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RESEARCH ARTICLE

Monetary policy and multiple equilibria with constrained investment and externalities

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Abstract This paper focuses on two mechanisms under which interest-rate feedback rules induce local indeterminacy in a closed economy with capital accumulation: arbitrage activity and the pricing channel. It shows that constrained investment, in the sense that it requires liquidity or that adjustment to the stock of capital is costly, is enough to induce indeterminacy if monetary policy follows a strictly passive interest rate rule. Determinacy of equilibrium is ensured under an active monetary policy stance. These results change when production externalities are introduced into the model so as to mimic the pricing channel in New Keynesian models. In this case, a policy stance that ensures determinacy is either active or strictly passive. In view of the contradicting results for the passive stance and the similar results for the active stance it is recommended that central banks act according to the active stance.

Keywords Interest rate rules · Determinacy · Investment · Monetary policy

JEL Classification E4 · E52 · E61 · E62 · E63

1 Introduction

The problem of equilibrium determinacy under endogenous capital accumulation and interest rate policy rules has drawn much attention in the past several years.

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One important issue concerning this current literature is that conclusions can be very different, even opposite, in continuous-time setting from a discrete-time setting. Dupor (2001) and Carlstrom and Fuerst (2005) are one example to the contrast in conclusions in a New Keynesian paradigm. While the former calls for a passive rule in order to ensure equilibrium determinacy in a continuous time setting, the latter shows that an active rule is necessary for ensuring equilibrium determinacy in a discrete-time setting. Carlstrom and Fuerst (2005) attribute the opposite conclusions to the difference in timing in the no-arbitrage condition of investing in bonds and capital between the two settings: while the continuous-time setting entails a contemporaneous no-arbitrage condition, the no-arbitrage condition in the discrete-time setting involves only future variables which bring a zero eigenvalue into the linearized dynamic system.

Li (2005) made a notable attempt to find the conditions that can ensure consistency between a continuous-time setting and a discrete-time setting. The motivation of Li (2005) is to resolve the specific conflict between the conclusions by Dupor (2001) and Carlstrom and Fuerst (2005), and her strategy is to identify some continuity properties that would hold in the limiting case of a discrete-time setting when the length of the time interval approaches zero. The main finding of Li (2005) is that, in general, macroeconomic continuous time modeling could be misleading in the sense that it does not correctly approximate the behavior of the discrete-time model of arbitrarily small periods. Therefore, special care should be taken with assumptions of the model that are not realistic for small period length. One way to overcome contemporaneous features of continuous time macroeconomic models that enter at the "back door" as the period length gets shorter is to impose assumptions that apply only to the continuous-time model and are not necessary in the discrete-time model. For example, one way to get rid of the contemporaneous feature of the no-arbitrage condition in the continuous-time setting is to introduce adjustment costs to capital. Li (2005) and Carlstrom and Fuerst (2005) share this view.

This paper illustrates a model of monetary policy in continuous time with the least amount of restrictions required to overcome the loss of generality pointed out in Li (2005) and Carlstrom and Fuerst (2005). The contribution of this paper is to provide a comprehensive exposition of the circumstances under which indeterminacy arises in a continuous-time model where the monetary policy follows a forward looking interest rate rule and prices are flexible. Here, the instantaneous no-arbitrage condition vanishes when adjustment-costs are introduced into the investment technology.[1] An alternative assumption is that investment is financially constrained. Either assumption implies that investment is instantaneously constrained. Therefore, the stock of capital cannot "jump" to the level implied by models with neoclassical investment. Results show that when prices are flexible and some frictions in investment induce an arbitrage channel, monetary policy should follow an active or a neutral stance so as to induce real determinacy. Whereas a mildly passive stance yields ambiguous results and a strictly passive stance induce real indeterminacy.

[1] A complete analysis along the line of Li (2005) so as to find the conditions that can ensure consistency between a continuous-time setting and a discrete-time setting in macroeconomic issues deserves a separate paper. I, therefore, implement the conclusions of Li (2005) and Carlstrom and Fuerst (2005) and assume frictions in investment.

The second issue of this paper is whether capital tend to cause macroeconomic instability through additional channels. In the New Keynesian paradigm, Huang and Meng (2007), Benhabib and Eusepi (2005) and Sveen and Weinke (2005) argue that capital tends to cause macroeconomic instability mainly through firms' pricing behavior in product markets. Benhabib and Eusepi (2005) and Sveen and Weinke (2005) study the indeterminacy issue under endogenous investment and current-looking interest-rate feedback policy. The forward-looking policy considered by Huang and Meng (2007) in their discrete-time model is more in line with the policy considered in the current paper, as the instantaneous rate of inflation in a continuous-time setting is the right derivative of the logged price level. Thus, the discrete-time counterpart of a continuous-time policy rule that sets the interest rate in response to instantaneous rate of inflation is a forward-looking policy that responds to changes in expected inflation. In light of the two alternative views in the New Keynesian framework, discussed in Carlstrom and Fuerst (2005) and Huang and Meng (2007), i.e., the arbitrage channel and the pricing channel, an examination of those views in the Neoclassical paradigm warrant attention. Blanchard and Kiyotaki (1987) identify the distortions that characterize a symmetric equilibrium in a monopolistic competition model as aggregate demand externalities. As an attempt to mimic the role of capital via a similar channel in a Neoclassical paradigm, I impose aggregate-supply externalities. Specifically, production externalities related to per capita capital are introduced into the production function. The upshot of this analysis is that the active policy stance ensure real determinacy. However, within the region of passive stance, it is the mildly passive stance that induces real indeterminacy, whereas the strictly passive stance induces real determinacy. Considering the opposite results obtained for the passive stance through the two alternative channels, it is recommended to stick to the active stance which, through both channels, ensure real determinacy.

The rest of the paper is organized as follows: in Sect. 2, the optimization problem of a representative household in a flexible-price, cash-in-advance capital-based production economy is specified. It is assumed that investment technology involves some convex cost of adjustment. In Sect. 3 this exercise is repeated for financially constrained investment. Section 4 discusses the effect that monetary policy has on the determination of the real variables when the production function exhibits a positive externality that increases with per capita capital. Section 5 discusses the main results and Sect. 6 concludes.

2 A model with frictions in investment

It is assumed throughout this paper that the central bank operates according to an interest rate feedback rule and responds to deviations of the instantaneous measure of inflation from an inflation target. Money enters the economy via a cash-in-advance constraint on consumption. To avoid steady state multiplicity, the analysis is restricted, following Benhabib et al. (2002), to steady states which are characterized by a positive inflation target, and consequently, a strictly positive nominal interest rate. Finally, it is assumed throughout that nominal prices are flexible.

