)$ the vertical part of the tangent bundle of $\widetilde{X}$. We now introduce almost complex structures compatible with the fibration. These are by definition almost complex structures $\widetilde{J}$ on $\widetilde{X}$ with the following properties:

(1) The projection π is $(\widetilde{J}, j)$–holomorphic.
(2) For every $z \in S^2$ the restriction J_z of $\widetilde{J}$ to Σ_z is compatible with the symplectic form Ω_z, i.e. $\Omega_z(J_z\xi, J_z\eta) = \Omega_z(\xi, \eta)$ for every $\xi, \eta \in T^v_z\widetilde{X}$, and $\Omega_z(\xi, J_z\xi) > 0$ for every $0 \neq \xi \in T^v_z\widetilde{X}$.

We denote the space of such almost complex structures by $\widetilde{\mathcal{J}}(\pi, \widetilde{\Omega})$.

Denote by $H_2^\pi \subset H_2(\widetilde{X}; \mathbb{Z})$ be the set of classes $\widetilde{A}$ such that $\pi_*(\widetilde{A}) = [S^2]$. Given $\widetilde{A}$ and $\widetilde{J} \in \widetilde{\mathcal{J}}(\pi, \widetilde{\Omega})$ denote by $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J})$ the space of $\widetilde{J}$–holomorphic sections in the class $\widetilde{A}$, i.e. the space of maps $\widetilde{u} : S^2 \longrightarrow \widetilde{X}$ with the following properties:

(1) $\widetilde{u}$ is $(j, \widetilde{J})$–holomorphic.
(2) $\widetilde{u}$ is a section, i.e. $\pi \circ \widetilde{u} = \mathrm{id}$.
(3) $\widetilde{u}_*[S^2] = \widetilde{A}$.

Fix $z_0 \in S^2$ and fix an identification $(\Sigma, \omega_\Sigma) \approx (\Sigma_{z_0}, \Omega_{z_0})$. The space of sections comes with an evaluation map:

$$ev_{\widetilde{J}, z_0} : \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}) \longrightarrow \Sigma, \quad ev_{\widetilde{J}, z_0}(\widetilde{u}) = \widetilde{u}(z_0).$$

4.1.1. *Transversality.* In order to obtain regularity and transversality properties for the moduli spaces of holomorphic sections and their evaluation maps we will need to work with so called *regular* almost complex structures. Moreover, since the moduli spaces of holomorphic sections are usually not compact they do not carry fundamental classes and so the evaluation maps do not induce in a straightforward way homology classes in their target (Σ in this case). The reason for non-compactness of these moduli spaces is that a sequence of holomorphic sections might develop bubbles in one of the fibers (see e.g. [MS2]). The simplest way to overcome this difficulty is to make some positivity assumptions on the fiber Σ (called *monotonicity*). Under such conditions the moduli

spaces of holomorphic sections admits a nice compactification which makes it possible to define homology classes induced by the evaluation maps. Here is the relevant definition.

Definition 4.1.A. Let (Σ, ω_Σ) be a symplectic manifold. Denote by $H_2^S(\Sigma) \subset H_2(\Sigma;\mathbb{Z})$ the image of the Hurewicz homomorphism $\pi_2(\Sigma) \longrightarrow H_2(\Sigma;\mathbb{Z})$. Denote by $c_1^\Sigma \in H^2(\Sigma;\mathbb{Z})$ the first Chern class of the tangent bundle $(T\Sigma, J_\Sigma)$, where J_Σ is any almost complex structure compatible with ω_Σ. The symplectic manifold (Σ, ω_Σ) is called spherically monotone if there exists a constant $\lambda > 0$ such that for every $A \in H_2^S(\Sigma)$ we have $\omega_\Sigma(A) = \lambda c_1^\Sigma(A)$. For example, if Σ is a Fano manifold and ω_Σ is a symplectic form with $[\omega_\Sigma] = c_1^\Sigma$ then obviously (Σ, ω_Σ) is spherically monotone.

From now on we assume that the fiber (Σ, ω_Σ) of $\pi : (\widetilde{X}, \widetilde{\Omega}) \longrightarrow S^2$ is spherically monotone. Denote by $c_1^v = c_1(T^v\widetilde{X}) \in H^2(\widetilde{X})$ the vertical Chern class, i.e. the first Chern class of the vertical tangent bundle of $\widetilde{X}$. The following is proved in [Sei, MS2]. There exists a *dense subset* $\widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega}) \subset \widetilde{\mathcal{J}}(\pi, \widetilde{\Omega})$ such that for every $\widetilde{J} \in \widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$ and every $\widetilde{A} \in H_2^\pi$ the following holds:

(1) For every $\widetilde{A} \in H_2^\pi$, the moduli space $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J})$ of $\widetilde{J}$-holomorphic sections in the class $\widetilde{A}$ is either empty or a smooth manifold of dimension

$$dim_{\mathbb{R}}\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}) = \dim_{\mathbb{R}} \Sigma + 2c_1^v(\widetilde{A}).$$

Moreover, $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J})$ has a canonical orientation.

(2) The evaluation map $ev_{\widetilde{J},z_0} : \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}) \longrightarrow \Sigma$ is a pseudo-cycle (see [MS2] for the definition). In particular, its Poincaré dual gives a cohomology class $\mathcal{S}(\widetilde{A}; \widetilde{J}) \in H^d(\Sigma;\mathbb{Z})_{fr} = H^d(\Sigma;\mathbb{Z})/\text{torsion}$, where $d = -2c_1^v(\widetilde{A})$. Moreover, the class $\mathcal{S}(\widetilde{A}; \widetilde{J})$ is independent of the regular $\widetilde{J}$ used to define it. Therefore we will denote it from now on by $\mathcal{S}(\widetilde{A})$.

We refer the reader to [MS2, Sei] for more general results on transversality.

The definition of regularity for $\widetilde{J} \in \widetilde{\mathcal{J}}(\pi, \widetilde{\Omega})$ involves three ingredients. The first is that the restriction J_{z_0} of $\widetilde{J}$ to $\Sigma = \Sigma_{z_0}$ is regular in the sense of Chapter 3 of [MS2], namely that the linearization of the $\overline{\partial}_{J_{z_0}}$-operator at every J_{z_0}-holomorphic curve in Σ is surjective. (In addition one has to require that certain evaluation maps for tuples of such curves are mutually transverse.) The second ingredient is that (the vertical part of) the $\overline{\partial}_{\widetilde{J}}$-operator at every $\widetilde{J}$-holomorphic section is surjective. The third one is that $ev_{\widetilde{J},z_0}$ is transverse to all J_{z_0}-holomorphic bubble trees in Σ.

In practice, we will have to compute cohomology classes of the type $\mathcal{S}(\widetilde{A}) = \mathcal{S}(\widetilde{A}; \widetilde{J})$ using a specific choice of $\widetilde{J}$ that naturally appears in our context. It is not an easy task to decide whether a given almost complex structure $\widetilde{J}$ is regular or not. However, in some situations it is possible to compute some of the classes $\mathcal{S}(\widetilde{A})$ by using almost complex

structures $\widetilde{J}$ that satisfy weaker conditions than regularity. Criteria for verification of these conditions have been developed in [Sei] (see Proposition 7.11 there) and in [MS2] (see Section 3.3 and 3.4 there). Below we will actually not appeal to such criteria and use simpler arguments.

4.2. **The Seidel representation.** Let (Σ, ω_Σ) be a closed monotone symplectic manifold (see Definition 4.1.A is §4.1.1). Denote by $C_\Sigma \in \mathbb{N}$ the minimal Chern number, i.e.

$$C_\Sigma = \min\{c_1^\Sigma(A) \mid A \in H_2^S, c_1^\Sigma(A) > 0\}.$$

Denote by $\Lambda = \mathbb{Z}[q^{-1}, q]$ the ring of Laurent polynomials. We endow Λ with a grading by setting $\deg(q) = 2C_\Sigma$. Let $QH^*(\Sigma;\Lambda) = (H^\bullet(\Sigma) \otimes \Lambda)^*$ be the quantum cohomology of Σ, where the grading is induced from both factors $H^\bullet(\Sigma)$ and Λ. We endow $QH(\Sigma;\Lambda)$ with the quantum product $*$. The unity will be denoted as usual by $1 \in QH^0(\Sigma;\Lambda)$. We refer the reader to Chapter 11 of [MS2] for the definitions and foundations of quantum cohomology. (Note however that our grading conventions are slightly different than the ones in [MS2]).) With our grading conventions we have:

$$QH^j(\Sigma;\Lambda) = \bigoplus_{l\in\mathbb{Z}} H^{j-2lC_\Sigma}(\Sigma)q^l.$$

We will need also a coefficients extension of $QH(\Sigma;\Lambda)$. Denote $\overline{\Lambda} = \mathbb{Z}[t^{-1}, t]$ the ring of Laurent polynomials in the variable t, graded so that $\deg(t) = 2$. Consider now $QH^*(\Sigma;\overline{\Lambda}) = (H^\bullet(\Sigma) \otimes \overline{\Lambda})^*$, endowed with the quantum product $*$. We can regard $\overline{\Lambda}$ as an algebra over Λ using the embedding of rings induced by $q \longmapsto t^{C_\Sigma}$. This also induces an embedding of rings

$$QH^*(\Sigma;\Lambda) \hookrightarrow QH^*(\Sigma;\overline{\Lambda}).$$

We will therefore view from now on $QH(\Sigma;\Lambda)$ as a subring of $QH(\Sigma;\overline{\Lambda})$ by setting $q = t^{C_\Sigma}$.

In [Sei] Seidel associated to a Hamiltonian fibration $\pi : \widetilde{X} \longrightarrow S^2$ with fiber Σ an invertible element $\widetilde{S}(\pi) \in QH^0(\Sigma;\overline{\Lambda})$. We refer the reader to [Sei, MS2] for a detailed account of this theory. Here is a brief review of the main construction.

Pick a regular almost complex structure $\widetilde{J} \in \widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$. Define a class:

$$\widetilde{S}(\pi) := \sum_{\widetilde{A}\in H_2^\pi} \mathcal{S}(\widetilde{A};\widetilde{J}) \otimes t^{c_1^v(\widetilde{A})} \in QH^0(\Sigma;\overline{\Lambda}). \tag{5}$$

Note that since the degree of $\mathcal{S}(\widetilde{A},\widetilde{J})$ is $-2c_1^v(\widetilde{A})$, a class $\widetilde{A} \in H_2^\pi$ contributes to the sum in (5) only if

$$2 - 2n \leq 2c_1^v(\widetilde{A}) \leq 0. \tag{6}$$

The class $\widetilde{S}(\pi)$ is called the Seidel element of the fibration $\pi : (\widetilde{X}, \widetilde{\Omega}) \longrightarrow S^2$.

In what follows it will be more convenient to work with the more "economical" ring Λ rather than $\overline{\Lambda}$. We will now define a normalized version of the Seidel element, denoted $S(\pi)$, which lives in $QH(\Sigma;\Lambda)$. Fix a reference class $\widetilde{A}_0 \in H_2^\pi$ and set $c_0(\pi) = c_1^v(\widetilde{A}_0)$. Define now

$$S(\pi) = t^{-c_0(\pi)}\widetilde{S}(\pi). \tag{7}$$

Since any two classes in H_2^π differ by a class in $H_2^S(\Sigma)$, there exists a uniquely defined function $\nu : H_2^\pi \to \mathbb{Z}$ such that

$$c_1^v(\widetilde{A}) = c_0(\pi) + \nu(\widetilde{A})C_\Sigma, \quad \forall \widetilde{A} \in H_2^\pi.$$

As $q = t^{C_\Sigma}$ we have:

$$S(\pi) := \sum_{\widetilde{A}\in H_2^\pi} \mathcal{S}(\widetilde{A};\widetilde{J}) \otimes q^{\nu(\widetilde{A})} \in QH^{-2c_0(\pi)}(\Sigma;\Lambda). \tag{8}$$

By abuse of terminology we will call $S(\pi)$ also the Seidel element of the fibration π. Of course the element $S(\pi)$ (as well as its degree) depends on the choice of the reference section $\widetilde{A}_0$, however different reference sections $\widetilde{A}_0$ will result in elements that differ by a factor of the type q^r for some $r \in \mathbb{Z}$. In particular, many algebraic properties of $S(\pi)$ (such as invertibility) do not depend on this choice. We will therefore ignore this ambiguity from now on.

