A Gel'fand-type spectral radius formula and stability of linear constrained switching systems☆

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Abstract

Using ergodic theory, in this paper we present a Gel'fand-type spectral radius formula which states that the joint spectral radius is equal to the generalized spectral radius for a matrix multiplicative semigroup $\boldsymbol{S}^+$ restricted to a subset that need not carry the algebraic structure of $\boldsymbol{S}^+$. This generalizes the Berger-Wang formula. Using it as a tool, we study the absolute exponential stability of a linear switched system driven by a compact subshift of the one-sided Markov shift associated to $\boldsymbol{S}$.

Keywords: Joint/generalized spectral radius, Gel'fand-type spectral-radius formula, linear switched system, asymptotic stability

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1. Introduction

In this paper, we study the Gel'fand-type spectral-radius formula and stability of a matrix multiplicative semigroup $\boldsymbol{S}^+$ restricted to a subset that does not need to carry the algebraic structure of the semigroup $\boldsymbol{S}^+$, using ergodic-theoretic and dynamical systems approaches.

1.1. *The Gel'fand-type formulae*

Let $d \geq 1$ be an integer and $\mathcal{I}$ a metrizable topological space. We consider a continuous matrix-valued function $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$; $i \mapsto S_i$. Let us denote by $\Sigma_{\mathcal{I}}^+$ the set of all the one-sided infinite switching signals $i(\cdot)\colon \mathbb{N} \to \mathcal{I}$ endowed with the standard infinite-product topology, where $\mathbb{N} = \{1, 2, \ldots\}$. For simplicity, we write $i(n) = i_n$ for all $n \in \mathbb{N}$. Then in the state space $\mathbb{C}^d$, we define the linear, discrete-time, switched dynamical system $\boldsymbol{S}_{i(\cdot)}$:

$$x_n = S_{i_n} \cdots S_{i_1} x_0 \qquad (x_0 \in \mathbb{C}^d,\ n \geq 1),$$

for any switching signal $i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$. For any word $w = (i_1, \ldots, i_n) \in \mathcal{I}^n = \overbrace{\mathcal{I} \times \cdots \times \mathcal{I}}^{n\text{-time}}$ of length $n \geq 1$, simply write $\boldsymbol{S}_w = S_{i_n} \cdots S_{i_1}$ and let $\|\boldsymbol{S}_w\|$ denote the operator norm of the linear

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transformation $x \mapsto \boldsymbol{S}_w x$ induced by any preassigned vector norm $\|\cdot\|$ on $\mathbb{C}^d$; that is to say, $\|\boldsymbol{S}_w\| = \sup_{x\in\mathbb{C}^d,\|x\|=1}\|\boldsymbol{S}_w x\|$.

The *joint spectral radius* of $\boldsymbol{S}$ (free of constraints) is introduced by G.-C. Rota and G. Strang in [37] as follows:

$$\hat{\rho}(\boldsymbol{S}) = \limsup_{n\to+\infty}\left\{\sup_{w\in\mathcal{I}^n}\sqrt[n]{\|\boldsymbol{S}_w\|}\right\} \quad \left(= \lim_{n\to+\infty}\left\{\sup_{w\in\mathcal{I}^n}\sqrt[n]{\|\boldsymbol{S}_w\|}\right\}\right).$$

Since

$$\log\left(\sup_{w\in\mathcal{I}^{\ell+m}}\|\boldsymbol{S}_w\|\right) \le \log\left(\sup_{w\in\mathcal{I}^m}\|\boldsymbol{S}_w\|\right) + \log\left(\sup_{w\in\mathcal{I}^\ell}\|\boldsymbol{S}_w\|\right)$$

for all $\ell, m \ge 1$, i.e., the subadditivity holds, the above limit always exists. On the other hand, the *generalized spectral radius* of $\boldsymbol{S}$ (free of constraints) is defined by I. Daubechies and J.C. Lagarias in [13] as

$$\rho(\boldsymbol{S}) = \limsup_{n\to+\infty}\left\{\sup_{w\in\mathcal{I}^n}\sqrt[n]{\rho(\boldsymbol{S}_w)}\right\},$$

where $\rho(A)$ denotes the usual spectral radius of the matrix $A \in \mathbb{C}^{d\times d}$.

Then, the so-called generalized Gel'fand spectral-radius formula, due to M.A. Berger and Y. Wang [2] and conjectured by I. Daubechies and J.C. Lagarias [13], can be stated as follows:

The Berger-Wang Formula 1.1 (See [2])**.** *If $\boldsymbol{S} = \{S_i\}_{i\in\mathcal{I}}$ is a bounded subset of $\mathbb{C}^{d\times d}$, then there holds the equality $\rho(\boldsymbol{S}) = \hat{\rho}(\boldsymbol{S})$.*

This formula was proved by using different approaches, for example, in [2, 15, 39, 8, 4, 9]. Recently, this formula has been generalized to sets of precompact linear operators constraint-free acting on a Banach space by Ian D. Morris in [33] using ergodic theory.

The above Gel'fand-type spectral-radius formula is an important tool in a number of research areas, such as in the theory of control and stability of unforced systems, see [1, 25, 20, 12] for example; in coding theory, see [32]; in wavelet regularity, see [13, 14, 22, 31]; and in the study of numerical solutions to ordinary differential equations, see, e.g., [19].

However, in many real-world situations, constraints on allowable switching signals often arise naturally as a result of physical requirements on a system. One often needs to consider some switching constraints imposed by some kind of uncertainty about the model or about environment in which the object operates, see [41, 27, 28, 29, 6] and so on. Consider in the control theory, for example, a proper subset Λ of $\Sigma_{\mathcal{I}}^+$ as the set of admissible switching signals, such as

$$\Lambda = \Sigma_{\mathbb{A}}^+ := \left\{ i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+ \,|\, a_{i_n i_{n+1}} = 1 \ \forall n \ge 1 \right\}$$

where $\mathcal{I} = \{1, \dots, \kappa\}$ consists of finitely many letters and where $\mathbb{A} = (a_{\ell m})$ is a $\kappa \times \kappa$ matrix of zeros and ones induced by a Markov transition matrix or a directed graph. A more general way to define Λ is via a language, as shown, for example in [42, 23, 29].

So, it is natural and necessary to introduce the definition of Gel'fand-type spectral radius under some switching constraints.

Hereafter, if Λ is a nonempty subset of $\Sigma_{\mathcal{I}}^+$, then $\boldsymbol{S}_{\upharpoonright\Lambda}$ is identified with the family of systems $\boldsymbol{S}_{i(\cdot)}$ over all switching signals $i(\cdot) \in \Lambda$, and called the switched system with constraint Λ.