2.1 The economic environment

Households The economy is populated by a continuum of identical infinitely long-lived households, with measure one. The representative household's lifetime utility function is given by

$$U = \int_0^\infty \mathrm{e}^{-\rho t} u(c)\,\mathrm{d}t$$

where $\rho > 0$ denotes the rate of time preference, c denotes consumption, $u(c)$ is twice differentiable, strictly increasing, and strictly concave. The household has no direct utility from fiat money. However, consumption is subject to a cash-in-advance constraint. In addition to fiat money, households can store wealth in government-issued non-indexed bonds and physical capital. Bonds pay a net nominal interest of $R > 0$. Capital is used for production and can be converted into consumption goods at a cost. The same cost of adjustment applies to investing and depends exclusively on the magnitude of investment. The household's budget constraint is therefore described as:

$$c + I + \dot{b} + \dot{m} = (R - \pi)b - \pi m + f(k) - \frac{\gamma}{2}I^2 - \tau \tag{2}$$

where I is the flow of investment, b is the real value of government bonds held by the household, m is the real value of money balances, k is the stock of capital, τ is a real lump-sum tax, γ is a positive coefficient and π is the rate of inflation. Note that all variables are time-dependent (the time argument is omitted to keep notation simple). Finally it is assumed that the production function, $f(k)$, is twice differentiable, strictly increasing and strictly concave. By defining $a \equiv b + m$ as the real value of non-capital wealth the household's budget constraint becomes:

$$\dot{a} = (R - \pi)a - Rm + f(k) - \frac{\gamma}{2}I^2 - c - I - \tau \tag{3}$$

The household's consumption is then subject to a cash-in-advance constraint,[2]

$$c \leq m \tag{4}$$

With positive nominal interest rates, holding money entails opportunity costs. Hence, the constraint in Eq. (4) is binding and the household's lifetime maximization problem becomes

[2] Money is introduced into the optimization plan of the agents via a cash-in-advance constraint. This version of CIA is similar to Rebelo and Xie (1999) and Wang and Yip (1992).

$$\text{Max} \int_0^\infty e^{-\rho t} u(c)\, dt$$

s.t.

$$\dot{a} = (R - \pi)a + f(k) - \frac{\gamma}{2} I^2 - c(1 + R) - I - \tau$$

$$\dot{k} = I$$

With the following no-Ponzi-game condition $\text{Lim}_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\, ds}[a(t)+k(t)] = 0$. The household chooses sequences of *{c,I}* so as to maximize its lifetime utility, taking as given the initial stock of capital $k(0)$, and the time paths of τ, R, and π.

An optimal program must choose c and I so as to maximize the current-value Hamiltonian $H_1 \equiv U(c) + \lambda\left[(R - \pi)a + f(k) - \frac{\gamma}{2} I^2 - c(1 + R) - I - \tau\right] + \mu I$ Thus, the necessary conditions for an interior maximum of the household's problem are

$$\lambda = \frac{U'(c)}{1 + R} \tag{5}$$

$$I = \frac{1}{\gamma}\left[\frac{\mu}{\lambda} - 1\right] \tag{6}$$

where λ_t and μ_t (in the current-value Hamiltonian the time subscripts are omitted to simplify notation) are co-state variables interpreted as the marginal valuation of a unit of financial assets and installed capital, respectively. Second, the co-state variables must evolve according to the law

$$\dot{\lambda} = \lambda[\rho + \pi - R] \tag{7}$$

$$\dot{\mu} = -\lambda f'(k) + \rho\mu \tag{8}$$

Finally, the law of motion for the real value of financial assets is

$$\dot{a} = (R - \pi)a + f(k) - \frac{\gamma}{2} I^2 - c(1 + R) - I - \tau$$

and the law of motion for capital is

$$\dot{k} = I \tag{9}$$

We study equilibria close to the steady state; therefore, the transversability condition always holds.

Let η denote the ratio between the two co-state variables, $\eta \equiv \frac{\mu}{\lambda}$. Accordingly,

$\dot{\eta} = \frac{\dot{\mu}}{\lambda} - \eta\frac{\dot{\lambda}}{\lambda}$. Substituting in Eqs. (7) and (8) yields the evolution of η:

$$\dot{\eta} = -f'(k) + \eta(R - \pi) \tag{10}$$

Finally, from (6) and (9), the evolution of the capital stock is according to:

$$\dot{k} = \frac{1}{\gamma}[\eta - 1] \tag{11}$$

Equations (7), (10)–(11) fully describe the optimal instantaneous decision making of the representative household as it takes the time paths of τ, R, and π as (exogenously) given. Equation (7) describes the evolution of the marginal valuation of a unit of the real good stored as a government liability. Equation (11) describes the optimal rate of investment and is familiar from the literature of investment. In the baseline model of investment with adjustment costs, the optimal flow of investment is the difference between the marginal valuation of a unit of installed capital (Tobin's q) and the cost of a unit of capital, divided by the adjustment cost coefficient.[3] In this model, unlike the baseline model of investment, households trade off physical capital with financial assets and, therefore, the ratio between the marginal valuation of a unit of installed of capital and the marginal valuation of a unit of an interest bearing government-liability plays the role of Tobin's marginal q in an otherwise cashless economy. And, as financial and real assets are perfect substitutes, investment flow is positive if and only if the marginal valuation of a unit of installed capital is greater than the marginal valuation of a unit stored in financial assets. The evolution of the ratio is described in Eq. (10).

The Government Following Dupor (2001) and Benhabib et al. (2001), it is assumed that monetary policy takes the form of an interest-rate feedback rule whereby the nominal interest rate is set as an increasing function of instantaneous inflation. Specifically, it is assumed that

$$R = R(\pi) \tag{12}$$

where $R(\pi)$ is continuous, non-decreasing and strictly positive, and there exists at least one $\pi^* > -\rho$ such that $R(\pi^*) = \rho + \pi^*$. Monetary policy is referred to as active if $R' > 1$ and as passive if $R' < 1$. Dupor (2001) and Benhabib et al. (2001) discuss the issue of local real determinacy in a continuous time model where the monetary authority sets a nominal interest rate as a function of the instantaneous rate of inflation. The policy considered here follows this line and is also in line with the forward-looking policy considered by Carlstrom and Fuerst (2005) and Huang and Meng (2007) in their discrete-time models. As we know, the instantaneous rate of inflation in a continuous-time setting is the right-derivative of the logged price level and thus, the discrete-time counterpart of a continuous-time policy rule that sets the interest rate in response to the instantaneous rate of inflation is characterized by forward-looking policy that responds to expected future inflation.