4.2.1. *Relations to Hamiltonian loops.* An important feature of the theory is the connection between Hamiltonian fibrations over S^2 with fiber (Σ, ω_Σ) and $\pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$. To a loop based at the identity $\lambda = \{\varphi_t\}_{t\in S^1}$ in $Ham(\Sigma, \omega_\Sigma)$ one can associate a Hamiltonian fibration $\pi_\lambda : \widetilde{M}_\lambda \to S^2$ as follows. Let D_+ and D_- be two copies of the unit disk in $\mathbb{C}$, where the orientation on D_- is reversed. Define:

$$\widetilde{M}_\lambda = \Big((\Sigma \times D_+)\coprod(\Sigma \times D_-)\Big)/\sim, \quad \text{where } (x, e_+^{2\pi it}) \sim (\varphi_t(x), e_-^{2\pi it}). \tag{9}$$

Identifying $S^2 \approx D_+ \cup_\partial D_-$ we obtain a fibration $\pi : \widetilde{M}_\lambda \longrightarrow S^2$. As the elements of λ are symplectic diffeomorphisms, the form ω_Σ gives rise to a family of symplectic forms $\{\Omega_z\}_{z\in S^2}$ on the fibers $\Sigma_z = \pi^{-1}(z)$ of π. Moreover, $\pi : (\widetilde{M}_\lambda, \{\Omega_z\}_{z\in S^2}) \longrightarrow S^2$ is locally trivial. Since the elements of λ are in fact Hamiltonian diffeomorphisms it follows that the family of fiberwise forms $\{\omega_z\}_{z\in S^2}$ can be extended to a closed 2-form $\widetilde{\Omega}$ on $\widetilde{M}_\lambda$, i.e. $\widetilde{\Omega}|_{\Sigma_z} = \Omega_z$ for every z. See [Sei, MS2] for the proofs. We therefore obtain from this construction a Hamiltonian fibration $\pi : (\widetilde{M}_\lambda, \widetilde{\Omega}) \longrightarrow S^2$.

From the construction one can see that homotopic loops in $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ give rise to isomorphic fibrations. We denote the isomorphism class of fibrations corresponding to an element $\gamma \in \pi_1(Ham(\Sigma, \omega_\Sigma))$ by π_γ.

Conversely, if $\pi : (\widetilde{M}, \widetilde{\Omega}) \longrightarrow S^2$ is a Hamiltonian fibration with fiber (Σ, ω_Σ) one can express $\widetilde{M}$ as a gluing of two trivial bundles over the two hemispheres in S^2. The gluing map would be a loop of Hamiltonian diffeomorphisms of (Σ, ω_Σ). Different trivializations lead to homotopic loops. Thus the fibration π determines a class $\gamma(\pi) \in \pi_1(Ham(\Sigma, \omega_\Sigma))$.

This correspondence has the following properties in relation to the Seidel elements (see [Sei] for the proofs):

$$\overline{S}(\pi_{\gamma_1 \cdot \gamma_2}) = \overline{S}(\pi_{\gamma_1}) * \overline{S}(\pi_{\gamma_2}), \quad \forall\, \gamma_1, \gamma_2 \in \pi_1(Ham(\Sigma, \omega_\Sigma)).$$

Here $*$ stands for the quantum product. The unit element $e \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$ corresponds to the trivial fibration $\pi_e : \Sigma \times S^2 \longrightarrow S^2$ and we have $\overline{S}(\pi_e) = 1 \in QH(\Sigma; \Lambda)$. It follows that $\overline{S}(\pi)$ is an invertible element in $QH(\Sigma; \overline{\Lambda})$ for every π. The corresponding homomorphism

$$\overline{S} : \pi_1(Ham(\Sigma, \omega_\Sigma)) \longrightarrow QH(\Sigma, \overline{\Lambda})^\times, \quad \gamma \longmapsto \overline{S}(\pi_\gamma)$$

(which by abuse of notation we also denote by $\overline{S}$), where $QH(\Sigma, \overline{\Lambda})^\times$ is the group of invertible elements in $QH(\Sigma, \overline{\Lambda})$, is called the Seidel representation.

As mentioned before, for our purposes it would be more convenient to work with the normalized version $S(\pi)$ of the Seidel element rather than with $\overline{S}(\pi)$. We claim that any normalized Seidel element $S(\pi)$ is invertible in $QH(\Sigma; \Lambda)$ (not just in $QH(\Sigma; \overline{\Lambda})$). To see this, denote by $\gamma \in \pi_1(\mathrm{Ham}(\Sigma))$ the homotopy class of loops corresponding to the fibration π (so that $\pi = \pi_\gamma$). Denote by $\pi' = \pi_{\gamma^{-1}}$ the fibration corresponding to the inverse of γ. Choose two reference sections $\widetilde{A}_0$ and $\widetilde{A'}_0$ for π and π' respectively. The corresponding normalized Seidel elements are $S(\pi) = t^{-c_0(\pi)}\widetilde{S}(\pi)$, $S(\pi') = t^{-c_0(\pi')}\widetilde{S}(\pi')$. Since $\widetilde{S}_\pi * \widetilde{S}_{\pi'} = 1$ we have

$$S(\pi) * S(\pi') = t^{-c_0(\pi) - c_0(\pi')}.$$

But $S(\pi)$ and $S(\pi')$ both belong to the subring $QH(\Sigma; \Lambda)$ of $QH(\Sigma; \overline{\Lambda})$, hence their product $S(\pi) * S(\pi') \in QH(\Sigma; \Lambda)$ too. Thus $t^{-c_0(\pi) - c_0(\pi')} = q^r$ for some $r \in \mathbb{Z}$. It follows that $S(\pi)$ is invertible in $QH(\Sigma; \Lambda)$.

5. From manifolds with small dual to Hamiltonian fibrations

Let $X \subset \mathbb{C}P^N$ be a projective manifold with small dual. Put $n = \dim_{\mathbb{C}} X$ and $k = \mathrm{def}(X) > 0$. Since $X^* \subset (\mathbb{C}P^N)^*$ has codimension $k + 1 \geq 2$ we can find a pencil of hyperplanes $\ell \subset (\mathbb{C}P^N)^*$ such that ℓ does not intersect X^*. Consider the manifold

$$\widetilde{X} = \{(x, H) \mid H \in \ell,\ x \in H\} \subset X \times \ell.$$

Identify $\ell \cong \mathbb{C}P^1 \cong S^2$ in an obvious way. Denote by

$$p : \widetilde{X} \longrightarrow X, \quad \pi_\ell : \widetilde{X} \longrightarrow \ell \cong S^2$$

the obvious projections. The map p can be considered as the blowup of X along the base locus of the pencil ℓ. The map π_ℓ is a honest holomorphic fibration (without singularities) over $\ell \cong \mathbb{C}P^1$ with fibers $\pi_\ell^{-1}(H) = X \cap H$.

Denote by $\omega_{_X}$ the symplectic form on X induced from the Fubini-Study Kähler form of $\mathbb{C}P^N$. Let ω_{S^2} be an area form on S^2 with $\int_{S^2} \omega_{S^2} = 1$. Endow $X \times S^2$ with $\omega_{_X} \oplus \omega_{S^2}$ and denote by $\widetilde{\Omega}$ the restriction of $\omega_{_X} \oplus \omega_{S^2}$ to $\widetilde{X} \subset X \times S^2$. The restriction of $\widetilde{\Omega}$ to the fibers $\widetilde{\Omega}|_{\pi_\ell^{-1}(H)}$, $H \in \ell$, coincides with the symplectic forms $\omega_{_X}|_{X \cap H}$. Thus $\pi_\ell : \widetilde{X} \longrightarrow S^2$ is a Hamiltonian fibration. Fix a point $H_0 \in \ell$, and set $(\Sigma, \omega_\Sigma) = (\pi_\ell^{-1}(H_0), \omega_{_X}|_{X \cap H_0})$.

Remark 5.A. Different pencils $\ell \subset (\mathbb{C}P^N)^*$ with $\ell \cap X^* = \emptyset$ give rise to isomorphic Hamiltonian fibrations. This is so because the *real* codimension of X^* is at least 4 hence any two pencils ℓ, ℓ' which do not intersect X^* can be connected by a *real* path of pencils in the complement of X^*. Thus the isomorphism class of the Hamiltonian fibration π_ℓ, the element $\gamma(\pi_\ell) \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$, as well as the corresponding Seidel element $S(\pi_\ell)$ can all be viewed as invariants of the projective embedding $X \subset \mathbb{C}P^N$.

Theorem 5.B. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $\dim_{\mathbb{C}}(X) = n \geq 2$ and $\mathrm{def}(X) = k > 0$. Denote by $H_2^S(X) = \mathrm{image}\left(\pi_2(X) \longrightarrow H_2(X;\mathbb{Z})\right) \subset H_2(X;\mathbb{Z})$ the image of the Hurewicz homomorphism. Denote by $h \in H^2(X)$ the class dual to the hyperplane section. Assume that there exists $0 < \lambda \in \mathbb{Q}$ such that $c_1^X(A) = \lambda h(A)$ for every $A \in H_2^S(X)$. Then the Seidel element of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$ is:*

$$S(\pi_\ell) = [\omega_\Sigma] \in QH^2(\Sigma; \Lambda).$$

The degree of the variable $q \in \Lambda$ is $deg(q) = \frac{n+k}{2}$.

The proof of this Theorem is given in §6.

Remark 5.C. The condition $c_1^X(A) = \lambda h(A)$, $\forall A \in H_2^S$, implies that $\lambda = \frac{n+k+2}{2}$. Indeed, as explained in §2, manifolds X with small dual contain projective lines $S \subset X$ (embedded linearly in $\mathbb{C}P^N$) with $c_1^X(S) = \frac{n+k+2}{2}$. As $h(S) = 1$ it follows that $\lambda = \frac{n+k+2}{2}$.

Examples. Theorem 5.B applies for example to algebraic manifolds $X \subset \mathbb{C}P^N$ with small dual that satisfy one of the following conditions:

(1) $b_2(X) = 1$.
(2) More generally, the free part of $H_2^S(X)$ has rank 1.

This is so because in both of these cases we must have $h = \lambda c_1^X$ for some $\lambda \in \mathbb{Q}$. The fact that $\lambda > 0$ follows from the existence of rational curves $S \subset X$ with $c_1^X(S) = \frac{n+k+2}{2}$ as explained in §2.

Here is a concrete class of examples with $b_2(X) > 1$ (hence different than those in §1) to which Theorem 5.B applies. Let $Y \subset \mathbb{C}P^m$ be any algebraic manifold with $\pi_2(Y) = 0$ (or more generally with $h_Y(A) = 0$ for every $A \in H_2^S(Y)$, where h_Y is the Poincaré dual of the hyperplane class on Y). Let $i : \mathbb{C}P^n \times \mathbb{C}P^m \longrightarrow \mathbb{C}P^{(n+1)(m+1)-1}$ be the Segre embedding and put $X = i(\mathbb{C}P^n \times Y)$. It is well known (see Theorem 6.5 in [Tev1]) that

$$\mathrm{def}(X) \geq n - \dim_{\mathbb{C}}(Y),$$

hence if $n > \dim_{\mathbb{C}}(Y)$, X will have a small dual. Note that the conditions of Theorem 5.B are obviously satisfied.

One could generalize this example further by replacing $\mathbb{C}P^n$ with any manifold Z satisfying $c_1^Z(A) = \lambda h_Z(A)$ for every $A \in H_2^S(Z)$ for some $\lambda > 0$ and such that $\mathrm{def}(Z) > \dim_{\mathbb{C}}(Y)$. (See [Tev1] for more on such examples.)

Corollary 5.D. *Under the assumptions of Theorem 5.B we have:*

$$\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X), \quad \widetilde{b}_j(\Sigma) = \widetilde{b}_{j+2}(\Sigma) \ \ \forall\, j \in \mathbb{Z},$$

where the definition of $\widetilde{b}_j$ is given in (1) in §1. Or, put in an unwrapped way, we have the following identities for X:

$$b_j(X) + b_{j+n+k+2}(X) = b_{j+2}(X) + b_{j+n+k+4}(X), \ \ \forall\, 0 \leq j \leq n+k-1,$$
$$b_{n+k}(X) = b_{n+k+2}(X) + 1, \quad b_{n+k+1}(X) = b_{n+k+3}(X) + b_1(X),$$

and the following ones for Σ:

$$b_j(\Sigma) + b_{j+n+k}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma), \ \ \forall\, 0 \leq j \leq n+k-3,$$
$$b_{n+k-2}(\Sigma) = b_{n+k}(\Sigma) + 1, \quad b_{n+k-1}(\Sigma) = b_{n+k+1}(\Sigma) + b_1(\Sigma).$$

The proof is given in §7

6. Proofs of theorem 5.B and Theorems B and D

As noted in the discussion after the statement of Theorem 5.B, Theorems B, D from §1 are immediate consequences of Theorem 5.B. Therefore we will concentrate in this section in proving the latter. We will make throughout this section the same assumptions as in Theorem 5.B and use here the construction and notation of §5.