Definitions 1.2. Let $\varLambda$ be a nonempty subset of $\varSigma_{\mathcal{I}}^{+}$ as the set of admissible switching signals. Define the *joint spectral radius of* $\boldsymbol{S}_{\restriction\varLambda}$ as

$$\hat{\rho}(\boldsymbol{S}_{\restriction\varLambda}) = \limsup_{n\to+\infty}\left\{\sup_{i(\cdot)\in\varLambda}\sqrt[n]{\|S_{i_n}\cdots S_{i_1}\|}\right\}.$$

The *generalized spectral radius of* $\boldsymbol{S}_{\restriction\varLambda}$ is defined as

$$\rho(\boldsymbol{S}_{\restriction\varLambda}) = \limsup_{n\to+\infty}\left\{\sup_{i(\cdot)\in\varLambda}\sqrt[n]{\rho(S_{i_n}\cdots S_{i_1})}\right\}.$$

We notice that if $\varLambda$ is *invariant* by the natural one-sided Markov shift $\theta_{+}\colon\, i(\cdot)\mapsto i(\cdot+1)$; that is, $i(\cdot+1)$ belongs to $\varLambda$ for any $i(\cdot)\in\varLambda$, then from the subadditivity, there follows that $\hat{\rho}(\boldsymbol{S}_{\restriction\varLambda})$ is well defined in the sense that

$$\hat{\rho}(\boldsymbol{S}_{\restriction\varLambda}) = \lim_{n\to+\infty}\left\{\sup_{i(\cdot)\in\varLambda}\sqrt[n]{\|S_{i_n}\cdots S_{i_1}\|}\right\}.$$

It is easily seen that there holds the inequality $\rho(\boldsymbol{S}_{\restriction\varLambda}) \le \hat{\rho}(\boldsymbol{S}_{\restriction\varLambda})$. Clearly, $\hat{\rho}(\boldsymbol{S}_{\restriction\varLambda}) = \hat{\rho}(\boldsymbol{S})$ and $\rho(\boldsymbol{S}_{\restriction\varLambda}) = \rho(\boldsymbol{S})$ for the special free-constraint case $\varLambda = \varSigma_{\mathcal{I}}^{+}$, if $\boldsymbol{S}$ is bounded in $\mathbb{C}^{d\times d}$.

Based on the recent work of Ian D. Morris [33] (see Theorem 2.6 below), in this paper, we present the following Gel'fand-type spectral-radius formula under switching constraints:

Theorem A (Spectral-radius formula with constraints)**.** *Let* $\boldsymbol{S}\colon\mathcal{I}\to\mathbb{C}^{d\times d};\ i\mapsto S_i$ *be continuous in* $i\in\mathcal{I}$ *where* $\mathcal{I}$ *is a metric space, and assume* $\varLambda\subset\varSigma_{\mathcal{I}}^{+}$ *is an invariant compact set of the one-sided Markov shift*

$$\theta_{+}\colon\varSigma_{\mathcal{I}}^{+}\to\varSigma_{\mathcal{I}}^{+};\quad i(\cdot)=(i_n)_{n=1}^{+\infty}\mapsto i(\cdot+1)=(i_{n+1})_{n=1}^{+\infty}.$$

Then there holds the equality $\rho(\boldsymbol{S}_{\restriction\varLambda}) = \hat{\rho}(\boldsymbol{S}_{\restriction\varLambda})$*.*

Let $\boldsymbol{S}_{\restriction\varLambda}^{+}$ be the set of all product matrices $S_{i_n}\cdots S_{i_1}$ where $n\ge 1$ and $i(\cdot)=(i_n)_{n=1}^{+\infty}\in\varLambda$. A technical problem is, for the constrained case $\varLambda\subsetneq\varSigma_{\mathcal{I}}^{+}$, that $\boldsymbol{S}_{\restriction\varLambda}^{+}$ does not need to carry the algebraic structure of a semigroup; otherwise, [4, Theorem B] works and implies Theorem A in our context. The compactness and θ_{+}-invariance of $\varLambda$ both are needed for our discussion of using ergodic theory.

We note that [41, Theorem 7.3] contains a "Gel'fand-type formula" with constraints which is for continuous time and in a special case, using Lyapunov function. Our theorem will be proved in Section 2 based on a recent theorem of Ian D. Morris in [33].

Theorem A is a generalization of the Berger-Wang formula. In fact, from it we could obtain concisely the Berger-Wang formula as follows.

Proof of the Berger-Wang formula*.* Let $\{S_i\,|\,i\in\mathcal{I}\}\subset\mathbb{C}^{d\times d}$ be an arbitrary bounded set. Write $\mathbb{I}=\mathrm{Cl}_{\mathbb{C}^{d\times d}}(\{S_i\,|\,i\in\mathcal{I}\})$, the closure of the set $\{S_i\colon i\in\mathcal{I}\}$ in $\mathbb{C}^{d\times d}$. Then, $\mathbb{I}$ is compact in $\mathbb{C}^{d\times d}$, and the function $\mathbb{S}\colon\mathbb{I}\to\mathbb{C}^{d\times d}$, defined by $\mathbf{i}\mapsto\mathbb{S}_{\mathbf{i}}$ where $\mathbb{S}_{\mathbf{i}}=\mathbf{i}\ \forall\mathbf{i}\in\mathbb{I}$, is continuous in $\mathbf{i}\in\mathbb{I}$. Since there holds that

$$\sup_{\mathbf{w}\in\mathbb{I}^n}\sqrt[n]{\|\mathbb{S}_{\mathbf{w}}\|}=\sup_{w\in\mathcal{I}^n}\sqrt[n]{\|\boldsymbol{S}_w\|}\quad\text{and}\quad\sup_{\mathbf{w}\in\mathbb{I}^n}\sqrt[n]{\rho(\mathbb{S}_{\mathbf{w}})}=\sup_{w\in\mathcal{I}^n}\sqrt[n]{\rho(\boldsymbol{S}_w)}$$

for all $n \geq 1$ from the fact $\mathbb{I}^n = \mathrm{Cl}_{\mathbb{C}^{d\times d}}(\{\boldsymbol{S}_w \,|\, w \in \mathcal{I}^n\})$, we can obtain that $\hat{\rho}(\boldsymbol{S}) = \hat{\rho}(\mathbb{S})$ and $\rho(\boldsymbol{S}) = \rho(\mathbb{S})$. So, applying Theorem A in the case $\Lambda = \Sigma_{\mathbb{I}}^+$, we have got that $\hat{\rho}(\mathbb{S}) = \rho(\mathbb{S})$. This completes the proof of the Berger-Wang formula (Theorem 1.1). $\square$

In addition, we define the Lyapunov exponent associated to an initial state $x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ and a switching signal $i(\cdot) = (i_n)_{n=1}^{+\infty}$ by

$$\chi(x_0, \boldsymbol{S}_{i(\cdot)}) = \limsup_{n\to+\infty} \frac{1}{n} \log \|S_{i_n} \cdots S_{i_1} x_0\|.$$

It is easily seen that $\hat{\rho}(\boldsymbol{S}) \geq \exp \chi(x_0, \boldsymbol{S}_{i(\cdot)})$ for all $i(\cdot) \in \Lambda$ and all $x_0 \in \mathbb{C}^d$. However, we will prove that $\hat{\rho}(\boldsymbol{S})$ might be achieved by some optimal pair $(x_0, i(\cdot)) \in \mathbb{C}^d \times \Lambda$; see Corollary 2.7 below, which generalizes a corresponding result in [1] in the free-constraints case.

Recall for any given $i(\cdot) \in \Sigma_{\mathcal{I}}^+$ that $\boldsymbol{S}$ is said to be $i(\cdot)$-exponentially stable, provided that there exists $\mathbf{c} \geq 1$ and $\chi < 0$ such that

$$\|S_{i_n} \cdots S_{i_1} x_0\| \leq \mathbf{c}\|x_0\| \exp(n\chi) \quad \forall x_0 \in \mathbb{C}^d \text{and } n \geq 1.$$

This is equivalent to

$$\chi(\boldsymbol{S}_{i(\cdot)}) := \limsup_{n\to+\infty} \frac{1}{n} \log \|S_{i_n} \cdots S_{i_1}\| < 0.$$

Moreover, this is also equivalent to $\chi(x_0, \boldsymbol{S}_{i(\cdot)}) < 0$ for all $x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$. Further, $\boldsymbol{S}$ is called to be *uniformly* $i(\cdot)$-exponentially stable, provided that there exists $C \geq 1$ and $\chi < 0$ such that

$$\|S_{i_{m+\ell}} \cdots S_{i_\ell} \cdots S_{i_1} x_0\| \leq C\|S_{i_\ell} \cdots S_{i_1} x_0\| \exp(m\chi) \quad \forall x_0 \in \mathbb{C}^d \text{and } m \geq 1,$$

uniformly for $\ell \geq 0$. This is equivalent to that $\boldsymbol{S}$ is exponentially stable over the closure of the orbit $\{i(\cdot + m) \colon m = 0, 1, 2, \ldots\}$ in $\Sigma_{\mathcal{I}}^+$.