It is assumed that government purchases are zero at all times, and that the only disbursement is interest payments over the outstanding debt. Thus, the government's

[3] The familiar equation for optimal flow of investment in the presence of adjustment costs is $I = \frac{1}{\gamma}[q - 1]$, where q is "*Tobin's q*".

nominal budget constraint is

$$RB = \dot{M} + \dot{B} + P\tau \tag{13}$$

where B and M are the nominal stocks of bonds and fiat money, respectively, and P is the level of nominal prices. The central bank imposes the desired interest rate by controlling the price of riskless nominal bonds and exchanging money for bonds at any quantities demanded at that price. In that sense, the nominal rate of interest is exogenous and M and B are endogenous. Simple algebraic manipulations of Eq. (13) yield that $a \equiv m + b$ evolves according to:

$$\dot{a} = (R - \pi)a - Rm - \tau \tag{14}$$

Equilibrium In equilibrium, the goods market clears

$$f(k) - \frac{\gamma}{2} I^2 = c + I \tag{15}$$

Rearranging and substituting Eq. (6) into (15) yields

$$c(\eta, k) = f(k) - \frac{\gamma}{2}\left[\frac{1}{\gamma}(\eta - 1)\right]^2 - \left[\frac{1}{\gamma}(\eta - 1)\right] \tag{16}$$

That is, the time path of consumption in equilibrium can be directly obtained from the time paths of k and η. It is assumed that $R = R(\pi)$. Taking the household's optimality conditions, $R = \frac{U'(c(\eta,k))}{\lambda} - 1 = R(\lambda, \eta, k)$. That is, the nominal interest rate in equilibrium is a mapping of (λ, η, k). It follows from $R(\pi) = R(\lambda, \eta, k)$ that $\pi = \pi(\lambda, \eta, k)$. That is, given the time path $\{\lambda, \eta, k\}$, the time path of $\{\pi\}$ can also be constructed.

2.2 Equilibrium dynamics

Partial derivatives of $\pi(\eta, \mu, k)$ are obtained from the equality $R(\pi(\lambda, \eta, k)) = \frac{U'(c(\eta,k))}{\lambda} - 1$, Thus:

$$\begin{aligned} \pi_\lambda &= -\frac{(1 + R(\pi))^2}{U'(c(\eta, k))R'(\pi)} < 0 \\ \pi_\eta &= -\frac{1}{\gamma}\frac{U''(c(\eta, k))}{R'(\pi)}\frac{\eta}{\lambda} > 0 \\ \pi_k &= \frac{U''(c(\eta, k))f'(k)}{R'(\pi)}\frac{1}{\lambda} < 0 \end{aligned} \tag{17}$$

The rate of inflation is negatively related to λ. According to Eq. (17) the rate of inflation is negatively related to marginal utility of consumption, and positively related to consumption. The rate of inflation is positively related to η and negatively related to the stock of capital. That implies that when the marginal valuation of capital increases relative to the marginal valuation of financial assets inflation rises. Such an event is associated with a decline in the stock of capital—or a positive shock to productivity—accompanied by an increase in the marginal productivity of capital and consequently with the real rate of interest. The increase in the real rate of interest implies an increase in government's expenses and consequently a budget deficit which entail a raise in inflation tax revenues.

The dynamics of all the variables in the economy can thus be described by (λ, η, k), the dynamics of which are given in the system:

$$\dot{\lambda} = F(\lambda, \eta, k)$$

$$\dot{\eta} = G(\lambda, \eta, k)$$

$$\dot{k} = H(\lambda, \eta, k)$$

$$\dot{a} = [R(\pi(\lambda, \eta, k)) - \pi(\lambda, \eta, k)]\, a(\lambda, \eta, k) - R(\pi(\lambda, \eta, k))m(c(\eta, k)) - \tau$$

The transversality condition is

$$\operatorname*{Lim}_{t\to\infty} e^{-\int_0^t [R(\pi(\lambda,\eta,k))-\pi(\lambda,\eta,k)]\,\mathrm{d}s}[a(\lambda, \eta, k) + k] = 0$$

where

$$F(\lambda, \eta, k) = \lambda[\rho + \pi(\lambda, \eta, k) - R\,(\pi(\lambda, \eta, k))] \tag{18}$$

$$G(\lambda, \eta, k) = -f'(k) + \eta[R\,(\pi(\lambda, \eta, k)) - \pi(\lambda, \eta, k)] \tag{19}$$

$$H(\lambda, \eta, k) = \frac{1}{\gamma}[\eta - 1] \tag{20}$$

2.3 Determinacy of equilibrium

The time paths of all the variables in the economy are spanned by $\{\lambda, \eta, k\}$. Accordingly, we define real indeterminacy as follows:

Definition 1 Equilibrium displays real indeterminacy if there exists an infinite number of equilibrium trajectories $\{\lambda, \eta, k\}$ that converge to the steady state.[4]

Real determinacy Equations (7), (18)–(20) imply that in the steady state $\lambda^* = \frac{U'(c^*)}{1+\rho+\pi^*}$, $\eta^* = 1$, $f'(k^*) = \rho$ and $R(\pi^*) = \rho + \pi^*$. Linear approximation of the dynamic

[4] Benhabib et al. (2001) consider the real part of the economy as indeterminate "…*.if there exists an infinite number of equilibrium sequences* π". In this paper $\pi = \pi(\lambda, \eta, k)$. Therefor, we explicitly define indeterminacy as the existence of an infinite number of equilibrium trajectories in the(λ, η, k)space.

system near the steady state (λ^*, η^*, k^*) is obtained through the system

$$\begin{bmatrix} \dot{\lambda} \\ \dot{\eta} \\ \dot{k} \end{bmatrix} = \underbrace{\begin{bmatrix} R(\pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)} & \frac{U''(c^*)}{\gamma}\frac{R'(\pi^*)-1}{R'(\pi^*)} & -U''(c^*)f'(k^*)\frac{R'(\pi^*)-1}{R'(\pi^*)} \\ -\frac{R(\pi^*)^2}{U'(c^*)}\frac{R'(\pi^*)-1}{R\prime(\pi^*)} & \rho + \frac{\psi_A(c^*)R(\pi^*)}{\gamma}\frac{R'(\pi^*)-1}{R'(\pi^*)} & -f''(k^*) - \rho\psi_A(c^*)R(\pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)} \\ 0 & 1/\gamma & 0 \end{bmatrix}}_{A} \times \begin{bmatrix} \lambda - \lambda^* \\ \eta - \eta^* \\ k - k^* \end{bmatrix}$$

where $\psi_A(c^*) \equiv -\frac{U''(c^*)}{U'(c^*)} > 0$ is the measure of absolute risk aversion in the steady state.