For a hyperplane $H \in (\mathbb{C}P^N)^*$ write $\Sigma_H = X \cap H$. For a pencil $\ell \subset (\mathbb{C}P^N)^*$ denote by $B_\ell = \Sigma_{H_0} \cap \Sigma_{H_1} \subset X$, $(H_0, H_1 \in \ell)$, its base locus. Recall that $p : \widetilde{X} \longrightarrow X$ can be viewed as the blowup of X along B_ℓ. Denote by $E \subset \widetilde{X}$ the exceptional divisor of this blowup.

The restriction $p|_E : E \longrightarrow B_\ell$ is a holomorphic fibration with fiber $\mathbb{C}P^1$. Denote the homology class of this fiber by $F \in H_2(\widetilde{X};\mathbb{Z})$. Since $\dim_{\mathbb{R}} B_\ell = 2n-4$, the map induced by inclusion $H_2(X \setminus B_\ell;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$ is an isomorphism, hence we obtain an obvious injection $j : H_2(X;\mathbb{Z}) \longrightarrow H_2(\widetilde{X};\mathbb{Z})$. The 2'nd homology of $\widetilde{X}$ is then given by

$$H_2(\widetilde{X};\mathbb{Z}) = j(H_2(X;\mathbb{Z})) \oplus \mathbb{Z}F.$$

The $(2n-2)$'th homology of $\widetilde{X}$ fits into the following exact sequence:

$$0 \longrightarrow \mathbb{Z}[E] \longrightarrow H_{2n-2}(\widetilde{X};\mathbb{Z}) \xrightarrow{p_*} H_{2n-2}(X;\mathbb{Z}) \longrightarrow 0,$$

where the first map is induced by the inclusion. We obviously have $p_* \circ j = \mathrm{id}$. Denote by $\widetilde{\Sigma} \subset \widetilde{X}$ the proper transform of Σ (with respect to p) in $\widetilde{X}$. The intersection pairing between H_{2n-2} and H_2 in $\widetilde{X}$ is related to the one in X as follows:

$$\begin{gathered} V \cdot j(A) = p_*(V) \cdot A, \quad \forall\, V \in H_{2n-2}(\widetilde{X};\mathbb{Z}),\ A \in H_2(X;\mathbb{Z}), \\ [\widetilde{\Sigma}] \cdot F = 1, \quad [E] \cdot F = -1, \quad [E] \cdot j(A) = 0, \quad \forall A \in H_2(X;\mathbb{Z}). \end{gathered} \tag{10}$$

Consider now the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$. The fiber over $H_0 \in \ell$ is precisely $\Sigma = \Sigma_{H_0}$. It follows from (10) that the set of classes $H_2^{\pi_\ell}$ that represent sections of π satisfies:

$$H_2^{\pi_\ell} \subset \{j(A) + dF \mid [\Sigma] \cdot A = 1 - d\}. \tag{11}$$

Denote by J_0 the standard complex structure of X (coming from the structure of X as an algebraic manifold). Denote by $\mathcal{R}(X) \subset H_2(X;\mathbb{Z})$ the positive cone generated by classes that represent J_0-holomorphic rational curves in X, i.e.

$$\mathcal{R}(X) = \Big\{ \sum a_i [C_i] \;\Big|\; a_i \in \mathbb{Z}_{\geq 0},\ C_i \subset X \ \text{ is a rational } J_0\text{-holomorphic curve} \Big\}.$$

Lemma 6.A. *Let $\widetilde{A} = j(A) + dF \in H_2^{\pi_\ell}$, with $A \in H_2(X;\mathbb{Z})$, $d \in \mathbb{Z}$. If $\mathcal{S}(\widetilde{A}) \neq 0$ then $A \in \mathcal{R}(X)$ and $d \leq 1$, with equality if and only if $A = 0$.*

Proof. Denote by $\widetilde{J_0}$ the standard complex structure on $\widetilde{X} \subset X \times \ell$, namely the complex structure induced from the standard complex structure $J_0 \oplus i$ on $X \times \ell$. Let $\widetilde{J_n}$ be a sequence of regular almost complex structures on $\widetilde{X}$ with $\widetilde{J_n} \longrightarrow \widetilde{J_0}$. Since $\mathcal{S}(\widetilde{A}, \widetilde{J_n}) \neq 0$, there exist $\widetilde{J_n}$-holomorphic sections $u_n \in \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J_n})$. After passing to the limit $n \longrightarrow \infty$ we obtain by Gromov compactness theorem a (possibly reducible) $\widetilde{J_0}$-holomorphic curve $D \subset \widetilde{X}$ in the class $\widetilde{A}$. As $p : \widetilde{X} \longrightarrow X$ is $(\widetilde{J_0}, J_0)$-holomorphic it follows that $p(D)$ is a J_0-holomorphic rational curve, hence $A = p_*([D]) \in \mathcal{R}(X)$.

Next, recall that $[\Sigma] \cdot A = 1 - d$. But $\Sigma \subset X$ is ample, hence $[\Sigma] \cdot A = 1 - d \geq 0$ with equality if and only if $A = 0$. □

The next lemma shows that when $d < 1$ the sections in the class $\widetilde{A}$ do not contribute to the Seidel element in (8).

Lemma 6.B. *Let $\widetilde{A} = j(A) + dF \in H_2^{\pi_\ell}$ with $A \in H_2(X;\mathbb{Z})$ and $d < 1$. Then $c_1^v(\widetilde{A}) > 0$. In particular, in view of (6), $\widetilde{A}$ does not contribute to $S(\pi_\ell)$.*

Proof. Denote by $c_1^{\widetilde{X}}$ the first Chern class of (the tangent bundle of) $\widetilde{X}$ and by c_1^X that of X. Since $\widetilde{X}$ is the blowup of X along B_ℓ, the relation between these Chern classes is given by:

$$c_1^{\widetilde{X}} = p^* c_1^X - PD([E]), \tag{12}$$

where $PD([E]) \in H^2(\widetilde{X})$ stands for the Poincaré dual of $[E]$. (See e.g. [GH1].)

Denote by c_1^ℓ the first Chern class of $\ell \cong \mathbb{C}P^1$. Since $\widetilde{A}$ represents sections of π_ℓ we have:

$$c_1^v(\widetilde{A}) = c_1^{\widetilde{X}}(\widetilde{A}) - \pi_\ell^*(c_1^\ell)(\widetilde{A}) = c_1^{\widetilde{X}}(\widetilde{A}) - 2.$$

Together with (12) and (10) this implies:

$$c_1^v(\widetilde{A}) = p^*(c_1^X)(\widetilde{A}) - [E]\cdot\widetilde{A} - 2 = c_1^X(A) + d - 2. \tag{13}$$

By Lemma 6.A, $A \in \mathcal{R}(X) \subset H_2^S(X)$, hence by Remark 5.C we have

$$c_1^X(A) = \frac{n+k+2}{2} h(A) = \frac{n+k+2}{2}([\Sigma]\cdot A) = \frac{n+k+2}{2}(1-d).$$

Together with (13) we obtain:

$$c_1^v(\widetilde{A}) = \frac{n+k}{2}(1-d) - 1 \geq \frac{n+k}{2} - 1 > 0,$$

because $d < 1$ and $n \geq 2$. □

We now turn to the case $\widetilde{A} = F$. Let $b \in B_\ell$. Define

$$\widetilde{u}_b : \ell \longrightarrow \widetilde{X}, \quad \widetilde{u}_b(z) = (b, z).$$

It is easy to see that $\widetilde{u}_b$ is a $\widetilde{J}_0$-holomorphic section of π_ℓ representing the class F.

Lemma 6.C. *The sections $\widetilde{u}_b$, $b \in B_\ell$, are the only $\widetilde{J}_0$-holomorphic sections in the class F, hence $\mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0) = \{\widetilde{u}_b \mid b \in B_\ell\}$. The evaluation map is given by*

$$ev_{\widetilde{J}_0, z_0}(\widetilde{u}_b) = b \in \Sigma$$

and is an orientation preserving diffeomorphism between $\mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0)$ and the base locus B_ℓ.

Proof. Let $\widetilde{u} : \ell \longrightarrow \widetilde{X}$ be a $\widetilde{J}_0$-holomorphic section in the class F. Write $\widetilde{u}(z) = (v(z), z) \in X \times \ell$. Due to our choice of $\widetilde{J}_0$, v is a J_0-holomorphic map. Since $p_*(F) = 0$ the map $v = p \circ u : \ell \longrightarrow X$ must be constant, say $v(z) \equiv b$, $b \in X$. But $v(z) \in \Sigma_z$ for every $z \in \ell$. It follows that $b \in \cap_{z\in\ell}\Sigma_z = B_\ell$. The rest of the statements in the lemma are immediate. □

We are now ready for the

Proof of Theorem 5.B. In view of (11) and Lemmas 6.A, 6.B, the only class that contributes to the Seidel element $S(\pi_\ell)$ is F, hence:

$$S(\pi_\ell) = \mathcal{S}(F) \in QH^2(\Sigma;\Lambda).$$

(We take F to be the reference class of sections and note that $c_1^v(F) = -1$.)

In order to evaluate $\mathcal{S}(F)$ we need to compute $\mathcal{S}(F,\widetilde{J})$ for a regular $\widetilde{J}$. We first claim that there exists a neighborhood $\mathcal{U}$ of $\widetilde{J}_0$ inside $\widetilde{\mathcal{J}}(\pi_\ell,\widetilde{\Omega})$ such that for every $\widetilde{J}\in\mathcal{U}$ the space $\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J})$ is compact.

To see this, first note that $\widetilde{\Omega}$ is a genuine symplectic form on $\widetilde{X}$ and that $\widetilde{J}_0$ is tamed by $\widetilde{\Omega}$ (i.e. $\Omega(v,\widetilde{J}_0 v)>0$ for all non-zero vectors $v\in T\widetilde{X}$ be they vertical or not). Hence there is a neighborhood $\mathcal{U}$ of $\widetilde{J}_0$ in $\widetilde{\mathcal{J}}(\pi_\ell,\widetilde{\Omega})$ such that every $\widetilde{J}\in\mathcal{U}$ is tamed by $\widetilde{\Omega}$. Next note that $\widetilde{\Omega}$ defines an integral (modulo torsion) cohomology class $[\widetilde{\Omega}]\in H^2(\widetilde{X})_{fr}$ and that $\widetilde{\Omega}(F)=1$ (see §5). It follows that F is a class of minimal positive area for $\widetilde{\Sigma}$. Therefore, for $\widetilde{J}$ tamed by $\widetilde{\Omega}$, a sequence of $\widetilde{J}$-holomorphic rational curves in the class F cannot develop bubbles. By Gromov compactness $\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J})$ is compact.

Next we claim that $\widetilde{J}_0$ is a regular almost complex structure in the sense of the general theory of pseudo-holomorphic curves (see Chapter 3 in [MS2]). To see this recall the following regularity criterion (see Lemma 3.3.1 in [MS2]): *let (M,ω) be a symplectic manifold and J an* integrable *almost complex structure. Then J is regular for a J-holomorphic curve $u:\mathbb{C}P^1\longrightarrow M$ if every summand of the holomorphic bundle $u^*TM\to\mathbb{C}P^1$ (in its splitting to a direct sum of line bundles) has Chern number ≥ -1.* Applying this to our case, a simple computation shows that for every $\widetilde{u}_b\in\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J}_0)$ we have

$$\widetilde{u}_b^*T\widetilde{X} = \mathcal{O}_\ell(2)\oplus\mathcal{O}_\ell^{\oplus(n-2)}\oplus\mathcal{O}_\ell(-1),$$

hence $\widetilde{J}_0$ is regular for all $\widetilde{u}\in\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J}_0)$.

Pick a regular almost complex structure $\widetilde{J}\in\widetilde{\mathcal{J}}_{\mathrm{reg}}(\pi,\widetilde{\Omega})\cap\mathcal{U}$ which is close enough to $\widetilde{J}_0$. By the standard theory of pseudo-holomorphic curves [MS2] the evaluation maps $ev_{\widetilde{J},z_0}$ and $ev_{\widetilde{J}_0,z_0}$ are cobordant, hence give rise to cobordant pseudo-cycles. Moreover by what we have seen before this cobordism can be assumed to be compact (and the pseudo-cycles are in fact cycles). It follows that the homology class $(ev_{\widetilde{J},z_0})_*[\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J})]$ equals to $(ev_{\widetilde{J}_0,z_0})_*[\mathcal{M}^{\mathfrak{s}}(F,\widetilde{J}_0)]=[B_\ell]$. Putting everything together we obtain:

$$S(\pi_\ell) = \mathcal{S}(F,\widetilde{J}) = PD([B_\ell]) = [\omega_\Sigma].$$

□

7. Proof of Corollary 5.D

The quantum cohomology of Σ can be written additively (as a vector space) as

$$QH^j(\Sigma;\Lambda) \cong \bigoplus_{l\in\mathbb{Z}} H^{j+2C_\Sigma l}(\Sigma).$$

By Theorem 5.B, $[\omega_\Sigma] \in QH^2(\Sigma;\Lambda)$ is invertible with respect to the quantum product $*$, hence the map

$$(-)*[\omega_\Sigma] : QH^j(\Sigma;\Lambda) \longrightarrow QH^{j+2}(\Sigma;\Lambda), \quad a \longmapsto a*[\omega_\Sigma]$$

is an isomorphism for every $j\in\mathbb{Z}$. The statement about $\widetilde{b}_j(\Sigma)$ follows immediately.