From [12] together with K.G. Hare *et al*. [21], one can construct an explicit counterexample to show that the $i(\cdot)$-exponential stability is essentially weaker than the uniform $i(\cdot)$-exponential stability of $\boldsymbol{S}$.

1.2. Stability criteria under switching-path constraints

As pointed out in D. Liberzon and A.S. Morse [30], there are three benchmark problems for switched systems: stabilization under arbitrary switching signals, stabilization under a switching path constraint, and construction of stabilizing switching signals. To the second problem, as another result of our spectral-radius formula, in the second part of this paper, we give the following criteria of the absolutely asymptotic stability for a linear system obeying switching constraints, which will be proved in Section 3.

Theorem B. *Let $\boldsymbol{S} \colon \mathcal{I} \to \mathbb{C}^{d\times d}$ be continuous and bounded with $\rho(\boldsymbol{S}) = 1$ and assume $\Lambda \subset \Sigma_{\mathcal{I}}^+$ is an invariant compact set of the one-sided Markov shift $\theta_+ \colon \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$. Then, the following conditions are mutually equivalent:*

(a) *$\boldsymbol{S}$ is "Λ-absolutely asymptotically stable", i.e.,*

$$S_{i_n} \cdots S_{i_1} \to \mathbf{0}_{d\times d} \ \textit{as } n \to +\infty \quad \forall i(\cdot) \in \Lambda,$$

where $\mathbf{0}_{d\times d}$ is the origin of $\mathbb{C}^{d\times d}$.

(b) *The generalized spectral radius* $\rho(\boldsymbol{S}_{|\Lambda}) < 1$.

(c) *There exists a constant* $0 < \gamma < 1$ *and an integer* $N \geq 1$ *such that*

$$\rho(S_{i_n} \cdots S_{i_1}) \leq \gamma \quad \forall n \geq N \textit{ and } i(\cdot) \in \Lambda.$$

The claim (a) $\Leftrightarrow$ (b) still holds without the assumption $\rho(\boldsymbol{S}) = 1$, by using the Fenichel uniformity theorem (Lemma 3.3 below) and Theorem A; see Lemmas 3.2 and 3.3 below. Here the compactness of Λ is important for the proof of Theorem B presented in this paper. Let us see a simple counterexample as follows:

Example 1.3. Let $\mathcal{I} = \{0, 1\}$, $\Lambda = \Sigma_{\mathcal{I}}^{+} \setminus \{(0, 0, 0, \ldots), (1, 1, 1, \ldots)\}$ and let $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{2\times 2}$ be defined by

$$0 \mapsto S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1 \mapsto S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easily seen that $\rho(\boldsymbol{S}) = 1$ and $\boldsymbol{S}$ is Λ-absolutely asymptotically stable. However, $\rho(\boldsymbol{S}_{|\Lambda}) = 1$. Moreover, for any $N \geq 1$, one can find some $i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Lambda$ such that $\rho(S_{i_N} \cdots S_{i_1}) = 1$. Note here that Λ is θ_+-invariant, but it is an open and noncompact subset of $\Sigma_{\mathcal{I}}^{+}$.

Remark 1.4. To any $\varepsilon > 0$, there always exists a norm $\|\cdot\|_{\varepsilon}$ on $\mathbb{C}^d$ such that

$$\|S_i\|_{\varepsilon} \leq \hat{\rho}(\boldsymbol{S}) + \varepsilon \quad \forall i \in \mathcal{I},$$

for example in [37], also see [15, 35, 39] for much shorter proofs. This implies that

$$\hat{\rho}(\boldsymbol{S}) = \inf_{\|\cdot\|\in\mathcal{N}} \left\{ \sup_{i\in\mathcal{I}} \|S_i\| \right\},$$

where $\mathcal{N}$ denotes the set of all possible vector norms on $\mathbb{C}^d$.

So, whenever $\hat{\rho}(\boldsymbol{S}) < 1$ one always can pick a pre-extremal norm $\|\cdot\|$ on $\mathbb{C}^d$ so that there exists a constant $\hat{\gamma}$ with

$$(\star) \qquad \|S_i\| \leq \hat{\gamma} < 1 \quad \forall i \in \mathcal{I}.$$

Thus, $\|S_{i_n} \cdots S_{i_1}\| \to 0$ as $n \to \infty$ uniformly for $i(\cdot) \in \Sigma_{\mathcal{I}}^{+}$ whenever $\hat{\rho}(\boldsymbol{S}) < 1$. However, this inequality $(\star)$ is not, in general, the case for the constrained case $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$ when $\Lambda \neq \Sigma_{\mathcal{I}}^{+}$ because of the lack of the semigroup structure of $\boldsymbol{S}_{|\Lambda}^{+}$ as mentioned before. In fact, the Λ-stability of $\boldsymbol{S}$ cannot imply the stability of every subsystems. This point causes an essential difference between the case free of any switching constraints and one obeying switching constraints.

Remark 1.5. For the case free of constraints, there holds the following identity:

$$(*) \qquad \rho(\boldsymbol{S}) = \sup_{n\geq 1} \left\{ \sup_{w\in\mathcal{I}^n} \sqrt[n]{\rho(\boldsymbol{S}_w)} \right\},$$

which is very important; this is because it simply implies the continuity of $\rho(\boldsymbol{S})$ with respect to $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$ under the C^0-topology [22]. For example, see [13, Lemma 3.1] and [4, Remark in Section 1]. Moreover, this is used in [26, 3, 39]. Here we present an other proof for this. Since for any $\varepsilon > 0$ one can pick out a norm $\|\cdot\|_\varepsilon$ on $\mathbb{C}^d$ such that $\|S_i\|_\varepsilon \leq \hat{\rho}(\boldsymbol{S}) + \varepsilon$ for all $i \in \mathcal{I}$, as mentioned in Remark 1.4. So, from the Berger-Wang formula, it follows that

$$\sqrt[n]{\rho(\boldsymbol{S}_w)} \leq \sqrt[n]{\hat{\rho}(\boldsymbol{S}_w)} \leq \rho(\boldsymbol{S}) + \varepsilon \qquad \forall w \in \mathcal{I}^n \text{ and } n \geq 1.$$

Thus, $\sup_{w\in\mathcal{I}^n} \sqrt[n]{\rho(\boldsymbol{S}_w)} \leq \rho(\boldsymbol{S})$ for any $n \geq 1$ and so $\sup_{n\geq 1} \left\{\sup_{w\in\mathcal{I}^n} \sqrt[n]{\rho(\boldsymbol{S}_w)}\right\} = \rho(\boldsymbol{S})$.

In our situation, however, the above $(*)$ does not need to hold restricted to Λ because of the lack of condition $(\star)$. We consider an explicit constrained system. Let $\boldsymbol{S}$ be defined as in Example 1.3 and let

$$\Lambda = \{i'(\cdot) = (0, 1, 0, 1, 0, 1, \ldots), \quad i''(\cdot) = (1, 0, 1, 0, 1, 0, \ldots)\}.$$

Since $\theta_+(i'(\cdot)) = i''(\cdot)$ and $\theta_+(i''(\cdot)) = i'(\cdot)$, Λ is a θ_+-invariant compact subset of $\Sigma_{\mathcal{I}}^+$. Clearly,

$$\rho(\boldsymbol{S}_{|\Lambda}) = 0 \lneqq \sup_{n\geq 1} \left\{\max_{i(\cdot)\in\Lambda} \sqrt[n]{\rho(S_{i_n} \cdots S_{i_1})}\right\} = 1.$$

This shows that the dynamics behavior of a constrained system is sometimes very different from that of a system free of any constraints.