Let $\theta_i\,(i = 1, 2, 3)$ denote the eigenvalues of matrix A, and note that the determinant of A equals the product of its eigenvalues and that the sum of eigenvalues equals the trace of A. Thus,

$$\theta_1\theta_2\theta_3 = \frac{1}{\gamma}(1+\rho+\pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)}f''(k^*) \tag{21}$$

$$\theta_1+\theta_2+\theta_3 = \rho + (1+\rho+\pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)}(1+\frac{1}{\gamma}\Psi_A(c^*)) \tag{22}$$

From Eqs. (21)–(22) we conclude the following:

Proposition 1 *Neutral or active monetary policy stances, that is $R'(\pi^*) \geq 1$, are sufficient to ensure real determinacy. A passive monetary policy stance such that $R'(\pi^*) \leq \frac{1}{1+\rho}$ is sufficient to ensure real indeterminacy. If monetary policy is "mildly passive", that is $\frac{1}{1+\rho} < R'(\pi^*) < 1$, the nature of equilibrium is ambiguous. However, as long as the monetary policy is "mildly passive", and if equilibrium is determinate, the only perfect-foresight equilibrium in which real allocation converges to the steady state is the steady state itself.*

Proof Consider a neutral policy stance Note the right-hand side of Eq. (21). Substituting in $R'(\pi^*) = 1$, we obtain that the determinant of matrix A equals zero. A neutral policy is linearly approximated near the steady state by $R(\pi) = \rho + \pi$. Substituting that feedback rule into the first-order condition, described by Eq. (7), implies that the costate λ is constant throughout [which clarifies why a neutral monetary stance is associated with a singular jacobian in the $\{\lambda, \eta, k\}$ space]. Thus, the dynamics of the economy under the neutral stance becomes

$$\dot{\eta} = -f'(k) + \rho\eta$$
$$\dot{k} = \frac{1}{\gamma}[\eta - 1].$$

With $\dot{\lambda} = 0$, $\lambda = \lambda^*$ in and off the steady state. In that case, and after substituting $R'(\pi^*) = 1$ into Eqs. (21) and (22), local dynamics of the economy near the underlying steady state is linearly approximated by $\begin{bmatrix} \dot{\eta} \\ \dot{k} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho & -f''(k^*) \\ 1/\gamma & 0 \end{bmatrix}}_{B} \begin{bmatrix} \eta - \eta^* \\ k - k^* \end{bmatrix}$. Note that, $\mathrm{Det}(B) = \frac{1}{\gamma} f''(k^*) < 0$ and with a predetermined capital stock, this also implies that the steady state is a saddle point and therefore determinate.

Consider an active policy stance Note the right-hand side of Eq. (21). Assuming $(1 + \rho + \pi^*) > 0$, $R'(\pi^*) > 0$ and $f''(k^*) < 0$, the sign of the product of eigenvalues is determined by the expression $(R'(\pi^*) - 1)$. Under the active stance, the right-hand side of Eq. (21) is negative. Thus, $\theta_1\theta_2\theta_3 < 0$. In a three-dimensional space, the product of the eigenvalues is negative if either one eigenvalue is negative and two eigenvalues have positive real parts, or if all three eigenvalues are negative. Note Eq. (22). An active policy rule implies that $\theta_1 + \theta_2 + \theta_3 > 0$, which rules out the possibility that all the eigenvalues are negative. Thus, when $R'(\pi^*) > 1$ there is exactly one negative eigenvalue, which implies that the steady state is a saddle point.

Consider a passive policy stance Under passive policy $\theta_1\theta_2\theta_3 > 0$. The product of the eigenvalues is positive if there are either two eigenvalues containing negative real parts and one positive eigenvalue, or if all three eigenvalues are positive.

If there are two eigenvalues containing negative real parts and one positive eigenvalue, then with only one predetermined state variable the equilibrium is indeterminate. Note Eq. (22). A necessary condition for all three eigenvalues to be positive is that $\theta_1 + \theta_2 + \theta_3 > 0$, which implies that $\rho + (1 + \rho + \pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)}(1 + \Psi_A(c^*)) > 0$. Rearranging yields the necessary condition $R'(\pi^*) > \frac{1}{1+\rho}$. When all eigenvalues are positive the only equilibrium path that converges to the steady state is the steady state itself. □

3 Equilibria with financially constrained investment

Carlstrom and Fuerst (2005) propose the capital adjustment-cost approach to eliminate the arbitrage activity of households from the continuous-time model. In this section, the neoclassical investment-technology assumption is maintained. However, instead of assuming that investment entails adjustment costs, a CIA constraint is imposed on all transactions, including investment. The main motivation of this section is to demonstrate that any mechanism that prevents households from performing arbitrage activity entails similar implications with respect to the regions of determinacy. Equation (23) describe the underlying assumption:

$$c + I \leq m \tag{23}$$

Assuming a binding CIA constraint and substituting it into the household's budget constraint yields the household's lifetime optimization problem:

$$\text{Max} \int_0^{\infty} e^{-\rho t} u(c)\, dt$$
s.t.
$$\dot{a} = (R - \pi)a + f(k) - (c + I)(1 + R) - \tau$$
$$\dot{k} = I$$

and the no-Ponzi-game condition $\text{Lim}_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\, ds}[a(t) + k(t)] = 0$.

The household chooses sequences of $\{c, I\}$ so as to maximize its lifetime utility, taking as given the initial stock of capital $k(0)$, and the time path of τ, R, and π.

Defining the current value hamiltonian as $H_2 \equiv U(c) + \lambda[(R - \pi)a + f(k) - (c + I)(1 + R) - \tau] + \mu I$, the optimality conditions associated with the household's problem are:

$$\lambda = \frac{U'(c)}{1 + R} \tag{24}$$

$$\mu = \lambda(1 + R) \tag{25}$$

$$\dot{\lambda} = \lambda[\rho + \pi - R] \tag{26}$$

$$\dot{\mu} = -\lambda f'(k) + \rho\mu \tag{27}$$

and the law of motion for the real value of financial assets is $\dot{a} = (R - \pi)a + f(k) - (c + I)(I + R) - \tau$ and the law of motion for capital is $k = \dot{I}$.

Let η denote the ratio between the two co-state variables. Than, a simple rearrangement of Eqs. (24)–(27), according to the rule $\dot{\eta} = \frac{\dot{\mu}}{\lambda} - \eta\frac{\dot{\lambda}}{\lambda}$, together with the goods market-clearing condition yields the dynamic system:

$$\dot{\lambda} = \lambda[\rho + \pi - R] \tag{28}$$

$$\dot{\eta} = -f'(k) + \eta(R - \pi) \tag{29}$$

$$\dot{k} = f(k) - c \tag{30}$$

Equilibrium dynamics

Note that (25) could be written as

$$1 + R(\pi) = \eta \tag{31}$$

It follows from Eq. (31) that $\pi = \pi(\eta)$; $\pi_\lambda = \pi_k = 0$; $\pi_\eta = \frac{1}{R'(\pi)}$.