We now turn to the proof of the statement about $\widetilde{b}_j(X)$. First recall that $2C_\Sigma = n+k$ and $2C_X = n+k+2$. We will show now that for every $0 \le j \le n+k+1$ we have $\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X)$.

Step 1. Assume $j \le n+k-4$. By Corollary 3.A, $b_j(\Sigma) = b_j(X)$ and $b_{j+2}(\Sigma) = b_{j+2}(X)$. We claim that

$$b_{j+n+k}(\Sigma) = b_{j+n+k+2}(X), \quad b_{j+n+k+2}(\Sigma) = b_{j+n+k+4}(X). \tag{14}$$

Indeed, by Corollary 3.A, $b_{n-j-k-2}(\Sigma) = b_{n-j-k-2}(X)$, hence the first equation in (14) follows from Poincaré duality for Σ and X. The proof of the second equality is similar.

It follows that

$$\begin{aligned}\widetilde{b}_j(X) &= b_j(X) + b_{j+n+k+2}(X) = b_j(\Sigma) + b_{j+n+k}(\Sigma) = \widetilde{b}_j(\Sigma)\\ &= \widetilde{b}_{j+2}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma) = b_{j+2}(X) + b_{j+n+k+4}(X) = \widetilde{b}_{j+2}(X).\end{aligned}$$

Step 2. Assume $n+k-3 \le j \le n+k-1$. In this case we have $\widetilde{b}_j(X) = b_j(X)$ and $\widetilde{b}_{j+2}(X) = b_{j+2}(X)$ and the equality between the two follows from Corollary 3.B.

Step 3. Assume $j = n+k$. We have to prove that $b_{n+k}(X) = b_0(X) + b_{n+k+2}(X)$. By Poincaré duality this is equivalent to showing that $b_{n-k}(X) = b_0(X) + b_{n-k-2}(X)$. The last equality is, by Corollary 3.A, equivalent to $b_{n-k}(\Sigma) = b_0(\Sigma) + b_{n-k-2}(\Sigma)$. Applying Poincaré duality on Σ the latter becomes equivalent to $b_{n+k-2}(\Sigma) = b_0(\Sigma) + b_{n+k}(\Sigma)$. But this has already been proved since $b_{n+k-2}(\Sigma) = \widetilde{b}_{n+k-2}(\Sigma) = \widetilde{b}_{n+k}(\Sigma) = b_0(\Sigma) + b_{n+k}(\Sigma)$.

Step 3. Assume $j = n+k+1$. The proof in this case is very similar to the case $j = n+k$. We omit the details. □

8. Further results

As we have seen above the algebraic geometry of manifolds with small dual is intimately connected with their symplectic topology. Here we add another ingredient which has to do with Lagrangian submanifolds. Below we will use the following notation. For an algebraic manifold $X \subset \mathbb{C}P^N$ and an algebraic submanifold $\Sigma \subset X$ we denote by $\omega_{_X}$ and $\omega_{_\Sigma}$ the restrictions of the standard Kähler form of $\mathbb{C}P^N$ to X and to Σ respectively.

The following theorem was observed in [Bir1] (Theorem K) and in [Bir2] (Theorem 2.1).

Theorem 8.A. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold and $\Sigma \subset X$ a hyperplane section. If $\mathrm{def}(X) = 0$ then $(\Sigma, \omega_{_\Sigma})$ contains a (embedded) Lagrangian sphere.*

Thus we can detect manifolds with small dual (i.e. $\mathrm{def} > 0$) by methods of symplectic topology e.g. by showing that their hyperplane sections do not contain Lagrangian spheres.

In some situations we also have the converse to Theorem 8.A.

Theorem 8.B. *Let $\Sigma \subset X \subset \mathbb{C}P^N$ be as in Theorem 5.B and assume in addition that $\dim_{\mathbb{C}}(\Sigma) \geq 3$. Then the symplectic manifold $(\Sigma, \omega_{_\Sigma})$ contains no Lagrangian spheres.*

Proof. Suppose by contradiction that $L \subset (\Sigma, \omega_{_\Sigma})$ is a Lagrangian sphere. We will use now the theory of Lagrangian Floer cohomology for in order to arrive at a contradiction. More specifically, we will use here a particular case of the general theory that works for so called *monotone Lagrangian submanifolds*. We will take $\mathbb{Z}_2$ as the ground ring and work with the self Floer cohomology of L, denoted $HF(L, L)$, with coefficients in the Novikov ring $\Lambda_{\mathbb{Z}_2} = \mathbb{Z}_2[q, q^{-1}]$. This ring is graded so that the variable q has degree $\deg(q) = N_L$, where N_L is the minimal Maslov number of L. We refer the reader to [Oh1, Oh2, BC5, BC4] for the foundations of this theory.

Since L is simply connected, the assumptions on Σ and X imply that $L \subset \Sigma$ is a monotone Lagrangian submanifold and its minimal Maslov number is $N_L = 2C_\Sigma = n + k$. (Here, as in Theorem 5.B, $k = \mathrm{def}(X) \geq 1$.) Under these circumstances it is well known that the self Floer homology of L, $HF(L, L)$ is well defined and moreover we have an isomorphism of graded $\Lambda_{\mathbb{Z}_2}$-modules:

$$HF^*(L, L) \cong (H^\bullet(L; \mathbb{Z}_2) \otimes \Lambda_{\mathbb{Z}_2})^*.$$

Since L is a sphere of dimension $\dim_{\mathbb{R}}(L) \geq 3$ this implies that

$$HF^0(L, L) \cong \mathbb{Z}_2, \quad HF^2(L, L) \cong H^2(L; \mathbb{Z}_2) = 0. \tag{15}$$

Denote by $QH(\Sigma;\Lambda_{\mathbb{Z}_2})$ the modulo-2 reduction of $QH(\Sigma;\Lambda)$ (obtained by reducing the ground ring $\mathbb{Z}$ to $\mathbb{Z}_2$). By Theorem 5.B, $[\omega_\Sigma]\in QH^2(\Sigma;\Lambda)$ is an invertible element, hence its modulo-2 reduction, say $\alpha\in QH^2(\Sigma;\Lambda_{\mathbb{Z}_2})$ is invertible too.

We now appeal to the quantum module structure of $HF(L,L)$ introduced in [BC5, BC4, BC3]. By this construction, $HF(L,L)$ has a structure of a graded module over the ring $QH(\Sigma;\Lambda_{\mathbb{Z}_2})$ where the latter is endowed with the quantum product. We denote the module action of $QH^*(\Sigma;\Lambda_{\mathbb{Z}_2})$ on $HF^*(L,L)$ by

$$QH^i(\Sigma;\Lambda_{\mathbb{Z}_2})\otimes_{\Lambda_{\mathbb{Z}_2}} HF^j(L,L)\longrightarrow HF^{i+j}(L,L),\quad a\otimes x\longmapsto a\circledast x,\ \ i,j\in\mathbb{Z}.$$

Since $\alpha\in QH^2(\Sigma;\Lambda_{\mathbb{Z}_2})$, α induces an isomorphism $\alpha\circledast(-):HF^*(L,L)\longrightarrow HF^{*+2}(L,L)$. This however, is impossible (e.g for $*=0$) in view of (15). Contradiction. □

Corollary 8.C. *Let Σ be an algebraic manifold with $\dim_{\mathbb{C}}(\Sigma)\geq 3$ and $b_2(\Sigma)=1$. Suppose that Σ can be realized as a hyperplane section of a projective manifold $X\subset\mathbb{C}P^N$ with small dual. Then in any other realization of Σ as a hyperplane section of a projective manifold $X'\subset\mathbb{C}P^{N'}$ we have $\mathrm{def}(X')>0$. In fact, $\mathrm{def}(X')=\mathrm{def}(X)$.*

Proof. Let ω_Σ be the restriction to Σ (via $\Sigma\subset X\subset\mathbb{C}P^N$) of the standard symplectic structure of $\mathbb{C}P^N$. Similarly let ω'_Σ the restriction to Σ (via $\Sigma\subset X'\subset\mathbb{C}P^{N'}$) of the standard symplectic structure of $\mathbb{C}P^{N'}$.

Since $b_2(\Sigma)=1$, it follows from Lefschetz theorem that $b_2(X)=1$. Thus X satisfies the conditions of Theorem 5.B (see the discussion after Theorem 5.B). By Theorem 8.B the symplectic manifold (Σ,ω_Σ) does not contain Lagrangian spheres.

Since $b_2(\Sigma)=1$ the cohomology classes $[\omega_\Sigma]$ and $[\omega'_\Sigma]$ are proportional, so there is a constant c such that $[\omega'_\Sigma]=c[\omega_\Sigma]$. Clearly we have $c>0$ (to see this, take an algebraic curve $D\subset\Sigma$ and note that both $\int_D\omega_\Sigma$ and $\int_D\omega'_\Sigma$ must be positive since both ω_Σ and ω'_Σ are Kähler forms with respect to the complex structure of Σ). We claim that the symplectic structures ω'_Σ and $c\omega_\Sigma$ are diffeomorphic, i.e. there exists a diffeomorphism $\varphi:\Sigma\longrightarrow\Sigma$ such that $\varphi^*\omega'_\Sigma=c\omega_\Sigma$. Indeed this follows from Moser argument [MS1] since all the forms in the family $\{(1-t)c\omega_\Sigma+t\omega'_\Sigma\}_{t\in[0,1]}$ are symplectic (since $c\omega_\Sigma$ and ω'_Σ are both Kähler with respect to the same complex structure) and all lie in the same cohomology class.

Since $(\Sigma,c\omega_\Sigma)$ has no Lagrangian spheres the same holds for (Σ,ω'_Σ) too. By Theorem 8.A we have $\mathrm{def}(X')>0$.

That $\mathrm{def}(X')=\mathrm{def}(X)$ follows immediately from the fact that for manifolds with positive defect the minimal Chern number C_Σ of a hyperplane section Σ is determined by

the defect. More specifically, we have (see §2):

$$\frac{n+\mathrm{def}(X)}{2} = C_\Sigma = \frac{n+\mathrm{def}(X')}{2},$$

where $n = \dim_{\mathbb{C}}(X)$. □

Theorem A says that the complement of a hyperplane section $X \setminus \Sigma$ of an algebraic manifold $X \subset \mathbb{C}P^N$ with small dual is subcritical. Here is a partial converse:

Theorem 8.D. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $n = \dim_{\mathbb{C}}(X) \geq 3$ and let $\Sigma \subset X$ be a hyperplane section. Assume that (Σ, ω_Σ) is spherically monotone with $C_\Sigma \geq 2$ and that $2C_\Sigma$ does not divide n. If $X \setminus \Sigma$ is subcritical then* $\mathrm{def}(X) > 0$.

Note that the spherical monotonicity of (Σ, ω_Σ) is automatically satisfied e.g. when Σ is Fano and $b_2(\Sigma) = 1$.

Proof. Suppose by contradiction that $\mathrm{def}(X) = 0$. By Theorem 8.A (Σ, ω_Σ) has a Lagrangian sphere, say $L \subset \Sigma$. Note that since Σ is spherically monotone, the Lagrangian $L \subset \Sigma$ is monotone too and since L is simply connected its minimal Maslov number is $N_L = 2C_\Sigma$.

Put $W = X \setminus \Sigma$ endowed with the symplectic form ω_W induced from X (which in turn is induced from $\mathbb{C}P^N$). We now appeal to the Lagrangian circle bundle construction introduced in [Bir3, BC2]. We briefly recall the construction. Pick a tubular neighborhood $\mathcal{U}$ of Σ in X whose boundary $\partial\mathcal{U}$ is a circle bundle over Σ. Denote this circle bundle by $\pi : \partial\mathcal{U} \to \Sigma$. Then $\Gamma_L = \pi^{-1}(L)$ is the total space of a circle bundle over L, embedded inside W. By [Bir3], for a careful choice of $\mathcal{U}$ the submanifold Γ_L is Lagrangian in W. Moreover, since L is monotone Γ_L is monotone too and has the same minimal Maslov number: $N_{\Gamma_L} = N_L = 2C_\Sigma$. (See [Bir3] for more details.)