Similar to the proof of the Berger-Wang formula presented before, it follows easily from Theorem B that if $\rho(\boldsymbol{S}) < 1$ then $\boldsymbol{S}$, free of constraints, is absolutely exponentially stable. So, this theorem extends Brayton-Tong [5, Theorem 4.1], Barabanov [1], Daubechies-Lagarias [13, Theorem 4.1], Gurvits [20, Theorem 2.3] and Shih-Wu-Pang [39, Theorem 1] for a discrete-time linear switched system that is free of any switching constraints to one which obeys some switching constraints.

Finally, the paper ends with some questions related closely to Theorems A and B for us to further study in Section 4.

2. The Gel'fand-type spectral-radius formula obeying constraints

In this section, we will devote our attention to proving Theorem A which asserts a Gel'fand-type spectral-radius formula of a set of matrices obeying some switching constraints, using ergodic-theoretic approaches.

2.1. Some ergodic-theoretic results

Let $T\colon \Omega \to \Omega$ be a continuous transformation of a compact topological space Ω. Let $\mathscr{B}_\Omega$ be the Borel σ-field of the space Ω, which is generated by all open sets of the topology space Ω.

Definition 2.1 (See [34]). A probability measure μ on the Borel measurable space $(\Omega, \mathscr{B}_\Omega)$ is said to be *T-invariant*, write as $\mu \in \mathcal{M}_{inv}(\Omega, T)$, if $\mu = \mu \circ T^{-1}$, i.e. $\mu(B) = \mu(T^{-1}(B))$ for all $B \in \mathscr{B}_\Omega$. A T-invariant probability measure μ is called *T-ergodic*, write as $\mu \in \mathcal{M}_{erg}(\Omega, T)$, provided that for $B \in \mathscr{B}_\Omega$, $\mu\left((B \setminus T^{-1}(B)) \cup (T^{-1}(B) \setminus B)\right) = 0$ implies $\mu(B) = 1$ or 0.

To prove Theorem A, we need several ergodic-theoretic lemmas. The first is the standard Kingman subadditive ergodic theorem.

Theorem 2.2 (See [24])**.** *Let $\langle f_n\rangle_{n=1}^{+\infty}\colon \Omega \to \mathbb{R}\cup\{-\infty\}$ be a sequence of upper-bounded Borel measurable functions such that $f_{m+n}(\omega)\le f_n(T^m(\omega))+f_m(\omega)$ for every $\omega\in\Omega$ and any $m,n\ge 1$. Then, for any $\mu\in\mathcal{M}_{erg}(\Omega,T)$, it holds that*

$$\lim_{n\to+\infty}\frac{1}{n}\int_\Omega f_n(\omega)\,d\mu(\omega)=\inf_{n\ge1}\frac{1}{n}\int_\Omega f_n(\omega)\,d\mu(\omega)=\lim_{n\to+\infty}\frac{1}{n}f_n(\omega)$$

for μ-a.s. $\omega\in\Omega$.

As usual, one can introduce a natural topology for $\mathbb{R}\cup\{-\infty\}$ under which $[0,+\infty)$ is homeomorphic to $\mathbb{R}\cup\{-\infty\}$ by a strictly increasing continuous function from $\mathbb{R}\cup\{-\infty\}$ onto $[0,+\infty)$ with $-\infty\mapsto 0$. The second lemma needed is the semi-uniform subadditive ergodic theorem, independently due to S. J. Schreiber [38] and R. Sturman and J. Stark [40], which could be stated as follows:

Theorem 2.3 (See [38, 40])**.** *Let $\langle f_n\rangle_{n=1}^{+\infty}\colon \Omega\to\mathbb{R}\cup\{-\infty\}$ be a sequence of continuous functions such that $f_{\ell+m}(\omega)\le f_\ell(T^m(\omega))+f_m(\omega)$ for every $\omega\in\Omega$ and any $\ell,m\ge1$. If there is a constant $\boldsymbol{\alpha}$ such that*

$$\lim_{n\to+\infty}\frac{1}{n}\int_\Omega f_n(\omega)\,d\mu(\omega)<\boldsymbol{\alpha}\quad \forall\mu\in\mathcal{M}_{erg}(\Omega,T),$$

then there exists an $N\ge1$ such that for any $\ell\ge N$, $\sup_{\omega\in\Omega}\frac{1}{\ell}f_\ell(\omega)<\boldsymbol{\alpha}$.

See [10] for an elementary and short proof of the above semi-uniformity theorem. Next, we put

$$\chi(\langle f_n\rangle_1^\infty)=\lim_{n\to+\infty}\left\{\sup_{\omega\in\Omega}\frac{1}{n}f_n(\omega)\right\}\quad\text{and}\quad \chi(\mu,\langle f_n\rangle_1^\infty)=\inf_{\ell\ge1}\frac{1}{\ell}\int_\Omega f_\ell(\omega)\,d\mu(\omega).$$

Clearly, $\chi(\langle f_n\rangle_1^\infty)\le\max_{\omega\in\Omega}f_1(\omega)<+\infty$ by the subadditivity and the continuity of $f_n(\omega)$ in $\omega\in\Omega$.

As a result of Theorem 2.3, we can simply obtain the following version of Theorem 2.3.

Lemma 2.4. *Let $\langle f_n\rangle_1^{+\infty}\colon\Omega\to\mathbb{R}\cup\{-\infty\}$ be be a T-subadditive sequence of continuous functions. Then*

$$\chi(\langle f_n\rangle_1^\infty)=\max_{\mu\in\mathcal{M}_{erg}(\Omega,T)}\chi(\mu,\langle f_n\rangle_1^\infty).$$

Proof. Let $\boldsymbol{\alpha}=\chi(\langle f_n\rangle_1^\infty)$. It is easy to see $\boldsymbol{\alpha}\ge\chi(\mu,\langle f_n\rangle_1^\infty)$ from Theorem 2.2. To prove the statement, suppose, by contradiction, that $\chi(\mu,\langle f_n\rangle_1^\infty)<\boldsymbol{\alpha}$ for all $\mu\in\mathcal{M}_{erg}(\Omega,T)$. Then from Theorem 2.3, it follows that there exists an $N\ge1$ such that $\sup_{\omega\in\Omega}\frac{1}{N}f_N(\omega)<\boldsymbol{\alpha}$. Since Ω is compact and f_N is continuous, one can find some constant $\alpha'<\boldsymbol{\alpha}$ such that $\frac{1}{N}f_N(\omega)\le\alpha'$ for all $\omega\in\Omega$. Combining this with the subadditivity of $\langle f_n\rangle_1^{+\infty}$ implies that $\chi(\langle f_n\rangle_1^\infty)\le\alpha'$, a contradiction. This proves Lemma 2.4. □

We notice here that the compactness of Ω is important for the statements of Theorem 2.3 and Lemma 2.4, but not necessary for Theorem 2.2.