From (24) and (31), it follows that $U'(c) = \lambda\eta$ and therefore $c_\lambda = \frac{\eta}{U''(c)}$, $c_\eta = \frac{\lambda}{U''(c)}$, $c_k = 0$

Thus, we can conclude that the dynamics of all the variables in the economy can be described through the dynamics of (λ, η, k), which is

$$\begin{aligned}\dot{\lambda} &= F(\lambda, \eta, k)\\ \dot{\eta} &= G(\lambda, \eta, k)\\ \dot{k} &= H(\lambda, \eta, k)\\ \dot{a} &= [R(\pi(\eta)) - \pi(\eta)]\, a(\lambda, \eta, k) - R(\pi(\eta))m(c(\lambda, \eta)) - \tau\end{aligned}$$

where $F(.)$, $G(.)$, $H(.)$ are the right-hand sides of Eqs. (28)–(30) respectively, and the transversality condition is $\text{Lim}_{t\to\infty} e^{-\int_0^t [R(\pi(\lambda,\eta,k))-\pi(\lambda,\eta,k)]\,\mathrm{d}s}[a(\lambda, \eta, k) + k] = 0$

Determinacy of equilibrium

In the steady state, $R(\pi^*) = \rho+\pi^*$, $\eta^* = 1+\rho+\pi^*$, $f'(k^*) = \rho(1+\rho+\pi^*)$, $\lambda^* = \frac{U'(c^*)}{1+R(\pi^*)}$ and $\mu^* = U'(c^*)$.[5] Linear approximation of the dynamic system near the steady state (λ^*, η^*, k^*) is obtained through the system

$$\begin{bmatrix}\dot{\lambda}\\ \dot{\eta}\\ \dot{k}\end{bmatrix} = \left[\underbrace{\begin{matrix} 0 & -\frac{U'(c^*)}{R(\pi^*)}\frac{R'(\pi^*)-1}{R'(\pi^*)} & 0\\ 0 & \rho + R(\pi^*)\frac{R'(\pi^*)-1}{R'(\pi^*)} & -f''(k^*)\\ -\frac{R(\pi^*)}{U''(c^*)} & \frac{1}{\psi_A(c^*)R(\pi^*)} & \rho R(\pi^*)\end{matrix}}_{C}\right]\begin{bmatrix}\lambda-\lambda^*\\ \eta-\eta^*\\ k-k^*\end{bmatrix}$$

where $\psi_A(c^*)$ is the measure of absolute risk aversion in the steady state. Let $\theta_i (i = 1, 2, 3)$ denote the eigenvalues of matrix C, then

$$\theta_1\theta_2\theta_3 = \frac{f''(k^*)[R'(\pi^*) - 1]}{\psi_A(c^*)R'(\pi^*)} \tag{32}$$

$$\theta_1 + \theta_2 + \theta_3 = \frac{R'(\pi^*) - 1}{R'(\pi^*)}(1 + \rho + \pi^*) + \rho(\rho + \pi^* + 2) \tag{33}$$

Proposition 2 *When investment is financially constrained, neutral or active monetary policy stances are sufficient to ensure real determinacy. A passive monetary policy stance such that* $R'(\pi^*) \leq \frac{1}{1+2\rho}$ *is sufficient to ensure real indeterminacy. If monetary policy is "mildly passive", that is* $\frac{1}{1+2\rho} < R'(\pi^*) < 1$, *the nature of equilibrium is ambiguous. However, as long as the monetary policy is "mildly passive", if equilibrium is determinate, the only perfect-foresight equilibrium in which real allocation converges to the steady state is the steady state itself.*

(See Appendix for the proof.)

[5] In the CIA constrained investment economy, the steady state stock of capital is lower than the steady state stock of capital in Sect. 2. This result is in the spirit of Stockman (1981).

Qualitatively, the results for the two cases are similar. In both cases an active or neutral monetary policy is a sufficient condition for equilibrium determinacy whereas a passive monetary policy induces equilibrium indeterminacy. Furthermore, this result is robust to specification of preferences.

4 Equilibria with capital accumulation and externalities

In previous sections capital induced macroeconomic instability under forward looking interest rate rule through affecting household arbitrage activity in asset markets. This section discusses the role of capital as a source of macroeconomic instability through factor markets and aggregate output.

Huang and Meng (2007) show that capital matters for determinacy as its share in value-added production cost affects firms' pricing behavior in the product markets and thus affects the New Phillips curve. The New Phillips curve, a central element in the New-Keynesian literature, crucially depends on the degree of price rigidity. Clearly, these features of the macroeconomy disappear once the degree of price rigidity approaches zero.[6] Thus, the pricing channel found in the recent New-Keynesian models is absent from the Neoclassical-flexible price setting. It should be noted that the economic story within the New-Keynesian models which gives rise to the New-Phillips curve is fundamentally an aggregate-demand externality. Blanchard and Kiyotaki (1987) emphasize this externality in models of monopolistic competitions. Thus, although aggregate-demand externalities are absent from the Neoclassical paradigm, other types of externalities can be incorporated so as to examine directly the role of capital as a source of macroeconomic instability in the present context. In this section all assumptions that allow for arbitrage in the asset markets are overlooked. Instead, it is assumed that the production function exhibits a positive externality with respect to the per capita capital stock. The discussion starts with a model with no-arbitrage and no-externalities. Then, production externalities are added to the model to examine the role of capital as a source of indeterminacies.

The representative household's lifetime utility function is $U = \text{Max} \int_0^\infty e^{-pt} u(c)\, \mathrm{d}t$, Capital is used for production. Adjustments to the stock of capital are performed instantly, at no cost. The household's budget constraint is therefore

$$c + \dot{k} + \dot{b} + \dot{m} = (R - \pi)b - \pi m + f(k) - \tau \tag{34}$$

The production function, $f(k)$, is twice differentiable, strictly increasing and strictly concave and at this point it is assumed that $f(k)$ displays no externalities. $w \equiv b + m + k$ denotes real wealth. Thus, the household's budget constraint can be written as:

$$\dot{w} = (R - \pi)w - Rm - (R - \pi)k + f(k) - c - \tau \tag{35}$$

Household consumption is then subject to a cash-in-advance constraint, $c \leq m$ and as in previous sections, the constraint is binding near the positive-nominal-interest-rate

[6] Equation (18) in Benhabib and Eusepi (2005) demonstrate this characteristic of the New-Phillips curve.

steady state. Along the equilibrium trajectory characterized by $c = m$ the household's lifetime maximization problem is

$$\text{Max} \int_0^\infty e^{-\rho t}\, U(c)\, \mathrm{d}t$$
s.t.
$$\dot{w} = (R-\pi)w - (R-\pi)k + f(k) - c(1+R) - \tau$$

and the no-Ponzi-game condition $\text{Lim}_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\,\mathrm{d}s}\, w(t) = 0$.
The household chooses sequences of $\{c, k\}$ so as to maximize its lifetime utility, taking as given the initial stock of capital $k(0)$, and the time path of τ, R, and π.