Denote by $(\widehat{W}, \widehat{\omega}_W)$ the symplectic completion of the symplectic Stein manifold (W, ω_W) (see [BC1, Bir3] for the details). By the results of [BC1], Γ_L is Hamiltonianly displaceable (i.e. there exists a compactly supported Hamiltonian diffeomorphism $h : (\widehat{W}, \widehat{\omega}_W) \longrightarrow (\widehat{W}, \widehat{\omega}_W)$ such that $h(\Gamma_L) \cap \Gamma_L = \emptyset$). In particular, $HF(\Gamma_L, \Gamma_L) = 0$.

One can arrive now at a contradiction by using an alternative method to compute $HF(\Gamma_L, \Gamma_L)$ such as the Oh spectral sequence [Oh2, Bir3]. (This is a spectral sequence whose initial page is the singular homology of Γ_L and which converges to $HF(\Gamma_L, \Gamma_L)$, which is 0 in our case.) We will not perform this computation here since the relevant part of it has already been done in [Bir3], hence we will use the latter.

Here are the details. We first claim that the bundle $\pi|_{\Gamma_L} : \Gamma_L \to L$ is topologically trivial. To see this denote by $N_{\Sigma/X}$ the normal bundle of Σ in X, viewed as a complex line bundle. Note that $\Gamma_L \to L$ is just the circle bundle associated to $N_{\Sigma/X}|_L$. Thus

it is enough to show that $N_{\Sigma/X}|_L$ is trivial. Denote by $c \in H^2(\Sigma;\mathbb{Z})$ the first Chern class of $N_{\Sigma/X}$ and by $c_{\mathbb{R}}$ its image in $H^2(\Sigma;\mathbb{R})$. Similarly, denote by $c|_L$ and by $c_{\mathbb{R}}|_L$ the restrictions of c and $c_{\mathbb{R}}$ to L. As $\Sigma \subset X$ is a hyperplane section we have $c_{\mathbb{R}} = [\omega_{\Sigma}]$. But $L \subset (\Sigma, \omega_{\Sigma})$ is Lagrangian hence $c_{\mathbb{R}}|_L = 0$. As $H^*(L;\mathbb{Z})$ has no torsion (L is a sphere) it follows that $c|_L = 0$ too. Thus the restriction $N_{\Sigma/X}|_L$ of $N_{\Sigma/X}$ to L has zero first Chern class. This implies that the line bundle $N_{\Sigma/X}|_L \to L$ is trivial (as a smooth complex line bundle). In particular $\Gamma_L \to L$ is a trivial circle bundle.

Since $\Gamma_L \approx L \times S^1$ we have $H^i(\Gamma_L;\mathbb{Z}_2) = \mathbb{Z}_2$ for $i = 0, 1, n-1, n$ and $H^i(\Gamma_L;\mathbb{Z}_2) = 0$ for every other i. By Proposition 6.A of [Bir3] we have $2C_{\Sigma} \mid n$. A contradiction. (Note that the conditions $n \geq 3$ and $C_{\Sigma} \geq 2$ in the statement of the theorem are in fact required for Proposition 6.A of [Bir3] to hold.) □

8.1. **Other approaches to proving Corollary C.** Here we briefly outline an alternative approach to proving Corollary C and possibly Theorem B, based on the subcriticality of $X \setminus \Sigma$ that was established in Theorem A.

Put $W = X \setminus \Sigma$ and ω_W be the symplectic form on W induced from that of X. Let $\mathcal{U}$ be a tubular neighborhood of Σ in X as in the proof of Theorem 8.D. The boundary $P = \partial\mathcal{U}$ of $\mathcal{U}$ is a circle bundle $\pi : P \longrightarrow \Sigma$ over Σ. Consider the embedding

$$i : P \longrightarrow W \times \Sigma, \quad i(p) = (p, \pi(p)).$$

Denote by $\Gamma_P = i(P) \subset W \times \Sigma$ the image of i. By the results of [Bir3], one can choose $\mathcal{U}$ in such a way that there exists a positive constant (depending on the precise choice of $\mathcal{U}$) such that $i(P)$ is a Lagrangian submanifold of $(W \times \Sigma, \omega_W \oplus -c\omega_{\Sigma})$. (Note the minus sign in front of ω_{Σ}.) Moreover, the Lagrangian Γ_P is monotone and its minimal Maslov number is $N_P = 2C_{\Sigma}$, where C_{Σ} is the minimal Chern number of Σ. So by the results recalled in §2 we have $N_P = n + k$. Note that $\dim_{\mathbb{R}} \Gamma_P = 2n + 1$.

As W is subcritical it follows that Γ_P can be Hamiltonianly displaced in the completion $(\widehat{W} \times \Sigma, \widehat{\omega}_W \oplus -c\omega_{\Sigma})$ and therefore

$$HF(\Gamma_P, \Gamma_P) = 0.$$

(See [Bir3] for the details. See also the proof of Theorem 8.D above.) Note that in order to use here Floer cohomology with ground coefficient ring $\mathbb{Z}$ we need to have Γ_P oriented and endowed with a spin structure. In our case, Γ_P carries a natural orientation and it is easy to see that it has a spin structure (in fact, it is easy to see that $H^1(P;\mathbb{Z}_2) = 0$ hence this spin structure is unique).

We now appeal to the Oh spectral sequence [Oh2, Bir3]. Recall that this is a spectral sequence whose first page is the singular cohomology of Γ_P and which converges to the Floer cohomology $HF(\Gamma_P, \Gamma_P)$. A simple computation shows that in our case, due to the

fact that $N_P = n+k$, this sequence collapses at the second page, and moreover since $HF(\Gamma_P, \Gamma_P) = 0$ this second page is 0 everywhere. By analyzing the differentials on the first page we obtain the following exact sequences for every $j \in \mathbb{Z}$:

$$H^{j-1+n+k}(\Gamma_P;\mathbb{Z}) \longrightarrow H^j(\Gamma_P;\mathbb{Z}) \longrightarrow H^{j+1-n-k}(\Gamma_P;\mathbb{Z}). \tag{16}$$

This implies many restrictions on the cohomology of $P \approx \Gamma_P$, e.g. that $H^j(P;\mathbb{Z}) = 0$ for every $n-k+3 \leq j \leq n+k-2$, that $H^j(P;\mathbb{Z}) \cong H^{j-1+n+k}(P;\mathbb{Z})$ for every $0 \leq j \leq n-k-2$ and more. We now substitute this information into the Gysin sequences of the bundle $P \longrightarrow \Sigma$ (whose Euler class is just the hyperplane class h corresponding to the embedding $\Sigma \subset \mathbb{C}P^N$). Combining the calculation via the Gysin sequences together with the Lefschetz theorem yields the desired periodicity for the cohomology of Σ. We omit the details as they are rather straightforward.

One could try to push the above argument further by using the methods of [BK] (see e.g. §14 in that paper) in order to prove Theorem B via Lagrangian Floer cohomology. However, this would require an extension of the methods of [BK] to coefficients in $\mathbb{Z}$ rather than just $\mathbb{Z}_2$.

9. What happens in the non-monotone case

Here we briefly explain what happens in Theorem 5.B when the condition "$c_1^X(A) = \lambda h(A)$ for some $\lambda > 0$" is not satisfied, e.g. when (Σ, ω_Σ) is not spherically monotone (see Definition 4.1.A).

We will need to change here a bit our coefficient ring for the quantum cohomology since (Σ, ω_Σ) is not spherically monotone anymore. Denote by $\mathcal{A}$ the ring of all formal series in the variables q, T

$$P(q,T) = \sum_{i,j} a_{i,j} q^i T^{s_j}, \quad a_{i,j} \in \mathbb{Z}, s_j \in \mathbb{R},$$

which satisfy that for every $C \in \mathbb{R}$

$$\#\big\{(i,j) \mid a_{i,j} \neq 0 \text{ and } s_j > C\big\} < \infty.$$

This ring is a special case of the more general Novikov ring commonly used in the theory of quantum cohomology. With this ring as coefficients, the definition of the quantum product $*$ on $QH(\Sigma;\mathcal{A})$ is very similar to what we have had before. Namely, the powers of the variable q will encode Chern numbers of rational curves involved in the definition of $*$ and the powers of T encode their symplectic areas. See [MS2] for more details.

We now turn to the Hamiltonian fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$. We will use here the construction and notation from §5 and §6. Additionally, denote by $\widetilde{i} : \Sigma \longrightarrow \widetilde{X}$ the inclusion of the fiber into the total space of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$. Recall also from §6 that we

have a canonical injection $j : H_2(X;\mathbb{Z}) \longrightarrow H_2(\widetilde{X};\mathbb{Z})$ which satisfies $j \circ p_* = \mathrm{id}$, where $p : \widetilde{X} \longrightarrow X$ is the blow down map. Denote by $B_\ell \subset X$ the base locus of the pencil ℓ. With this notation we have:

$$\widetilde{i}_*(\alpha) = j(\alpha) - ([B_\ell]\cdot\alpha)F = j(\alpha) - \langle[\omega_\Sigma],\alpha\rangle F \quad \forall \alpha \in H_2(\Sigma;\mathbb{Z}). \tag{17}$$

The symplectic form $\widetilde{\Omega}$ satisfies:

$$\begin{aligned} &[\widetilde{\Omega}] = 2p^*[\omega_X] - e, \quad \text{where } e \in H^2(\widetilde{X}) \text{ is the Poincaré dual of } E,\\ &\langle[\widetilde{\Omega}], j(A)\rangle = 2\langle[\omega_X], A\rangle \quad \forall A \in H_2(X;\mathbb{Z}),\\ &\langle[\widetilde{\Omega}], F\rangle = 1. \end{aligned} \tag{18}$$

The Seidel element of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$ will now be:

$$S(\pi_\ell) = \sum_{\widetilde{A}\in H_2^\pi} \mathcal{S}(\widetilde{A};\widetilde{J}) \otimes q^{\nu(\widetilde{A})} T^{\langle[\widetilde{\Omega}],\widetilde{A}\rangle} \in QH^{-2c_0(\pi_\ell)}(\Sigma;\mathcal{A}).$$

Some parts of the proof of Theorem 5.A go through in this new setting. More specifically, Lemma 6.A as well as Lemma 6.C continue to hold (with the same proofs) and it follows that the contribution of the class F to the Seidel element is as before, namely

$$\mathcal{S}(F) = [\omega_\Sigma]. \tag{19}$$

If we choose as before the reference class of sections to be F then the total degree of the Seidel element $S(\pi_\ell)$ continues to be 2.

In contrast to the above, Lemma 6.B does not hold anymore since we might have holomorphic sections in the class $\widetilde{A} = j(A) + dF$ with $d \leq 0$. (We will see in §9.1 an example in which this is indeed the case.) Nevertheless we can still obtain some information on $S(\pi_\ell)$ beyond (19). Let $d \in \mathbb{Z}$ and put $\widetilde{A} = j(A) + dF$ where $A \in H_2^S(X)$. Recall from Lemma 6.A that $\widetilde{A}$ might contribute to $S(\pi_\ell)$ only if the following three conditions are satisfied:

(1) $d \leq 1$.
(2) $[\Sigma]\cdot A = 1 - d$.
(3) $A \in \mathcal{R}(X)$ where $\mathcal{R}(X) \subset H_2^S(X)$ is the positive cone generated by those classes that can be represented by J_0-holomorphic rational curves. (See §6.)

Moreover, $d = 1$ iff $A = 0$.

The case $d = 1$ has already been treated in (19). Assume that $d \leq 0$. A simple computation shows that

$$\langle[\widetilde{\Omega}], \widetilde{A}\rangle = 2 - d, \quad \langle c_1^v, \widetilde{A}\rangle = -1 + \langle c_1^X - h, A\rangle.$$

Here $h \in H^2(X)$ is the hyperplane class corresponding to the embedding $X \subset \mathbb{C}P^N$, i.e. $h = PD([\Sigma])$. This proves the following theorem:

Theorem 9.A. *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual and $\Sigma \subset X$ a hyperplane section. Then the Seidel element $S(\pi_\ell)$ corresponding to the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$ is given by:*

$$S(\pi_\ell) = [\omega_\Sigma]T + \sum_{d \leq 0, A} \mathcal{S}(j(A) + dF) T^{2-d} q^{(c_1^X(A) - h(A))/C_\Sigma}, \tag{20}$$

where the sum is taken over all $d \leq 0$ and $A \in \mathcal{R}(X)$ with:

(1) $h(A) = 1 - d$.
(2) $3 - d - n \leq c_1^X(A) \leq 2 - d$.

In particular, if $-K_X - \Sigma$ is nef and $\min\{(-K_X - \Sigma) \cdot A \mid A \in \mathcal{R}(X)\} \geq 2$ then

$$S(\pi_\ell) = [\omega_\Sigma]T.$$

Note that the powers of T in the second summand of (20) are always ≥ 2 and the powers of q in the second summand are always ≤ 1 (but might in general be also negative).