We call the numbers $\chi(\langle f_n\rangle_1^\infty)$ and $\chi(\mu,\langle f_n\rangle_1^\infty)$, defined above, the *joint growth rate* and *growth rate at* μ, of the subadditive sequence $\langle f_n\rangle_1^\infty$, respectively. In addition, put

$$\chi(\omega,\langle f_n\rangle_1^\infty)=\limsup_{n\to+\infty}\frac{1}{n}f_n(\omega).$$

Then from Theorem 2.2, it follows that

$$\chi(\omega, \langle f_n\rangle_1^\infty) = \chi(\mu, \langle f_n\rangle_1^\infty) \qquad \mu\text{-a.s. } \omega \in \Omega.$$

So, for any T-subadditive sequence $\langle f_n\rangle_1^\infty$ as in Theorem 2.3, by Lemma 2.4 we have

$$\chi(\langle f_n\rangle_1^\infty) = \max_{\omega\in\Omega} \chi(\omega, \langle f_n\rangle_1^\infty).$$

Thus, we can obtain the following optimization result for the subadditive function sequence $\langle f_n(\omega)\rangle_1^\infty$ given as in Theorem 2.3.

Lemma 2.5. *Let $\langle f_n\rangle_1^\infty$ be arbitrary given as in Theorem 2.3. Then there can be found some $\boldsymbol{\mu}_* \in \mathcal{M}_{erg}(\Omega, T)$ such that $\chi(\langle f_n\rangle_1^\infty) = \chi(\boldsymbol{\mu}_*, \langle f_n\rangle_1^\infty)$. This also implies that $\chi(\langle f_n\rangle_1^\infty) = \chi(\omega, \langle f_n\rangle_1^\infty)$ for $\boldsymbol{\mu}_*$-a.s. $\omega \in \Omega$.*

This result is an extension of [11, Theorem 3.1] from finite set $\boldsymbol{S}$ to infinite case. For the case that $\langle f_n\rangle_1^\infty \colon \Omega \to \mathbb{R}$, the statement of Lemma 2.5 can be read in Y.-L. Cao [7].

On the growth of the spectral radius, the following result is due to Ian D. Morris, which has been proved based on the multiplicative ergodic theorem (cf. [18, 36, 17]) using invariant cone.

Theorem 2.6 (See [33]). *Let $T\colon (\Omega, \mathscr{B}_\Omega, \mu) \to (\Omega, \mathscr{B}_\Omega, \mu)$ be a measure-preserving continuous transformation of a metrizable topological space Ω, and $\mathcal{L}\colon \Omega\times\mathbb{Z}_+ \to \mathbb{C}^{d\times d}$ a Borel measurable linear cocycle driven by T, i.e.,*

$$\mathcal{L}(\omega, 0) = \mathrm{Id}_{\mathbb{C}^d}, \quad \mathcal{L}(\omega, \ell + m) = \mathcal{L}(T^m(\omega), \ell)\mathcal{L}(\omega, m) \quad \forall \omega \in \Omega \text{ and } \ell, m \geq 1.$$

If $\int_\Omega \log^+ \|\mathcal{L}(\omega, 1)\| d\mu(\omega) < \infty$ where $\log 0 = -\infty$ and $\log^+ x = \max\{0, \log x\}$ for any $x \geq 0$, then one can find a T-invariant Borel subset Υ_μ of Ω with $\mu(\Upsilon_\mu) = 1$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \rho(\mathcal{L}(\omega, n)) = \lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{L}(\omega, n)\|$$

for all $\omega \in \Upsilon_\mu$.

Particularly, let $\Omega = \Sigma_{\mathcal{I}}^+, T = \theta_+$ and $\mathcal{L}(\omega, n) = S_{i_n} \cdots S_{i_1}$ for $\omega = i(\cdot)$. Then, this theorem tells us that there holds:

$$\limsup_{n\to+\infty} \sqrt[n]{\rho(S_{i_n} \cdots S_{i_1})} = \lim_{n\to+\infty} \sqrt[n]{\|S_{i_n} \cdots S_{i_1}\|} \quad \mu\text{-a.s. } i(\cdot) \in \Sigma_{\mathcal{I}}^+,$$

for every θ_+-invariant probability measure μ on $\Sigma_{\mathcal{I}}^+$.

2.2. Proof of Theorem A and an optimization result

Let $\Lambda \subset \Sigma_{\mathcal{I}}^+$ be a θ_+-invariant closed set and $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$ be continuous. Then, the Λ-stability of the linear switched system given by

$$x_n = S_{i_n} \cdots S_{i_1} x_0 \qquad (n \geq 1,\ x_0 \in \mathbb{C}^d,\ i(\cdot) \in \Sigma_{\mathcal{I}}^+),$$

is equivalent to the stability of the linear cocycle defined as follows:

$$\mathcal{L}\colon \Lambda \times \mathbb{Z}_+ \to \mathbb{C}^{d\times d}; \quad (i(\cdot), k) \mapsto \mathcal{L}(i(\cdot), k) = \begin{cases} \mathrm{Id}_{\mathbb{C}^d} & \text{if } k = 0, \\ S_{i_k} \cdots S_{i_1} & \text{if } k \geq 1. \end{cases}$$

Under the product topology of $\Sigma_{\mathcal{I}}^+$, the cocycle $\mathcal{L}(i(\cdot), k)$ is continuous, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ is endowed with the discrete topology. In addition, note that $\Sigma_{\mathcal{I}}^+$ is metrizable.

Now, we are ready to prove our Gel'fand-type spectral-radius theorem.

Proof of Theorem A. Since Λ is a compact subset and $\mathcal{L}(i(\cdot),1)$ is continuous with respect to $i(\cdot) \in \Lambda$, $\log^+ \|\mathcal{L}(i(\cdot),1)\|$ is bounded uniformly for $i(\cdot) \in \Lambda$. Applying Theorem 2.6 in the case $\Omega = \Lambda$ and $T = \theta_{+\restriction\Lambda}$, we could define a θ_+-invariant subset $\Upsilon \subset \Lambda$ such that $\mu(\Upsilon) = 1$ for all $\mu \in \mathcal{M}_{erg}(\Lambda, \theta_{+\restriction\Lambda})$ and that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \rho(\mathcal{L}(i(\cdot),n)) = \lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{L}(i(\cdot),n)\| \quad \forall i(\cdot) \in \Upsilon.$$

In fact, for each $\mu \in \mathcal{M}_{erg}(\Lambda, \theta_{+\restriction\Lambda})$ we can define a set Υ_μ by Theorem 2.6 and then let $\Upsilon = \bigcup \Upsilon_\mu$. Then from the definition of the generalized spectral radius, there holds the inequality

$$\rho(\boldsymbol{S}_{\restriction\Lambda}) \geq \limsup_{n\to+\infty} \sqrt[n]{\rho(\mathcal{L}(i(\cdot),n))} \quad \forall i(\cdot) \in \Upsilon.$$

Theorem 2.6 implies that

$$\rho(\boldsymbol{S}_{\restriction\Lambda}) \geq \lim_{n\to+\infty} \sqrt[n]{\|\mathcal{L}(i(\cdot),n)\|} \quad \forall i(\cdot) \in \Upsilon.$$

Since $f_n(i(\cdot)) = \log \|\mathcal{L}(i(\cdot),n)\|$ is continuous with respect to $i(\cdot) \in \Lambda$ and the sequence $\langle f_n \rangle_1^{+\infty}$ is θ_+-subadditive, from Theorem 2.2 it follows that

$$\begin{aligned} \log \rho(\boldsymbol{S}_{\restriction\Lambda}) &\geq \inf_{n\geq 1} \left\{ \int_\Lambda \log \sqrt[n]{\|\mathcal{L}(i(\cdot),n)\|}\, d\mu(i(\cdot)) \right\} \\ &= \lim_{n\to+\infty} \int_\Lambda \log \sqrt[n]{\|\mathcal{L}(i(\cdot),n)\|}\, d\mu(i(\cdot)) \end{aligned}$$

for all $\mu \in \mathcal{M}_{erg}(\Lambda, \theta_{+\restriction\Lambda})$. Now, applying Theorem 2.3 one can obtain that

$$\log \rho(\boldsymbol{S}_{\restriction\Lambda}) \geq \lim_{n\to+\infty} \left\{ \sup_{i(\cdot)\in\Lambda} \log \sqrt[n]{\|\mathcal{L}(i(\cdot),n)\|} \right\}.$$

Thus, from the definition of $\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$ there holds the inequality $\rho(\boldsymbol{S}_{\restriction\Lambda}) \geq \hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$ and further there follows that $\rho(\boldsymbol{S}_{\restriction\Lambda}) = \hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$ from $\rho(\boldsymbol{S}_{\restriction\Lambda}) \leq \hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$. This completes the proof of Theorem A. □

As a consequence of Lemma 2.5 and Theorem A, we could obtain at once the following optimization result.