Let the current value hamiltonian be defined as $H_3 \equiv U(c) + \lambda[(R-\pi)w - (R-\pi)k + f(k) - c(1+R) - \tau]$; hence, the optimality conditions associated with the household's problem are

$$\lambda = \frac{U'(c)}{1+R} \tag{36}$$

$$\dot{\lambda} = \lambda[\rho + \pi - R] \tag{37}$$

$$f'(k) = R - \pi \tag{38}$$

$$\dot{k} = f(k) - c \tag{39}$$

The Government Equations (14) and (15) describe the government's budget constraint.

Equilibrium dynamics

Note (38). Accordingly,

$$f'(k) + \pi = R(\pi) \tag{40}$$

Thus, $\pi = \pi(k)$. Deriving both sides of (40) with respect to k yields $\pi_k = \frac{f''(k)}{R'(\pi)-1}$. It follows from (36) that $c = c(\lambda, k)$; $c_\lambda = \frac{1+R(\pi(k))}{U''(c)}$; $c_k = \frac{\lambda R'(\pi(k))}{U''(c)} \frac{f''(k)}{R'(\pi)-1}$. Thus the dynamics of all the variables in the economy can be described through the dynamics of (λ, k) that are given by the system

$$\dot{\lambda} = \lambda[\rho - f'(k)]$$
$$\dot{k} = f(k) - c(\lambda, k)$$
$$\dot{w} = (R(\pi(k)) - \pi(k))w - (R(\pi(k)) - \pi(k))k + f(k) - c(\lambda, k)(1 + R(\pi(k))) - \tau$$

and the transversality condition $\text{Lim}_{t\to\infty} e^{-\int_0^t [R(\pi(k))-\pi(k)]\,\mathrm{d}s}\, w = 0$. Linear approximation of the dynamic system near the steady state (λ^*, k^*) is obtained through the system

$$\begin{bmatrix} \dot{\lambda} \\ \dot{k} \end{bmatrix} = \left[\underbrace{\begin{matrix} 0 & -\lambda^* f''(k^*) \\ -c_\lambda(\lambda^*, k^*) & f'(k^*) - c_k(\lambda^*, k^*) \end{matrix}}_{D} \right] \begin{bmatrix} \lambda - \lambda^* \\ k - k^* \end{bmatrix}.$$

Proposition 3 *In a CIA economy with a neoclassical investment technology, the equilibrium displays real determinacy regardless of the monetary policy stance.*

Proof With the underlying assumptions, we obtain $\det(D) = \frac{f''(k^*)}{\psi_A(c^*)} < 0$. This implies that the jacobian matrix D has two eigenvalues of opposite signs. There is one predetermined state variable (the stock of capital). Thus the steady state is a saddle point regardless of the stance of monetary policy. This result was obtained in Meng and Yip (2004) with a money-in-the-utility model, implying that unlike the results obtained in previous sections the equilibrium displays real determinacy whether the monetary policy is "active" or "passive" whenever households engage in arbitrage activity. The effects of arbitrage activity, demonstrated by the no-arbitrage Eq. (38), on the regions of indeterminacy was extensively discussed in Meng and Yip (2004) for the Neoclassical case and Carlstrom and Fuerst (2005) for the New-Keynesian case.

To examine explicitly the role of capital in a Neo-Classical model as a source of indeterminacies, consider an economy with a production externality. Following Kehoe et al. (1992) and Rebelo and Xie (1999), suppose that the production function, $f(k, k_a)$, exhibits a positive externality where k_a is the per capita capital stock in the whole economy. The production function $f(k, k_a)$ is strictly increasing in both arguments and concave in k and continuously differentiable. To simplify the analysis it is assumed that $f(k, k_a) \equiv A(k_a)k^\alpha$ with $0 < \alpha < 1. A(k_a)$ describes total factor productivity and exhibits spillovers that increase with per capita capital.α measures the share of capital in production as we assumed throughout that labor is supplied elastically. With these assumptions Proposition 3 no longer obtains. The first order conditions associated with the household's problem become

$$\lambda = \frac{U'(c)}{1+R} \tag{41}$$

$$\dot{\lambda} = \lambda[\rho + \pi - R] \tag{42}$$

$$A(k)\alpha k^{\alpha-1} = R - \pi \tag{43}$$

$$\dot{k} = A(k)k^\alpha - c \tag{44}$$

where the equilibrium condition $k_a = k$ is substituted into Eqs. (43) and (44) only after the derivative of H_3 with respect to k is taken. Even so, consumption per capita and the rate of inflation are set in general equilibrium. Furthermore, the stance of monetary policy is set so as so obtain an efficient solution to the central planner's problem.[7]

[7] We use the local stability analysis of the steady to demonstrate how the central planner chooses the efficient policy stance.

Accordingly, these magnitudes are derived as if the central planner internalizes the externality:
If follows from (43) that $\pi_k = \frac{A'(k)\alpha k^{\alpha-1}+A(k)\alpha(\alpha-1)k^{\alpha-2}}{R'(\pi)-1}$ and from (41) that

$$U'(c(\lambda,k)) = \lambda(1+R(\pi(k)) \Rightarrow c_\lambda = \frac{1+R(\pi(k))}{U''(c)};$$

$$c_k = \frac{\lambda R'(\pi(k))}{U''(c)} \frac{A'(k)\alpha k^{\alpha-1} + A(k)\alpha(\alpha-1)k^{\alpha-2}}{R'(\pi)-1}$$

Thus, the linear approximation of the dynamic system near the steady state (λ^*, k^*) at the economy with spillover externalities is obtained through the system

$$\begin{bmatrix} \dot{\lambda} \\ \dot{k} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\left[A'(k^*)\alpha k^{*\alpha-1} + A(k^*)\alpha(\alpha-1)k^{*\alpha-2}\right]\frac{U'(c^*)}{1+R(\pi^*)} \\ -\frac{1+R(\pi^*)}{U''(c*)} & \rho - \frac{U'(c^*)}{1+R(\pi^*)}\frac{R'(\pi^*)}{U''(c^*)}\frac{A'(k^*)\alpha k^{*\alpha-1}+A(k^*)\alpha(\alpha-1)k^{*\alpha-2}}{R'(\pi^*)-1} \end{bmatrix}}_{E} \begin{bmatrix} \lambda-\lambda^* \\ k-k^* \end{bmatrix} \tag{45}$$