Here is a non-monotone example, not covered by Theorem 5.B but to which Theorem 9.A does apply. Let $X = \mathbb{C}P^{m+r} \times \mathbb{C}P^m$ with $m \geq 2$ and $r \geq 1$ be embedded in $\mathbb{C}P^{(m+1)(m+r+1)-1}$ by the Segre embedding. It is well known that $\mathrm{def}(X) = r$ (see Theorem 6.5 in [Tev1]). It is easy to see that $c_1^X - h$ is ample and since $m \geq 2$ its minimal value on $\mathcal{R}(X)$ is $m \geq 2$. It follows that $S(\pi_\ell) = [\omega_\Sigma]T \in QH^2(\Sigma; \mathcal{A})$.

This calculation fails to be true when $m = 1$, as will be shown in §9.1 below.

9.1. **A non-monotone example.** Consider the algebraic manifold $\Sigma = \mathbb{C}P^1 \times \mathbb{C}P^1$. Denote by $f, s \in H_2(\Sigma; \mathbb{Z})$ the classes

$$f = [\mathrm{pt} \times \mathbb{C}P^1], \quad s = [\mathbb{C}P^1 \times \mathrm{pt}]. \tag{21}$$

We have $H_2^S(\Sigma) = H_2(\Sigma; \mathbb{Z}) = \mathbb{Z}s \oplus \mathbb{Z}f$. Denote by $\alpha, \beta \in H^2(\Sigma)$ the Poincaré duals of f, s respectively, i.e.:

$$\langle \alpha, s \rangle = 1, \ \ \langle \alpha, f \rangle = 0, \quad \langle \beta, s \rangle = 0, \ \ \langle \beta, f \rangle = 1. \tag{22}$$

A simple computation shows that

$$c_1^\Sigma = 2\alpha + 2\beta.$$

Before we continue, a small remark about our algebro-geometric conventions is in order. For a complex vector space V we denote by $\mathbb{P}(V)$ the space of complex lines through 0 (*not* the space of hyperplanes or 1-dimensional quotients of V). Similarly, for a vector bundle $E \to B$ we denote by $\mathbb{P}(E) \to B$ the fiber bundle whose fiber over $x \in B$ is $\mathbb{P}(E_x)$,

as just defined, i.e. the space of lines through 0 in E_x. We denote by $T \to \mathbb{P}(E)$ the tautological bundle, which by our convention, is defined as the line bundle whose fiber over $l \in \mathbb{P}(E_x)$ is the line l itself. We denote by T^* the dual of T, i.e. $T_l^* = \hom(l, \mathbb{C})$. For example, with these conventions, for $E = \mathbb{C}^{n+1}$ (viewed as a bundle over $B = \mathrm{pt}$) we have $T^* = \mathcal{O}_{\mathbb{C}P^n}(1)$, and T^* is ample.

Consider now the bundle $\mathcal{O}_{\mathbb{C}P^1}(-1)$ over $\mathbb{C}P^1$. There is an obvious inclusion

$$\iota : \mathcal{O}_{\mathbb{C}P^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}$$

coming from viewing an element $l \in \mathbb{C}P^1$ as a subspace $l \subset \mathbb{C} \oplus \mathbb{C}$. Consider now the inclusion:

$$\mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(-1) \xrightarrow{\mathrm{id}\oplus\iota} \mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}. \tag{23}$$

Denote by E the bundle on the right-hand side of this inclusion and by E' the bundle on the left-hand side. Put

$$X = \mathbb{P}(E)$$

and denote by $\mathrm{pr} : X \longrightarrow \mathbb{C}P^1$ the bundle projection. Note that $\mathbb{P}(E') \cong \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}) = \mathbb{C}P^1 \times \mathbb{C}P^1 = \Sigma$ hence (23) induces an embedding $i_{\Sigma,X} : \Sigma \longrightarrow X$. Let $T \to X$ be the tautological bundle (as previously defined) and consider the bundle

$$\mathcal{L} = T^* \otimes p^*\mathcal{O}_{\mathbb{C}P^1}(1).$$

Theorem 9.1.A. *The line bundle $\mathcal{L}$ is very ample and the projective embedding of X induced by it has* $\mathrm{def} = 1$. *The embedding of Σ, $i_{\Sigma,X}(\Sigma) \subset X$, is a smooth hyperplane section of the projective embedding of X induced by $\mathcal{L}$. Moreover if ω_X is the symplectic structure on X induced by the projective embedding of $\mathcal{L}$ and $\omega_\Sigma = i_{\Sigma,X}^*\omega_X$ then we have:*

$$[\omega_\Sigma] = 2\alpha + \beta.$$

If ℓ is a pencil in the linear system $|\mathcal{L}|$ lying in the complement of the dual variety X^* then the Seidel element of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$ associated to ℓ is:

$$S(\pi_\ell) = (2\alpha + \beta)T + \beta T^2.$$

The proof is given in §9.2 below. One can easily generalize the above example to other projective bundles and also to higher dimensions.

Note that $[\omega_\Sigma]$ and c_1^Σ are not proportional hence the conditions of Theorem 5.B are not satisfied. It is also easy to see that (for homological reasons) (Σ, ω_Σ) does not contain any Lagrangian spheres (c.f. Theorems 8.A, 8.B).

The quantum product for (Σ, ω_Σ) is given by (see [MS2]):

$$\alpha * \alpha = qT^2, \quad \beta * \beta = qT, \quad \alpha * \beta = \alpha \cup \beta, \ \text{ where } \deg(q) = 4.$$

The inverse of $S(\pi_\ell)$ in quantum cohomology is given by

$$S(\pi_\ell)^{-1} = \tfrac{1}{qT^2(1-T)^2}\big(-2\alpha + (1+T)\beta\big).$$

Here we have written $\frac{1}{(1-T)^2}$ as an abbreviation for $(\sum_{q=0}^{\infty} T^j)^2$.

In contrast to the situation in Theorem 9.1.A we can exhibit the *same* manifold $\Sigma = \mathbb{C}P^1 \times \mathbb{C}P^1$ as a hyperplane section of a different projective manifold X' with $\mathrm{def} = 0$, but with different induced symplectic structure (c.f. Theorem 8.C). This goes as follows. Let $X' \subset \mathbb{C}P^5$ be the image of the degree–2 Veronese embedding of $\mathbb{C}P^3$. It is well known that $\mathrm{def}(X') = 0$. A simple computation shows that a smooth hyperplane section of X' is isomorphic to $\Sigma = \mathbb{C}P^1 \times \mathbb{C}P^1$. The symplectic form ω'_Σ on Σ induced from the standard symplectic structure of $\mathbb{C}P^5$ satisfies

$$[\omega'_\Sigma] = \alpha + \beta.$$

Note that (Σ, ω'_Σ) has Lagrangian spheres. To see that, note that (e.g. by Moser argument) ω'_Σ is diffeomorphic to the split form $\omega_0 = \sigma \oplus \sigma$, where σ is the standard Kähler form on $\mathbb{C}P^1$. The symplectic manifold (Σ, ω_0) obviously has Lagrangian spheres, for example $L = \{(z,w) \mid w = \bar{z}\}$, hence so does (Σ, ω'_Σ). (c.f. Theorems 8.A, 8.B).

Finally, note that $[\omega'_\Sigma] = \alpha + \beta$ is not invertible in the quantum cohomology $QH(\Sigma;\Lambda)$. In fact, a simple computation shows that $(\alpha+\beta) * (\alpha - \beta) = 0$.

9.2. **Proof of Theorem 9.1.A.** Consider the inclusion

$$\mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1} \xrightarrow{\iota \oplus \mathrm{id} \oplus \mathrm{id}} \mathcal{O}_{\mathbb{C}P^1}^{\oplus 4}. \tag{24}$$

Put $Y = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}^{\oplus 4}) = \mathbb{C}P^1 \times \mathbb{C}P^3$ and denote by $\mathrm{pr}_1 : Y \to \mathbb{C}P^1$ and $\mathrm{pr}_2 : Y \to \mathbb{C}P^3$ the projections. The inclusion (24) gives us an obvious inclusion $i_{X,Y} : X \longrightarrow Y$. The bundle T^* naturally extends to Y as $\mathrm{pr}_2^*\mathcal{O}_{\mathbb{C}P^3}(1)$, and $\mathrm{pr}^*\mathcal{O}_{\mathbb{C}P^1}(1)$ extends to Y too as $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{C}P^1}(1)$. It follows that

$$\mathcal{L} = \widetilde{\mathcal{L}}|_X, \text{ where } \widetilde{\mathcal{L}} = \mathrm{pr}_2^*\mathcal{O}_{\mathbb{C}P^3}(1) \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{C}P^1}(1).$$

The bundle $\widetilde{\mathcal{L}}$ is obviously very ample, hence so is $\mathcal{L}$. Moreover it is well known that the embedding $Y \subset \mathbb{C}P^7$ induced by $\widetilde{\mathcal{L}}$ (the Segre embedding) has $\mathrm{def}(Y) = 2$ (see e.g. [Tev1]). A straightforward computation shows that $i_{X,Y}(X) \subset Y$ is indeed a hyperplane section corresponding to the embedding $Y \subset \mathbb{C}P^7$ induced by $\widetilde{\mathcal{L}}$. If we denote by $[x_0 : x_1]$ homogeneous coordinates on $\mathbb{C}P^1$ and by $[w_0 : w_1 : w_2 : w_3]$ homogeneous coordinates on $\mathbb{C}P^3$ then $i_{X,Y}(X) \subset Y$ is given by the equation

$$i_{X,Y}(X) = \{x_0 w_1 - x_1 w_0 = 0\} \subset \mathbb{C}P^1 \times \mathbb{C}P^3.$$

As for the defect of the projective embedding of X, we have by (2) that $\mathrm{def}(i_{X,Y}(X)) = \mathrm{def}(Y) - 1 = 1$.

It remains to show that $i_{\Sigma,X}(\Sigma) \subset X$ is indeed a hyperplane section corresponding to $\mathcal{L}$. Using the coordinates x, w just introduced, we can write a point in X as $([x_0 : x_1], [w_0 : w_1 : w_2 : w_3])$ with $(w_0, w_1) \in \mathbb{C}(x_0, x_1)$, $w_2, w_3 \in \mathbb{C}$. A smooth hyperplane section of X (with respect to $\mathcal{L}$, or alternatively with respect to the embedding $X \subset \mathbb{C}P^6$) is given for example by the equation

$$\Sigma_0 = \{x_0 w_2 - x_1 w_3 = 0\}. \tag{25}$$

A simple computation shows that $\Sigma_0 = i_{\Sigma,X}(\Sigma)$.

Next we construct a pencil ℓ of divisors in the linear system $|\mathcal{L}|$ lying in the complement of the dual variety X^* (corresponding to the projective embedding induced by $\mathcal{L}$). For this end, we first construct a linear embedding

$$\iota' : \mathcal{O}_{\mathbb{C}P^1}(-2) \longrightarrow \mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1}$$

as follows. Write elements of the fiber of $\mathcal{O}_{\mathbb{C}P^1}(-1)$ over $[x_0 : x_1] \in \mathbb{C}P^1$ as pairs $(v_0, v_1) \in \mathbb{C}(x_0, x_1) \subset \mathbb{C}^2$ (or in coordinates: $v_0 x_1 = v_1 x_0$). Similarly, write elements of $\mathcal{O}_{\mathbb{C}P^1}(-2) = \mathcal{O}_{\mathbb{C}P^1}(-1) \otimes \mathcal{O}_{\mathbb{C}P^1}(-1)$ as $\big((u_0, u_1) \otimes (u_0', u_1')\big)$ with $(u_0, u_1), (u_0', u_1') \in \mathcal{O}_{\mathbb{C}P^1}(-1)$. Define

$$\iota'\big((u_0, u_1) \otimes (u_0', u_1')\big) = \big((u_1 u_0', u_1 u_1'), u_0 u_0'\big).$$

We get an embedding $\mathcal{O}_{\mathbb{C}P^1}(-2) \oplus \mathcal{O}_{\mathbb{C}P^1} \xrightarrow{\iota' \oplus \mathrm{id}} \mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{P}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}$ hence also an embedding

$$i' : \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(-2) \oplus \mathcal{O}_{\mathbb{C}P^1}) \longrightarrow X.$$

Put $\Sigma_1 = i'(\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(-2) \oplus \mathcal{O}_{\mathbb{C}P^1}) \subset X$.

A simple computation shows that Σ_1 is given by the following equation

$$\Sigma_1 = \{x_0 w_0 - x_1 w_2 = 0\} \subset X.$$

It follows that Σ_1 lies in the linear system $|\mathcal{L}|$. Consider now the pencil $\ell \subset |\mathcal{L}|$ generated by Σ_0 (see (25)) and Σ_1. A straightforward computation shows that ℓ lies in the complement of the dual variety X^*. Note that a generic element in the pencil ℓ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, however finitely many elements in ℓ are isomorphic to the Hirzebruch surface $F_2 = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(-2) \oplus \mathcal{O}_{\mathbb{C}P^1})$.