Corollary 2.7. *Let $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$ be continuous and assume $\Lambda \subset \Sigma_{\mathcal{I}}^+$ is an invariant compact set of the one-sided Markov shift $\theta_+ \colon \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$. Then, for the linear switched system*

$$x_n = S_{i_n} \cdots S_{i_1} x_0 \qquad (n \geq 1,\ x_0 \in \mathbb{C}^d,\ i(\cdot) \in \Lambda),$$

there holds that

$$\rho(\boldsymbol{S}_{\restriction\Lambda}) = \max_{\mu\in\mathcal{M}_{erg}(\Lambda,\theta_{+\restriction\Lambda})} \{\exp \chi(\mu, \boldsymbol{S})\} = \max_{i(\cdot)\in\Lambda} \left\{\exp \chi(\boldsymbol{S}_{i(\cdot)})\right\} = \max_{(x_0,i(\cdot))\in\mathbb{C}^d\times\Lambda} \left\{\exp \chi(x_0, \boldsymbol{S}_{i(\cdot)})\right\}.$$

Here $\chi(\boldsymbol{S}_{i(\cdot)})$ is defined as Section 1.1, and

$$\chi(\mu, \boldsymbol{S}) := \limsup_{n\to+\infty} \frac{1}{n} \log \|S_{i_n} \cdots S_{i_1}\| \quad \text{for } \mu\text{-a.s. } i(\cdot) \in \Lambda$$

is called the (maximal) Lyapunov exponents of $\boldsymbol{S}$ at μ.

Proof. Applying Lemma 2.5 to the case that $f_n(i(\cdot)) = \log\|S_{i_n}\cdots S_{i_1}\|$ for $i(\cdot) \in \Omega = \Lambda$ and $T = \theta_{+\restriction\Lambda}$, one can find some θ_+-ergodic probability, say μ_*, on Λ such that

$$\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda}) = \exp\chi(\mu, \boldsymbol{S}) = \exp\chi(\boldsymbol{S}_{i(\cdot)}) \quad \text{for } \mu_*\text{-a.s. } i(\cdot) \in \Lambda.$$

Furthermore, from the multiplicative ergodic theorem [18, 36], it follows that there always are unit vectors $x_0 \in \mathbb{C}^d$ satisfying $\chi(\boldsymbol{S}_{i(\cdot)}) = \chi(x_0, \boldsymbol{S}_{i(\cdot)})$. Thus, the statement follows at once from Theorem A. □

Thus, there holds the following.

Corollary 2.8. *Let $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$ be continuous and assume $\Lambda \subset \Sigma_{\mathcal{I}}^+$ is an invariant compact set of the one-sided Markov shift $\theta_+\colon \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$. Then, the following statements are equivalent to each other.*

(1) $\rho(\boldsymbol{S}_{\restriction\Lambda}) < 1$.

(2) *$\boldsymbol{S}$ is Λ-absolutely exponentially stable.*

(3) *$\boldsymbol{S}$ is "Λ-pointwise exponentially stable", i.e., $\chi(x_0, \boldsymbol{S}_{i(\cdot)}) < 0$ for all $x_0 \in \mathbb{C}^d$ and any $i(\cdot) \in \Lambda$.*

This statement will be useful for proving Theorem B in Section 3.

3. Criteria for stability under switching constraints

In this section, we will prove Theorem B stated in Section 1.2, using Theorem A and Corollary 2.8 that have been proved in Section 2. As before, we let $\Sigma_{\mathcal{I}}^+$ denote the space of all switching signals $i(\cdot)\colon \mathbb{N} \to \mathcal{I}$. Let $\theta_+\colon \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$ be the one-sided Markov shift defined as in Theorem A, that is to say,

$$\theta_+\colon i(\cdot) \mapsto i(\cdot + 1) \quad \forall i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+.$$

Let Λ be an arbitrary, θ_+-invariant, closed, and nonempty subset of $\Sigma_{\mathcal{I}}^+$ and $\boldsymbol{S}\colon \mathcal{I} \to \mathbb{C}^{d\times d}$ continuous with respect to $i \in \mathcal{I}$. Recall that the linear switched system with constraint Λ

$$\boldsymbol{S}_{\restriction\Lambda} \qquad x_n = S_{i_n}\cdots S_{i_1}x_0 \qquad (n \geq 1,\ x_0 \in \mathbb{C}^d,\ i(\cdot) \in \Lambda)$$

is called *Λ-absolutely asymptotically stable* in case

$$S_{i_n}\cdots S_{i_1} \to \mathbf{0}_{d\times d} \text{ as } n \to \infty \qquad \forall i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Lambda,$$

where $\mathbf{0}_{d\times d}$ is the origin of $\mathbb{C}^{d\times d}$. Let $\|\cdot\|_2$ be the matrix norm on $\mathbb{C}^{d\times d}$ induced by the usual Euclidean vector norm on $\mathbb{C}^d$.

3.1. A criterion of Λ-stability

First, we present a criterion of Λ-absolute asymptotic stability (Lemma 3.1), which is an extension of [5, Theorem 4.1] from the case free of any constraints to a system which obeys switching constraints.

Lemma 3.1. *Let Λ be a θ_+-invariant compact subset of $\Sigma_{\mathcal{I}}^+$ and let*

$$\boldsymbol{S}_{\restriction\Lambda}^+(0) = \{\mathrm{Id}_{\mathbb{C}^d}\}, \quad \boldsymbol{S}_{\restriction\Lambda}^+(\ell) = \left\{S_{i_\ell}\cdots S_{i_1};\ i(\cdot) \in \Lambda\right\} \text{ for } \ell \geq 1 \quad \text{and} \quad \boldsymbol{S}_{\restriction\Lambda}^+ = \bigcup\nolimits_{\ell\geq 0}\boldsymbol{S}_{\restriction\Lambda}^+(\ell).$$

Then, $\boldsymbol{S}$ is Λ-absolutely asymptotically stable if and only if

(1) $\boldsymbol{S}_{\restriction\Lambda}^{+}$ *is bounded in* $\mathbb{C}^{d\times d}$*,i.e.,* $\exists\beta>0$ *such that* $\|A\|_2\le\beta\ \forall A\in\boldsymbol{S}_{\restriction\Lambda}^{+}$*; and*

(2) *there exists a constant* $\gamma>0$ *and an integer* $N\ge1$ *such that*

$$\rho(A)\le\gamma<1\quad\forall A\in\boldsymbol{S}_{\restriction\Lambda}^{+}(\ell),$$

for any $\ell\ge N$*.*

The condition (1) in Theorem 3.1 means that $\boldsymbol{S}$ is Lyapunov stable restricted to Λ. This theorem is itself very interesting and it is a key step towards the proof of Theorem B. Comparing to the case that is free of any switching constraints, now $\boldsymbol{S}_{\restriction\Lambda}^{+}$ is not a semigroup. This might cause an essential difficulty described as follows: if $\Lambda=\Sigma_{\mathcal{I}}^{+}$, i.e., free of any switching constraints, then condition (1) above implies that there can be found a pre-extremal vector norm $\|\cdot\|$ on $\mathbb{C}^d$ for $\boldsymbol{S}$ such that $\|A\|\le1$ for all $A\in\boldsymbol{S}_{\restriction\Lambda}^{+}$; But now in our context, this does not need to be true.