Note that $\text{Det}(E) = \frac{\rho}{\psi_A(c*)}\left[\frac{A'(k^*)}{A(k^*)} + \frac{\alpha-1}{k^*}\right]$ and for some specifications of the externality, A(k), the expression in brackets can be positive. To further simplify the analysis, it is assumed that the externality has the form $A(k) \equiv Ak^{1-\alpha+\varepsilon}$. In this case $\forall\varepsilon \geq 0$ $\text{Det}(E) = \frac{\rho}{\psi_A(c^*)}\frac{\varepsilon}{k^*} \geq 0$. If Det($E$) equals zero, one eigenvalue is zero which implies that with one predetermined variable the steady state is determinate. However, when Det(E) is positive, the real part of the two eigenvalues have the same sign. Therefore, in order to ensure determinacy, the monetary policy should be conducted so that the eigenvalues of the dynamic system turn out to be with positive real parts. In that case the steady state is an unstable node and the only trajectory that converges to the steady state is the steady state itself. Proposition 4 present the necessary and sufficient condition for local real determinacy for the spillover-externalities model.

Proposition 4 *When consumption is cash-in-advance constrained and the production function, $f(k,k_a) \equiv A(k_a)k^\alpha$ exhibits a positive externality where k_a is the per capita capital stock in the whole economy and the TFP coefficient, $A(k) \equiv Ak^{1-\alpha+\varepsilon}$, exhibits spillovers that increase with per capita capital, the stance of monetary policy that ensure determinacy depends on the measure of ε. If $\varepsilon \leq 0$, equilibrium displays real determinacy regardless of the monetary policy stance. If $\varepsilon > 0$, equilibrium displays real determinacy if and only if $R'(\pi^*) > 1$ or $R'(\pi^*) < \frac{1}{1+\frac{\varepsilon}{k^*\psi_A(c^*)(1+R(\pi^*))}}$ and the only perfect-foresight equilibrium in which real allocation converges to the steady state is the steady state itself.*

(See Appendix for the proof.)

5 Main results

Meng and Yip (2004) show that in a money-in-the-utility setup with a neoclassical investment technology, determinacy of equilibrium is ensured regardless of the monetary policy stance. However, the characteristics of the neoclassical economy fade once we depart from neoclassical assumptions. Here, the departure is offered in two different ways: first, by assuming frictions in investment, and second, by assuming that the production function exhibits a positive externality that increase with per capita capital.

With frictions in investment, indeterminacies arise under the passive monetary stance, whereas the neutral and active stances ensure real determinacy: consider an economy where investment bears adjustment costs.[8] Suppose that the economy shifts from the steady state as a result of a positive shock to productivity. In terms of the model, the stock of capital is now below its steady state level, and the marginal product of capital is higher than its steady state level. The nominal interest rate would consequently rise because initially, the real interest rate has increased. In order to finance the increase in payments following the rise of the real interest rate, inflation tax revenues must increase which in turn further increases the nominal interest rate.

At the next instant, the stance of the monetary authority is carried out in the open market, subject to the government's budget constraint. Under the passive (active) stance, the central bank increases the rate of money (bond) creation relative to the rate effective prior to the shock, thus driving the real interest rate below (above) its steady state level.

Note the first-order condition of the household optimization problem, $\dot{\lambda} = \lambda[\rho + \pi - R]$; in equilibrium, λ measures the nominal-interest-rate-distorted marginal utility of consumption. Under the active stance the real interest rate is above its steady state level. Thus, the expression in the brackets is negative meaning that $\frac{\dot{\lambda}}{\lambda} < 0$. Similarly, under a neutral monetary policy $\frac{\dot{\lambda}}{\lambda} = 0$, and under a passive policy $\frac{\dot{\lambda}}{\lambda} > 0$. Hence, when off the steady state, an active stance induces negative growth rate in the marginal utility of consumption, which implies that the active policy stance drives the households to increase consumption. This implies eating away the capital stock, which further distances the economy from the steady state. A neutral policy induces a zero growth rate in the marginal utility of consumption, which implies that households are driven to consume at the steady state level which also implies eating away the capital stock.

A passive monetary stance induces positive growth in the marginal utility of consumption, which implies that this policy drives households to decrease consumption and consequently accumulate capital. However, the presence of adjustment costs and absolute risk aversion exert diametric effects on capital accumulation. To induce an intertemporal rate of substitution sufficiently high to motivate capital accumulation, the real interest rate needs to be below a respective threshold. With sufficiently low real

[8] Focusing on this economy causes no loss of generality relative to the model with financially constrained investment. The mechanism that brings about macroeconomic instability is similar in these cases as demonstrated in Sects. 2 and 3.

yields, households reduce consumption and accumulate capital. As a result, the capital stock returns to its steady state level and any trajectory is consistent with equilibrium. When the no-arbitrage equation does not obtain, as a result of frictions in investing, the central bank should operate according to the non-passive stance. That is, $R'(\pi)$ should exceed $\frac{1}{1+\rho}$ and $\frac{1}{1+2\rho}$ for the adjustment cost case and the financially constrained investment, respectively. Thus, for plausible values of ρ, $R'(\pi) > 0.91$ is a necessary condition to avoid indeterminacies but not sufficient. $R'(\pi^*) > 1$ is sufficient for both cases.

In the economy with spillover externalities an active policy ensures determinacy, whereas the effects of a passive policy achieve almost the opposite effect relative to its affect over the economy with constrained investment. To reveal the economic intuition we turn, for the purpose of exposition, to a simple model of the Phillips curve. Note that in the economy with no frictions in investment $\pi = \pi(k)$. This follows directly from the no-arbitrage Eq. (40). Thus, linearizing π near the steady state yield $\pi(k) \cong \pi(k^*) + \pi_k(k^*)(k - k^*)$, also note that in the economy with spillover externalities $\pi_k = \frac{A'(k)\alpha k^{\alpha-1} + A(k)\alpha(\alpha-1)k^{\alpha-2}}{R'(\pi)-1}$ where $A(k) \equiv Ak^{1-\alpha+\varepsilon}$. Thus, substituting in k^* and rearranging we obtain that in the economy with production externalities

$$\hat{\pi} = \frac{\rho\varepsilon}{R'(\pi^*) - 1}\hat{k} \tag{46}$$

where $\hat{\pi} \equiv \pi - \pi^*$, $\hat{k} \equiv \frac{k-k^*}{k^*}$ are percentage deviations of the variables from their steady state values. Equation (46) resembles a Phillips-curve and it prevails only when aggregate output displays increasing returns to scale and the central bank follows an active stance. Here marginal costs depend positively on the interest rate because capital is paid its marginal product, and, in the households' budget financial assets are perfect substitutes to capital. Thus, an active stance, such that increases the real rate of interest when inflation exceeds its steady state value, has also a positive effect over firms' marginal costs. Capital here matters for determinacy through how its per capita level affects firms' productivity, and therefore, aggregate output. It should be noted that the positive relation between output and inflation prevails when monetary policy is active. This relation ceases to exist when either ε approaches zero or when the monetary policy is passive. Furthermore, Proposition 4 implies that when ε approaches zero Eq. (46) no longer obtains and real determinacy is ensured regardless of monetary policy stance. This result is in accordance with Proposition 3.