Denote by $s_X \in H_2(X; \mathbb{Z})$ the homology class represented by the rational curve

$$\mathbb{P}(0 \oplus 0 \oplus \mathcal{O}_{\mathbb{C}P^1}) \subset X.$$

Denote by $f_X \in H_2(X; \mathbb{Z})$ the class represented by a projective line lying in a fiber of the projective bundle $X \longrightarrow \mathbb{C}P^1$. Clearly $(i_{\Sigma,X})_*(f) = f_X$ and $(i_{\Sigma,X})_*(s) = s_X + f_X$.

(See (21).) The base locus B_ℓ of the pencil ℓ is a smooth algebraic curve whose homology class in Σ is $[B_\ell] = s + 2f$, and when viewed in X we have

$$(i_{\Sigma,X})_*([B_\ell]) = s_X + 3f_X.$$

Denote by $\pi_\ell : \widetilde{X} \longrightarrow \ell$ the fibration associated to ℓ. We endow $\widetilde{X}$ with the symplectic form $\widetilde{\Omega}$ as in §5. We will now compute the Seidel element $S(\pi_\ell)$.

Denote by $\omega_{_Y}$ the symplectic form on Y induced from the standard symplectic form on $\mathbb{C}P^7$ via the embedding $Y \subset \mathbb{C}P^7$. Denote also $\omega_{_X} = i^*_{X,Y}\omega_{_Y}$ and $\omega_{_\Sigma} = i^*_{\Sigma,X}\omega_{_X}$ the induced forms on X and Σ. A simple computation shows that $[\omega_{_Y}] = \mathrm{pr}_1^* a + \mathrm{pr}_2^* b$, where $a \in H^2(\mathbb{C}P^1;\mathbb{Z})$ and $b \in H^2(\mathbb{C}P^3;\mathbb{Z})$ are the positive generators. A straightforward computation shows now that

$$[\omega_{_\Sigma}] = 2\alpha + \beta.$$

We now go back to the situation of Lemma 6.B (which does not hold in our case) and try to find the contribution of holomorphic sections of $\pi_\ell : \widetilde{X} \longrightarrow \ell$ in the class $\widetilde{A} = j(A) + dF$ with $d \leq 0$.

A simple computation shows that $j(s_X) \in H_2^{\pi_\ell}$ and that $c_1^v(j(s_X)) = -1$. We also have (see (17))

$$\widetilde{i}_*(f) = j(f_X) - F, \quad \widetilde{i}_*(s) = j(s_X) + j(f_X) - 2F.$$

The degree of the Seidel element $S(\pi_\ell)$ is in our case 2, and as $C_\Sigma = 2$ it follows that the only classes $\widetilde{A}$ that might contribute to $S(\pi_\ell)$ are classes $\widetilde{A}$ that differ from s_X by an element coming from $\widetilde{i}_* : H_2(\Sigma;\mathbb{Z}) \longrightarrow H_2(\widetilde{X};\mathbb{Z})$ with zero c_1^Σ. This means that

$$\widetilde{A} = j(s_X) + r\widetilde{i}_*(s - f) = (r+1)j(s_X) - rF, \;\; r \in \mathbb{Z}.$$

As $\widetilde{A}$ represents a holomorphic section we must have $0 < \langle[\widetilde{\Omega}], \widetilde{A}\rangle = r + 2$, hence $r \geq -1$.

Lemma 9.2.A. *If $\mathcal{S}(\widetilde{A}) \neq 0$ then either $r = -1$ or $r = 0$.*

We postpone the proof for later in this section and continue with the proof of Theorem 9.1.A.

The case $r = -1$ is when $\widetilde{A} = F$. This has already been treated at the beginning of §9 and we have $\mathcal{S}(F) = [\omega_{_\Sigma}] = 2\alpha + \beta$.

We turn to the case $r = 0$, i.e. $\widetilde{A} = j(s_X)$. Denote by $\widetilde{J}_0$ the standard complex structure on $\widetilde{X} \subset X \times \ell$, namely the complex structure induced from the standard complex structure $J_0 \oplus i$ on $X \times \ell$.

Consider for every $[u : v] \in \mathbb{C}P^1$ the J_0-holomorphic rational curve $\sigma_{[u:v]} : \mathbb{C}P^1 \longrightarrow X$ defined by $\sigma_{[u:v]}([x_0 : x_1]) = [0 : u : v]$, where $[0 : u : v]$ on the right-hand side lies in the fiber over $[x_0 : x_1]$, i.e. $\sigma_{[u:v]}$ is a section of the projective bundle $X \to \mathbb{C}P^1$ and $[\sigma_{[u:v]}] = s_X$. In the family of curves $\{\sigma_\xi\}_{\xi \in \mathbb{C}P^1}$ there are precisely two curves, say $\sigma_{\xi'}$ and

$\sigma_{\xi''}$, that intersect the base locus B_ℓ of the pencil ℓ. (We assume here that the pencil ℓ was chosen to be generic.) Thus, each of the curves σ_ξ, $\xi \in \mathbb{C}P^1 \setminus \{\xi', \xi''\}$, lifts to a holomorphic curve $\widetilde{\sigma}_\xi : \mathbb{C}P^1 \longrightarrow \widetilde{X}$. (Recall that $\widetilde{X}$ is the blow up of X along the base locus B_ℓ.) The homology class of $\widetilde{\sigma}_\xi$ is $j(s_X)$. A simple computation shows that each $\widetilde{\sigma}_\xi$, $\xi \neq \xi', \xi''$, is a section of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell \cong \mathbb{C}P^1$ and moreover these are all the $\widetilde{J}_0$-holomorphic sections in the class $j(s_X)$.

It is interesting to note that the moduli space $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_0)$ of sections in the class $\widetilde{A} = j(s_X)$ is not compact. Indeed $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_0)$ can be identified with $\mathbb{C}P^1 \setminus \{\xi', \xi''\}$ and as $\xi \to \xi'$ or ξ'' we obtain bubbling in the fiber. More precisely, when $\xi \to \xi_*$ with $\xi_* \in \{\xi', \xi''\}$ the sections $\widetilde{\sigma}_\xi$ converge to a reducible curve consisting of two components: the first is a $\widetilde{J}_0$-holomorphic section in the class F and the other one is a holomorphic curve in the class $\widetilde{i}_*(s - f)$ lying in one of the fibers of π_ℓ. The latter is a (-2)–curve hence this can occur only in one of the fibers that is isomorphic to the Hirzebruch surface F_2 (obviously not in any of the fibers that are isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, since those are Fano).

The corresponding evaluation map $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_0) \longrightarrow \Sigma$ gives a pseudo-cycle whose homology class is $s \in H_2(\Sigma; \mathbb{Z})$. Moreover for each $\widetilde{\sigma}_\xi \in \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_0)$ we have

$$\widetilde{\sigma}_x^*(T\widetilde{X}) \cong \mathcal{O}_\ell(2) \oplus \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell$$

hence by the regularity criterion from [MS2] (see Lemma 3.3.1 in that book) $\widetilde{J}_0$ is regular for all the elements in $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_0)$. Consequently we have:

$$\mathcal{S}(\widetilde{A}) = PD(s) = \beta.$$

Putting everything together we see that

$$S(\pi_\ell) = [\omega_\Sigma] T^{\widetilde{\Omega}(F)} + \beta T^{\widetilde{\Omega}(j(S_X))} = (2\alpha + \beta)T + \beta T^2.$$

The proof of Theorem 9.1.A is now complete modulo the proof of Lemma 9.2.A. □

Proof of Lemma 9.2.A. Write $\widetilde{A} = (r+1)j(s_X) - rF$. We have $\widetilde{\Omega}(\widetilde{A}) = r + 2$. Since $[\widetilde{\Omega}]$ is an integral cohomology class we must have $r \geq -1$. Thus we have to prove that it is impossible to have $\mathcal{S}(\widetilde{A}, \widetilde{J}) \neq 0$ for generic $\widetilde{J}$ if $r \geq 1$.

Claim 1. There exist no $\widetilde{J}_0$-holomorphic sections in the class $\widetilde{A}$ for $r \geq 1$.

Indeed, assume by contradiction that $\widetilde{u}_0$ is such a section. Recall that we have the blow down projection $p : \widetilde{X} \longrightarrow X$ which is $(\widetilde{J}_0, J_0)$-holomorphic. Therefore $p \circ \widetilde{u}_0$ is a J_0-holomorphic curve in X. Its homology class is $(r+1)s_X$. Recall that we also have the inclusion $i_{X,Y} : X \longrightarrow Y = \mathbb{C}P^1 \times \mathbb{C}P^3$ which is holomorphic. Hence $u := i_{X,Y} \circ p \circ \widetilde{u}_0$ is a holomorphic curve in the class $(r+1)[\mathbb{C}P^1 \times \mathrm{pt}] \in H_2(Y; \mathbb{Z})$. As the projection $\mathrm{pr}_2 : Y \longrightarrow \mathbb{C}P^3$ is holomorphic it follows that u is an $(r+1)$-multiply covered curve

whose image is $\mathbb{C}P^1 \times \mathrm{pt}$. Note that $\widetilde{u}_0 : \mathbb{C}P^1 \longrightarrow \widetilde{X}$ is injective (since it is a section) and it is easy to see (by an explicit calculation) that $\widetilde{u}_0$ cannot be entirely contained inside the exceptional divisor E (of the blow up $p : \widetilde{X} \longrightarrow X$). Therefore $p \circ \widetilde{u}_0$ must be generically injective (i.e. it restricts to an injective map over an open dense subset of $\mathbb{C}P^1$), and the same should hold also for u. But we have seen that u is $(r+1)$-multiply covered curve. We thus obtain a contradiction if $r \geq 1$. This proves Claim 1.

Next, consider $\widetilde{J}_0$-holomorphic *cusp sections* in the class $\widetilde{A}$. By this we mean a bubble tree of $\widetilde{J}_0$-holomorphic curves $(\widetilde{u}_0, v_1, \ldots, v_q)$ consisting of one holomorphic section $\widetilde{u}_0$ together with holomorphic rational curves $v_i : \mathbb{C}P^1 \longrightarrow \widetilde{X}$ each lying in a fiber of π_ℓ. Moreover we have

$$[\widetilde{u}_0] + [v_1] + \cdots + [v_q] = \widetilde{A}, \tag{26}$$

where $[\widetilde{u}_0] = (\widetilde{u}_0)_*([\mathbb{C}P^1])$ and similarly for the $[v_i]$'s. Note however, that some of the v_i's might not be reduced, i.e. they might be multiply covered.

Claim 2. For $r \geq 1$ the only possible cusp sections $(\widetilde{u}_0, v_1, \ldots, v_q)$ in the class $\widetilde{A}$ are such that $\widetilde{u}_0$ is a curve in the class F which is a fiber of the fibration $p|_E : E \longrightarrow B_\ell$ and each v_i being a (possible multiple cover of a) (-2)–curve in its corresponding fiber of π_ℓ. (Thus the bubbles v_i may occur only in the special fibers of π_ℓ which are Hirzebruch surfaces different than $\mathbb{C}P^1 \times \mathbb{C}P^1$.) Moreover, there exist only finitely many cusp sections in the class $\widetilde{A}$.

To prove this, let $(\widetilde{u}_0, v_1, \ldots, v_q)$ be such a cusp section. We first claim that $\widetilde{u}_0$ has its image inside the exceptional divisor E. Indeed, suppose otherwise, and consider the curve $u := i_{X,Y} \circ p \circ \widetilde{u}_0$ as well as the curves $w_i := i_{X,Y} \circ p \circ v_i$. The sum of the curves u and $w_1, \ldots, w_q$ represent together the class $(r+1)j(s_X)$. As in the proof of Claim 1 it follows that the union of their images is a curve of the type $(r+1)(\mathbb{C}P^1 \times \mathrm{pt})$. By assumption u is not constant hence u is a curve of the type $m(\mathbb{C}P^1 \times \mathrm{pt})$, and as $\widetilde{u}_0$ is a section the curve u must be reduced (as p is $1-1$ outside of E). So $m = 1$. Similarly each of the curves $w_i := i_{X,Y} \circ p \circ v_i$ must be either constant or with the same image as u, i.e. $\mathbb{C}P^1 \times \mathrm{pt}$ (but w_i might be a multiple cover of u). As $r \geq 1$ at least one of the w_i's is not constant. However this is impossible since each of the curves v_i lies in a fiber of π_ℓ hence the images of the non-constant w_i's cannot coincide with that of u. A contradiction. This proves that the image of $\widetilde{u}_0$ lies inside E.