We note here that if the joint spectral radius $\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})<1$ then $\boldsymbol{S}$ is obviously Λ-absolutely asymptotically stable from Corollary 2.8. In fact, there holds the following stronger result.

Lemma 3.2. *Let* Λ *be a* θ_+*-invariant compact subset of* $\Sigma_{\mathcal{I}}^{+}$*. Then* $\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})<1$ *if and only if* $\boldsymbol{S}$ *is "*Λ*-uniformly exponentially stable"; that is, there exists a number* $0<\lambda<1$ *and an integer* $N\ge1$ *such that*

$$\|S_{i_n}\cdots S_{i_1}\|_2\le\lambda^n\quad\forall i(\cdot)\in\Lambda \textit{ and } n\ge N.$$

Proof. Let $1>\lambda>\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$. Then from the definition of $\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})$, there is some integer $N\ge1$ such that

$$\sup_{i(\cdot)\in\Lambda}\sqrt[n]{\|S_{i_n}\cdots S_{i_1}\|_2}\le\lambda\quad\forall n\ge N.$$

So, $\boldsymbol{S}$ is Λ-uniformly exponentially stable. Conversely, if there exists a constant $0<\lambda<1$ and an integer $N\ge1$ such that

$$\|S_{i_n}\cdots S_{i_1}\|_2\le\lambda^n\quad\forall i(\cdot)\in\Lambda \text{ and } n\ge N,$$

then

$$\hat{\rho}(\boldsymbol{S}_{\restriction\Lambda})=\inf_{n\ge1}\left\{\sup_{i(\cdot)\in\Lambda}\sqrt[n]{\|S_{i_n}\cdots S_{i_1}\|_2}\right\}\le\lambda<1,$$

as desired. This proves Lemma 3.2. □

At the first glance, Λ-absolute asymptotic stability is weaker than the Λ-absolute exponential stability for the switching system $\boldsymbol{S}$. However, they are equivalent to each other as is shown in the case free of any switching constraints (cf. [13, Theorem 4.1] and [20, Theorem 2.3]). In fact, the Λ-absolute asymptotic stability is equivalent to the Λ-uniform exponential stability from the Fenichel uniformity theorem [16], stated as follows:

Lemma 3.3 (See N. Fenichel [16])**.** *Let* Λ *be a* θ_+*-invariant compact subset of* $\Sigma_{\mathcal{I}}^{+}$*. Then,* $\boldsymbol{S}$ *is* Λ*-absolutely asymptotically stable if and only if it is* Λ*-uniformly exponentially stable.*

Remark 3.4. For Lemma 3.3, the hypothesis that Λ is "compact" is important, as shown by Example 1.3 in Section 1.2.

Now, we can readily prove Lemma 3.1 using the statements of Lemmas 3.2 and 3.3.

Proof of Lemma 3.1. If $\boldsymbol{S}$ is Λ-absolutely asymptotically stable, then from Lemmas 3.3 and 3.2 there follows that conditions (1) and (2) in Lemma 3.1 are trivially fulfilled. Next, let conditions (1) and (2) in Lemma 3.1 both hold. We proceed to prove that $\boldsymbol{S}$ is Λ-absolutely asymptotically stable.

Assume, by contradiction, that $\boldsymbol{S}$ were not Λ-absolutely asymptotically stable; then one can find some switching signal, say $i(\cdot) = (i_n)_{n=1}^{+\infty}$, in Λ such that $\|S_{i_n} \cdots S_{i_1}\|_2 \not\to 0$ as $n \to \infty$. Using the boundedness of $\boldsymbol{S}_{\restriction\Lambda}^{+}$ in $\mathbb{C}^{d\times d}$, we can pick out an increasing positive integer sequence, say $\{j_\ell\}_{\ell=1}^{+\infty}$, with $j_\ell \to +\infty$ as $\ell \to +\infty$, such that

$$C_\ell := S_{i_{j_\ell}} \cdots S_{i_1} \to C \neq \mathbf{0}_{d\times d} \quad \text{as } \ell \to \infty.$$

Now, define $B_\ell := S_{i_{j_{\ell+1}}} \cdots S_{i_{j_\ell+1}}$ and so $C_{\ell+1} = B_\ell C_\ell$. Since $\theta_+^{j_\ell}(i(\cdot)) = i(\cdot + j_\ell) \in \Lambda$, i.e., $(i_{n+j_\ell})_{n=1}^{+\infty}$ lies in Λ, by the θ_+-invariance of Λ, one could obtain $B_\ell \in \boldsymbol{S}_{\restriction\Lambda}^{+}$. Using the boundedness again, we can pick out a subsequence

$$B_{\ell_k} \to B \in \mathbb{C}^{d\times d} \quad \text{as } k \to \infty.$$

Then, $C = BC$, $C \neq \mathbf{0}_{d\times d}$, and $\rho(B) = \lim_{k\to\infty} \rho(B_{\ell_k}) \leq \gamma < 1$ by condition (2) of Lemma 3.1. But

$$B(\mathrm{Im}C) = \mathrm{Im}C \neq \{\mathbf{0}\},$$

so $B_{\restriction \mathrm{Im}C}$ is the identity. Thus, $\rho(B) \geq 1$; it is a contradiction to condition (2).

This therefore proves the statement of Lemma 3.1. □

3.2. *A reduction lemma*

To prove Theorem B stated in Section 1.2, we need an important reduction theorem, which is due to L. Elsner [15, Lemma 4] and simply proved in X. Dai [9].

Lemma 3.5 (See [15])**.** *If $\hat{\rho}(\boldsymbol{S}) = 1$ and $\boldsymbol{S}$ is product unbounded in $\mathbb{C}^{d\times d}$, then there is a nonsingular $P \in \mathbb{C}^{d\times d}$ and $1 \leq d_1 < d$ such that*

$$P^{-1} S_i P = \begin{bmatrix} S_i^{(2)} & \clubsuit_i \\ \mathbf{0}_{d_1\times(d-d_1)} & S_i^{(1)} \end{bmatrix} \quad \forall i \in \mathcal{I},$$

where $S_i^{(1)} \in \mathbb{C}^{d_1\times d_1}$.

Here $\boldsymbol{S}$ is said to be product unbounded, if the multiplicative semigroup $\boldsymbol{S}^{+}$ defined in the manner as in Lemma 3.1 in the case $\Lambda = \Sigma_{\mathcal{I}}^{+}$ is unbounded in $\mathbb{C}^{d\times d}$ under an arbitrary induced operator norm.

3.3. *Proof of Theorem B*

Let $\Lambda \subset \Sigma_{\mathcal{I}}^{+}$ be an invariant compact set of the one-sided Markov shift $\theta_+ \colon \Sigma_{\mathcal{I}}^{+} \to \Sigma_{\mathcal{I}}^{+}$, which gives rise to the constrained linear switched system

$$\boldsymbol{S}_{\restriction\Lambda} \qquad x_n = S_{i_n} \cdots S_{i_1} x_0 \qquad (n \geq 1,\ x_0 \in \mathbb{C}^d,\ i(\cdot) \in \Lambda),$$

where $\boldsymbol{S} \colon \mathcal{I} \to \mathbb{C}^{d\times d};\ i \mapsto S_i$ is as in the assumption of Theorem B.