6 Conclusion

The present paper addresses the issue of interest rate feedback rules by deriving the necessary and sufficient conditions for local real determinacy in a continuous-time economy with perfect competition and endogenous capital accumulation, where the monetary authority sets a nominal interest rate as a function of the instantaneous rate of inflation. The instantaneous rate of inflation in a continuous-time setting is the right-derivative of the logged price level and is the counterpart of a discrete-time one-period forward-looking policy that responds to expected future inflation. Thus, the policy considered here is in one line with the forward-looking policy considered by Carlstrom and Fuerst (2005) and Huang and Meng (2007).

This paper examines two different channels through which capital can potentially become the source of macroeconomic instability. The first channel is the arbitrage channel found in economies with frictions in investing. In such economies arbitrage activity of the households is ineffective. As a result, the central bank is incapable of controling the stock of capital by setting its marginal product. The second channel is a Neoclassical counterpart of the pricing channel discussed in Huang and Meng (2007) and in Benhabib and Eusepi (2005). In the present paper, production externalities mimic the role of aggregate-demand externalities, as a channel throughwhich capital may become a source of multiple equiliobria.

In the economy with convex costs of adjustment or with financially constrained investment, monetary policy can ensure equilibrium determinacy by adopting an active or a neutral stance. A mildly passive stance, that is a response within the passive region but above a respective threshold, induces ambiguous results. An interest rate feed back rule such that the elasticity of nominal interest rate is below the respective threshold, induces equilibrium indeterminacy. For a wide range of parameters this threshold is at least 0.91.

In the economy with production externalities where the total factor productivity exhibits spillovers that increase with per capita capital, results show that monetary policy can ensure equilibrium determinacy by adopting an active stance. However, in this case, a mildly passive stance usually induces real indeterminacy, whereas a strictly passive rule induces real determinacy. In view of the contradicting results for the passive policy stance, it is recommended to take on the active monetary policy stance. In view of recent developments, it is also recommended to elaborate this result to an interest rate rule that incorporates a response to movements in current output in addition to movements in instantaneous inflation. I leave this analysis to a separate paper.

Appendix

Proof of Proposition 2 *Consider an active stance* Note the right hand side of Eq. (32). When monetary policy is active, $\theta_1 \theta_2 \theta_3 < 0$ which imply that either there is one negative eigenvalue and two eigenvalues with positive real parts, or all three eigenvalues are negative. Note Eq. (33). Under an active stance $\theta_1 + \theta_2 + \theta_3 > 0$ which rules out the possibility that all the eigenvalues are negative. With one negative eigenvalue and one predetermined state variable the dynamic equilibrium is unique.
Consider a neutral stance The time path of all the variables in the economy is reduced to a span of $\{\eta, k\}$. The explanation is detailed in the proof of Proposition 1. The local dynamics of the economy under a neutral monetary policy is described by a two dimensional system in the $\{\eta, k\}$ space. Under the neutral stance $\mathrm{Det}(B) = \rho^2(1 + \rho + \pi^*) + \frac{f''(k^*)}{\Psi_A(c^*)(1+\rho+\pi^*)}$ and $\mathrm{Tr}(B) = \rho(2+\rho+\pi^*) > 0$. If Det(*B)* is negative, then the steady state is a saddle point. If Det(*B)* is positive, than the positive trace implies that the eigenvalues are both positive and that the steady state is an unstable node.

Consider the passive stance The passive policy implies that $\theta_1\,\theta_2\,\theta_3 > 0$. The product of eigenvalues is positive if either there are two eigenvalue with negative real parts and one positive eigenvalue, or if all three eigenvalues are positive.

If there are two eigenvalues with negative real parts and one positive eigenvalue, then with only one predetermined state variable the equilibrium is indeterminate. A necessary condition that the underlying equilibrium is determinate is that the trace of the jacobian matrix is positive. Note Eq. (33). Rearranging the right hand side of Eq. (33) yields that the necessary condition is

$$R'(\pi^*) > \frac{1+\rho+\pi^*}{(1+\rho+\pi^*)+\rho(\rho+2+\pi^*)} \Rightarrow R'(\pi^*) > \frac{1}{1+\frac{\rho(\rho+2+\pi^*)}{(1+\rho+\pi^*)}}$$
$$\approx \frac{1}{1+\frac{2\rho}{(1+\rho+\pi^*)}} > \frac{1}{1+2\rho}.$$

When all eigenvalues are positive, the steady state is an unstable node. □

Proof of Proposition 4 $\mathrm{Det}(E) = \frac{\rho}{\psi_A(c*)}\frac{\varepsilon}{k*}$. Therefore, If $\varepsilon \leq 0$, Det(E) is non positive and with one predetermined variable the equilibrium displays real determinacy regardless of the monetary policy stance. If $\varepsilon > 0$ Det(E) is positive. Here, the role of monetary policy is to make sure that the real-part of the eigenvalues assumes a positive sign. We therefore require that the magnitude of response to instantaneous inflation is such that $Tr(E) = \rho - \frac{U'(c^*)}{1+R(\pi^*)}\frac{R'(\pi^*)}{U''(c*)}\frac{A'(k^*)\alpha k^{*\alpha-1}+A(k^*)\alpha(\alpha-1)k^{*\alpha-2}}{R'(\pi^*)-1} > 0$ further rearranging the inequality yields the condition $1+\frac{1}{\psi_A(c^*)[1+R(\pi^*)]}\frac{R'(\pi^*)}{R'(\pi^*)-1}\frac{\varepsilon}{k^*} > 0$.

This condition is fulfilled when $R'(\pi^*) > 1$ or when $R'(\pi^*) < \frac{1}{1+\frac{\varepsilon}{k^*\,\psi_A(c^*)(1+R(\pi^*))}}$. In both cases Det ($E$) and Trace($E$) are positive which implies that both eigenvalues have positive real parts. □

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