Next we have to prove that $\widetilde{u}_0$ is one of the fibers of the projective bundle $E \longrightarrow B_\ell$ (hence represents the class F). Indeed assume the contrary, then the projection $p \circ \widetilde{u}_0$ is not constant and, as $\widetilde{u}_0$ is contained inside E, $p \circ \widetilde{u}_0$ must be the base locus B_ℓ or a multiple cover of it (recall that B_ℓ is the center of the blow-up $p : \widetilde{X} \longrightarrow X$). Recall that

$[B_\ell] = s_X + 3f_X$. But by the preceding arguments $p \circ \widetilde{u}_0$ must be a curve in the class s_X or a multiple of it. A contradiction. This proves that $p \circ \widetilde{u}_0$ is constant. It follows that $\widetilde{u}_0$ is either a fiber of $p|_E : E \longrightarrow B_\ell$ or a multiple of it. But $\widetilde{u}_0$ is a injective (because it is a section of π_ℓ), so it is precisely a fiber of $E \longrightarrow B_\ell$.

Next we prove that each of the bubbles v_i is a (-2)–curve or a multiple of it. To see this note that each v_i when viewed as a curve in a fibre of π_ℓ must be in the homology class $a_i s - b_i f$ with $a_i, b_i \in \mathbb{Z}$ and a simple computation shows that

$$\widetilde{i}_*([v_i]) = a_i j(s_X) + (a_i - b_i) j(f_X) - (2a_i - b_i)F.$$

This implies that $a_i \geq b_i$. But we also have $\sum_{i=1}^q (a_i - b_i) = 0$ because of (26), hence $a_i = b_i$ for every i. It follows that each v_i is a curve in the class $a_i(s-f)$. As $\langle c_1^\Sigma, s-f\rangle = 0$ it easily follows by adjunction that $[v_i]$ is an a_i-cover of a (-2)–curve in the class $s - f$.

Finally, we prove that we have only a finite number of cusp sections in the class $\widetilde{A}$. Inside the pencil ℓ we have only a finite number of elements $z \in \ell$ over which the fiber $\Sigma_z = \pi_\ell^{-1}(z)$ is not biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ (namely the ones that are Hirzebruch surfaces F_2). Inside each of these there is a unique (-2)-curve say C_z. Consider the evaluation map

$$ev_{\widetilde{J}_0,z} : \mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0) \longrightarrow \Sigma_z, \quad ev_{\widetilde{J}_0,z}(\widetilde{u}) = \widetilde{u}(z).$$

It is easy to see that the image of $ev_{\widetilde{J}_0,z}$ is just the base locus B_ℓ (or more precisely its image inside the proper transform of Σ_z inside $\widetilde{X}$). As B_ℓ is an irreducible curve in Σ_z with positive self intersection it intersects C_z at finitely many points. Therefore the number of cusp sections in the class $\widetilde{A}$ is finite. This completes the proof of Claim 2.

We are now ready to complete the proof of the lemma. Suppose by contradiction that $\mathcal{S}(\widetilde{A}) \neq 0$. Consider the fiber Σ_0 of π_ℓ, say lying over the point $z_* \in \ell$ (recall that Σ_0 is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$). Consider all possible $\widetilde{J}_0$-holomorphic cusp sections in the class $\widetilde{A}$. By Claim 2 we have only a finite number of them and each of them intersects Σ_0 exactly at one point (the bubbles cannot be inside Σ_0 as they are all (-2)–curves). We thus obtain a finite number of points $p_1, \ldots, p_\nu \in \Sigma_0$. As $\mathcal{S}(\widetilde{A}) \neq 0$ we can find a real 2-dimensional cycle (actually a real smooth closed surface) $Q \subset \Sigma_0$ lying in the complement of $p_1, \ldots, p_\nu$ and such that $\langle \mathcal{S}(\widetilde{A}), [Q]\rangle \neq 0$. This implies that for every regular almost complex structure $\widetilde{J} \in \widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$ we have a $\widetilde{J}$-holomorphic section $\widetilde{u}$ in the class $\widetilde{A}$ which intersects Q. Take a sequence $\widetilde{J}_n \in \widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$ with $\widetilde{J}_n \longrightarrow \widetilde{J}_0$ as $n \longrightarrow \infty$ and a corresponding sequence $\widetilde{u}_n \in \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}_n)$ with $\widetilde{u}_n(z_*) \in Q$. By Gromov compactness the sequence $\widetilde{u}_n$ either has a subsequence that converges to a genuine $\widetilde{J}_0$-holomorphic section in the class $\widetilde{A}$ or there is a subsequence that converges to a $\widetilde{J}_0$-holomorphic cusp section $(\widetilde{u}_0, v_1, \ldots, v_q)$ in the class $\widetilde{A}$, and by our construction we must have $\widetilde{u}_0(z_*) \in Q$. However, both cases are impossible. The first case is ruled out by Claim 1 and the second

case is impossible since Q lies in the complement of $p_1, \ldots, p_\nu$. The proof of the lemma is now complete. □

10. Discussion and questions

Here we briefly discuss further directions of study arising from the results of the paper.

10.1. **Questions on the symplectic topology of manifolds with small dual.** Consider the class of manifolds Σ that appear as hyperplane sections of manifolds X with small dual, viewed as symplectic manifolds. Does the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(\Sigma)$ of such manifolds Σ have sepecial properties (from the Geomeric, or algebraic viewpoints) ? This question seems very much related to the subcriticality of $X \setminus \Sigma$, and results in this direction have been recently obtained by Borman [Bor] who found a relation between quasi-morphisms on $\mathrm{Ham}(\Sigma)$ and quasi-morphisms on $\mathrm{Ham}(X)$.

The structure of the fundamental group $\pi_1(\mathrm{Ham})$ of the group of Hamiltonian diffeomorphisms of a symplectic manifold has been the subject of many studies in symplectic topology. Still, relatively little is known about the structure of these fundamental groups. (e.g. the pool of known examples of symplectic manifolds with non-simply connected Ham is quite limited.) It would be interesting to ask whether manifolds with small dual and their hyperplane sections exhibit special properties in terms of $\pi_1(\mathrm{Ham})$ or more generally in terms of the topology of Ham.

Here are more concrete questions in this direction. Let $X \subset \mathbb{C}P^N$ be a manifold with small dual. Denote $k = \mathrm{def}(X)$ and let $\Sigma \subset X$ be a smooth hyperplane section, endowed with the symplectic structure ω_Σ induced from $\mathbb{C}P^N$. Denote by $\lambda \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma)$ the non-trivial element coming from the fibration in §5 using the recipe of §4.2.1.

- What can be said about the minimal Hofer length of the loops in $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ in the homotopy class λ ? More generally, what can be said in general about the length spectrum of $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ with respect to the Hofer metric ? Preliminary considerations seem to indicate that at least when $b_2(X) = 1$ the positive part of the norm of $\lambda \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma)$ satisfies $\nu_+(\lambda) \leq \frac{1}{\dim_{\mathbb{C}}(\Sigma)+1}$. It would be interesting to verify this, and more importantly to obtain a bound on $\nu(\lambda)$. (See [Pol2, Pol1] for the definition of these norms on $\pi_1(\mathrm{Ham})$ and ways to calculate them.)
- Can the homotopy class λ be represented by a Hamiltonian circle action ? Several examples of manifolds with small dual indicate that this might be true. In case a Hamiltonian circle action does exist, is it true that it can be deformed into a holomorphic circle action (i.e. an action of S^1 by biholomorphisms of Σ) ?
- In which cases is the element λ of finite order ? Whenever this is the case, does the order of λ has any relation to $k = \mathrm{def}(X)$?

- In case the order of λ is infinite, what can be said about the value of the Calabi homomorphism $\widetilde{Cal}$ on λ ? (We view here λ as an element of the universal cover $\widetilde{\mathrm{Ham}}(\Sigma, \omega_\Sigma)$.) See [EP] for the definition of $\widetilde{Cal}$ etc.

Of course, one could ask the same questions also about X itself (rather than Σ). It is currently not known what are the precise conditions insuring that an algebraic manifold X with small dual can be realized as a hyperplane section in an algebraic manifold Y (of one dimension higher).

Another question, lying at the border between symplectic topology and algebraic geometry is the following. The main results of this paper show that an algebraic manifold $X \subset \mathbb{C}P^N$ with small dual and $b_2(X) = 1$ gives rise to a distinguished *non-trivial* element $\lambda \in \pi_1(\mathrm{Ham}(\Sigma))$ where Σ is a hyperplane section of X. On the other hand every homotopy class of loops $\gamma \in \pi_1(\mathrm{Ham}(\Sigma))$ gives rise to a Hamiltonian fibration $\pi_\gamma : \widetilde{M}_\gamma \longrightarrow S^2$ with fiber Σ. Consider now (positive as well as negative) iterates $\gamma = \lambda^r$, $r \in \mathbb{Z}$, of λ and the Hamiltonian fibrations corresponding to them $\pi_{\lambda^r} : \widetilde{M}_{\lambda^r} \longrightarrow S^2$. Do these fibrations correspond to an embedding of Σ as a hyperplane section in some algebraic manifold with positive defect ? Or more generally, do the fibrations π_{λ^r} correspond to some geometric framework involving the algebraic geometry of Σ and its projective embeddings ? It seems tempting to suspect that λ^2 for example corresponds to a fibration similar to $\pi_\ell : \widetilde{X} \longrightarrow \ell \approx S^2$ (see §5) but instead of taking ℓ to be a line in the complement of X^* one takes ℓ to be a degree 2 curve in the complement of X^*.

Finally, here is another general question motivated by analogies to algebraic geometry. Can the concept of manifolds with small dual be generalized to symplectic manifolds ? Can one define a meaningful concept of defect ? The motivation comes from the following framework. Let (X, ω) be a closed integral symplectic manifold (integral means that $[\omega]$ admits a lift to $H^2(X; \mathbb{Z})$). By a theorem of Donaldson [Don1] X admits symplectic hyperplane sections, i.e. for $k \gg 0$ there exists a symplectic submanifold Σ representing the Poincaré dual to $k[\omega]$. (Moreover, the symplectic generalization of the notion of Lefschetz pencil, exists too [Don2].) Suppose now that for some such Σ the manifold $X \setminus \Sigma$ is subcritical. Does this imply on Σ and X results similar to what we have obtained in this paper ? (e.g. is $[\omega|_\Sigma]$ invertible in $QH(\Sigma)$?) One of the difficulty in this type of questions is that the concept of dual variety (of a projective embedding, or of a linear system) does not exist in the realm of symplectic manifolds and their Donaldson hyperplane sections. Note that we are not aware of examples of pairs (X, Σ) with $X \setminus \Sigma$ subcritical that are not equivalent (e.g. symplectomorphic) to algebraic pairs (X', Σ').

10.2. **Questions about the algebraic geometry of manifolds with small dual.** We have seen that for hyperplane sections Σ of manifolds with small dual $X \subset \mathbb{C}P^N$,

$[\omega_\Sigma] \in QH^2(\Sigma;\Lambda)$ is invertible. Is the same true for X, i.e. is $[\omega_X] \in QH^2(X;\Lambda)$ an invertible element ? The 2-periodicity of the Betti number of X in Corollary C indicates that this might be the case. Note that our proof of the 2-periodicity for X was based on the 2-periodicity for Σ (which in turn comes from the invertibility of $[\omega_\Sigma]$), together with some Lefschetz-type theorems, and did not involve any quantum cohomology considerations for X.

Another circle of questions has to do with Theorem 9.A. It would be interesting to figure out more explicitly the terms with $d \leq 0$ in formula (20). This might be possible to some extent of explicitness using Mori theory in the special case of manifolds with small dual (see e.g. [BFS, Tev2, BS] and the references therein). In the same spirit it would be interesting to see if there are any topological restrictions on Σ and X coming from the invertibility of $S(\pi_\ell)$ in the non-monotone case. We remark that when (Σ, ω_Σ) is not spherically monotone one should work with a more complicated Novikov ring $\mathcal{A}$ as explained in §9.

Another interesting question has to do with the structure of the quantum cohomology $QH^*(\Sigma;\Lambda)$ of hyperplane sections Σ of manifolds with small dual X. As a corollary of Theorem B we have obtained that in the monotone case $QH^*(\Sigma;\Lambda)$ satisfies the relation $[\omega_\Sigma] * \alpha = q$ for some $\alpha \in H^{n+k-2}(\Sigma)$. In some examples this turns out to be the only relation. Thus it is tempting to ask when do we have a ring isomorphism

$$QH^*(\Sigma;\Lambda) \cong \frac{(H^\bullet(\Sigma) \otimes \Lambda)^*}{\langle \omega * \alpha = q \rangle}.$$

In a similar context, it is interesting to note that the algebraic structure of quantum cohomology of uniruled manifolds has been studied in a recent paper of McDuff [McD]. In particular, in [McD] McDuff proves a general existence result for non-trivial invertible elements of the quantum cohomology of uniruled manifolds using purely algebraic methods. One can view part of the results in this paper as a direct computation in the case of manifolds with positive defect.

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