We now proceed to prove Theorem B.

Proof of Theorem B. Clearly, (a) $\Rightarrow$ (b) follows from Lemma 3.3 and Corollary 2.8, and (b) $\Rightarrow$ (c) follows from Theorem A, Corollary 2.8 and Lemma 3.1.

So, to prove Theorem B, we need to prove only (c) $\Rightarrow$ (a). According to the definition of $\rho(\boldsymbol{S}_{|\Lambda})$ and from Theorem A, condition (c) implies that

$$\rho(\boldsymbol{S}_{|\Lambda}) = \hat{\rho}(\boldsymbol{S}_{|\Lambda}) \leq 1.$$

If $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$, then from Lemma 3.2, there holds condition (a). So, we proceed, by induction on the dimension d of the state-space $\mathbb{C}^d$, to prove $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$.

Note that if $\boldsymbol{S}_{|\Lambda}^{+}$, defined as in Lemma 3.1, is bounded in $\mathbb{C}^{d\times d}$, then condition (a) follows from Lemma 3.1 together with condition (c). So, the assertion is true for $d = 1$; this is because $\rho(S_{i_n}\cdots S_{i_1}) = \|S_{i_n}\cdots S_{i_1}\|_2 = |S_{i_n}|\cdots|S_{i_1}| \leq \gamma < 1$ for any $n \geq N$, for any $i(\cdot) \in \Lambda$ in this case.

Let $m \geq 1$ be an arbitrarily given integer. Assume the assertion is true for all dimensions $d \leq m$. We claim that the assertion holds for $d = m + 1$.

Suppose, by contradiction, that $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) = 1$ for dimension $d = m + 1$. If $\boldsymbol{S}_{|\Lambda}^{+}$ is bounded in $\mathbb{C}^{(m+1)\times(m+1)}$, by Lemma 3.1 and condition (c), $\boldsymbol{S}_{|\Lambda}$ is Λ-absolutely asymptotically stable so that $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$ from Lemmas 3.3 and 3.2, a contradiction. Therefore $\boldsymbol{S}_{|\Lambda}^{+}$ is unbounded in $\mathbb{C}^{(m+1)\times(m+1)}$ and further $\boldsymbol{S}$ is product unbounded in $\mathbb{C}^{(m+1)\times(m+1)}$. Then from Lemma 3.5, one can find a nonsingular $P \in \mathbb{C}^{(m+1)\times(m+1)}$ and $1 \leq n_1 \leq m$ such that

$$P^{-1}S_iP = \begin{bmatrix} S_i^{(2)} & \clubsuit_i \\ \mathbf{0} & S_i^{(1)} \end{bmatrix} \quad \forall i \in \mathcal{I},$$

where $S_i^{(1)} \in \mathbb{C}^{n_1\times n_1}$ and $\mathbf{0}$ is the origin of $\mathbb{C}^{n_1\times(m+1-n_1)}$. Set

$$\boldsymbol{S}^{(r)} = \left\{ S_i^{(r)} \,|\, i \in \mathcal{I} \right\}, \quad r = 1, 2.$$

Then, by condition (c)

$$\rho(A) \leq \gamma < 1 \quad \forall A \in \boldsymbol{S}_{|\Lambda}^{(r)+} \quad \text{for } r = 1, 2,$$

where $\boldsymbol{S}_{|\Lambda}^{(r)+}$ is defined similarly to $\boldsymbol{S}_{|\Lambda}^{+}$ based on $\boldsymbol{S}_{|\Lambda}^{(r)}$. As the switched systems $\boldsymbol{S}_{|\Lambda}^{(r)}$ have dimension less than $m + 1$ for $r = 1, 2$, by the induction assumption and Theorem A

$$\rho\left(\boldsymbol{S}_{|\Lambda}^{(r)}\right) < 1 \quad \text{for } r = 1, 2.$$

Therefore

$$\rho(\boldsymbol{S}_{|\Lambda}) = \max\left\{ \rho\left(\boldsymbol{S}_{|\Lambda}^{(1)}\right), \rho\left(\boldsymbol{S}_{|\Lambda}^{(2)}\right) \right\} < 1,$$

and $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$ by Theorem A, contradicting the hypothesis that $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) = 1$.

This contradiction shows that $\hat{\rho}(\boldsymbol{S}_{|\Lambda}) < 1$, completing the proof of Theorem B. □

4. Concluding remarks and further questions

In this paper, using ergodic theory we have studied the relationship of the joint spectral radius and the generalized spectral radius of a linear switched system obeying some type of switching constraints, and presented several stability criteria. We now raise some questions to further study.

Theorem A asserts a Gel'fand-type spectral-radius formula for a linear switched system obeying some switching constraints. Let $\Lambda \subsetneq \Sigma_{\mathcal{I}}^{+}$ be an invariant closed set of the one-sided Markov shift θ_+. Clearly, for any $i(\cdot) = (i_n)_{n=1}^{+\infty} \in \Lambda$ and any $n \geq 1$, the sub-word $w = (i_1, \dots, i_n)$ of length n does not need to be extended to a permissive periodic switching signal, i.e., although

$$(\overbrace{i_1, \dots, i_n}^{w}, \overbrace{i_1, \dots, i_n}^{w}, \dots) \in \Sigma_{\mathcal{I}}^+$$

is a periodic point of θ_+, but it need not belong to the given subset Λ. For any $n \geq 1$, put

$$W_{\mathrm{per}}^n(\Lambda) = \left\{ w = (i_1, \dots, i_n) \in \mathcal{I}^n \,|\, (\overbrace{i_1, \dots, i_n}, \overbrace{i_1, \dots, i_n}, \dots) \in \Lambda \right\},$$

called the set of all Λ-periodic words of length n. It is natural to ask the following question:

Question 1. *If the periodical switching signals are dense in Λ then, does there hold the following equality:*

$$\limsup_{n\to+\infty} \left\{ \sup_{w \in W_{\mathrm{per}}^n(\Lambda)} \sqrt[n]{\rho(\boldsymbol{S}_w)} \right\} = \limsup_{n\to+\infty} \left\{ \sup_{w \in W_{\mathrm{per}}^n(\Lambda)} \sqrt[n]{\|\boldsymbol{S}_w\|} \right\} ?$$

Here $\boldsymbol{S}_w = S_{i_n} \cdots S_{i_1}$ for any word $w = (i_1, \dots, i_n)$ of length $n \geq 1$ as before.

In our proof of Theorem A, the compactness of Λ plays a role. So, we naturally ask the following question:

Question 2. *If Λ is a θ_+-invariant closed subset of $\Sigma_{\mathcal{I}}^+$ not necessarily compact, does the statement of Theorem A still hold when $\boldsymbol{S} = \{S_i\}_{i\in\mathcal{I}}$ is bounded in $\mathbb{C}^{d\times d}$?*

In the statement of Theorem B, from the results proved in Section 3 there can still be deduced without the assumption $\rho(\boldsymbol{S}) = 1$ that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). This assumption imposed there is used in the proof of (c) $\Rightarrow$ (a) where we need to employ Lemma 3.5.

So, we ask the following question:

Question 3. *Does the statement of Theorem B still hold without the assumption $\rho(\boldsymbol{S}) = 1$?*

Furthermore, we believe that it is very possible to have a positive solution to Question 3.

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